What Situation Is This?
Shared Frames and Collective Performance

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A Online Appendix

Throughout this appendix, we use majuscules to denote random variables and minuscules to denote their realizations.

Proposition A.1. *Under the benchmark, the expected payoff to a party when both players use their dominant strategies is* $1/6$.

Proof. When both players use their dominant strategies, the payoff to an agent is PR if $R + P > 1$, and $-PR$ if $R + P < 1$. Write this payoff as

$$V = -PR + 2PR \cdot \mathbb{1}_{\{R \geq 1-P\}}$$

Given the independence of R and P, its expectation is

$$E(V) = E(-PR) + E\left(2PR \cdot \mathbb{1}_{\{R \geq 1-P\}}\right) = -\frac{1}{4} + \frac{5}{12} = \frac{1}{6}$$

□

As discussed in the main text, the payoff matrix for a game $G(r, p)$ is

	H	L
H	pr, pr	$-(1-p)(1-r), (1-p)(1-r)$
L	$(1-p)(1-r), -(1-p)(1-r)$	$-pr, -pr$

A frame bundles several games as a single situation. We compute the expected payoffs for each strategy profile over all the games categorised in the same situation. For generality, let $K = (\alpha, \beta) \times (\gamma, \delta)$ be the cell including the games $G(R, P)$ with $r \in (\alpha, \beta)$ and $P \in (\gamma, \delta)$, where $0 \leq \alpha < \beta \leq 1$ and $0 \leq \gamma < \delta \leq 1$; see Figure A.1.

We prove that each party has a dominant choice for almost every cell $K = (\alpha, \beta) \times (\gamma, \delta)$. We begin with two lemmas characterizing best replies.

Lemma A.2. *Suppose that* i*'s opponent plays* H *over the cell* $K = (\alpha, \beta) \times (\gamma, \delta)$*. Then* i*'s best reply over* K *is* H *if and only if*

$$\alpha + \beta + \gamma + \delta \geq 2 \tag{1}$$

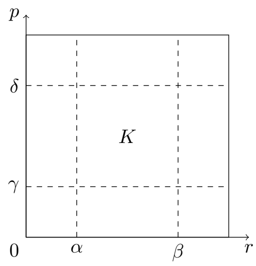


Figure A.1: A cell in the game space.

Proof. The expected payoff for i over K under the strategy profile (H, H) is $E_K\Big(PR\Big)$. Because the two uniform random variables r and P are stochastically independent, we have

$$E_K\Big(PR\Big) = E_K\Big(P\Big)\cdot E_K\Big(R\Big) = \frac{(\alpha+\beta)(\gamma+\delta)}{4}$$

Analogously, under the strategy profile LH, the expected payoff for i over K is

$$E_K\Big((1-P)(1-R)\Big) = E_K\Big(1-R\Big)\cdot E_K\Big(1-P\Big) = \frac{(2-\alpha-\beta)(2-\gamma-\delta)}{4}$$

Hence, when the opponent plays H over K, H is preferred to L if and only if

$$\frac{(\alpha+\beta)(\gamma+\delta)}{4} \geq \frac{(2-\alpha-\beta)(2-\gamma-\delta)}{4}$$

Simplifying and rearranging, we obtain the inequality in (1). □

Lemma A.3. *Suppose that i's opponent plays L over the cell $K = (\alpha, \beta)\times(\gamma, \delta)$. Then i's best reply over K is H if and only if (1) holds.*

Proof. Under the strategy profile HL, the expected payoff for i over K is $E_K\Big[-(1-P)(1-R)\Big]$. Under the strategy profile LL, the expected payoff for i over K is $E_K\Big(-PR\Big)$. Hence, H is preferred to L if and only if $E_K\Big(PR\Big) \geq E_K\Big[-(1-P)(1-R)\Big]$, which leads back to the inequality in (1). □

Proposition A.4. *Given a cell $K = (\alpha, \beta)\times(\gamma, \delta)$, let $\overline{r} = (\beta+\alpha)/2$ and $\overline{p} = (\delta+\gamma)/2$. Then i has a (strictly) dominant strategy if $\overline{r}+\overline{p} \neq 1$ and is indifferent between H and L if equality holds. Moreover, the dominant strategy is H if $\overline{r}+\overline{p} > 1$, and it is L if $\overline{r}+\overline{p} < 1$.*

