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# Regions, extensions, distances, diameters

**Claudio Calosi**

Department of Philosophy, University of Geneva

**Correspondence**
Claudio Calosi, Department of Philosophy, University of Geneva.
Email: claudio.calosi@unige.ch

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**Abstract**

Extended simple regions have been the focus of recent developments in philosophical logic, metaphysics, and philosophy of physics. However, only a handful of works provides a rigorous characterization of an extended simple region. In particular, a recent paper in this journal defends a definition based on an extrinsic notion of *least distance*. Call it the *Least Distance* proposal. This paper provides the first assessment of it. It argues that *Least Distance* faces difficulties and drawbacks. The paper then goes on to suggest a different proposal, the *Diameter* proposal that is able to handle such drawbacks and difficulties.

KEYWORDS
diameter, extended simples, extension, least distance

## 1 | EXTENSION, EXTENDED SIMPLES, EXTENDED SIMPLE REGIONS

*Extension* plays a crucial role in our conceptual system and in our overall metaphysics of the concrete world. One of the most basic disagreement throughout scholastic and modern philosophy was about whether the substratum of change was extended.[1] In effect, extension has been suggested and defended as the very hallmark of materiality or physicality.[2] And these are only two examples.

[1] See Pasnau (2011: 539).

[2] By Hobbes and Descartes for example. For a recent discussion see Markosian (2000).

*Extended Simples* are credited as providing the bedrock of one of the three main theories on the ultimate constitution of matter—the other theories focusing on unextended atoms and gunk.[3] Recently extended simples have been the focus of investigations in philosophical logic,[4] metaphysics,[5] and philosophy of physics.[6] Extended Simples come (at least) in two varieties: extended simple regions and extended simple objects. If one has a suitable locative notion such as "exact location",[7] one could easily define an extended simple object as a simple material object that is exactly located at an extended region.[8]

*Extended Simple Regions* have been used crucially in recent arguments against e.g., *parsimonious theories of location*,[9] and the possibility of *heterogeneous simples*.[10]

Oddly enough, there are only a few works that focus precisely on the notion of extension in general and on a precise characterization of an extended simple region in particular.[11] A recent paper in this journal—Goodsell, Duncan and Miller (2020)—proposes a definition of extension in terms of the notion of *least distance* that explicitly allows for extended simple regions. Let me call it, unimaginatively, the *Least Distance* proposal. The proposal is not without its difficulties. It defines extension in terms of an extrinsic notion of distance that is metrical in nature. In this paper I suggest a different notion of extension that is (i) broadly metrical in nature, (ii) intrinsic, (iii) allows for extended simple regions, and (iv) does not face the same difficulties as *Least Distance*.

The structure is as follows. I first present the *Least Distance* proposal (§2), and then provide the first thorough assessment of it (§3). The resulting critique is not important just on its own. It reveals what I take to be the correct spirit of that proposal. And this in turn paves the way for an alternative that, for reasons that will be obvious shortly, I call the *Diameter* proposal. First I introduce such a proposal (§4), and then I offer one significant development (§5). A brief look ahead at further possible developments concludes the paper (§6).

## 2 | THE *LEAST DISTANCE* PROPOSAL

Let me start with a few notions that I will use throughout. The most general notion is that of a partially ordered set, or poset $(\mathcal{R}, \leq)$. On the intended interpretation $\mathcal{R}$ is the domain of regions. I will use $r_i$ terms to refer to its members. The $\leq$ order on $\mathcal{R}$ is just parthood. In general, I only assume it is a partial order. The following are common mereological definitions. A proper part ($\ll$) of a region is a distinct part of that region: $r_1 \ll r_2 \equiv_{\text{df}} r_1 \leq r_2 \wedge r_1 \neq r_1$. Two regions overlap ($\circ$) iff they share a part: $r_1 \circ r_2 \equiv_{\text{df}} \exists r_3(r_3 \leq r_1 \wedge r_3 \leq r_2)$. Two regions are disjoint ($\wr$) iff they do not overlap: $r_1 \wr r_2 \equiv_{\text{df}} \neg(r_1 \circ r_2)$. A region is simple or atomic ($A$) iff it does not have proper parts: $A(r) \equiv_{\text{df}} \neg\exists r_1(r_1 \ll r)$). A fusion of things that satisfy the open formula $\phi$ ($F_\phi$) is something that has all the $\phi$-ers as parts, and whose parts overlap at least a $\phi$-er: $F_\phi(x) \equiv_{\text{df}} \forall y(\phi(y) \rightarrow y \leq x) \wedge$

