Strategy Proofness and Coalitional Stability in Assignment Problems with Externalities*

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Abstract

We study assignment problems with externalities where agents have expectations about the reactions of other agents to group deviations. We present notions of core consistent with such expectations and identify the largest and smallest cores. We restrict the domain of preferences to study the relationship between essentially single valued cores and the existence of strategy-proof, individually rational, and efficient mechanisms.

Keywords: Assignment problems, Externalities, Matching theory, SIE-mechanisms.

Economic Literature Classification Numbers: C71, C72, C78, D62, D78.

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Introduction

When externalities are relevant, the standard concepts of core and core-consistent solutions are often empty (see among others Ehlers, 2018; Mumcu and Saglam, 2010; Roth and Sotomayor, 1990; Sasaki and Toda, 1986, 1996). The introduction of expectations allows us to model alternative ways in which the literature has dealt with the concept of coalitional stability with externalities, such as the ones studied in Ehlers (2018), Mumcu and Saglam (2010), Sasaki and Toda (1986, 1996).[1]

We define a new concept of core called *expectational core*, which is consistent with the idea that when making deviation decisions, agents take into account their expectations about other agents' reactions. We assume that a coalition of agents deviates from an assignment if any of the possible outcomes of the deviation make all the members of the coalition better off, and at least one agent of the coalition strictly improves their situation.

The introduction of expectations allows us to model alternative ways in which the literature has dealt with externalities. Two examples of expectations are "*prudent* expectations" and "*optimistic* expectations". When expectations are *prudent*, agents do not have any information on the possible reactions of the other agents, thus they expect that any reassignment outside the coalition is feasible.[2] Prudent expectations are consistent the stability concepts considered in Sasaki and Toda (1986, 1996) for marriage markets.[3][4] In contrast, under *optimistic* expectations the deviating agents expect that the assignment they propose will be enacted. Under *prudent* expectations the agents belonging to a coalition will deviate only if the worse assignment that can result by the deviation is better than the previous assignment. Under *optimistic* expectations a coalition will

[1] Previous authors have analyzed the effects of externalities, Echenique and Yenmez (2007) and Dutta and Masso (1997) in college admission problems, Alcalde and Revilla (2004) in the formation of research teams, Klaus and Klijn (2005) and Bando (2012) in labor market, Chowdhury (2004) in marriage market, and Salgado-Torres (2011) in housing market. Bando et al. (2016) is a useful survey about two-sided matching with externalities.

[2] In the literature *prudent agents* are sometimes called *pessimistic agents*.

[3] See also Contreras and Torres-Martínez (2019), for *prudent* expectations in roommate problems. Prudent expectations are also related with the α-effective concept used by Aumann and Peleg (1960).

[4] Sasaki and Toda (1986, 1996) assume expectations are exogenously given, Hafalir (2008) introduced endogenously generated beliefs.

deviate if the assignment that can result by the deviation is better than the previous assignment. Thus, the expectational core generated by *optimistic* expectations coincides with the standard definition of a core. We prove that *optimistic* expectations and *prudent* expectations generate the smallest and the largest expectational cores, respectively (Proposition 1). The models studied in Ehlers (2018) and Mumcu and Saglam (2010) are consistent with specific single-valued expectations, we call them "*dissolving* expectations" and "*myopic* expectations", respectively.[5] They describe situations where the agents in a coalitions can perfectly forecast the reaction of other agents. We show the relationship between the expectational cores generate by all these kinds of expectations (Corollary 1).

We concentrate our attention on the relationship between expectational cores and strategy-proof, individually rational, and efficient mechanism (SIE-mechanism, for short) in contexts with externalities. The seminal contribution by Sönmez (1999) shows that if a SIE-mechanism exists, then the core of the game is essentially single valued, whenever it is not empty. Furthermore, if the core is externally stable, non empty, and essentially single valued in a given preference domain, then any selection of the core is strategy-proof, efficient, and individually rational.[6] Sönmez's model allows for externalities. However, his blocking concept is overly restrictive. He assumes that, when groups of agents renegotiate a given application, they do not take into account the behavior of agents outside the coalition. However, it might be the case that the other agents renegotiate their assignment in reaction to the deviation (which is not relevant without externalities). We show that we need additional and mild restrictions to connect the existence of an SIE-mechanism with coalitional stability. More specifically, we introduce conditions under which the existence of SIE-mechanisms is equivalent to having an essentially single valued expectational core (Theorem 3 and Theorem 2).

We work with blocking coalitions where each member has expectations about the reaction of the agents outside the coalition. However, we do not model explicitly the process of coalition formation.[7] Instead, we assume the coalition structure is given exogenously and

[5]Bando (2012) extends the analysis of Mumcu and Saglam (2010) to many-to-one matching market.

[6]Takamiya (2003) obtains the same result replacing the external stability condition for a richness condition on the preference domain.

[7]Some works related with endogenous formation of coalitions with externalities are Bloch (1996),

we focus on the existence of SIE-mechanisms and its relation with the core.

The work is organized as follows. In Section 1 we present the model. In Section 2 we study the relationship between expectational cores and SIE-mechanisms. In section 3 we compare the resulting expectational cores. Finally, Section 4 concludes.

Huang and Sjöström (2003), Hafalir (2007), and Kóczy (2009).

1 The Model

Consider an assignment problem with externalities $(N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i\in N})$ where N is a finite set of *agents*, e is the *endowments*, $\mathcal{A}^f$ is the set of *feasible assignments*. Every agent i has *preferences*, R_i, belonging to a class $\mathcal{R}_i$ of complete and transitive binary relations over $\mathcal{A}^f$.[8] Additionally, every agent i has *admissible expectations*, Θ_i, about the reaction of other agents to group deviations. Formally, the expectation of agent i is a correspondence $\Theta_i : \mathcal{A}^f \times \mathcal{A}^f \times 2^N \times \mathcal{R} \rightrightarrows \mathcal{A}^f$, with $i \in N$. More precisely, $\Theta_i(a, b, T, R)$ is the set of assignments that i expects that could be effectively attained when the coalition T announces that will deviate from a to match their members as in b.[9] Usually, we refer to a as "the previous assignment" and to b as "the announcement". We assume that the definition of expectation is part of the model. An **environment** is $\mathcal{E} = \big(N, e, \mathcal{A}^f, \mathcal{R}, (\Theta_i)_{i\in N}\big)$, and an **assignment problem** is a tuple $(N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i\in N})$.

We impose two restriction on admissible expectations. First, we require that the agreement taken by a coalition is credible in the sense that the redistribution agreement agreed by the coalition members will be respected by them. Second, we require that the agents within a coalition can redistribute their endowments, otherwise the previous assignment is implemented. Formally we say that expectations are **admissible** if:

(i) $c(k) = b(k), \forall c \in \Theta_i(a, b, T, R), \forall k \in T$ if $\bigcup_{i\in T} b(i) \subseteq \bigcup_{i\in T} e_i$; and

(ii) $\Theta_i(a, b, T, R) = \{a\}$ if $\bigcup_{i\in T} b(i) \nsubseteq \bigcup_{i\in T} e_i$.

The following will assume admissible expectations.

The idea is that, when a coalition of agents renegotiate an assignment using their own endowments, other individuals outside the coalition can react. If there are externalities, this behavior can affect the welfare of the agents in T. Then we next incorporate the admissible expectations in the blocking concept. A coalition T **blocks** an assignment

[8] For convenience, in examples we use the symbol $\succ$ to denote strict preference, and $\backsim$ to denote indifference.

[9] Expectations are equivalent to *beliefs*, *estimations* or *conjectural valuations* used in Sasaki and Toda (1996) to marriage market.

$a \in \mathcal{A}^f$ under a preference profile $(R_i)_{i\in N}$ when there is an announcement $b \in \mathcal{A}^f$ such that:

(i) For every $i \in T$ we have that cR_ia for all $c \in \Theta_i(a, b, T, R)$.

(ii) There exists $i \in T$ such that cP_ia for all $c \in \Theta_i(a, b, T, R)$.

That is, to block an assignment, a coalition requires to redistribute their own endowments and the agreement taken by the coalition be credible (condition of admissible expectations) in such form that at least one of its members expects that she will strictly improve her situation in all assignments that expect could be attained, without harming any other member of the coalition (conditions (i) and (ii)). Notice that this formulation is non Bayesian because players do not assign probabilities to the different matchings, but rather deviate only if the deviation is profitable for *all* matchings that they expect could be obtained.

A mechanism Γ is **weakly coalitionally strategy-proof** if for all $R \in \mathcal{R}$, for all $T \subseteq N$, and for all $\widetilde{R}_T \in \mathcal{R}_T$ there exists an agent $i \in T$ such that $\Gamma(R)\, R_i\, \Gamma(\widetilde{R}_T, R_{-T})$.

An assignment $a \in \mathcal{A}^f$ belongs to the **expectational core** of $(N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i\in N})$ when it cannot be blocked by any coalition $T \subseteq N$. The **expectational core correspondence** $\mathscr{C} : \mathcal{R} \rightrightarrows \mathcal{A}^f$ is the set-valued mapping that assigns to each preference profile $R \in \mathcal{R}$ the expectational core of $(N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i\in N})$. The **individually-rational expectational core correspondence** $\mathscr{C}^* : \mathcal{R} \rightrightarrows \mathcal{A}^f$ is the set-valued mapping that assigns to each preference profile $R \in \mathcal{R}$ the set $\mathscr{C}(R) \cap \mathscr{I}(R)$. We say that in this economy there exists **veto power** if the agent $i \in N$ unilaterally announces that she will block the assignment $a \in \mathcal{A}^f$ with another assignment $\{e_i, b_{-i}\}$ where i keeps her endowment, everyone expects that the market will disintegrate and each agent retains their endowment, formally $\Theta_i(a, \{e_i, b_{-i}\}, \{i\}, R) = \{e\}$ for each $i \in N$. Notice that in this context $\mathscr{C}(R) \subseteq \mathscr{I}(R)$ and then $\mathscr{C}(R) = \mathscr{C}^*(R)$.

