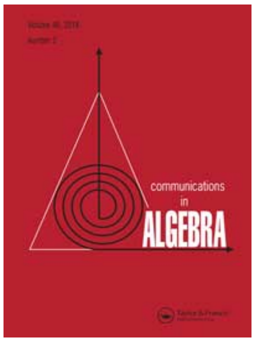

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On sharply 2-transitive groups with point stabilizer of exponent $2^n \cdot 3$

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ABSTRACT
We describe sharply 2-transitive groups whose point stabilizer is a nilpotent $\{2, 3\}$-group without elements of order 9 and, more generally, in which the third power of each element belongs to the FC-center. In particular, we will prove that these groups are finite.

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1. Introduction

Let G be a sharply 2-transitive permutation group acting on a set Ω (finite or infinite, with $|\Omega| \geq 2$), that is, G is transitive on Ω and only the identity of G fixes more than one element of Ω.

In the finite case, sharply 2-transitive groups have been classified by Zassenhaus [15], in particular, they are *split*, that is, they have always a normal abelian subgroup N which is regular on Ω.

In the infinite case the situation is more complex and recently examples were built of *non-split* sharply 2-transitive groups [11].

In some cases, imposing special conditions on the structure of a point stabilizer $G_\alpha = \{g \in G \mid g(\alpha) = \alpha\}$ of G ($\alpha \in \Omega$), it can be shown that G is split. This is the case in which every conjugacy class of G_α is finite ([6], Theorem 9.6), G_α is a 2-group [12] and G_α has exponent 3 or 6 [7]. In this note we generalize Mayr's result proving the following

Theorem 1.1. *Let G be a sharply 2-transitive permutation group on a set Ω, and let $H = G_\alpha$ be the stabilizer of an element $\alpha \in \Omega$. If H is nilpotent and has exponent $2^n \cdot 3$ with $n \geq 1$, then G is finite.*

If X is a group and $g \in X$, by $g^X = \{g^x \mid x \in X\}$, we denote the conjugacy class of g in X. We define the *FC-center* of X as the set

$$\widehat{Z}(X) = \{g \in X \mid |g^X| < \infty\},$$

which one can easily prove to be a characteristic subgroup of X. A group X is said to be a FC-group if $X = \widehat{Z}(X)$.

Theorem 1.1 is consequence of a more general result.

Theorem 1.2. *Let G be a sharply 2-transitive permutation group on a set Ω, and let $H = G_\alpha$ be the stabilizer of an element $\alpha \in \Omega$. If H is a $\{2, 3\}$-group and $H/\widehat{Z}(X)$ has exponent dividing 3, then $|\Omega| \in \{5^2, 7^2, 17^2\}$ or Ω has prime order, in particular, G is finite.*

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2. Notation and Preliminary Results

In the following, G denotes a sharply 2-transitive permutation group on a set Ω, α a fixed element of Ω and $H = G_\alpha$ the stabilizer in G of α. An element $g \in G$ is called regular if g displaces all elements of Ω or, equivalently,

$$g \in G \setminus \bigcup_{x\in G} H^x = G \setminus \bigcup_{\omega\in\Omega} G_\omega.$$

Clearly H is *malnormal* in G, that is, $H \cap H^g = 1$ for every $g \in G \setminus H$.

Theorem 2.1 ([3], Theorem 20.7.1). *Let $\omega_1, \omega_2 \in \Omega$ and suppose that at most one element taking ω_1 in ω_2 is regular. Then the identity and the regular elements of G form a transitive normal abelian subgroup N.*

Lemma 2.2 ([3], Lemma 20.7.1). *There exists one and only one involution in G which interchanges a specified pair of distinct elements $\omega_1, \omega_2 \in \Omega$.*

Lemma 2.3 ([3], Lemmas 20.7.2 and 20.7.4). *The involutions of G are in a single conjugacy class. The product of two different involutions is a regular element of G.*

Lemma 2.4 ([3], Lemma 20.7.3 and Theorem 12.5.2). *If the involutions of G are not regular, then in H there is a unique involution, which belongs to the center of H. In particular, a 2-subgroup of finite exponent of H is cyclic or quaternion, and hence finite.*

Let J be the set of involutions of G and put $J^2 = \{jk \mid j,k \in J\}$. If X is a subset of G, we define $X^\# = X \setminus \{1\}$.