[3] See e.g., Simons (2004) and Dumsday (2015).

[4] See e.g., Calosi (2023).

[5] See among others Scala (2002), McDaniel (2007a), Gilmore (2018), and Rettler (2019).

[6] See Braddon-Mitchell and Miller (2006), Baker (2016), and Baron and LeBihan (2022).

[7] See e.g., Parsons (2007), and Gilmore (2018).

[8] Or, more generally, that *pervades* an extended region. For the locative notion of pervasion see e.g., Gilmore (2018).

[9] Roughly, theories of location that employs only one locative notion as a primitive. See Kleinschmidt (2016).

[10] Roughly, a simple that exhibits qualitative variation. See Spencer (2010).

[11] Two recent exceptions are Goodsell, Duncan and Miller (2020) and Calosi (2023). See also Ehrlich (2022).

$\forall y(y \leq x \rightarrow \exists z(\phi(z) \wedge y \circ z))$. I assume that regionhood is closed under fusion and parthood. In other words, fusions of regions are regions, parts of regions are regions. Two particular examples of fusions of regions are the *binary fusion*, which I will abbreviate as $x = y \oplus z$, and the *binary product* of two regions—the fusion of all the parts those two regions have in common—which I will abbreviate as $x = y \otimes z$. I also assume that when fusions exist they are unique.

Goodsell, Duncan and Miller (2020) defines extension in terms of a *primitive notion* of *least distance*. They do not provide many details about such primitive notion but it is fair to assume that it is a function $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{R}^{\geq 0}$, that obeys the following, for any $r_1, r_2$ in $\mathcal{R}$:

$$d(r, r) = 0 \tag{1}$$

$$d(r_1, r_2) = d(r_2, r_1) \tag{2}$$

$$r_1 \leq r_2 \rightarrow d(r_1, r_2) = 0 \tag{3}$$

$$r_1 \circ r_2 \rightarrow d(r_1, r_2) = 0 \tag{4}$$

The list above is redundant, in that e.g., (3) follows from (4). I will return to these principles shortly. As of now, let us press on. Goodsell, Duncan and Miller then go on to put forward what I called the *Least Distance* proposal which is based on the following definition of extension:[12]

**Definition 1. Extension$_d$**: A region $r$ is extended$_d$ ($E_d$) iff there are distinct regions $r_1$ and $r_2$ such that the least distance $d$ between $r_1$ and $r_2$ is greater than the sum of the least distance from $r_1$ and $r$, and $r$ and $r_2$: $E_d(r) \equiv \exists r_1 \exists r_2 (d(r_1, r_2) > d(r_1, r) + d(r, r_2))$.

Note that, for the sake of readability, I left out from the "formal" rendition in **Definition** 1 the distinctness conjuncts $r \neq r_1$, $r \neq r_2$, and $r_1 \neq r_2$, but Goodsell, Duncan and Miller are explicit that they hold—and this plays a role. Extension$_d$ is explicitly *extrinsic*. Goodsell, Duncan and Miller concede that having a definition of extension that makes it an *intrinsic* property of regions is "certainly appealing" (Goodsell, Duncan and Miller 2020: 650). However, the extrinsicness cost is outweighed by the payoffs of Extension$_d$. These include:

*Parsimony*: Extension$_d$ is defined in terms of another primitive, so that it cuts down the primitive notions by one (Goodsell, Duncan and Miller 2020: 652).