The expectational core correspondence is **essentially single-valued** if it has non-empty values and, for any preference profile R, two assignments in $\mathscr{C}(R)$ are indifferent to all agents. Similarly, $\mathscr{C}^*$ is essentially single-valued when, for each $R \in \mathcal{R}$, $\mathscr{C}^*(R) \neq \emptyset$ and $a, b \in \mathscr{C}^*(R) \Longrightarrow aI_ib \; \forall i \in N$.

A set of assignments $\mathcal{S} \subseteq \mathcal{A}^f$ is **externally stable** if for every $a \in \mathcal{A}^f \backslash \mathcal{S}$ there exists $T \subseteq N$ which blocks a by announcing some $b \in \mathcal{A}^f$ such that $\mathcal{S} \cap \Theta_i(a, b, T, R) \neq \emptyset$ for all $i \in T$, which meas that for each assignment outside S there exists a coalition which blocks it and everyone expect that at least one assignment that could be attain is in the set S. The core correspondence is externally stable if it has externally stable values.

1.1 Specifications of expectational cores

The flexibility of the correspondences $\Theta_i : \mathcal{A}^f \times \mathcal{A}^f \times 2^N \times \mathcal{R} \rightrightarrows \mathcal{A}^f$, with $i \in N$, allows us to model situations previously studied in the literature to deal with assignment problems in markets with externalities.

Let's start defining the following expectational correspondence:

$$\Theta_i^{\mathcal{P}}(a, b, T, R) = \begin{cases} \{c \in \mathcal{A}^f : c(k) = b(k), \forall k \in T\} \text{ if } \bigcup_{i \in T} b(i) \subseteq \bigcup_{i \in T} e_i, \\ \{a\} \text{ if } \bigcup_{i \in T} b(i) \nsubseteq \bigcup_{i \in T} e_i. \end{cases}$$

In this case, we say that agents have ***prudent* expectations**. The model represents a situation where agents belonging to a coalition, say T, do not have any information about how the agents outside T will react to their deviation. Then they expect that any assignment which respects the redistribution proposed in b for T's members could be attained. Furthermore, before joining a coalition T to block assignment a, they want to be sure that nobody in T will loose and at least one agent will gain with respect to assignment a, whatever the reaction of the agents outside the coalition. The *prudent* expectation notion extends to generalized assignment problems the blocking concept introduced in Sasaki and Toda (1996) in marriage markets with externalities.[10]

We denote $\mathscr{C}_{\mathcal{P}}$ and $\mathscr{C}^*_{\mathcal{P}} = \mathscr{C}_{\mathcal{P}} \cap \mathscr{I}$ the *prudent* expectational core correspondence the individually-rational *prudent* expectational core correspondence, respectively. Notice

[10]See also Contreras and Torres-Martínez (2019) to roommate problems with externalities.

that, under *prudent* expectations, any individually rational assignment is participative but the reverse is not true. While we do not model uncertainty in this model, the expectational core is consistent with an extreme form of ambiguity aversion (see Schmeidler and Gilboa, 2004).

On the other extreme there are the ***optimistic* expectations**.

$$\Theta_i^{\mathcal{O}}(a,b,T,R) = \begin{cases} \{b\} & \text{if } \bigcup_{i\in T} b(i) \subseteq \bigcup_{i\in T} e_i, \\ \{a\} & \text{if } \bigcup_{i\in T} b(i) \nsubseteq \bigcup_{i\in T} e_i. \end{cases}$$

In this case, each agent expects that the exact assignment announced by the coalition T will be attained, if in the announce agents in T are redistributing their own endowments. Notice that the agents within the coalition are determining also the objects the agents outside the coalition will receive. Under *optimistic* expectations, agents in T block assignment a if there exists a feasible assignment such that nobody loose and at least one agent will gain upon assignment a. Under *optimistic* expectations we find the traditional definition of core.

The expectational core correspondence under this specification is denoted by $\mathscr{C}_{\mathcal{O}}$, and $\mathscr{C}^*_{\mathcal{O}} \equiv \mathscr{C}_{\mathcal{O}} \cap \mathscr{I}$. Under *optimistic* expectations, any participative assignment is individually rational. Thus, $\mathscr{C}_{\mathcal{O}} = \mathscr{C}^*_{\mathcal{O}} \equiv \mathscr{C}_{\mathcal{O}} \cap \mathscr{I}$.

Next, we show alternative notions of cores.

For every (i,a,b,T), set $c(k) = b(k)$, for all $k \in T$, $c(k) = (a(k) \setminus e(T)) \cup (a(T) \cap e(k))$ for all $k \notin T$. Set

$$\Theta_i^{\mathcal{M}}(a,b,T,R) = \begin{cases} \{c\} \text{ if } \bigcup_{i\in T} b(i) \subseteq \bigcup_{i\in T} e_i \text{ and } c \in \mathcal{A}^f, \\ \{a\} \text{ if } \bigcup_{i\in T} b(i) \nsubseteq \bigcup_{i\in T} e_i \text{ and } c \in \mathcal{A}^f. \end{cases}$$

In this case, we say that agents have ***myopic* expectations**. They expect that after the announcement of a blocking coalition T, the agents who do not belong to the coalition will not react. Each $i \in N \setminus \{T\}$

Let $\mathscr{C}_{\mathcal{M}}$ be the expectational core correspondence that arises with this specification and denote by $\mathscr{C}^*_{\mathcal{M}} = \mathscr{C}_{\mathcal{M}} \cap \mathscr{I}$ the associated individually-rational expectational core. This core concept has been introduced by Sasaki and Toda (1986) to analyze coalitional stability in marriage markets with externalities, and by Mumcu and Saglam (2010) to the housing market problems with externalities.

In the case of *myopic* expectations the individual rationality is not stronger nor weaker than the participative constraint as shown by Example 8 in Appendix 5.

Next, for every (i, a, b, T), set $c(k) = b(k)$, for all $k \in T$, $c(k) = e(k)$ for all $k \notin T$.

$$\Theta_i^{\mathcal{D}}(a, b, T, R) = \begin{cases} \{c\} \text{ if } \bigcup_{i\in T} b(i) \subseteq \bigcup_{i\in T} e_i \text{ and } c \in \mathcal{A}^f; \\ \{a\} \text{ if } \bigcup_{i\in T} b(i) \subseteq \bigcup_{i\in T} e_i \text{ and } c \in \mathcal{A}^f. \end{cases}$$

In this case, we say that agents have ***dissolving* expectations**. After the announcement of a blocking coalition T, the coalition of the agents not belonging to T dissolves and each one of them receives her endowment.

Let $\mathscr{C}_{\mathcal{D}}$ be the expectational core correspondence that arises with this specification and denote by $\mathscr{C}^*_{\mathcal{D}} = \mathscr{C}_D \cap \mathscr{I}$. This core concept coincide with the IR-core studied by Ehlers (2018).[11] When expectations are *dissolving*, each agent has veto power on the assignment, which is $\Theta_i(a, e, \{i\}, R) = \{e\}$ for each agent $i \in N$ and each feasible assignment a, thus the set of individually rational assignments and the set of participative assignments coincide. In particular, $\mathscr{C}_{\mathcal{D}} = \mathscr{C}^*_{\mathcal{D}} = \mathscr{C}_D \cap \mathscr{I}$.

Example 1 help to clarify the differences between the four specifications seen.

Example 1 *Let* $N = \{i_1, i_2, i_3, i_4, i_5\}$, $e = (e_1, e_2, e_3, e_4, e_5)$ *and the following assignments:*

$$a = (e_1, e_2, e_4, e_5, e_3); \qquad b = (e_2, e_1, e_5, e_3, e_4);$$

$$c = (e_2, e_1, e_3, e_4, e_5); \qquad d = (e_2, e_1, e_4, e_5, e_3);$$

$$f = (e_2, e_1, e_3, e_5, e_4).$$

[11] See also Hart and Kurz (1983).

Consider the coalition $T = \{i_1, i_2\}$. The preferences are:

$$R_{i_1} : b \succ c \sim d \succ a \sim e \succ f,$$

$$R_{i_2} : b \succ c \succ d \succ f \succ a \sim e.$$

Let's review each specification of expectations we have seen:

> $\Theta_i^{\mathcal{P}}(a, b, T, R) = \{b, c, d, f\}$. *Then if members of the coalition T are prudent they will not block the assignment a announcing b.*
>
> $\Theta_i^{\mathcal{O}}(a, b, T, R) = \{b\}$, $\Theta_i^{\mathcal{M}}(a, b, T, R) = \{d\}$, $\Theta_i^{\mathcal{D}}(a, b, T, R) = \{c\}$. *Then if members of the coalition T are optimistic, myopic or dissolving they will block the assignment a announcing b.*

Example 1 suggests that under *prudent* expectations is more difficult to block. In fact, we prove in Section 3 that *prudent* expectations generate the largest core.