Lemma 2.5 ([6], II.4.1.b and II.9.2). *If the involutions of G are not regular, then $(J^2)^\#$ is a conjugacy class in G. Moreover, every element of $(J^2)^\#$ has prime order $p \neq 2$ or infinite order.*

The following is a standard definition.

Definition I. Let G be a sharply 2-transitive permutation group.

If an involution (and hence any involution) of G is not regular, we define char(G), the characteristic of G, to be p if an element of $(J^2)^\#$ has order p and char$(G) = 0$ if an element of $(J^2)^\#$ has infinite order.

If the involutions of G are regular, we define char$(G) = 2$.

Lemma 2.6 ([6], II.9.2). *If* char$(G) = p > 0$, *then H contains a cyclic subgroup of order* $p - 1$.

Remark A. If char$(G) = 0$, then we can prove that H contains elements of infinite order. Since we will consider only the case in which H is periodic and contains elements of even order, from now we will assume that char$(G) = p > 2$.

Lemma 2.7 ([9], see also [1], Lemma 11.50 and Proposition 11.51). *Let $j, k \in J$ be distinct involutions. Then*

(a) *$C_G(jk) = jJ \cap kJ$ is abelian and inverted by j;*

(b) *the set $\{C_G(x)^\# \mid x \in (J^2)^\#\}$ forms a partition of $(J^2)^\#$;*

(c) *$N_G(C_G(jk)) = C_G(jk) \rtimes N_H(C_G(jk))$ is a split sharply 2-transitive group.*

If $G = NH$ is split, then the group H acts freely on N, that is for all $\nu \in N^\#$ and all $h \in H^\#$, $\nu^h \neq \nu$.

Lemma 2.8 ([4], Theorem 1.1 and Corollary 1.2). *Let N be an abelian group, and let H be a group of automorphisms of N. If H has exponent $2^m \cdot 3^n$ for $0 \leq m$ and $0 \leq n \leq 2$ and H acts freely on N, then H is finite. Moreover, if NH is a sharply 2-transitive permutation group and $n > 0$, then $|N| \in \{5^2, 7^2, 17^2\}$ or N has prime order.*

Lemma 2.9. *Let N be an abelian group, and let H be a group of automorphisms of N acting freely on N. If H is locally finite and has finite exponent, then H is finite.*

Proof. Denote by $\pi(H)$ the set of prime numbers that divide the order of some element of H. Since H is locally finite, if $p \in \pi(H)$, then every Sylow p-subgroup of H is cyclic or, if $p = 2$, quaternion ([2] Theorem 10.3.1). By hypothesis H has finite exponent and hence $\pi(H)$ is finite, moreover, every Sylow p-subgroup of H is finite and hence also H is finite. □

3. The $\lambda\rho$–Method

Let t be the unique involution of H and fix $\vartheta \in J$, $\vartheta \neq t$. Since G is doubly transitive, we known that $G = H \cup H\vartheta H$ ([2], Theorem 2.7.2). In particular, by the sharply 2-transitivity of G, for every $h \in H^\#$, there is a unique $\lambda(h) \in H^\#$ and a unique $\rho(h) \in H^\#$ such that

$$\vartheta h \vartheta = \lambda(h)\vartheta\rho(h). \tag{1}$$

Thus this defines two maps $\lambda, \rho : H^\# \longrightarrow H^\#$ as in [10]. We define also

$$\Delta(h) = \lambda(h)\rho(h) \qquad \text{and} \qquad \nabla(h) = \rho(h)\lambda(h) \tag{2}$$

if $h \in H^\#$ and we extend Δ, ∇ to all H putting $\Delta(1) = \nabla(1) = t$.

