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Drift criteria for persistence of discrete stochastic processes on the line

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Abstract

We provide sufficient conditions for the persistence or transience of stochastic processes on the line based on the sign of the conditional drift. Our findings extend previous results in the literature to the large class of discrete time processes with bounded increments.

Keywords: Discrete-time stochastic processes; asymptotic behavior; persistence and transience.

JEL Classification: C02

1 Introduction

Consider a discrete process on the real line $\{x_t\}$ and let $(P, \Sigma, \Im)$ be its underlying filtered probability space where Σ is a subset of sequences of real numbers σ, $\{\Im_t\}$ is a filtration of Borel σ-fields on Σ, and P is the associated probability measure. We are interested to investigate whether, in the long run, the process persistently visit a finite set or, alternatively, it diverges to infinity. In particular, we want to investigate if the asymptotic sign of the conditional drift allows us to decide between to the two alternatives. The results we present concerns specifically the following[1]

Definition 1.1. A stochastic process $\{x_t\}$ is *persistent* if there exists a real interval $A = (a, b)$ such that for any t it is $\text{Prob}\{x_{t'} \in A \text{ for some } t' > t\} = 1$. If a process is not persistent it is *transient*.

A process is persistent when there exists a recurrent set A, a set that is visited in finite time with full probability. If a process is not persistent, then there is a positive probability that $\lim_{t\to\infty} |x_t| = +\infty$. Notice that in general it is not sufficient to characterize the supremum or infimum limit of a process to know if it is persistent or transient. In fact, according to our definition, all the following processes are persistent

[1] In what follows, the expression *almost surely* (a.s.) means "a part from a set of histories of measure zero with respect to P".

- a convergent process, $\text{Prob}\{\lim_{t\to\infty} x_t = x^*\} = 1$ with x^* finite; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t = x^*\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t = x^*\} = 1$, any set A that contains x^* can be used to show that the definition applies;

- a process for which there exists an A and a T such that $\text{Prob}\{x_t \in A \,\text{when}\, t > T\} = 1$; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t \leq \sup A\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t \geq \inf A\} = 1$;

- the symmetric random walk on the line, $x_{t+1} = x_t + b_t$ where b_t is a Bernoulli variable taking values 1 and -1 with the same probability; in this case any set A (with $b - a > 1$) can be used since $\text{Prob}\{\limsup_{t\to\infty} x_t = +\infty\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t = -\infty\} = 1$;

- a sub-martingale bounded from above; in this case the Martingale Converge Theorem guarantees that the process converges to a finite random variable $\hat{x}$.

Examples of transient processes are instead

- the asymmetric random walk on the line, $x_{t+1} = x_t + b_t$ where b_t is a Bernoulli variable taking values 1 and -1 with different probabilities; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t = \liminf_{t\to\infty} x_t\} = 1$ and the limits are $\pm\infty$ depending on the sign of the drift;

- the process with exploding increments $x_{t+1} = x_t + 2^t b_t$, where b_t is a Bernoulli variable taking values 1 and -1 with fixed probabilities; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t = \liminf_{t\to\infty} x_t\} = 0$ but $\text{Prob}\{\limsup_{t\to\infty} |x_t| = +\infty\} = 1$.

The last process, with exploding fluctuations, is somehow bizarre and we are happy to constraint our analysis to processes that comply with the following

Definition 1.2. A process $\{x_t\}$ has *bounded increments* if there exists a $B > 0$ such that $\text{Prob}\{|x_{t+1} - x_t| < B\} = 1$.

Definition 1.1 is different from the one provided in Lamperti (1960) for a recurrent process. The reason is that there only processes which are positive, $x_t \geq 0$, and such that $\text{Prob}\{\limsup_{t\to\infty} x_t = +\infty\} = 1$, are considered. While the two definitions are similar in spirit, the scope of the present investigation is larger as it includes general real processes.

2 Persistent processes

Let $\mu_t(x) = \text{E}[x_{t+1}|x_t = x, \Im_t] - x$ be the drift of the process in x, that is, conditional on the event $\{x_t = x\}$. The first result clarifies that a process that has a positive drift for sufficiently small realizations is bounded away from minus infinity.