2 Cores and the existence of *SIE*-mechanisms

In this section we connect expectational cores with the existence of strategy-proof assignments. More precisely we verify whether the relation between essentially singleton cores and the existence of strategy proof assignments, already studied in Sönmez (1999) and Ehlers (2018) holds in our setup. It is of particular interest since when cores are empty, the results in Sönmez (1999) and Ehlers (2018) have no grip.[12]

2.1 From *SIE*-mechanisms to essentially single valued cores

We investigate whether an essentially single valued core is a necessary condition for the existence of a SIE-mechanism. This is true if agents have either *optimistic* or *dissolving*

[12] In Section 3 we study the relationship among the cores generate by each specification of expectations presented and we show that the *prudent* expectation generates the largest core.

expectations as shown by Sönmez (1999) and Ehlers (2018), respectively. Intuitively, generalize the result to more permissive expectations requires more exigent conditions. We start introducing some conditions on the preferences domain.

Assumption 1 *For each* $i \in N$, $R_i \in \mathcal{R}_i$, *and* $a \in \mathcal{A}^f$, $aI_ie \iff a(i) = e_i$.

If Assumption 1 holds, an agent i is indifferent between an assignment and the endowment e if and only if she receives her endowment e_i. Then, Assumption 1 limits the externalities related to the endowments. If a domain satisfies Assumption 1 individual rationality and participation are equivalent requirements over an assignment, independently on expectations.

Assumption 2 *For each* $i \in N$ *and* $R_i \in \mathcal{R}_i$, *if an assignment* $a \in \mathcal{A}^f$ *satisfies* aR_ie, *then there exists a preference relation* $\widetilde{R}_i \in \mathcal{R}_i$ *such that*

(i) for each $b \in \mathcal{A}^f$, $bR_ia \Leftrightarrow b\widetilde{R}_ia$ *and* $aR_ib \Leftrightarrow a\widetilde{R}_ib$;

(ii) for each $b \in \mathcal{A}^f$, $aP_ib \Leftrightarrow a\widetilde{P}_ib$ *and* $a\widetilde{R}_ie\widetilde{R}_ib$.

If Assumption 2 holds, for each agent $i \in N$, preference profile $R_i \in \mathcal{R}_i$ and individually rational assignment a, there exists a preferences relation $\widetilde{R}_i$ which lifts e just below a, maintaining the relative ranking of all other assignments with respect to a.

Sönmez (1999) and Ehlers (2018) employ Assumptions 1 and 2.

Notice that if Assumptions 1 and 2 hold, when passing from R_i to an $\widetilde{R}_i$, not only e goes up, but also all assignments where agent i keeps her endowment.

Next, we introduce an invariance assumption on expectations.

Assumption 3 *For each* $a, b, c \in \mathcal{A}^f$, $i \in N$, $R \in \mathcal{R}$ *and* $T \subseteq N$, *we have that* $aI_ib \Longrightarrow \Theta_i(a, c, T, R) = \Theta_i(b, c, T, R)$.

The idea is that whenever an agent is indifferent between two assignments, she has the same expectation about the results of the deviation. The Assumption 3 is satisfied by any expectation correspondence which is independent on the starting assignment a. In particular it is satisfied by *optimistic*, *prudent*, and *dissolving* expectations. Thus it is implicitly satisfied in the models studied by Sönmez (1999) and Elhers (2018), because in their models, expectations depend only on the deviating coalition and on the proposed assignment.

We also employ the following assumption.

Assumption 4 *For each $i \in N$, and $R \in \mathcal{R}$, $b \in \mathscr{P}(R) \cap \mathscr{I}(R) \Longrightarrow \Theta_i(a, b, T, R) = \{b\}$ for all $a, b \in \mathcal{A}^f$ and $T \subseteq N$.*

Assumption 4 requires that, whenever a deviating coalition propose an individually rational and Pareto Efficient assignment, no agent will renegotiate that assignment.

Notice that Assumption 4 trivially holds when agents are optimists in the sense that individuals expect that those outside the blocking coalition will behave in the way that suits the members of the coalition. It implies that when the announcement is Pareto-efficient and individually rational, each agent acts as she was *optimistic.*

Finally, we consider an assumption on the set of feasible assignments which strengthens Assumption 1.

Assumption 5 *If $a \in \mathcal{A}^f$ and $a(i) = e(i)$ for some $i \in N$, then $a = e$.*

If this Assumption holds, the only assignment in which an agent has her endowment is where everyone keeps her endowment too.

Theorem 1 *Assume that there exists a Pareto efficient, individually rational, and strategy-proof mechanism, Γ, and any of the following sets of Assumptions is satisfied*

1. *Assumptions 1,2,3,5, or*

2. Assumptions 1,2,4.

Then, in the domain $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$, $\Gamma(R) \subseteq \mathscr{C}^*(R)$ and the individually-rational expectational core correspondence $\mathscr{C}^*$ is essentially single-valued.

Proof. [13]

1. Let $R \in \mathcal{R}^*$ and let $a \in \mathscr{C}^*(R)$. For every $i \in N$, consider preferences $\widetilde{R}_i \in \mathcal{R}_i$ satisfying the requirements (i)-(ii) of Assumption 2. Hence, $a \in \mathscr{C}^*(\widetilde{R})$, because a is individually rational and any $T \subseteq N$ that blocks a under $\widetilde{R} = (\widetilde{R}_i)_{i \in N}$ blocks a under R, too. Let Γ be a SIE-mechanism, and let $b \in \mathscr{P}(\widetilde{R}) \cap \mathscr{I}(\widetilde{R})$, we next prove $b\widetilde{I}_i a$ for all $i \in N$. By contradiction assume there exist an agent $i \in N$ such that $b\widetilde{P}_i a$. Define the coalition $T := \{i \in N : b(i) \neq e(i)\}$. If $T = \emptyset$ then $b = e$. Since a is individually rational, then $a\widetilde{R}_i b$ which generates a contradiction. If $T \neq \emptyset$, Assumption 5 guarantees that $T = N$. Notice that for any admissible expectations, $\Theta_i(a, b, N, R) = b$ for all $i \in N$. Individual rationality of b, construction of T, and Assumption 1 implies $b\widetilde{P}_i e$ thus

(i) For every $i \in T$ we have that $c\widetilde{R}_i a$ for all $c \in \Theta_i(a, b, T, \widetilde{R})$.

Moreover, by the contradiction assumption

(ii) There exists $i \in T$ such that $cP_i a$ for all $c \in \Theta_i(a, b, T, \widetilde{R})$.

Then by (i) and (ii) T blocks a under $\widetilde{R}$ announcing b which contradicts with $a \in \mathscr{C}^*(\widetilde{R})$. Thus $a\widetilde{R}_i b$ for all $i \in N$, but b is Pareto efficient, then $a\widetilde{I}_i b$ for all $i \in N$. Since b is an arbitrary assignment we conclude $a\widetilde{I}_i \Gamma(\widetilde{R})$ for all $i \in N$.

As strategy-proofness does not depends on the characteristics of $\{\Theta_i\}_{i \in N}$, Claim 2 in Sönmez (1999) implies that $\Gamma(R) I_i a$ for all $i \in N$. Therefore, as a is an arbitrary element of $\mathscr{C}^*(R)$, the transitivity of preferences implies that any pair of assignments in $\mathscr{C}^*(R)$ are indifferent to all agents. It follows that $\Gamma(R) I_i a$ for all $(i, R) \in N \times \mathcal{R}^*$ and $a \in \mathscr{C}^*(R)$. Thus, by Assumption 3 $\Theta_i(\Gamma(R), \cdot) = \Theta_i(a, \cdot)$ and we conclude that $\Gamma(R) \in \mathscr{C}^*(R)$.

[13] The proof follows close arguments used by Sönmez (1999) in the proof of Claim 1.

2. Following of notation of part (1), by Assumption 4 $\Theta_i(a, b, T, R) = \{b\}$ for each $i \in N$ and $T \subset N$. Fix a preference profile $R \in \mathcal{R}^*$ and $a \in \mathscr{C}^*(R)$. Since a is individually rational we have that aR_ie for each $i \in N$. Then there exists a preference profile $\widetilde{R}_i \in \mathcal{R}$ that fulfill items of Assumption 2. Moreover, by construction of $\widetilde{R}$, $a \in \mathscr{C}^*(\widetilde{R})$. Let $b \in \mathscr{P}(\widetilde{R}) \cap \mathscr{I}(\widetilde{R})$, and define $T := \{i \in N : b(i) \neq e(i)\}$, then by Assumption 4 we have $\Theta_i(a, b, T, R) = \{b\}$ for each $i \in N$ which coincides with used by Sönmez (1999) and Ehlers (2018). Therefore the proof follows for identical arguments that the mentioned papers.

■

As we mentioned before, Assumption 5 is too restrictive for Myopic and Dissolving expectations because under this Assumption is not possible that only part of agents keeps their endowments, as a consequence, $\mathscr{C}^*_M$ and $\mathscr{C}^*_D$ are usually the largest core but because the feasible assignments set is restrictive.[14] On the other hand, Assumption 4 limits the variety of expectations because when an announcement is Pareto-efficient and individually rational, each agent acts as an Optimistic person. However, in this case, all models are affected in the same way and then their relation does not change, and the largest core is $\mathscr{C}^*_P$ as in the context without restrictions.