It is an easy matter to verify that $\rho(t^{-1}) = \lambda(t)^{-1}$; we will put $\rho(t) = u$.

By Lemma 2.5, $|\langle t\vartheta\rangle| = \text{char}(G) = p > 2$. By Lemma 2.7.(c) $N_G(C_G(t\vartheta))$ is a split sharply 2-transitive group with complement $N_H(C_G(t\vartheta))$, in particular, G is split if and only if $N_H(C_G(t\vartheta)) = H$. Since the subgroup $N_H(C_G(t\vartheta))$ of H assumes some importance in our arguments, then we will put

$$\mathcal{E}_\vartheta(H) = N_H(C_G(t\vartheta))$$

in order to simplify the notation; further, if there is no loss of clarity, we simply write $\mathcal{E}(H)$ in place of $\mathcal{E}_\vartheta(H)$.

Lemma 3.1. *Let $h \in H^\#$, then*

$$\lambda(\lambda(h)) = \rho(\rho(h)) = \Delta(\Delta(h)) = h, \tag{3}$$

in particular λ, ρ and Δ are bijections form $H^\#$ to $H^\#$. Moreover,

$$\lambda(\rho(h)) = \lambda(h)^{-1}, \qquad \rho(\lambda(h)) = \rho(h)^{-1}, \tag{4}$$

$$\lambda(h^{-1}) = \rho(h)^{-1}, \qquad \rho(h^{-1}) = \lambda(h)^{-1}, \tag{5}$$

$$\Delta(h^{-1}) = \Delta(h)^{-1}, \qquad \nabla(h^{-1}) = \nabla(h)^{-1}. \tag{6}$$

Proof. From $\vartheta h\vartheta = \lambda(h)\vartheta\rho(h)$ we obtain $\vartheta\lambda(h)\vartheta = h\vartheta\rho(h)^{-1}$, so $\lambda(\lambda(h)) = h$ and $\rho(\lambda(h)) = \rho(h)^{-1}$. Similarly $\rho(\rho(h)) = h$ and $\lambda(\rho(h)) = \lambda(h)^{-1}$.

The proof of (5) is obtained by considering the equality

$$\rho(h)^{-1}\vartheta\lambda(h)^{-1} = \left(\lambda(h)\vartheta\rho(h)\right)^{-1} = \left(\vartheta h\vartheta\right)^{-1} = \vartheta h^{-1}\vartheta$$

and from (5) we deduce (6).

In order to prove (3), consider

$$\lambda(\Delta(h))\vartheta\rho(\Delta(h)) = \vartheta\Delta(h)\vartheta = \vartheta\lambda(h)\rho(h)\vartheta = \vartheta\lambda(h)\vartheta\vartheta\rho(h)\vartheta = \\ h\vartheta\rho(h)^{-1}\lambda(h)^{-1}\vartheta h = h\vartheta\Delta(h)^{-1}\vartheta h = h\lambda(\Delta(h)^{-1})\vartheta\rho(\Delta(h)^{-1})h,$$

so, by equating the left part of the first and the last terms of the previous equality, we obtain $\lambda(\Delta(h)) = h\lambda(\Delta(h)^{-1}) = h\rho(\Delta(h))^{-1}$ and hence $\Delta(\Delta(h)) = \lambda(\Delta(h))\rho(\Delta(h)) = h$. □

Remark B. By Lemma 3.1, we can deduce that $\langle\lambda, \rho\rangle$ is a permutation group on the set $H^{\#}$ isomorphic to S_3 and we have $\lambda(\rho(\lambda(h))) = h^{-1} = \rho(\lambda(\rho(h)))$ for every $h \in H^{\#}$ (see also Section 2 in [10]).