Theorem 2.1. *Consider a bounded increments process x_t. If there exist $M > 0$ and $\epsilon > 0$ such that, for all $x < -M$ and t, $\mu_t(x) > \epsilon$ a.s., then* $\text{Prob}\{\limsup_{t\to\infty} x_t > -M\} = 1$.

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > B$. For any fixed integer T consider the process

$$Y_t^T = \begin{cases} x_{T+t} & \text{if} \quad x_l < -M \quad \text{for} \quad T \leq l \leq T+t-1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The state 0 is clearly absorbing, so that if $Y_t^T = 0$, then $Y_{t+1}^T = Y_t^T = 0$. If $Y_t^T < -M$ then $x_{T+t} = Y_t^T < -M$ and consequently $Y_{t+1}^T = x_{T+t+1} < 0$ almost surely. Let $I(.)$ be the indicator function, that is $I(x)$ is equal to 1 if $x > 0$ and 0 otherwise. On the events such that $Y_t^T = X_{T+t}$ one has

$$\mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] = \mathrm{E}[I(-x_{T+t+1} - M)\, x_{T+t+1} | x_{T+t}, \Im_t] \geq \mathrm{E}[x_{T+t+1} | x_{T+t} < -M, \Im_t].$$

The latter is greater than $x_{T+t} = Y_t^T$ by the assumption on the drift. In general it is thus $\mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] \geq Y_t^T$ and the process Y_t^T is a sub-martingale bounded from above by 0. By the Martingale Convergence Theorem there exists a finite random variable $\hat{Y}^T$ such that $\lim_{t\to\infty} Y_t^T = \hat{Y}^T$ almost surely.

Assume that for some T it is $\hat{Y}^T < -M$ with positive probability. Then on a positive measure set of realizations it would be $\{Y_t^T\} = \{x_{t+T}\}$ and $\lim_{t\to\infty} x_{T+t} = \hat{Y}^T < -M$. The latter is absurd given the strictly positive drift of the process when $x < -M$. It follows that for any T it is $\hat{Y}^T = 0$ with probability 1. This implies that for any T there exists a t such that $x_{T+t} > -M$ a.s. and proves the assertion. □

The previous result proves that the event $x_t > -M$ is recurrent. Along the same lines one can easily prove the opposite: a process with negative drift for sufficiently large realizations is bounded away from plus infinity.

Corollary 2.1. *Consider a bounded increments process x_t. If there exist $M > B$ and $\epsilon > 0$ such that, a.s., $\mu_t(x) < -\epsilon$ for all $x > M$, then* $\mathrm{Prob}\{\liminf_{t\to\infty} x_t < M\} = 1$.

Together Theorem 2.1 and Corollary 2.1 provide a sufficient condition for persistence.

Theorem 2.2. *Consider a bounded increments process x_t. If there exist $M > 0$ and $\epsilon > 0$ such that, for any t and almost surely, it is $\mu_t(x) < -\epsilon$ if $x > M$ and $\mu_t(x) > \epsilon$ if $x < -M$, then the process is persistent.*

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > B$. For any positive integer T define the process

$$Y_t^T = \begin{cases} |x_{T+t}| & \text{if} \quad |x_l| > M \quad \text{for} \quad T \leq l \leq T+t-1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The state 0 is clearly absorbing so that if $Y_t^T = 0$, then $Y_{t+1}^T = Y_t^T = 0$. Let $I(.)$ be the indicator function, that is $I(x)$ is equal to 1 if $x > 0$ and 0 otherwise. If $Y_t^T > M$, then either $x_{T+t} > M$ or $x_{T+t} < -M$. In the first case it is $x_{T+t+1} > 0$ almost surely and on the events such that $Y_t^T = x_{T+t}$ one has

$$\begin{aligned} \mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] &= \mathrm{E}[I(x_{T+t+1} - M)\, x_{T+t+1} | x_{T+t}, \Im_t] \\ &\leq \mathrm{E}[x_{T+t+1} | x_{T+t}, \Im_t] < x_{T+t} = Y_t^T\ . \end{aligned}$$