2.2 From *SIE*-mechanisms to essentially single valued cores using the weak blocking concept

In this section, we present an alternative approach that it could be used to investigate whether an essentially single valued core is a necessary condition for the existence of a SIE-mechanism.

As we saw in the subsection 2.1, generalize the result to more permissive expectations requires more exigent conditions, to avoid that, in this section we work with a weak blocking concept. Specifically, we do not need that a member of the coalition improves

[14]An example of this point is presented in Appendix 5.

strictly her situation in *all* assignments that she expects that could be attained, instead, it is enough she improve strictly in only *one* expected assignment. Formally,

A coalition T **weak blocks** an assignment $a \in \mathcal{A}^f$ under a preference profile $(R_i)_{i\in N}$ when there is an announcement $b \in \mathcal{A}^f$ such that:

(i) For every $i \in T$ we have that $cR_i a$ for all $c \in \Theta_i(a, b, T, R)$.

(ii) There exists $i \in T$ such that $cP_i a$ for some $c \in \Theta_i(a, b, T, R)$.

That is, to block an assignment, a coalition requires to redistribute their own endowments and the agreement taken by the coalition be credible (condition of admissible expectations) in such form that at least one of its members expects that she will strictly improve her situation in at least one assignment that expect could be attained, without harming any other member of the coalition (conditions (i) and (ii)).

Notice that when individuals' expectations are single-valued (e.g., *optimistic*, *myopic*, and *dissolving* expectations) this blocking concept is equivalent to the one used previously. However, when expectation sets have more than one element (e.g., *prudent* expectations) the blocking concept used in this section is weakly than the used before. As a consequence, the *prudent* core is no more the biggest one, but neither is it contained in any of the others. Examples 2 and 3 show the intuition about these two facts.[15]

Example 2 *Consider the following problem with externalities where* $N = \{i_1, i_2, i_3, i_4\}$ *and the endowment is* $e = (e_1, e_2, e_3, e_4)$*. Denote* $a = (e_3, e_4, e_1, e_2)$*,* $b = (e_2, e_1, e_4, e_3)$ *and* $c = (e_2, e_1, e_3, e_4)$*. Consider the following preferences profiles:*

$$R_{i_1} : b \succ c \sim a$$

$$R_{i_2} : b \sim c \sim a$$

Then $\Theta^{\mathcal{P}}_{i_1,i_2}(a, b, \{i_1, i_2\}, R) = \{b, c\}$*, and* $\Theta^{\mathcal{D}}_{i_1,i_2}(a, b, \{i_1, i_2\}, R) = \{c\}$*.*

[15]Notice that the participative definition change. Specifically, assignment $a \in \mathcal{A}^f$ is **participative** if for all $i \in N$ either (i) $aP_i c$ for some $c \in \Theta_i(a, e_i, \{i\}, R)$, or (ii) $aI_i c$ for all $c \in \Theta_i(a, e_i, \{i\}, R)$.

If we consider the *weak blocking concept*, over *prudent* expectations the coalition $T = \{i_1, i_2\}$ blocks the assignment a announcing b. However, over *dissolving* expectations the coalition T does not do it.

Example 3 *Consider the following problem with externalities where* $N = \{i_1, i_2, i_3, i_4\}$ *and the endowment is* $e = (e_1, e_2, e_3, e_4)$. *Denote* $a = (e_3, e_4, e_1, e_2)$, $b = (e_2, e_1, e_4, e_3)$ *and* $c = (e_2, e_1, e_3, e_4)$. *Consider the following preferences profiles:*

$$R_{i_1} : c \succ a \succ b$$

$$R_{i_2} : c \sim b \sim a$$

Then $\Theta^{\mathcal{P}}_{i_1,i_2}(a, b, \{i_1, i_2\}, R) = \{b, c\}$, *and* $\Theta^{\mathcal{D}}_{i_1,i_2}(a, b, \{i_1, i_2\}, R) = \{c\}$.

If we consider the *weak blocking concept*, over *dissolving* expectations the coalition $T = \{i_1, i_2\}$ blocks the assignment a announcing b. However, over *prudent* expectations the coalition T does not do it.

Next, we show that under the *weak blocking concept*, additional to Assumption 1 and Assumption 2, we only require the following mild condition to prove that essentially single valued core is a necessary condition for the existence of a SIE-mechanism.

Assumption 6 *Let* $a \in \mathcal{A}^f$, $T \subseteq N$, *and* $R \in \mathcal{R}$. *If* $b \in \mathscr{I}(R)$, *then* $b \in \Theta_i(a, b, T, R)$ *and* cR_ie *for all* $c \in \Theta_i(a, b, T, R)$, *for all* $i \in T$.

Assumption 6 requires that when the coalition T announces a deviation toward an individually rational assignment, b, they expect that the process of renegotiation will lead to an individually rational assignment, and in particular the announcement b. This is consistent with the idea that any renegotiation process can be hindered by any agent leaving the coalition she belongs to with her endowment.

Next result shows that essentially single valued core is a necessary condition for the existence of an SIE-mechanism under *weak blocking concept* .

Theorem 2 *Assume that Assumptions 1,2,6 are satisfied. If there exists a Pareto efficient, individually rational, and strategy-proof mechanism,* Γ. *Then, in the domain* $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$, $\Gamma(R) \subseteq \mathscr{C}^*(R)$ *and the individually-rational expectational core correspondence* $\mathscr{C}^*$ *is essentially single-valued.*

Proof.

Fix an assignment problem with externalities $(N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i \in N})$ where $R_i \in \mathcal{R}^* = \prod_{i \in N} \mathcal{R}_i^*$. First, we prove that for a specific preferences profile any assignment in the core and the outcome of SIE-mechanism are indifferent to everyone. After, we show that the result holds for the true preferences.

Let $a \in \mathscr{C}^*(R)$ be an assignment in the core. For each $i \in N$, aR_ie by definition of the expectational core correspondence. For each $i \in N$, consider preferences $\widetilde{R}_i \in \mathcal{R}_i^*$ satisfying the requirements (i)-(ii) of Assumption 2. Hence, $a \in \mathscr{C}^*(\widetilde{R})$, because a is individually rational and any $T \subseteq N$ that blocks a under $\widetilde{R} = (\widetilde{R}_i)_{i \in N}$ blocks a under R, too.

We next prove that for every $b \in \mathscr{P}(\widetilde{R}) \cap \mathscr{I}(\widetilde{R})$ $b\widetilde{I}_ia$ for all $i \in N$. By contradiction suppose that

$$\exists j \in N, b\widetilde{P}_ja. \tag{1}$$

Consider a coalition T where each member is assigned to an assignment different to her endowment. Formally, $T = \{i \in N | b(i) \neq e(i)\}$. Since members outside the coalition keep their endowments,

$$\bigcup_{i \in T} b(i) = \bigcup_{i \in T} e(i).$$

Notice that

$$b\widetilde{P}_ja, \Rightarrow j \in T \tag{2}$$

because $b(i) = e(i)$ for all $i \in N \setminus T$ and Assumption 1 holds. Also, $\Theta_i(a, b, T, R) \neq \emptyset$ and $c(i) = b(i), \forall c \in \Theta_i(a, b, T, R)$ and $\forall i \in T$, because expectations are admissible. That is, each member of T conserves the assignment obtained in b, which is different from her endowment in all assignments each agent expects that could be effectively attained after the blocking announcement. This, jointly with Assumption 1 and Assumption 6, implies that $c\widetilde{P}_i e$ for all $i \in T$ and for all $c \in \Theta_i(a, b, T, R)$. It follows that by construction of $\widetilde{R}_i$

$$c\widetilde{R}_i a \text{ for all } i \in T \text{ and for all } c \in \Theta_i(a, b, T, R). \tag{3}$$

By (3), in particular, $b\widetilde{R}_i a$ for all $i \in T$. From (2) there exists $j \in T$ such that $b\widetilde{P}_j a$. Since (3) holds, coalition T blocks a by announcing b, which contradicts the fact that $a \in \mathscr{C}^*(\widetilde{R})$.

We conclude that $\nexists i \in N$ for which $b\tilde{P}_i a$, and hence $a\tilde{R}_i b$ for all $i \in N$. Assume that $a\tilde{P}_i b$ for some $i \in N$, then assignment a Pareto-dominates to assignment b but $b \in \mathscr{P}(\tilde{R})$ thus $a\tilde{I}_i b$ for all $i \in N$, in particular $a\tilde{I}_i \Gamma(\tilde{R})$ for all $i \in N$.

Next, we need to prove that results hold over original preferences R. As strategy-proofness does not depends on the characteristics of $\{\Theta_i\}_{i \in N}$, Claim 2 in Sönmez (1999) implies that $\Gamma(R) I_i a$ for all $i \in N$. ■

The intuition of the proof is similar to the one used at Sönmez (1999), but in our case is central the use of Assumption 6 to incorporate different types of expectations and thus allow less *optimistic* agents to be considered, which is more appropriate in contexts with externalities as was widely discussed earlier.

2.3 From essentially single valued cores to *SIE*-mechanisms

Now, we investigate the relation between core stability and the existence of a *SIE*-mechanism (see also Sönmez, 1999 and Ehlers, 2018).

Define $L(R_i, c) := \{d \in \mathcal{A}^f : cR_i d\}$ as the lower contour set of c relative to R_i, and $L^*(R_i, c) := \{d \in \mathcal{A}^f : cP_i d\}$ as the strict lower contour set of c relative to R_i. Before we show the main results of this section we establish an additional condition about the richness of the preference domain (see also Takamiya, 2003).