Proof. If we divide by 2 the expressions on either side of the inequality in (1), the result follows immediately from Lemma A.2 and Lemma A.3. □

Proposition A.5. *Given a threshold pair $(\hat{r}, \hat{p})$, the expected payoff to each agent when they both use dominant strategies is*

$$\frac{1 - 2\hat{r}^2\hat{p}^2}{4} \qquad \textit{when } \hat{r} + \hat{p} > 1 \tag{2}$$

and

$$\frac{1 - 2\hat{r}^2 - 2\hat{p}^2 + 2\hat{r}^2\hat{p}^2}{4} \qquad \textit{when } \hat{r} + \hat{p} < 1 \tag{3}$$

Proof. Suppose $\hat{r} + \hat{p} > 1$. The rational rule of behavior is Cooperation by Default, yielding a random payoff PR in situations S_1, S_2, S_4 and $-PR$ in S_3. Therefore,

$$E(V) = P(S_1)E_{S_1}(PR) + P(S_2)E_{S_2}(PR) - P(S_3)E_{S_3}(PR) + P(S_4)E_{S_4}(PR)$$

where $P(S_i)$ denotes the probability that the situation S_i occurs. The proof of Lemma A.2 provides general expressions for $E_K(Pr)$. Substituting these and dropping hats for simplicity, we find

$$\begin{aligned} E(V) &= (1-p)(1-r)E_{S_1}(PR) + p(1-r)E_{S_2}(PR) - prE_{S_3}(PR) + (1-p)rE_{S_4}(PR) \\ &= \frac{(1-r^2)(1-p^2)}{4} + \frac{p^2(1-r^2)}{4} - \frac{r^2p^2}{4} + \frac{r^2(1-p^2)}{4} = \frac{1-2r^2p^2}{4} \end{aligned}$$

Suppose instead $\hat{r} + \hat{p} < 1$. The rational rule of behavior is Defection by Default, yielding a random payoff PR in situation S_1, and $-PR$ in situations S_2, S_3, S_4. Proceeding similarly, we find

$$\begin{aligned} E(V) &= P(S_1)E_{S_1}(PR) - P(S_2)E_{S_2}(PR) - P(S_3)E_{S_3}(PR) - P(S_4)E_{S_4}(PR) \\ &= (1-p)(1-r)E_{S_1}(PR) - p(1-r)E_{S_2}(PR) - prE_{S_3}(PR) - (1-p)rE_{S_4}(PR) \\ &= \frac{(1-p^2)(1-r^2)}{4} - \frac{p^2(1-r^2)}{4} - \frac{r^2p^2}{4} - \frac{r^2(1-p^2)}{4} = \frac{1-2r^2-2p^2+2r^2p^2}{4} \end{aligned}$$

□

Proposition A.6. *When Coordination by Default prevails, there is fog of conflict if*

$$\hat{r}^2 \cdot \hat{p}^2 > 1/6 \tag{4}$$

and fog of cooperation if the opposite (strict) inequality holds.

When Defection by Default prevails, there is fog of conflict if

$$\hat{r}^2 + \hat{p}^2 - \hat{r}^2 \cdot \hat{p}^2 > 1/6 \tag{5}$$

and fog of cooperation if the opposite (strict) inequality holds.

Proof. By Proposition A.1, the expected payoff to a party under the benchmark is 1/6. Given a threshold pair $(\hat{r}, \hat{p})$, Proposition A.5 characterizes the expected payoff to a party under the rational rule of behavior. There is fog of conflict (or cooperation) when this payoff is lower (or greater) than 1/6. If $\hat{r} + \hat{p} > 1$ and Cooperation by Default applies, the expected payoff in (2) is lower (or greater) than 1/6 when (4) (or its opposite) holds. The argument is similar using (3) when Defection by Default applies. □

Frames and fog. Beyond the special case of $\hat{r} = \hat{p} = x$ discussed in Section 3 and shown in Figure 7, Proposition A.6 characterizes which frames generate which kind of fog. Its main message is conveyed in Figure A.2. Note that the unit square in Figure A.2 differs importantly from those in Figures 4 and 6: instead of depicting the situations created by one particular frame as in the earlier figures, now Figure A.2 illustrates the space of all possible frames—each frame is determined by a threshold pair $(\hat{r}, \hat{p})$ in $(0, 1)^2$.