Lemma 3.2. *The map $\nabla : H \to H$ is injective and $\nabla(h)$ is conjugate to $\Delta(h)$ for every $h \in H$. If C is a conjugacy class in H, then $\nabla(\Delta(C)) \subseteq C$ and, if C is finite, $\nabla(\Delta(C)) = C$. In particular, $\widehat{Z}(H) \subseteq \nabla(H)$.*

Proof. If $h \in H$, then

$$\vartheta\nabla(h) = \vartheta\rho(h)\lambda(h) = \big(\lambda(h)\vartheta\rho(h)\big)^{\lambda(h)} = (\vartheta h\vartheta)^{\lambda(h)} = h^{\vartheta\lambda(h)}. \tag{7}$$

Suppose $\nabla(h_1) = \nabla(h_2)$ with $h_1, h_2 \in H$, then, by (7), we can write $h_1^{\vartheta\lambda(h_1)} = h_2^{\vartheta\lambda(h_2)}$ and $h_1^{\vartheta\lambda(h_1)\lambda(h_2)^{-1}\vartheta} = h_2 \in H$. Since H is malnormal $\vartheta\lambda(h_1)\lambda(h_2)^{-1}\vartheta \in H$, so $\lambda(h_1) = \lambda(h_2)$ and $h_1 = h_2$ by Lemma 3.1.

Clearly, $\Delta(h) = \nabla(h)^{\lambda(h)}$, so $\Delta(h)$ and $\nabla(h)$ are conjugate. Let C be a conjugacy class of G, then, by Lemma 3.1, $\Delta(\Delta(C)) = C$ and since $\Delta(h)$ and $\nabla(h)$ are conjugate, we have $\nabla(\Delta(C)) \subseteq C$. Since ∇ is injective, if C is finite, then $\nabla(\Delta(C)) \subseteq C$ and this implies that $\widehat{Z}(H) \subseteq \nabla(H)$. □

Lemma 3.3. *If the map $\nabla : H \to H$ is surjective, then G is split.*

Proof. By Theorem 2.1, it is sufficient to prove that the unique regular element in $H\vartheta$ is $t\vartheta$. Let $h \in H \setminus \{1, t\}$ and $k \in H$ with $h = \nabla(k)$. The element $h\vartheta = \rho(k)\lambda(k)\vartheta$ is conjugate to $\lambda(k)\vartheta\rho(k) = \vartheta k\vartheta \in H^{\vartheta}$, and so $h\vartheta$ fixes an element of Ω. □

Remark C. One can prove that ∇ is surjective if and only if G is split and is *planar*, that is, $G = NH$ and for every $h \in H^{\#}$ the map

$$T_h : N \longrightarrow N \qquad \nu \mapsto \nu^{-1}\nu^h$$

is surjective (see Proposition 5.3).

There are, in each characteristic, examples of sharply 2-transitive groups that are split and in which ∇ is not surjective. In the case where H is periodic, it can be shown that this case can not happen (see Proposition 5.4).

Remark D. By Lemma 3.3 we deduce that if H is a FC-group, then G is split. This provides a more direct proof of Theorem 9.6 in [6].

The special case where H is abelian has a curious history in what it has been proved at least four times. In 1952 by Tits ([13], "hidden" in the Remark 2, p. 47), in 1961 by Zemmer [16], in 1990 by Mazurov [8] and by Károlyi et al. [5].

Lemma 3.4. $\mathcal{E}(H) = \big\{h \in H \mid \Delta(h) = th\big\}$.

Proof. We prove that $\Delta(h) = th$ if and only if $h \in E = N_H(C_G(t\theta))$. To do this, by Lemma 2.7 it sufficies to prove that $[t\vartheta, (t\vartheta)^h] = 1$ for all $h \in H$. The claim is obvious if $h = 1$ or $h = t$, so we assume $h \notin \{1, t\}$. Since $\vartheta t\vartheta = u^{-1}\vartheta u$ we can also write $\vartheta u^{-1}\vartheta = t\vartheta u^{-1}$ and $\vartheta u\vartheta = \vartheta t$ and hence, keeping in mind Lemma 3.1, if $\Delta(h) = th$, we obtain