Assumption 7 *Let $i \in N$, $R_i \in \mathcal{R}_i$, and $b, c \in \mathcal{A}^f$ be such that cP_ib. Then for all $R'_i \in \mathcal{R}_i$, there exist $R^*_i \in \mathcal{R}_i$ such that*

(i) $L^*(b, R_i) \subseteq L^*(b, R^*_i)$, *and* $L(b, R_i) \subseteq L(b, R^*_i)$*; and*

(ii) $L^*(c, R'_i) \subseteq L^*(c, R^*_i)$, *and* $L(c, R'_i) \subseteq L(c, R^*_i)$.

Assumption 7 requires that, if i prefers assignment c to assignment b, then, for each R'_i, the domain $mathcalR_i$ contains another preference profile R^*_i such that:

(i) b improves in i's ranking (and no assignment below b reaches it) moving from R_i to R^*_i;

(ii) c improves in i's ranking (and no assignment below b reaches it) moving from R'_i to R^*_i.

Preference profile R^*_i can be interpreted as a mixture of R_i and R'_i in the sense that, moving from R_i and R'_i to R^*_i, both alternatives b and c improve upon without affecting their relative ranking.

We next show that if the expectational core correspondences is non-empty and essentially singled valued, then, any selection from it is an SIE-mechanism, if the expectational core is externally stable or, the preference domain satisfies Assumption 7.

Theorem 3 *Assume that the individually-rational expectational core correspondence $\mathscr{C}^*$ is essentially single-valued on $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$, and one of the following conditions holds:*

1. *The individually-rational expectational core is externally stable; or*

2. *Assumption 7 is satisfied.*

Then, any selection from $\mathscr{C}^*$ is a Pareto efficient, individually rational, participative, and weakly coalitional strategy-proof (then strategy-proof) mechanism on $\mathcal{R}$.

See the proof in Appendix 5.

Let Γ be a selection of the individually-rational expectational core correspondence. By definition Γ is individually rational, participative, and Pareto efficient. The proof of rest of the claim is by contradiction. We show that if the core is externally stable and a coalition T can manipulate Γ by making every member of T strictly better off, then there is a second coalition U blocking a core assignment, which yields a contradiction. Alternatively, if Assumption 7 holds and a coalition T can manipulate Γ by making every member of T strictly better off, we show that Γ is not Pareto efficient.

The reader might be tempted to infer that Proposition 1 in Sönmez (1999) (or Theorem 1 in Takamiya (2003) or Proposition 2 in Ehlers (2018)) implies the claim of Theorem 3 for the case of *prudent* expectations. Indeed the core under *optimistic* (resp. *myopic* resp. *dissolving*) expectations is always a subset of the core under *prudent* expectations (see Proposition 1). However, it might be the case that the core under *optimistic* (resp. *myopic* resp. *dissolving*) expectations is empty while the core under *prudent* expectations is not. (see Proposition 1). Thus, Theorem 3 generalizes Sönmez (1999, Proposition 1), Takamiya (2003, Theorem 1), and Ehlers (2018, Proposition 2).

When the core is empty or contains two allocation that are not indifferent to all agents, Theorem 3 is silent. If agents have *prudent* expectations, the expectational core is more likely to be not empty than if agents have, for instance, *optimistic* expectations. However it is also more likely that it contains multiple non indifferent assignments. We next present two examples that make it evident this trade-off.[16]

First, we present a an example where $\mathscr{C}_P^*$ is essentially single-valued and externally stable, then by Theorem 3 there exists an SIE mechanism, but $\mathscr{C}_O^*$ is empty for some preference profile.

Example 4 *Consider the following house assignment problem with externalities where* $N = \{i_1, i_2, i_3, i_4\}$ *and the endowment is* $e = (h_1, h_2, h_3, h_4)$*. Denote* $a = (h_1, h_2, h_4, h_3)$

[16] In the examples we use $(\dots)$ to point out that it doesn't matter which order the preferences follow.

and $b = (h_1, h_3, h_2, h_4)$. Consider the following preferences profiles:

$$R_{i_1} : b \succ a \succ e \ldots,$$

$$R_{i_2} = R_{i_3} = R_{i_4} : a \succ \ldots,$$

Additional consider

$$R'_{i_1} : a \succ \ldots.$$

Define $R = (R_{i_1}, R_{i_2}, R_{i_3}, R_{i_4})$ and $R' = (R'_{i_1}, R_{i_2}, R_{i_3}, R_{i_4})$. Assume that $\mathcal{R}_{i_1} = \{R_{i_1}, R'_{i_1}\}$, $\mathcal{R}_{i_2} = \{R_{i_2}\}$, $\mathcal{R}_{i_3} = \{R_{i_3}\}$, $\mathcal{R}_{i_4} = \{R_{i_4}\}$. Then $\mathcal{R} = \{R, R'\}$. It is straight to see $\mathscr{C}^*_P(R) = \mathscr{C}^*_P(R') = \mathscr{C}^*_S(R') = \{a\}$ but $\mathscr{C}^*_S(R) = \emptyset$. Notice that $\mathscr{C}^*_P$ is essentially single-valued and expectational core is externally stable. Then any selection from $\mathscr{C}^*_P$ is a SIE-mechanism on $\mathcal{R}$.

Next, we present a case where $\mathscr{C}^*_O$ is essentially single-valued and externally stable thus there exists a SIE-mechanism but $\mathscr{C}^*_P$ is not essentially single-valued.

Example 5 *Consider the following house assignment problem with externalities where $N = \{i_1, i_2, i_3\}$ and the endowment is $e = (h_1, h_2, h_3)$. Denote $a = (h_1, h_3, h_2)$, $b = (h_2, h_1, h_3)$, $c = (h_2, h_3, h_1)$, $d = (h_3, h_1, h_2)$, and $f = (h_3, h_2, h_1)$. Consider the following preferences profiles:*[17]

$$R_{i_1} = R_{i_3} : a \succ c \succ e \succ \ldots,$$

$$R_{i_2} : c \succ a \succ e \succ \ldots.$$

Additional consider

$$R'_{i_3} : c \succ a \succ e \succ \ldots.$$

Define $R = (R_{i_1}, R_{i_2}, R_{i_3})$ and $R' = (R_{i_1}, R_{i_2}, R'_{i_3})$. Assume that $\mathcal{R}_{i_1} = \{R_{i_1}\}$, $\mathcal{R}_{i_2} = \{R_{i_2}\}$, and $\mathcal{R}_{i_3} = \{R_{i_3}, R'_{i_3}\}$. Then $\mathcal{R} = \{R, R'\}$. It's easy to see that $\mathscr{C}^*_S(R) = \mathscr{C}^*_S(R') = \{a\}$ and $\mathscr{C}^*_P(R) = \mathscr{C}^*_P(R') = \{a, c\}$. Notice that $\mathscr{C}^*_S$ is essentially single-valued and

[17] The symbol $\succ$ denote strict preference, and $(\ldots)$ in the preferences profiles means that the rest of the assignments can follow in any order.

externally stable. Then, it follows from Theorem 3 any selection from $\mathscr{C}_S^*$ is a SIE-mechanism on $\mathcal{R}$.

3 The largest and the minimal cores: *prudent* and *optimistic* expectations

For every system of expectations, the core generated by *optimistic* and *prudent* expectation yield the smallest and the largest cores, respectively.

Proposition 1 *Let $\{\Theta_i(\cdot)\}_{i\in N}$ be a system of admissible expectations and be $\mathscr{C}$ and $\mathscr{C}^* = \mathscr{C} \cap \mathscr{I}$, be the core and the individually rational core supported by $\{\Theta_i(\cdot)\}_{i\in N}$, then*

1. $\mathscr{C}_{\mathcal{O}} \subseteq \mathscr{C} \subseteq \mathscr{C}_{\mathcal{P}}$,
2. $\mathscr{C}_{\mathcal{O}}^* \subseteq \mathscr{C}^* \subseteq \mathscr{C}_{\mathcal{P}}^*$.

Proof. We prove 1. It suffices to show: (i) if coalition T blocks $a \in \mathcal{A}^f$ under $\left\{\Theta_i^{\mathcal{P}}(\cdot)\right\}_{i\in N}$ through $b \in \mathcal{A}^f$, then coalition T blocks a under $\{\Theta_i(\cdot)\}_{i\in N}$ through b; (ii) if coalition T blocks $a \in \mathcal{A}^f$ under $\{\Theta_i(\cdot)\}_{i\in N}$ through b, then there exists $c \in \mathcal{A}^f$ such that coalition T blocks a under $\left\{\Theta_i^{\mathcal{O}}(\cdot)\right\}_{i\in N}$ through c. Part (i) follows from $\Theta_i(\cdot) \subseteq \Theta_i^{\mathcal{P}}(\cdot)$ for all $i \in N$. Let's prove (ii). Assume that coalition T blocks a under $\{\Theta_i(\cdot)\}_{i\in N}$ through b, then coalition T blocks a under $\left\{\Theta_i^{\mathcal{O}}(\cdot)\right\}_{i\in N}$ through b. Then for all $c \in \Theta_i^{\mathcal{O}}(a,b,T,R)$, cR_ia for every $i \in T$ and cP_ja for some $j \in T$. Thus coalition T blocks a under $\left\{\Theta_i^{\mathcal{O}}(\cdot)\right\}_{i\in N}$ through any $c \in \Theta_i^{\mathcal{O}}(a,b,T,R)$. Now, completing the proof to show 2. is straightforward.
■

The intuition for the result is simple: the larger the expectational correspondence, the more difficult is to block an assignment; the smaller the expectational correspondence, the easier is to block an assignment. In fact, notice that under *prudent* expectation $c \in \Theta_i^{\mathcal{P}}(a,b,T,R) \iff c(k) = b(k), \forall k \in T$. Then, among the admissible expectations, the individually-rational expectational core correspondence under *prudent* expectations are the largest core consistent with the idea that the agent within a deviating coalition can use only the resources of the same coalition.[18]

From Proposition 1, we have.