In Figure A.2 cooperating by default prevails when the parties' shared frame is a threshold pair $(\hat{r}, \hat{p})$ above the diagonal, while defecting by default prevails when it is below. The

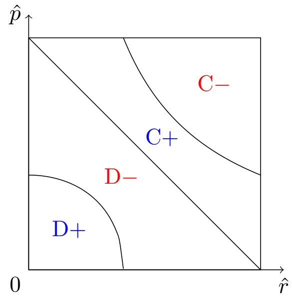


Figure A.2: Frames engendering fog of cooperation (+) or fog of conflict (−).

curve above the diagonal separates the cooperating-by-default region into the frames $(\hat{r}, \hat{p})$ generating a fog of conflict (marked −) versus those generating a fog of cooperation (marked +). Similarly, the curve below the diagonal separates the defecting-by-default region into fog of conflict (marked −) versus fog of cooperation (marked +). Consistent with the special case of $\hat{r} = \hat{p} = x$ depicted in Figure 7, moving from northeast to southwest within a rule in Figure A.2 improves payoffs continuously; on the other hand, crossing the boundary from Cooperation by Default into Defection by Default causes a discontinuous drop in payoffs.

Proposition A.7. *Suppose $x > 1/2$. Full Cooperation on (H, H) can be supported by a subgame perfect equilibrium based on a Nash-reversion strategy if and only if*

$$\delta \geq \frac{2 - 2x}{2 - 2x + x^4} \tag{6}$$

Proof. Cooperation on (H, H) is a dominant strategy for the three static situations S_1, S_2, S_4. Thus, we need to compare the short-term temptation against the long-term benefit only for S_3. Assuming $\hat{r} = \hat{p} = x$, the payoff matrix perceived by the agents in S_3 is depicted on the left of Figure A.3. (Payoffs are rescaled by a factor of 4.)

S_3	H	L
H	x^2	$-(2-x)^2$
L	$(2-x)^2$	$-x^2$

S_2	H	L
H	$(1+x)x$	$-(1-x)(2-x)$
L	$(1-x)(2-x)$	$-(1+x)x$

Figure A.3: The one-shot games associated with S_3 (left) and S_2 (right) for $\hat{r} = \hat{p} = x$.

The short-term temptation $\mathrm{ST}(S_3)$ for S_3 is the difference in payoffs from playing L instead of H in situation S_3 when the other party plays H:

$$\mathrm{ST}(S_3) = (2 - x)^2 - x^2 = 4(1 - x)$$

The long-term benefit $\mathrm{LB}(S_3)$ is the discounted sum of the (expected) incremental payoffs from sustaining cooperation in S_3. Since S_3 occurs with probability x^2, we find

$$\mathrm{LB}(S_3) = \frac{\delta}{1-\delta}\left[x^2 \cdot \left(x^2 - (-x^2)\right)\right] = \frac{2\delta x^4}{1-\delta}$$

Imposing $\mathrm{LB}(S_3) \geq \mathrm{ST}(S_3)$ gives

$$\frac{2\delta x^4}{1-\delta} \geq 4(1 - x)$$

which yields (6). □

Proposition A.8. *Suppose $x < 1/2$. Full Cooperation on (H, H) can be supported by a subgame perfect equilibrium based on a Nash-reversion strategy if and only if*

$$\delta \geq \frac{2 - 2x}{2 - 2x + 2x^2 - x^4} \tag{7}$$

Cooperation by Default on (H, H) can be supported by a subgame perfect equilibrium based *on a Nash-reversion strategy if and only if*

$$\delta \geq \frac{1-2x}{1-2x+2x^2-2x^4} \tag{8}$$

Proof. Consider Full Cooperation. Because cooperation on (H, H) is a dominant strategy only in the static situation S_1, we need to compare the short-term temptation against the long-term benefit across the other three situations S_2, S_3 and S_4. Under the assumption $\hat{r} = \hat{p} = x$, the payoffs for S_2 and S_4 are the same so we restrict attention to S_3 and S_2 in the following. Their payoff matrices are depicted above in Figure A.3.