In the second case it is $x_{T+t+1} < 0$ almost surely and on the events such that $Y_t^T = x_{T+t}$ one has

$$\begin{aligned} \mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] &= -\,\mathrm{E}[I(-x_{T+t+1} - M)\, x_{T+t+1} | x_{T+t}, \Im_t] \\ &\leq -\,\mathrm{E}[x_{T+t+1} | x_{T+t}, \Im_t] < -x_{T+t} = Y_t^T\ . \end{aligned}$$

Summarizing, it is always the case that $\mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] \leq Y_t^T$. Hence Y_t^T is a super-martingale bounded from below by 0 and by the Martingale Convergence Theorem there exists a random variable $\hat{Y}^T$, such that $\lim_{t\to\infty} Y_t^T = \hat{Y}^T$ almost surely.

If for some T it is $\hat{Y}^T > M$ with finite probability, then on a positive measure set of realizations it would be $\lim_{t\to\infty} |x|_{T+t} = \hat{Y}^T$. The latter is absurd given the non-zero drift of the process in those regions. Thus $\hat{Y}^T = 0$ with probability 1. This means that for any T there exists a t such that $y_{T+t} \in (-M, M)$ and the assertion is proved. □

3 Transient processes

Having derived sufficient conditions for the process to be persistent, we want to do the same for transience. For this purpose, we need to restrict our investigation to processes that do not have absorbing points.

Definition 3.1. A process $\{x_t\}$ has *finite positive increments* if there exists a $\epsilon_L > 0$ such that a.s. $\mathrm{Prob}\{x_{t+1} > x_t + \epsilon_L\} > \epsilon_L$. A process $\{x_t\}$ has *finite negative increments* if there exists a $\epsilon_L > 0$ such that a.s. $\mathrm{Prob}\{x_{t+1} < x_t - \epsilon_L\} > \epsilon_L$.

For the present discussion, the essential feature of a process with finite positive increments is that its asymptotic supremum (limsup) cannot be a finite number, apart, possibly, for a zero-measure set of realizations. In the case of a process with finite negative increments, the same is true for the asymptotic infimum (liminf).

The following result shows that a bounded process with finite positive increments and positive drift outside a finite set diverges to positive infinity and, consequently, is transient.

Theorem 3.1. *Consider a bounded increments process x_t with finite positive increments. If there exist $M > 0$ and $\epsilon > 0$ such that, for any t, it is $\mu_t(x) > \epsilon$ a.s. if $x > M$ or $x < -M$, then the process is transient and* $\mathrm{Prob}\{\lim_{t\to\infty} x_t = +\infty\} = 1$.

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > \max\{B, 1\}$. Since the process satisfies the condition of Th. 2.1, $\mathrm{Prob}\{\limsup_{t\to\infty} x_t > -M\} = 1$. Together with the fact that the process has finite positive increments, the latter implies that $\limsup_{t\to\infty} x_t = +\infty$.

Let $F_{x,t}(u)$ denote the conditional distribution of the increment $u_t = x_{t+1} - x_t$ if $x_t = x$ and $\mu_{n,t}(x)$ its n-th moment. Since the process has bounded increments, all moments are finite and, by hypothesis, $\mu_1(x) > \epsilon$ if $x > M$. Consider the process

$$Y_t = \begin{cases} 1 - \frac{1}{x_t} & \text{if} \quad x_t > M \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Since $M \geq 1$, the process $\{Y_t\}$ is bounded in $[0, 1]$. Moreover $\mathrm{Prob}\{\limsup_{t\to\infty} Y_t = 1\} = 1$. Let $K \geq 1 - 1/(M + B)$. If $Y_t > K$, then $x_t > 1/(1 - K) \geq M + B$ and, with probability one, $x_{t+1} > M$, thus

$$\mathrm{E}[Y_{t+1} - Y_t | Y_t > K, \Im_t] = \int_{-B}^{B} \mathrm{d}F_{x_t,\Im_t}(u) \, \frac{1}{x_t} - \frac{1}{x_t + u} \ .$$

Expanding the integrand around x_t with respect to u one obtains

$$\mathrm{E}[Y_{t+1} - Y_t | Y_t > K, \Im_t] = \frac{1}{x_t^2} \left(\mu_1(x_t) + h(x_t) \right)$$

where $h(x) = x^2/(x^3 + \bar{u})$ for $\bar{u}$ between zero and u, so that $\lim_{x\to\infty} h(x) = 0$.