Corollary 1 *For all $R \in \mathcal{R}$:*

1. $\mathscr{C}_{\mathcal{O}}(R) \subseteq \mathscr{C}_{\mathcal{M}}(R) \cap \mathscr{C}_{\mathcal{D}}(R)$ *and* $\mathscr{C}^*_{\mathcal{O}}(R) \subseteq \mathscr{C}^*_{\mathcal{M}}(R) \cap \mathscr{C}^*_{\mathcal{D}}(R)$.

2. $\mathscr{C}_{\mathcal{M}}(R) \cup \mathscr{C}_{\mathcal{D}}(R) \subseteq \mathscr{C}_{\mathcal{P}}(R)$ *and* $\mathscr{C}^*_{\mathcal{M}}(R) \cup \mathscr{C}^*_{\mathcal{D}}(R) \subseteq \mathscr{C}^*_{\mathcal{P}}(R)$.

3. $\mathscr{C}^*_{\mathcal{P}}(R) \subseteq \mathscr{P}(R) \cap \mathscr{I}(R)$.

The following diagram illustrates these relationships.

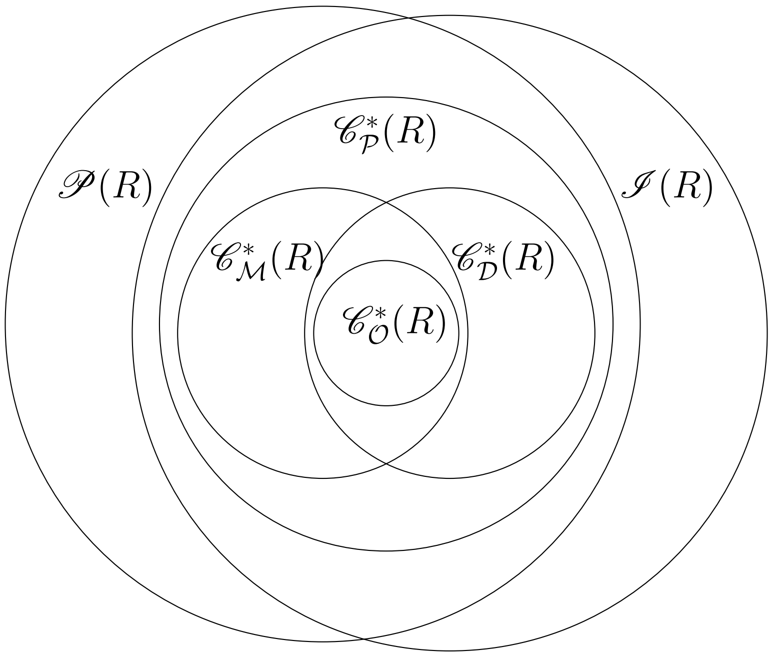

Figure 1. Relationship between individually-rational expectational cores

Proof.

1. Fix $R \in \mathcal{R}$, assume that $a \in \mathscr{C}^*_{\mathcal{O}}(R)$, then there is no announcement $b \in \mathcal{A}^f$ and $T \subseteq N$ such that (i) $\cup_{i \in T} b(i) = \cup_{i \in T} e(i)$, (ii) $bR_i a$ for all $i \in T$ and (iii) $bP_i a$ for some $i \in T$. This is true in particular for the following two assignments:

[18] In Section 3 we explain more about de largest core notion.

- c such that $c(k) = b(k)$ for all $k \in T$ and $(a(k) \setminus e(T)) \cup (a(T) \cap e(k))$ for all $k \notin T$.

- c such that $c(k) = b(k)$ for all $k \in T$ and $c(k) = e(k)$ for all $k \notin T$

Then $a \in \mathscr{C}^*_{\mathcal{M}}(R) \cap \mathscr{C}^*_{\mathcal{D}}(R)$. Now, to see that inclusions could be strict, consider the following example. Assume that $N = \{i_1, i_2, i_3\}$, and $e = (e_{i_1}, e_{i_2}, e_{i_3})$ is the endowment. Let $a = (e_{i_1}, e_{i_3}, e_{i_2})$ and $b = (e_{i_2}, e_{i_1}, e_{i_3})$ be two assignments. Assume that for all $c \in \mathcal{A}^f \setminus \{a, b\}$ the preferences are $aP_{i_1}bP_{i_1}c$ to agent i_1; $bP_{i_2}aP_{i_2}c$ to agent i_2; and $bP_{i_3}aP_{i_3}c$ to agent i_3. Then, $b \in \mathscr{C}^*_{\mathcal{M}}(R) \cap \mathscr{C}^*_{\mathcal{D}}(R)$ but $b \notin \mathscr{C}^*_{\mathcal{O}}(R)$ because i_1 blocks b announcing a since $\Theta^{\mathcal{O}}_{i_1}(b, a, \{i_1\}, R) = \{a\}$.[19] The relations hold when cores have no individually-rational assignments since expectations specifications don't depend on the individually-rational notion.

2. $\mathscr{C}^*_{\mathcal{D}}(R) \subseteq \mathscr{C}^*_{\mathcal{P}}(R)$ and $\mathscr{C}^*_{\mathcal{M}}(R) \subseteq \mathscr{C}^*_{\mathcal{P}}(R)$ come from Proposition 1.

3. $\mathscr{C}^*_{\mathcal{P}}(R) \subset \mathcal{I}(R)$ by definition (remember that $\mathscr{C}^*_{\mathcal{P}}(R) = \mathscr{C}_{\mathcal{P}}(R) \cap \mathcal{I}(R)$). On the other hand, since $\Theta^{\mathcal{P}}_i(\cdot)$ is an admissible expectations, $\mathscr{C}^*_{\mathcal{P}}(R) \subset \mathscr{P}(R)$. ■

Under Assumptions 1 and 2, and using Dissolving expectations, Ehlers (2018) showed necessary conditions to the existence of an SIE-mechanism.[20] In addition, he demonstrated that the individually-rational core is the largest one in that context. We show that by imposing certain *additional* conditions on expectations and feasible assignments, it is possible that results holds for larger cores, which allows us to include others blocking concepts widely used in assignment problems with externalities. Then notice that our results do not contradict the Ehlers' results.

The following examples show the advantage of having a model that allows larger cores for contexts with externalities. The first example shows an environment where there exists an SIE-mechanism, but only *prudent* specification generates a core different from empty in which the fulfillment of the Theorem 1 is nontrivial. On the other hand, the

[19]Notice that $\Theta^{\mathcal{P}}_{i_1}(b, a, \{i_1\}, R) = \Theta^{\mathcal{M}}_{i_1}(b, a, \{i_1\}, R) = \{e\}$ and since $bP_{i_1}e$ the agent i_1 decide does not block. Nobody else blocks since b is the best option for i_2 and i_3.

[20]In Ehlers (2018) also is present sufficient condition for which is used an additional Assumption, similar to our Assumption 7 but with upper contour instead of a lower one.

second example shows a case where the core produced by *prudent* expectations is not single-valued, and then we can conclude by Theorem 1 that there is no SIE-mechanism, however, the core generated with other specifications are empty and therefore useless to conclude something about the existence of SIE-mechanism.

Example 6 *Assume that* $N = \{i_1, i_2, i_3, i_4, i_5\}$ *and the endowment is* $e = (h_1, h_2, h_3, h_4, h_5)$. *Let* $a = (h_4, h_3, h_2, h_5, h_1)$, $b = (h_2, h_1, h_3, h_4, h_5)$, $c = (h_2, h_1, h_5, h_4, h_3)$, $d = (h_3, h_4, h_1, h_5, h_2)$, *and* $f = (h_1, h_2, h_4, h_3, h_5)$. *The set of feasible assignments is* $\mathcal{A}^f = \{a, b, c, d, e, f\}$. *The preferences are:*

$$\widetilde{R}_{i_1} = \widetilde{R}_{i_2} : d \succ b \succ a \succ c \succ e \backsim f;$$

$$\widetilde{R}_{i_3} = \widetilde{R}_{i_4} : a \backsim f \succ d \succ b \backsim e \succ c;$$

$$\widetilde{R}_{i_5} : a \backsim d \succ b \backsim e \backsim f \succ c.$$

Based on the preferences $\widetilde{R}$ *we generate* $\mathcal{R}$. *More precisely, each* $R_i \in \mathcal{R}_i$ *is formed by improving the position of the endowment* e *(along with the other assignments that leave* i *with her initial endowment) and maintaining the relative position of all other assignments.* $\mathcal{R}$ *fulfills Assumptions 1 and 2. We work we admissible expectations and Assumption 4 is satisfied.*

For all $R \in \mathcal{R}$, consider the following mechanism:

$$\widetilde{\Gamma}(R) = \begin{cases} a, & \text{when } a \text{ Pareto-dominates } e \text{ under } R, \\ f, & \text{in any other case.} \end{cases}$$

Notice that $\widetilde{\Gamma}$ is an SIE-mechanism. Indeed, $\widetilde{\Gamma}(R) \subseteq \mathscr{C}^*(R)$ in the domain $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$.[21]

By Theorem 1, the individually-rational expectational core correspondence $\mathscr{C}^*$ is essentially single-valued in the domain $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$. However notice that

[21] More details are in the Appendix 5.

under $\widetilde{R}$ preferences, $a \in \mathscr{C}_P^*(R)$ while $\mathscr{C}_S^*(R) = \mathscr{C}_E^*(R) = \mathscr{C}_M^*(R) = \emptyset$.