*Liberality*: Extension$_d$ is liberal in that it allows for some regions to be extended and simple.

*Explanatory Power, Part I*: As Goodsell, Duncan and Miller rightly note Extension$_d$ bears some similarity with the Triangle Inequality.[13] Indeed, they write:

[12] They call it **Extrinsic Extendedness (EE)** for reasons that will be clear shortly. See Goodsell, Duncan and Miller (2020: 652).

[13] A precise definition is in §4.

> [O]ne substantive benefit of EE [Extension$_d$] is that it explains why points satisfy the triangle inequality if they didn't, then they wouldn't be points. *If one takes extendedness to be intrinsic, then this fact is difficult to explain* (Goodsell, Duncan and Miller 2020: 653, italics added).

*Explanatory Power, Part II*: Extension$_d$ not only yields that extended simple regions are possible. It also

> [E]xplains what they are *like*. For under EE, the only difference between an ESR and an ordinary point-sized simple region is extrinsic. So an ESR is intrinsically the same as a point given EE. For someone who believes that extendedness is intrinsic, however, ESRs are thoroughly mysterious—what makes the difference, on the intrinsic view of extendedness, between an ESR and a point? (Goodsell, Duncan and Miller 2020: 653).

Undeniably, Extension$_d$ has a lot going for it.[14]

## 3 | THE *LEAST DISTANCE* PROPOSAL: AN ASSESSMENT

Goodsell, Duncan and Miller themselves concede that their *Least Distance* Proposal rules out cases (or worlds as I will call them) where there is a single extended region, or two extended regions. They call the *Lonely ESR* world the former and the *Sociable ESR* world the latter. It is instructive to cite their words:

> On EE, Lonely ESR is impossible because there aren't enough regions for $r$ to *violate the triangle inequality*, so $r$ must be unextended. (So too Sociable ESRs) (Goodsell, Duncan and Miller, 2020: 654, italics added).

Unfortunately there's plenty of cases in which we do have enough regions—at least three—and are still ruled out by the proposal. In effect, there are (at least) two ways of looking at these cases, both of which spell some trouble for *Least Distance*: either (i) one claims that the proposal rules out the problematic worlds as *analytically* impossible, or (ii) the proposal allows such worlds but the verdicts it delivers are "intuitively" wrong. In what follows I will actually opt to phrase my worries using (ii) but the reader is free to choose whatever they want. Indeed, I will talk of "counterexamples" to the proposal for the sake of brevity.

The first counterexample is given by a world $w_1$ in which there are three extended regions $r_1$, $r_2$ and $r_3$ such that the only mereological relations are given by $r_1 \ll r_2 \ll r_3$—Figure 1:

[14] Extension$_d$ also sheds new light on the metaphysical arguments that crucially depend on there being extended simple regions I mentioned in §1, namely the argument against parsimonious theories of location and against the possibility of heterogeneous simples. I will not discuss this in the rest of the paper.

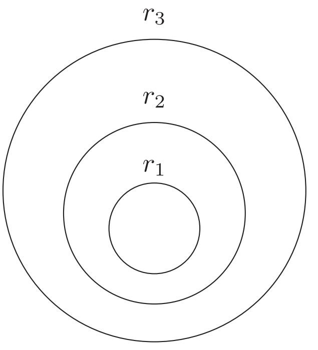


**Figure 1** World $w_1$.

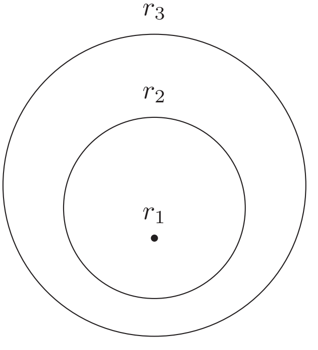


**Figure 2** World $w_2$.

It is clear that world $w_1$ provides a counterexample to the proposal because it follows from (3) that:

$$d(r_1, r_2) = d(r_1, r_3) = d(r_2, r_3) = 0 \tag{5}$$

According to the *Least Distance* proposal each region $r_i$ is unextended. Note that it would not do to amend the definition slightly by substituting $\geq$ instead of $>$ in **Definition** 1, thus obtaining: $E^*_d(r) \equiv \exists r_1 \exists r_2 (d(r_1, r_2) \geq d(r_1, r) + d(r, r_2))$. This will indeed take care of the $w_1$ counterexample. Yet, consider a world $w_2$ whose only difference from $w_1$ is that $r_1$ is an unextended, e.g., point-sized region—Figure 2:[15]