$$t\vartheta\left(t\vartheta\right)^h = t\vartheta h^{-1}t\vartheta h = t\left(\vartheta h^{-1}\vartheta\right)\left(\vartheta t\vartheta\right)h = t\lambda(h^{-1})\vartheta\rho(h^{-1})u^{-1}\vartheta uh =$$
$$t\lambda(h^{-1})\left(\vartheta u^{-1}\vartheta\right)\left(\vartheta\rho(h^{-1})\vartheta\right)uh = t\lambda(h^{-1})t\vartheta u^{-1}\lambda(\rho(h^{-1}))\vartheta\rho(\rho(h^{-1}))uh =$$
$$\lambda(h^{-1})\vartheta\rho(h)u^{-1}\vartheta u = \lambda(h^{-1})\vartheta\rho(h)\vartheta\vartheta u^{-1}\vartheta u = \lambda(h^{-1})\lambda(\rho(h))\vartheta h\vartheta t\vartheta =$$
$$\rho(h)^{-1}\lambda(h)^{-1}\vartheta h\vartheta t\vartheta = \Delta(h^{-1})\vartheta h\vartheta t\vartheta = h^{-1}t\vartheta h\vartheta t\vartheta = \left(t\vartheta\right)^h t\vartheta,$$

that is $[t\vartheta, (t\vartheta)^h] = 1$.

If $[t\vartheta, (t\vartheta)^h] = 1$, we develop both members of $t\vartheta\left(t\vartheta\right)^h = \left(t\vartheta\right)^h t\vartheta$ obtaining

$$t\vartheta\left(t\vartheta\right)^h = t\lambda(th^{-1})\vartheta\rho(th^{-1})h$$

and

$$\left(t\vartheta\right)^h t\vartheta = th^{-1}\lambda(th)\vartheta\rho(th).$$

So $\rho(th^{-1})h = \rho(th)$, $\Delta(th) = \lambda(th)\rho(th) = \rho(th^{-1})^{-1}\rho(th) = h$ and finally $\Delta(h) = \Delta(\Delta(th)) = th$. □

We also prove the following proposition that is not required for the proof of our theorems.

Proposition 3.5. *If $u \in Z(H)$, then G is split. In particular, if $\mathrm{char}(G) = 3$, then G is split.*

Proof. Let h be an element in $H^\#$; we have

$$\lambda(th)\vartheta\rho(th) = \vartheta th\vartheta = \vartheta t\vartheta\vartheta h\vartheta = u^{-1}\vartheta u\lambda(h)\vartheta\rho(h) = u^{-1}\vartheta\lambda(h)u\vartheta\rho(h) =$$
$$u^{-1}\vartheta\lambda(h)\vartheta\vartheta u\vartheta\rho(h) = u^{-1}h\vartheta\rho(h)^{-1}u\vartheta t\rho(h) = hu^{-1}\vartheta u\rho(h)^{-1}\vartheta\rho(h)t =$$
$$h\vartheta t\vartheta\rho(h)^{-1}\vartheta\rho(h)t = h\vartheta th^{-1}\vartheta\lambda(h)\rho(h)t = h\lambda(th^{-1})\vartheta\rho(th^{-1})\Delta(t)$$

and hence $\lambda(th) = h\lambda(th^{-1})$, that is, $\Delta(th) = h$ and $\Delta(h) = th$. By Lemma 3.4 we obtain $H = \mathcal{E}(H)$ and hence G is split.

If $\mathrm{char}(G) = 3$, then $(\vartheta t)^3 = 1$ and $\vartheta t\vartheta = t\vartheta t$, so $u = t \in Z(H)$. □

Other proofs that a sharply 2-transitive group G with $\mathrm{char}(G) = 3$ is split can be found in [6] (Theorem 8.7) and in [14].

The following two lemmas are a direct consequence of (7).