Example 7 highlights something important. Before the inclusion of *prudent* notion, an economist could not answer if there exists SIE-mechanisms for some kinds of environments.

Example 7 *Assume that* $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ *and the endowment is* $e = (h_1, h_2, h_3, h_4, h_5, h_6)$. *Denote* $a = (h_2, h_6, h_4, h_5, h_3, h_1)$, $b = (h_2, h_1, h_3, h_4, h_5, h_6)$, $c = (h_2, h_1, h_5, h_4, h_3, h_6)$, $d = (h_3, h_4, h_1, h_5, h_2, h_6)$, $f = (h_3, h_4, h_5, h_6, h_1, h_2)$, *and* $g = (h_2, h_1, h_4, h_5, h_3, h_6)$. *The set of feasible assignments is* $\mathcal{A}^f = \{a, b, c, d, e, f\}$. *The preferences are:*

$$\widetilde{R}_{i_1} = \widetilde{R}_{i_2} : b \backsim d \backsim g \succ f \succ a \succ c \succ e$$

$$\widetilde{R}_{i_3} : a \backsim f \succ d \succ b \backsim e \succ c \succ g \qquad \widetilde{R}_{i_4} : f \succ a \succ d \succ b \backsim e \backsim c \succ g$$

$$\widetilde{R}_{i_5} : a \backsim d \succ f \succ e \backsim b \succ c \succ g \qquad \widetilde{R}_{i_6} : a \succ f \succ e \backsim b \backsim c \backsim d \backsim g$$

Based on the preferences $\widetilde{R}$ *we generate* $\mathcal{R}$. *More precisely, each* $R_i \in \mathcal{R}_i$ *is formed by improving the position of the endowment* e *(along with the other assignments that leave* i *with her initial endowment) and maintaining the relative position of all other assignments.* $\mathcal{R}$ *fulfills Assumptions 1 and 2. We work we admissible expectations and Assumption 4 is satisfied.*

In this case, $a, f \in \mathscr{C}_P^*(R)$ (i.e., $\mathscr{C}_P^*(R)$ is not essentially single-valued) while $\mathscr{C}_S^*(R) = \mathscr{C}_E^*(R) = \mathscr{C}_M^*(R) = \emptyset$.

Then, by Theorem 1, there is no SIE-mechanism.

4 Conclusions

We show that core essentially single-valued is a necessary and sufficient condition to the existence of SIE-mechanism. Then, we managed to relate a property motivated by non-cooperative behavior as is strategy-proofness, with one of the most important concepts in cooperative game theory: the core. Moreover, these results hold for a very wide variety of blocking notions for contexts with externalities.

5 Appendix

Example to show that in the case of *myopic* expectations the individual rationality is not stronger nor weaker than the participative constraint.

Example 8 *Let* $N = \{i_1, i_2, i_3, i_4\}$ *be the set of agents, and let* $e = (e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ *be the endowment. Consider the following assignments:* $a = (e_{i_2}, e_{i_1}, e_{i_4}, e_{i_3})$ *and* $b = (e_{i_1}, e_{i_2}, e_{i_4}, e_{i_3})$. *Suppose that* $bP_{i_1}aP_{i_1}eP_{i_i}$ *to agent* i_1*, and assignment* a *is the first top raking for all others agents. Clearly* a *is individually rational. But if agent* i_1 *has myopic expectations she will block* a *since* $\Theta^{\mathcal{M}}_{i_1}(a, e, \{i_1\}, R) = \{b\}$ *and* b *is her most preferred assignment. Then assignment* a *is no participative. Now assume that agent* i_1*'s preferences are* $eP'_{i_1}aP'_{i_1}bP'_{i_i}$ *and define* $R' = (R_{-i_1}, R'_{i_1})$. *In this case the assignment* a *is no individually rational because for agent* i_1 *it is worse than endowment. But assignment* a *fulfills with the individual participation constraint because* $\Theta^{\mathcal{M}}_{i_1}(a, e, \{i_1\}, R') = \Theta^{\mathcal{M}}_{i_1}(a, b, \{i_1\}, R') = \{b\}$*, and* a *is better than* b *to agent* i_1 *and it is the best to other agents.*

Proof of Theorem 3.

Assume that $\mathscr{C}^*$ is essentially single-valued on $\mathcal{R}$. Let $\Gamma : \mathcal{R} \to \mathcal{A}^f$ be a selection of the individually-rational expectational core correspondence, that is $\Gamma(R) \in \mathscr{C}^*(R)$ for all $R \in \mathcal{R}$. By definition, Γ is individually rational, Pareto efficient, and participative. We next prove that Γ is weakly coalitional strategy-proof if either condition 1 or 2 holds. By contradiction, suppose that Γ is not weakly coalitionally strategy-proof. Then, there are preference profiles $R, \widetilde{R} \in \mathcal{R}$ and a coalition $T \subseteq N$ such that $\Gamma(\widetilde{R}_T, R_{-T})\, P_i\, \Gamma(R)$ for all $i \in T$. Since $\mathscr{C}^*$ is essentially single-valued, we have that $\Gamma(\widetilde{R}_T, R_{-T}) \notin \mathscr{C}^*(R)$.

1. Suppose that individually-rational expectational core is externally stable. Then there exists a coalition U that block $\Gamma(\widetilde{R}_T, R_{-T})$ announcing an assignment b such that

(a) $\bigcup_{k\in U} b(k) = \bigcup_{k\in U} e_k$ and $\Theta_i(\Gamma(\widetilde{R}_T, R_{-T}), b, U, R) \neq \emptyset$, for every $i \in N$;

(b) for each $i \in U$ we have that $cR_i\Gamma(\widetilde{R}_T, R_{-T})$ for all $c \in \Theta_i(\Gamma(\widetilde{R}_T, R_{-T}), b, U, R)$;

(c) there exists $i \in U$ such that $cP_i\Gamma(\widetilde{R}_T, R_{-T})$ for all $c \in \Theta_i(\Gamma(\widetilde{R}_T, R_{-T}), b, U, R)$;

(d) for each $i \in U$, $\mathscr{C}^*(R) \cap \Theta_i(\Gamma(\widetilde{R}_T, R_{-T}), b, U, R) \neq \emptyset$.

From (b) and (d) follows $\Gamma(R)\, R_i\, \Gamma(\widetilde{R}_T, R_{-T})$ for all $i \in U$ because the core is essentially single-valued. Since $\Gamma(\widetilde{R}_T, R_{-T})\, P_i\, \Gamma(R)$ for all $i \in T$, it follows that $T \cap U = \emptyset$, which in turn implies that U blocks $\Gamma(\widetilde{R}_T, R_{-T})$ when preferences are $(\widetilde{R}_T, R_{-T})$. This yields a contradiction since $\Gamma(\widetilde{R}_T, R_{-T})$ belongs to $\mathscr{C}^*(\widetilde{R}_T, R_{-T})$.

2. Assume that the domain $\mathcal{R}$ satisfies Assumption 7. For each $i \in T$, choose a preference relation R_i^* such that

(a) $L^*(\Gamma(R), R_i) \subseteq L^*(\Gamma(R), R_i^*)$, and $L(\Gamma(R), R_i) \subseteq L(\Gamma(R), R_i^*)$; and

(b) $L^*(\Gamma(\widetilde{R}_T, R_{-T}), \widetilde{R}_i) \subseteq L^*(\Gamma(\widetilde{R}_T, R_{-T}), R_i^*)$, and $L(\Gamma(\widetilde{R}_T, R_{-T}), \widetilde{R}_i) \subseteq L(\Gamma(\widetilde{R}_T, R_{-T}), R_i^*)$.

Assumption 7 assures that $\mathcal{R}_i$ includes R_i^* for all $i \in T$. Consider the preference profile (R_T^*, R_{-T}). By construction of (a), the assignment $\Gamma(R)$ keeps or improves her relative ranking from R_i to (R_{-T}, R_T^*) for all $i \in N$. Then assuming that agents' expectations do not change from R_i to (R_T^*, R_{-T}), if some coalition $S \subseteq N$ blocks $\Gamma(R)$ over (R_T^*, R_{-T}) also blocks over R which contradicts that $\Gamma(R) \in \mathscr{C}^*(R)$, then $\Gamma(R) \in \mathscr{C}^*(R_T^*, R_{-T})$. Moreover, since $\mathscr{C}^*$ is essentially single-valued, $\Gamma(R_{-T}, R_T^*)I_i^*\Gamma(R)$ for all $i \in T$, and $\Gamma(R_T^*, R_{-T})I_i\Gamma(R)$ for all $i \in N \setminus T$. Similarly, from (b), we have that $\Gamma(R_T^*, R_{-T})I_i^*\Gamma(\widetilde{R}_T, R_{-T})$ for all $i \in T$, and $\Gamma(R_T^*, R_{-T})I_i\Gamma(\widetilde{R}_T, R_{-T})$ for all $i \in N \setminus T$. Then, $\Gamma(R)I_i^*\Gamma(\widetilde{R}_T, R_{-T})$ for all $i \in T$, and $\Gamma(R)I_i\Gamma(\widetilde{R}_T, R_{-T})$ for all $i \in N \setminus T$. Since $\Gamma(\widetilde{R}_T, R_{-T})P_i\Gamma(R)$ for all $i \in T$, the grand coalition N blocks $\Gamma(R)$ announcing $\Gamma(\widetilde{R}_T, R_{-T})$, which contradicts $\Gamma(R) \in \mathscr{C}^*(R)$. ■

Claim 1 *Consider the House Assignment Problem with Externalities shown in Example 6. The following statements are true:*

(i) The mechanism $\widetilde{\Gamma}$ is an SIE-mechanism.