Equation (5) still holds. Hence, according to the amended *Least Distance** proposal $r_1$ is extended. Naturally one can add clauses that prohibit the "mereological arrangements" in worlds $w_1$ and $w_2$. The move smells of ad-hocness, but it should be conceded that $w_1$ and $w_2$ violate many mereological decomposition principles.[16] However, a slight modification of those worlds still delivers a counterexample that is actually a model of the strongest mereological theory, classical extensional mereology. To rigorously describe it, we actually need the topological primitive of "connectedness" ($C$)—that, as we shall see, Goodsell, Duncan and Miller themselves admit.

[15] According to the somewhat orthodox account points are indeed unextended. However this has been interestingly challenged. See e.g., Ehrlich (2022).

[16] For discussion of such decomposition principles see e.g., Cotnoir and Varzi (2021).

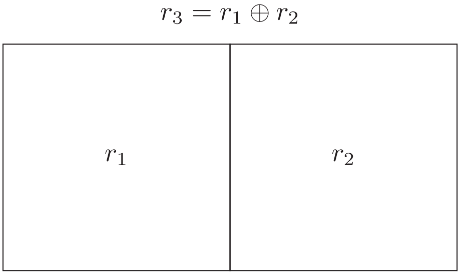


**Figure 3** World $w_3$.

I will give a precise characterization of $C$ in §5. For now, a few details suffice. $C$ is supposed to capture the intuitive notion of "touching". It should at least satisfy:

$$r_1 \circ r_2 \rightarrow C(r_1, r_2) \tag{6}$$

$$C(r_1, r_2) \rightarrow d(r_1, r_2) = 0 \tag{7}$$

Claim (6) is absolutely standard. In effect, it will be a theorem of the proposal in §5.[17] And Goodsell, Duncan and Miller subscribe to claim (7) explicitly.[18] Now, the description of the world $w_3$ is as follows: there are two extended simple connected regions $r_1$, and $r_2$, and their mereological fusion $r_3 = r_1 \oplus r_2$—Figure 3:[19]

It is not difficult to see that given (6) and (7), equation (1) holds for $w_3$ as well, so that $w_3$ presents yet another counterexample to *Least Distance*.

One could insist that the three regions need to be pairwise disjoint, and indeed disconnected. This would rule out $w_1 - w_3$. However, this is still not enough. Interestingly in the literature on (pointless) pseudo-metric spaces, it is pointed out that extended regions might *violate the triangle inequality*. Given the aforementioned, explicitly acknowledged similarity between the triangle inequality and Extension$_d$ it is no surprise that the world $w_4$ below that violates the inequality provides a counterexample to *Least Distance* as well—Figure 4:

World $w_4$ only contains three extended simple regions that are pairwise disconnected. Yet, **Definition** 1 yields that both $r_1$ and $r_2$ are unextended—see equations (8) and (9) respectively. The only extended region turns out to be $r_3$—see equation (10) below.This is because the following hold:

$$d(r_2, r_3) < d(r_2, r_1) + d(r_1, r_3) \tag{8}$$

$$d(r_1, r_3) < d(r_1, r_2) + d(r_2, r_3) \tag{9}$$

[17] See also, e.g., Casati and Varzi (1999).

[18] They write:

> For consider two objects in contact with one another—it doesn't make sense to say that the distance (simpliciter) between them is anything more than zero (Goodsell, Duncan and Miller, 2020: 653).

[19] More precisely: in the terminology of §5, $r_1$ and $r_2$ are tangentially connected.