Lemma 3.6. *Let h be an element of $H^\#$, then h and $\vartheta\nabla(h)$ have the same order.*

Proof. $\vartheta\nabla(h) = \vartheta\rho(h)\lambda(h)$ is conjugate to $\lambda(h)\vartheta\rho(h) = \vartheta h\vartheta$. □

Lemma 3.7. *Let w be an element of H, $w \neq t$. If $w \in \nabla(H)$, then ϑw cannot be regular.*

Proof. Let $h \in H$ be such that $w = \nabla(h)$. Then

$$\vartheta w = \vartheta\rho(h)\lambda(h) = \left(\lambda(h)\vartheta\rho(h)\right)^{\lambda(h)} = h^{\vartheta\lambda(h)}$$

fixes a point of Ω. □

4. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.2. In H there is a unique involution t and hence, by Lemma 2.4, the Sylow 2-subgroups of H are finite. In order to prove that H is finite we just prove that $T = \{h \in H \mid h^3 = 1\}$ is finite. Let $h \in T^{\#}$ and $w = \nabla(h)$. By Lemma 3.6 we have $(\vartheta w)^3 = 1$, that is

$$1 = \vartheta w \vartheta w \vartheta w = \vartheta w \lambda(w) \vartheta \rho(w) w$$

or $\vartheta = w \vartheta w \vartheta w = w \lambda(w) \vartheta \rho(w) w$ which implies $\lambda(w) = w^{-1} = \rho(w)$. Since $(\vartheta w)^3 = 1$, we have also $(\vartheta w^{-1})^3 = 1$ and hence $\vartheta \vartheta^w \vartheta^{w^2} = w^3$ that is

$$\vartheta w^3 = \vartheta^w \vartheta^{w^2}$$

and ϑw^3 should be a regular element of G. By hypothesis $w^3 \in \widehat{Z}(H)$ and by Lemma 3.2 $\widehat{Z}(H) \subseteq \nabla(H)$, hence, by Lemma 3.7, $\nabla(w^3) = 1$ and $w^3 = t$. Now $\Delta(w) = w^{-2} = w^{-3}w = tw$ and, by Lemma 3.4, $w \in \mathcal{E}(H)$. By Lemma 2.8 $\mathcal{E}(H)$ is finite and hence, by Lemma 3.2, T is finite. Thus G is finite and the structure of G is as described in Lemma 2.8. □

Proof of Theorem 1.1. Suppose H is nilpotent and of exponent $2^m \cdot 3$ for some $m \geq 1$. Let S be a Sylow 2-subgroup of H, since S contains a unique involution, then S is finite and $S \leq \widehat{Z}(H)$. Hence $H/\widehat{Z}(H)$ has exponent dividing 3, Theorem 1.2 applies. □

As one can check (using for instance Theorem 20.7.2 in [3] and the list of exceptionals Zassenhaus' near-fields provided in [3], p. 391), if G is a group that satisfies the hypotheses of the Theorem 1.1, then H is necessarily cyclic. If $G = NH$ satisfies the hypotheses of the Theorem 1.2 and N is not cyclic, then one of the following cases can occur:

- $N \simeq C_5 \times C_5$ and $H \simeq C_{24}$, or $H \simeq C_3 \rtimes C_8$, or $H \simeq SL(2,3)$;
- $N \simeq C_7 \times C_7$ and $H \simeq C_{48}$, or $H \simeq C_3 \rtimes C_{16}$, or $H \simeq GL(2,3)$;
- $N \simeq C_{17} \times C_{17}$ and $H \simeq C_{288}$, or $H \simeq C_9 \rtimes C_{32}$.

5. Appendix: Near-Fields and Near-Domains

Definition II. A *near-domain* is a set $\mathbf{F}$ equipped with two binary operations $\oplus$ and $\odot$ such that

(II.1) $(\mathbf{F}, \oplus)$ is a *loop* with neutral element 0;
(II.2) if $a \oplus b = 0$, then $b \oplus a = 0$;
(II.3) $(\mathbf{F}^{\#}, \odot)$ is a *group* with neutral element 1;
(II.4) $0 \odot a = 0$ for all $a \in \mathbf{F}$;
(II.5) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ for all $a, b, c \in \mathbf{F}$;
(II.6) for every $a, b \in \mathbf{F}$ there is $\partial_{a,b} \in \mathbf{F}^{\#}$ such that

$$a \oplus (b \oplus x) = (a \oplus b) \oplus (\partial_{a,b} \odot x)$$

for all $x \in \mathbf{F}$ ($\partial_{a,b}$ is independent from x).