Upon choosing K large enough, the drift condition implies that $\mathrm{E}[Y_{t+1} - Y_t | Y_t > K, \Im_t] > 0$. The process Y_t satisfies all the requirements of Theorem 2.2 in Lamperti (1960) and, consequently, $\mathrm{Prob}\left\{\lim_{t\to\infty} Y_t = 1\right\} = 1$. The assertion immediately follows. □

Along the same lines it is possible to prove the divergence to negative infinity of a process with finite negative increments and asymptotically negative drift.

Corollary 3.1. *Consider a bounded increments process with finite negative increments. If there exist $M > 0$ and $\epsilon > 0$ such that, for any t, it is $\mu_t(x) < -\epsilon$ a.s. if $x > M$ or $x < -M$, then the process is transient and* $\mathrm{Prob}\left\{\lim_{t\to\infty} x_t = -\infty\right\} = 1$.

The case of a homogeneous random walk with non-negative drift falls in one of the two previous cases.

When the drifts of the process outside a bounded set have a different sign, as long as they point away from the origin, the process is still transient as clarified by the following

Theorem 3.2. *Consider a bounded increments process x_t with positive and negative finite increments. If there exist $M > 0$ and $\epsilon > 0$ such that, for any t, it is $\mu_t(x) > \epsilon$ if $x > M$ and $\mu_t(x) < -\epsilon$ if $x < -M$ a.s., then the process is transient and either* $\lim_{t\to\infty} x_t = +\infty$ *or* $\lim_{t\to\infty} x_t = -\infty$.

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > B$. Consider any $K > M$ and the process $Y_t^K = \min\{|x_t|, K\}$. Since the process has finite increments, it is $\limsup_{t\to\infty} |x|_t = +\infty$ and consequently $\limsup_{t\to\infty} Y_t^K = K$. Then the process Y_t^K satisfies the hypothesis of Theorem 2.2 in Lamperti (1960) implying that $\lim_{t\to\infty} Y_t^K = K$. Since this is true for any sufficiently large K, we can conclude that $\lim_{t\to\infty} |x|_t = +\infty$ almost surely.

Assume that on a positive measure of trajectories it is $\liminf_{t\to\infty} x_t = -\infty$ and $\limsup_{t\to\infty} x_t = +\infty$. Then for any t for which $x_t > M$ there exists a t' fro which $x_{t'} < -M$. Since $M > B$, this implies that there is a t'' such that $t < t'' < t'$ and $x_{t''} \in [-M, M]$, that is $|x_{t''}| \in [0, M]$. But this is absurd given the previous result on the process $\{Y_t^K\}$. The statement follows directly. □

According to the previous theorem, if one defines two sets $\Sigma_{-\infty}$ and $\Sigma_{+\infty}$ of trajectories converging, respectively, to minus and plus infinity, it is $P(\Sigma_1 \cup \Sigma_2) = 1$.

Two remarks before we conclude. All the results can also be applied to diverging processes, for instance by removing an unconditional drift $\bar{\mu}$. In this case it is the sign of the relative drifts $\mu_t(x) - \bar{\mu}$ for large and small x that can be used to decide whether the trajectories of the process visit with probability one a neighborhood of $\bar{\mu}\, t$ or accumulate far away from it.

The previous analysis suggests that the qualitative behavior of a process with bounded and finite increments is equivalent to that of its diffusive limit, which can be obtained, for instance, through the Kramer-Moyal expansion of the homogeneous arrival process in which the “jumps” are not discrete in time but follow a Poisson process.

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