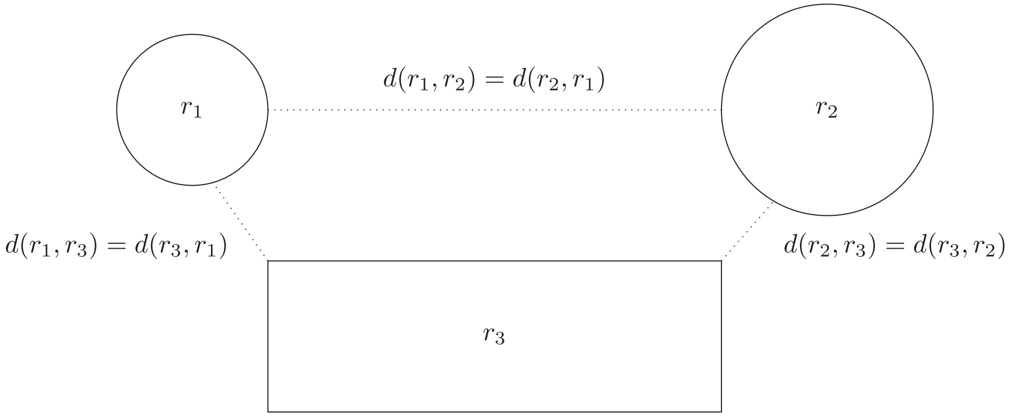


**Figure 4** World $w_4$.

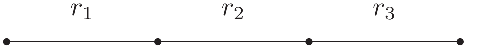


**Figure 5** World $w_5$.

$$d(r_1, r_2) > d(r_1, r_3) + d(r_3, r_2) \tag{10}$$

Equation (10) rightly delivers that $r_3$ is extended. In effect, a simple algebraic argument establishes that then $r_1$ and $r_2$ cannot be extended according to **Definition** 1. Here's the argument for $r_1$. The argument for $r_2$ is exactly similar.[20] Suppose $r_1$ is extended. Then, according to **Definition** 1 we should have:

$$d(r_2, r_3) > d(r_2, r_1) + d(r_1, r_3) \tag{11}$$

We only need to argue that (10) and (11) cannot both be true. From (10) and (2) we get:

$$d(r_1, r_2) - d(r_1, r_3) > d(r_2, r_3) \tag{12}$$

By transitivity of > applied to (11) and (12):

$$d(r_1, r_2) - d(r_1, r_3) > d(r_2, r_1) + d(r_1, r_3) \tag{13}$$

which, given (2) again, implies the following false inequality:

$$-d(r_1, r_3) > d(r_1, r_3) \tag{14}$$

The previous argument seems pretty general. In effect it applies to a case that is very similar to one Goodsell, Duncun and Miller themselves discuss.[21] The case is that of a world $w_5$ which contains three connected intervals. Suppose these intervals are extended simple regions—Figure 5:

As Goodsell, Duncun and Miller correctly note, **Definition** 1 rightly classifies $r_2$ as extended, in that $d(r_1, r_3) > d(r_1, r_2) + d(r_2, r_3)$. But it then misclassifies both $r_1$ and $r_3$ as unextended, given

[20] Just as a reminder. All $d$-s in $w_4$ are strictly > 0.

[21] Goodsell, Duncun and Miller (2020: 651).


**Figure 6** World $w_6$.

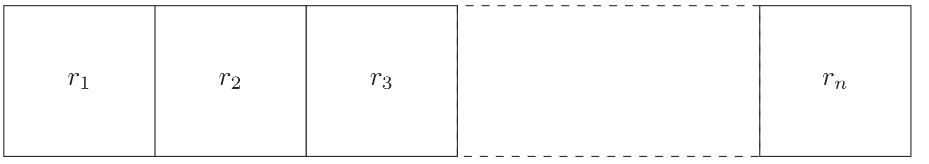


**Figure 7** World $w_7$.

that $d(r_1, r_2) = d(r_2, r_3) = 0$—by equation (7). Indeed, there is a sense in which the argument generalizes further. The generalization involves $n$-extended simple regions such that each region $r_i$ is connected to region $r_{i+1}$, each region $r_i$ is between regions $r_{i-1}$ and $r_{i+1}$, and regions $r_1$ and $r_n$ are not between any regions, that is, there are no regions $r_i$ and $r_j$ such that $r_1$ is between them—ditto for $r_n$. Figures 6 and 7 provide two models of different dimensionality.[22]