Definition III. A *near-field* is a near-domain such that $(\mathbf{F}, \oplus)$ is a group.

It is clear that a near-domain $\mathbf{F}$ is a near-field if and only if $\partial_{a,b} = 1$ for every $a, b \in \mathbf{F}$.

Definition IV. A near-domain $\mathbf{F}$ is *planar* if for every $a, m \in \mathbf{F}$ with $m \notin \{0, 1\}$ there exists an $x \in \mathbf{F}$ such that $a + mx = x$.

A planar near-domain is necessarily a near-field ([6], II.3.7 and II.5.11).

Theorem 5.1 ([6] II.6.1, II.7.1, II.7.2). *Let* $\mathbf{F}$ *be a near-domain, then the set of one-dimensional affine transformations on* $\mathbf{F}$

$$\mathrm{T}_2(\mathbf{F}) = \left\{ \mathbf{F} \longrightarrow \mathbf{F} \ \ x \mapsto a \oplus m \odot x \,\middle|\, a, m \in \mathbf{F},\ m \neq 0 \right\}$$

is a group under the composition of maps. $\mathbf{T}_2(\mathbf{F})$ *operates sharply 2-transitivity on the elements of* $\mathbf{F}$ *and moreover*
(a) $\mathbf{T}_2(\mathbf{F})$ *is split if and only if* $\mathbf{F}$ *is a near-field;*
(b) $\mathbf{T}_2(\mathbf{F})$ *is planar if and only if* $\mathbf{F}$ *is a planar near-field.*

We can interpret the $\lambda\rho$-method in the language of the near-domains.

Proposition 5.2. *Let G be a sharply 2-transitive group on a set Ω. Assume* $\mathrm{char}(G) \neq 2$, *let $H = G_\alpha$ be the stabilizer of an element $\alpha \in \Omega$ and let t be the unique involution in H. Put $\mathbf{F} = H\dot{\cup}\{0\}$ and in $\mathbf{F}$ define two operations in the following way:*

$$a \oplus b = \begin{cases} a\lambda(ta^{-1}b) & \text{if } a,b \in H \text{ and } a \neq tb \\ 0 & \text{if } a,b \in H \text{ and } a = tb \\ a & \text{if } b = 0 \\ b & \text{if } a = 0 \end{cases}$$

$$a \odot b = \begin{cases} ab & \text{if } a,b \in H \\ 0 & \text{if } a = 0 \text{ or } b = 0. \end{cases}$$

Then $\left(\mathbf{F}, \oplus, \odot\right)$ is a near-domain and G is isomorphic, as permutation group, to $\mathbf{T}_2(\mathbf{F})$.

Moreover, if $a, b \in \mathbf{F} \setminus \{0\} = H$, then $\partial_{a,b} = a\Delta(ta^{-1}b)b^{-1}$ and hence $\mathbf{F}$ is a near-field if and only if $\Delta(h) = th$ for every $h \in H$.

Proof. A tedious but easy computation. □

Proposition 5.3. *Let G be a sharply 2-transitive group with* $\mathrm{char}(G) \neq 2$ *and point stabilizer H and let $\mathbf{F}$ be the associated near-domain. Then $\mathbf{F}$ is a planar near-field if and only if $H = \nabla(H)$.*

Proof 1. (near-field style). Suppose $H = \nabla(H)$ and $a, m \in \mathbf{F}$ with $m \notin \{0,1\}$. If $m \neq t$ then, by hypothesis, there is $h \in H$ such that $ta^{-1}ma = \nabla(h)$ and define $x = a\lambda(h)^{-1}$. We have

$$\begin{aligned} a \oplus m \odot x = a \oplus mx = a\lambda(ta^{-1}mx) &= a\lambda\left(ta^{-1}ma\lambda(h)^{-1}\right) \\ &= a\lambda\left(\nabla(h)\lambda(h)^{-1}\right) = a\lambda(\rho(h)) = a\lambda(h)^{-1} = x. \end{aligned}$$

If $m = t$ then, remembering that $\vartheta t\vartheta = u^{-1}\vartheta u$ and $\lambda(u) = u$, we define $x = au$ and we obtain $a \oplus t \odot x = a\lambda(a^{-1}x) = a\lambda(u) = au = x$.