The short-term temptations in S_3 and in S_2 (or S_4) are $\text{ST}(S_3) = 4 - 4x$ and $\text{ST}(S_2) = \text{ST}(S_4) = 2 - 4x$, respectively. Because $\text{ST}(S_3) \geq \text{ST}(S_2)$ for any x, Full Cooperation across all situations obtains if and only if the long-term benefit across S_2, S_3 and S_4

$$\text{LB}(S_2S_3S_4) = \frac{\delta}{1-\delta}\left[x^2 \cdot \left(x^2 - (-x^2)\right) + 2x(1-x) \cdot \left(x + x^2 - (-x - x^2)\right)\right] = \frac{2\delta(2x^2 - x^4)}{1-\delta}$$

from cooperation in the three situations S_2, S_3 and S_4 exceeds the (higher) temptation $\text{ST}(S_3)$. Rearranging $\text{LB}(S_2S_3S_4) \geq \text{ST}(S_3)$ yields (7).

Consider now improved cooperation, when Defection by Default in the static model is upgraded to Cooperation by Default in the repeated interaction. Because the two rules offer matching prescriptions over the two consonant situations (cooperation in S_1, defection in S_3), it suffices to check that the long-term benefit

$$\text{LB}(S_2S_4) = \frac{\delta}{1-\delta}\left[2x(1-x) \cdot \left(x + x^2 - (-x - x^2)\right)\right] = \frac{2\delta(2x^2 - 2x^4)}{1-\delta}$$

from cooperation in S_2 and S_4 exceeds the short-term temptation $\text{ST}(S_2) = \text{ST}(S_4) = 2 - 4x$ in each dissonant situation. Rearranging $\text{LB} \geq \text{ST}(S_2)$ yields (8). $\square$

Proposition A.9. *Suppose that the parties perceive any two games* $(r_1, p_1) \neq (r_2, p_2)$ *in* $\mathcal{G}$ *as distinct. Then cooperation on* (H, H) *across all games in* $\mathcal{G}$ *can be supported by a subgame perfect equilibrium based on a Nash reversion strategy if and only if*

$$\delta \geq \frac{12}{13} \tag{9}$$

Proof. Recall the payoff matrix for a game $G(r, p)$ from Figure 2. Let the *short-term temptation*

$$\text{ST}(r, p) = (1-p)(1-r) - pr = 1 - p - r$$

be the difference in payoffs from choosing L versus H when the other party plays H in the one-shot game $G(r,p)$.

Let the *long-term benefit* LB be the discounted sum of the incremental payoffs from sustaining cooperation against defection across all games in $\mathcal{G}$. Because cooperation on (H,H) is a dominant strategy for $G(r,p)$ when $r+p \geq 1$, it suffices to consider the complementary event $D^- = \{(r,p) : \ r+p<1\}$. Then

$$\mathrm{LB} = \left(\frac{\delta}{1-\delta}\right) E\left[PR - (-PR)\cdot \mathbb{1}_{D^-}\right] = \left(\frac{\delta}{1-\delta}\right) E\left[2PR\mathbb{1}_{\{R+P<1\}}\right] = \frac{1}{12}\left(\frac{\delta}{1-\delta}\right)$$

Cooperation on (H,H) can be supported across all games only if the long-term benefit LB is never smaller than the short-term temptation $\mathrm{ST}(r,p)$ for all (r,p) in D^-; that is, only if $\mathrm{LB} \geq \mathrm{ST}(r,p)$. Rearranging this expression, we find

$$\delta \geq \frac{12(1-p-r)}{1+12(1-p-r)}$$

that holds for any (r,p) in D^- if and only if (9) holds. □

5.2.3. Fictitious exemplars. We provide a simple example to illustrate the tradeoffs behind using stories as fictitious exemplars. Let the exemplar sets be $E_\ell = \{0, 0.2\}$ and $E_h = \{0.6, 1\}$, with $\bar{e}_\ell = 0.1$ and $\bar{e}_h = 0.8$. The middle ground is $[0.2, 0.6]$. The current threshold is $t = 0.45$: the followers play Defection by Default and $V(t) \approx 0.272$.