Suppose for the sake of simplicity that the least distance between any two regions $r_i$ and $r_{i+1}$ is fixed. Without loss of generality we might set $d = 1$. In this case, the least distance between any two regions simply records the number of regions in between the two. In effect, we can simply take the least distance between any two regions to be the number $n$ of the regions that are in between them. Note that for any $r_i$ and $r_j$ with $j > i$ this is given by:

$$n_{ij} = (j - i - 1) \tag{15}$$

The argument has it that neither $r_1$ nor $r_n$ in $w_6$ (or $w_7$) is extended according to *Least Distance*. Here is the argument for $r_1$, the argument for $r_n$ being entirely similar. According to *Least Distance* $r_1$ is extended iff there are regions $r_i$ and $r_j$ with $j > i$ ranging over $\{2, \dots, n\}$ such that:

$$d(r_i, r_j) > d(r_i, r_1) + d(r_1, r_j) \tag{16}$$

However note that

$$d(r_i, r_j) < d(r_1, r_j) \tag{17}$$

To see this, note that substituting we get:

$$n_{ij} = j - i - 1 < n_{1j} = j - 1 - 1 \tag{18}$$

which yields the following truth:

$$i > 1 \tag{19}$$

Given that $d(r_i, r_1) \geq 0$, we do have that

$$d(r_i, r_j) < d(r_i, r_1) + d(r_1, r_j) \tag{20}$$

[22] One could rigorously describe such a scenario with topological resources together with a primitive three place relation of "*betweenness*".

Thus $r_1$ is not extended. Here is a way of looking at the previous argument a little less "formally". As I noted, in worlds $w_6$ and $w_7$ the least distance basically counts the number of regions in between any two given regions $r_i$ and $r_j$. But there are more regions between $r_1$ ($r_n$) and $r_j$ ($r_i$) than in between any regions $r_i$ and $r_j$ with $i, j \neq 1 (\neq n)$. Hence the tension with *Least Distance*.

## 4 | THE *DIAMETER* PROPOSAL

As we saw the *Least Distance* proposal centers around a distance function $d$ that obeys equations (1)–(4). Furthermore, as we saw it is explicitly tightly connected with the so called *triangle inequality*. In general terms, for a given distance function $d$, for any $x, y, z$ in the relevant set:

$$d(x, y) \leq d(x, z) + d(z, y) \tag{21}$$

Consider now the following, once again in general terms:

$$d(x, y) = 0 \rightarrow x = y \tag{22}$$

If we were to add (21) and (22) to (the general counterparts of) (1) and (2) we would indeed get a *metric space*:[23]

**Definition 2. Metric Space**: A metric space $(S, d)$ is a set $S$ together with a distance function $d : S \times S \rightarrow \mathbb{R}^{\geq 0}$ such that the following hold:

(i) $d(x, y) = 0 \rightarrow x = y$ [Eq. 1, Eq. 22],
(ii) $d(x, y) = d(y, x)$ [Eq. 2]
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ [Eq. 21].

As we saw there are reasons to think that extended simple regions violate the triangle inequality. But the basic insight to use spaces where distance functions are defined to characterize extension can be preserved. For instance, Baron and LeBihan (2022) suggests something like that. They write:

> [A] spatiotemporal region is extended when the metric distance between any two points in the manifold representation of the region is non-zero (Baron and LeBihan, 2022: 25).