Suppose $\mathbf{F}$ planar; by definition $\nabla(1) = t$ and $\nabla(t) = 1$. Let $k \in H \setminus \{1,t\}$ and let $x \in H$ be such that $t \oplus k \odot x = x$. Then we can verify that $h = \lambda(tx^{-1})$ is an element such that $\nabla(h) = k$. □

Proof 2. (group-theoretic style). By hypothesis $\nabla(H) = H$ and hence, by Lemma 3.3, G is split. So we can write $G = NH$ with $N \trianglelefteq G$ abelian and t acting by conjugation as the inversion on N. Let $\nu \in N$, since $\nu^t = \nu^{-1}$, there is an involution ϑ in G such that $\nu = t\vartheta$. Fix $h \in H^\#$, we have to show that the map $T_h : N \longrightarrow N$, $y \mapsto y^{-1}y^h$ is surjective. If $h = t$, the claim is trivial and hence we suppose $h \notin \{1,t\}$. Our assertion is proved if we can find an element $k \in H$ such that $(t\vartheta^k)^{-1}(t\vartheta^k)^h = t\vartheta$. If we choose k such that $h = t\nabla(\lambda(tk))^{-1}$, then, remembering that, since G is split, is $\Delta(k) = tk$, we obtain

$$\begin{aligned} &\left(t\vartheta^k\right)^{-1}\left(t\vartheta^k\right)^h = k^{-1}\vartheta kh^{-1}k^{-1}\vartheta kh = k^{-1}\vartheta kt\nabla(\lambda(tk))k^{-1}\vartheta kt\nabla(\lambda(tk))^{-1} = \\ &k^{-1}\vartheta tk\rho(\lambda(tk))\rho(\rho(tk))k^{-1}\vartheta tk\left(\rho(\lambda(tk))\rho(\rho(tk))\right)^{-1} = k^{-1}\vartheta k\rho(tk)^{-1}\vartheta\rho(tk) = \\ &k^{-1}\vartheta\left(\lambda(tk)\vartheta\rho(tk)\right) = k^{-1}\vartheta\left(\vartheta tk\vartheta\right) = t\vartheta, \end{aligned}$$

and the proof is complete. □

Remark E. The two proofs of Proposition 5.3 show that the definitions given in Remark C and Definition IV are actually equivalent.

Remark F. If char$(G) = 2$, then we can choose and involution $\vartheta \in G$ and, if $H = G_\alpha$, we can define as above the maps λ, ρ, Δ and ∇. In this case, if we put $t = 1$, then it is not difficult to verify that Propositions 5.2 and 5.3 are still true. Moreover, G is split if and only if $H = \mathcal{E}(H) = \{h \in H \mid \Delta(h) = h\}$.

We conclude this short appendix with the following result.

Proposition 5.4. *Let G be a split sharply 2-transitive group. If the point stabilizer H is periodic, then G is planar.*

Proof. Write $G = NH$ with $N \trianglelefteq G$ and $N \cap H = \{1\}$. Since H is periodic, then, by Remark A, char$(G) = p > 0$ and hence N is an elementary abelian p-group acted freely by H. Let $h \in H^\#$ be an element of order ℓ, then $(p, \ell) = 1$ and hence there is a positive integer δ such that $p^\delta \equiv 1 \mod \ell$. Let $\overline{h}$ be the automorphism induced by conjugation by h on N and let $q = p^\delta$. In the ring End(N), we have

$$(T_h)^q = (-1 + \overline{h})^q = -1 + \overline{h}^q = -1 + \overline{h} = T_h$$

that is, $T_h^{q-1} = \mathrm{id}_N$, and hence T_h is a bijection. □

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