The leader is considering a story from the middle ground. We represent her uncertainty about its interpretation by assuming that the followers will eventually attribute to the story a value x that is uniformly distributed over the entire middle ground $[0.2, 0.6]$. Once the interpretation is settled, the followers behave according to the new threshold.

If the story shifts the new threshold above $1/2$, the followers switch to Cooperation by Default and achieve a radical change for the better. Otherwise, the change enacted by the story is incremental and V decreases. Given an interpretation x, the new threshold is at $t' = (1.3 + 0.5x)/3$. Therefore, $t' > 0.5$ if and only if $x > 0.4$: there is an equal chance of radical change ($x > 0.4$) or incremental change ($x < 0.4$). The expected payoff of telling a story

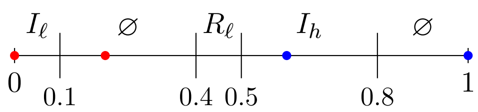
$$\frac{1}{2}\mathrm{E}\left(V(t')|x>0.4\right) + \frac{1}{2}\mathrm{E}\left(V(t')|x<0.4\right) \approx 0.516$$

is higher than the current value. So the leader finds it optimal to tell a story, even though there is a risk that the followers interpret the story differently from her intended goal.

5.2.4. Acting now or later. We provide a simple example to illustrate a real-option tradeoff when using exemplars. We use the same initial values as in the previous example, with exemplar sets be $E_\ell = \{0, 0.2\}$ and $E_h = \{0.6, 1\}$.

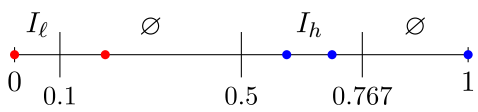
The leader can act only in the first two periods of her tenure and she can add at most one exemplar per period. In each period, exemplars are uniformly and independently distributed on $[0, 1]$. We assume that the leader can add an exemplar an exemplar $x < 1/2$ only to E_ℓ and an exemplar $x > 1/2$ only to E_h. The leader maximizes her expected discounted payoff over an infinite horizon, using the discount factor ρ.

Consider the static one-period problem. The optimal strategy for the leader depends on the value of x: if $x > 0.8$, the leader takes no action (because adding x to E_h would cause an incremental change that is detrimental); if $0.5 < x < 0.8$, she pursues incremental change and adds x to E_h; if $0.4 < x < 0.5$, she pursues radical change and adds x to E_ℓ; if if $0.1 < x < 0.4$, she takes no action; and if $0 < x < 0.1$, she pursues incremental change and add x to E_ℓ. We summarize the optimal strategy as follows, where I_k and R_k denote incremental and radical change by adding an exemplar to E_k (for $k = \ell, h$), $\varnothing$ denotes no action, and the dots represent the existing exemplars.



The expected value for this first-period strategy given the initial threshold $t = 0.45$ is $V_1 \approx 0.355$.

Consider now that infinite-horizon problem (recall that, for simplicity, the leader can act only in the first two periods). Suppose that in the first period the exemplar $x = 0.7$ becomes available: then the leader must choose whether to add it to E_h and pursue incremental change, or to ignore it and wait for another exemplar to arrive in the next period. If the leader adds $x = 0.7$ to E_h, the new threshold moves to $t' \approx 0.433$ and $V(t') \approx 0.320$. The short-term improvement in V affects the leader's optimal strategy in the second period; in particular, *there are no longer occurrences of x for which radical change is beneficial.* The optimal strategy in the second period can be summarized as follows, including the new exemplar at $x = 0.7$.



The expected value for this second-period strategy given the initial threshold $t' \approx 0.433$ is $V_2 \approx 0.326$.

Consider the two-period problem when $x = 0.7$ has occurred in the first period. If the leader adds $x = 0.7$ to E_h in the first period, this changes her optimal strategy in the second period and her discounted payoff is $0.320 + (0.326\rho)/(1 - \rho)$. If she takes no action and waits one period using the forthcoming occurrence in the best possible way, her expected discounted payoff is $0.272 + (0.355\rho)/(1 - \rho)$. She prefers to postpone action and preserve the option of radical change in the second period if $0.320 + (0.326\rho)/(1 - \rho) < 0.272 + (0.355\rho)/(1 - \rho)$; that is, if $\rho > 0.623$.