(ii) $\widetilde{\Gamma}(R) \subseteq \mathscr{C}^*(R)$ *in the domain* $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$.

Proof of Claim 1.

First, we prove that $\widetilde{\Gamma}$ is an SIE-mechanism.[22]

(i) Strategy-proof. Over true preferences R, $\widetilde{\Gamma}(R) = a$. Then, the only agents who might have incentives to lie are i_1 and i_2, since both have exactly preferences it is enough analyzed one of them, without loss of generality, let's analyze i_1. Assignments that have a better ranking than a to i_1 are b and d, but there is no way in which i_1 can lie that generates such assignments as an output of $\widetilde{\Gamma}$.

(ii) Individually rational. Let $\widetilde{R} \in \mathcal{R}$ be any preferences profile admissible If $\widetilde{\Gamma}(\widetilde{R}) = a$ by definition of $\widetilde{\Gamma}$ we know that a Pareto-dominates to e then for all $i \in N$, $a\widetilde{R}_i e$, and then $\widetilde{\Gamma}(\widetilde{R}) = a$ is individually rational. On the other hand, notice that $\forall \widetilde{R} \in \mathcal{R}$ and $i \in N$, $f\widetilde{R}_i e$ because by Assumption 1 $f\widetilde{I}_i e$ for all $i \in \{i_1, i_2, i_5\}$ and since $\forall i \in N$ and all pair $s, r \in \mathcal{A}^f$ such that $s(i) \neq e(i)$ and $r(i) \neq e(i)$, $\widetilde{R}$ keeps their relative order, we know that $f\widetilde{P}_i e$ for all $i \in \{i_3, i_4\}$. Then, $\widetilde{\Gamma}(\widetilde{R}) = f$ is individually rational.

(iii) Pareto-Efficient. By contradiction, assume that for some $\widetilde{R} \in \mathcal{R}$ we have that there exists an assignment $s \in \mathcal{A}^f \setminus \{\widetilde{\Gamma}(\widetilde{R})\}$ that Pareto-dominates $\widetilde{\Gamma}(\widetilde{R})$ over $\widetilde{R}$.

- Case $\widetilde{\Gamma}(\widetilde{R}) = a$. By definition of $\widetilde{\Gamma}$ we know that a Pareto-dominates to e, then $s \neq e$. Since $\forall i \in N$ and all pair of assignments $s, r \in \mathcal{A}^f$ such that $s(i) \neq e(i)$ and $r(i) \neq e(i)$, $\widetilde{R}$ keeps their relative order, the only candidate could be the assignment f, but by Assumption 1 it is impossible that $f\widetilde{R}_{i_5} a$, contradiction.
- Case $\widetilde{\Gamma}(\widetilde{R}) = f$. Since for all admissible preference profile f is the top-ranking for i_3 and i_4, the only candidate is the assignment a but then $a\widetilde{P}_i f$

[22]Additionally to $\widetilde{R}$, according to the description, the possible preferences to agents i_1 and i_2 are: $R'_{i_1,i_2} : d \succ f \sim e \succ b \succ a \succ c$, $R''_{i_1,i_2} : d \succ b \succ f \sim e \succ a \succ c$, $R'''_{i_1,i_2} : d \succ b \succ a \succ f \sim e \succ c$; agents i_3 and i_4 is $R'_{i_3,i_4} : a \succ e \sim b \succ d \succ c$; agent i_5 is $R'_{i_5} : a \succ e \sim f \succ b \succ c$.

for $i \in \{i_1, i_2, i_5\}$ and by Assumption 1 $a\tilde{P}_i e$ for $i \in \{i_1, i_2, i_5\}$ and since for all admissible preference profile a is the top-ranking for $i \in \{i_3, i_4, i_5\}$ then a Pareto-dominates e, contradiction.

Second, we proof that $\widetilde{\Gamma}(R) \subseteq \mathscr{C}^*(R)$ in the domain $\mathcal{R}^* = \{R \in \mathcal{R} : \mathscr{C}^*(R) \neq \emptyset\}$. To verify this we prove that if we assume that for some $\widetilde{R} \in \mathcal{R}$, $\widetilde{\Gamma}(\widetilde{R}) = a$ and $a \notin \mathscr{C}_P^*(\widetilde{R})$ implies that $\mathscr{C}_P^*(\widetilde{R}) = \emptyset$. By contradiction assume that $\widetilde{\Gamma}(\widetilde{R}) = a$, $a \notin \mathscr{C}_P^*(\widetilde{R})$ and there exists an assignment $s \in \mathcal{A}^f \setminus \{a\}$ such that $s \in \mathscr{C}_P^*(\widetilde{R})$. We know that $s \neq e$ because a Pareto-dominates e and then the grand coalition N blocks a. Also, $s \notin \{b, c, d\}$ because for all $\tilde{R} \in \mathcal{R}$, $\{i_3, i_4\}$ blocks b, c or d announcing f since f is their top-ranked and in f they redistribute their endowments. Then $s = f$. But, since a Pareto-dominates e we know that a Pareto-dominates f also, because by Assumption 1 aP_ie then aP_if for $i \in \{i_1, i_2, i_5\}$ and aI_if for $i \in \{i_3, i_4\}$ by how the set of admissible preferences is constructed. Therefore grand coalition blocks f announcing a, then $\mathscr{C}_P^*(\widetilde{R}) = \emptyset$, contradiction. Now assume that for some $\widetilde{R} \in \mathcal{R}$, $\widetilde{\Gamma}(\widetilde{R}) = f$, $f \notin \mathscr{C}_P^*(\widetilde{R})$ and there exists an assignment $s \in \mathcal{A}^f \setminus \{f\}$ such that $s \in \mathscr{C}_P^*(\widetilde{R})$. ■

Examples about the restriction of Assumption 5 to Myopic and Dissolving expectations

(i) *Case where $\mathscr{C}_E^*$ is trivially the largest core.* Assume that $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and the endowment is $e = (h_1, h_2, h_3, h_4, h_5, h_6)$ of some indivisible good (think in houses for example). Denote $a = (h_3, h_1, h_2, h_6, h_4, h_5)$, $b = (h_2, h_3, h_1, h_6, h_4, h_5)$, and $c = (h_2, h_3, h_1, h_5, h_6, h_4)$. The set of feasible assignments is $\mathcal{A}^f = \{a, b, c, e\}$. The preferences are: $bP_icP_iaP_ie$ for $i \in \{i_1, i_2, i_3\}$ and $aP_ibP_icP_ie$ for $i \in \{i_4, i_5, i_6\}$. This example fulfills Assumption 5. First, notice that over P the assignments e and c are blocked by N announcing b for any $\Theta_i \in \Omega_i$. On the other hand, $T = \{i_1, i_2, i_3\}$ blocks a announcing b when agents have Prudent, Myopic or Optimistic expectations. Nevertheless, when agents have Dissolving expectations T does not block a because $\Theta_i^E(a, b, T, R) = \emptyset$ (since the assignment $(h_2, h_3, h_1, h_4, h_5, h_6)$ does not exist), indeed $a \in \mathscr{C}^*_E(R)$. Finally, assignment b is blocked by $T = \{i_4, i_5, i_6\}$ announcing a when agents have Optimistic expectations, but no when they have Prudent, Myopic, and Dissolving expectations. Then, in this example: $\mathscr{C}^*_S(R) = \emptyset$, $\mathscr{C}^*_P(R) = \mathscr{C}^*_M(R) = \{b\}$ and $\mathscr{C}^*_E(R) = \{a, b\}$.

(ii) *Case where $\mathscr{C}^*_M$ is trivially the largest core.* Assume that $N = \{i_1, i_2, i_3\}$ and the endowment is $e = (h_1, h_2, h_3, h_4)$. Denote $a = (h_2, h_1, h_4, h_3)$. Assume that $\mathcal{A}^f = \{a, e\}$. The preferences are eP_ia for $i \in \{i_1, i_4\}$ and aP_ie for $i \in \{i_2, i_3\}$. This example fulfills Assumption 5. The assignment a is blocked by $\{i_1\}$, announcing e when i_1 has Prudent, Optimistic, or Dissolving expectations, but no if i_1 has Myopic expectations because $\Theta^M_i(a, e, \{i_1\}, P) = \emptyset$ (since (h_1, h_2, h_4, h_3) does not exist. Specifically $\mathscr{C}^*_S(R) = \mathscr{C}^*_P(R) = \mathscr{C}^*_E(R) = \{e\}$ and $\mathscr{C}^*_M(R) = \{e, a\}$.

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