This is a very natural suggestion indeed. However it is slightly problematic for our purposes. This is because we want a definition of extension that meets the *Liberality* requirement in §2. That is, the definition should not rule out extended simple regions. More needs to be said about the notion of "being in the manifold representation" of a region to adjudicate whether the proposal is liberal in the sense at stake here. Clearly, it would not be were the two points mentioned in the characterization of extension to be *part* of the relevant region. My suggestion is to take a cue from *pseudo-metric spaces*, and define:[24]

[23] A general counterpart because this is definable for every poset, not just for a set of regions ordered by parthood.

[24] See e.g., Weihrauch and Schreiber (1981), and Gerla (1990).

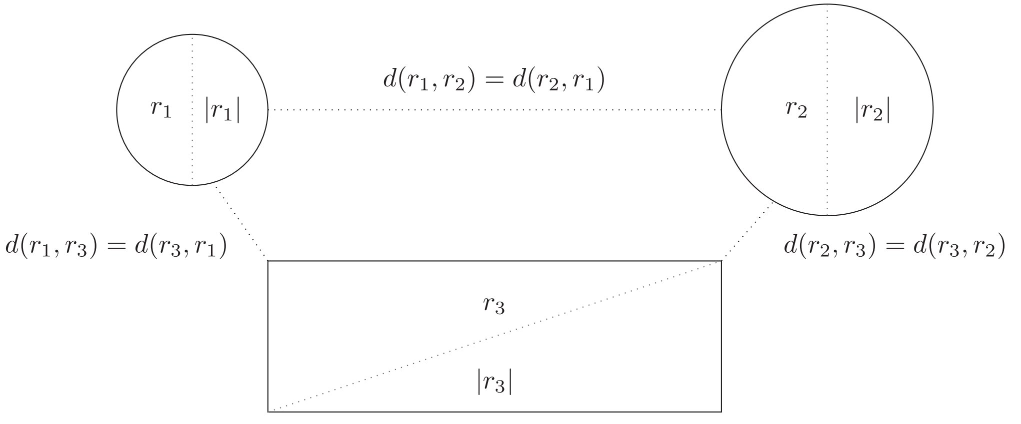


**Figure 8** World $w_4$—with *diameters*.

**Definition 3. Pseudometric Space**: A pseudometric space is a structure $\mathscr{R} = (\mathcal{R}, \leq, d, |\ |)$ where $(\mathcal{R}, \leq)$ is a set of regions partially ordered by parthood, and $d$ and $|\ |)$ are functions $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}^{\geq 0}$, $|\ | : \mathcal{R} \to \mathbb{R}^{\geq 0}$ such that, for every $r_1$, $r_2$, and $r_3$, they satisfy:

(i) $d(r, r) = 0$
(ii) $r_1 \leq r_2 \to |r_1| \leq |r_2|$
(iii) $r_1 \leq r_2 \to d(r_2, r_3) \leq d(r_1, r_3)$
(iv) $d(r_1, r_2) \leq d(r_1, r_3) + d(r_3, r_2) + |r_3|$

We call $d$ the *pseudo-distance* function and $|\ |$ the *diameter* function. The latter will eventually turn out to be the crucial one. To get an intuitive handle on it, consider orthodox pointy space. Then, the diameter of a region is the least upper bound of the distances between points of that region.

Now go back to **Definition** 3. In plain English the clauses above say that (i) every region is at pseudo-distance 0 from itself, (ii) the diameter function is *order-preserving*, in that if a region is part of another the diameter of the former is smaller or equal to that of the latter, (iii) if a region is part of another, the pseudo-distance of a region from the latter is smaller or equal than the pseudo-distance from the former. The fourth clause (iv) will play a somewhat significant role later on so it's worth spending a few words on it. It says that the distance between two regions $r_1$ and $r_2$ is less or equal than the sum of the distances of those regions from a third region $r_3$ *plus the diameter of said region*. This last detail is the crucial one. It is what distinguishes it from the *triangle inequality*. In effect, the triangle inequality is basically the *restriction* of condition (iv) to regions that have diameters = 0. Thus, condition (iv) is really a *generalization* of the triangle inequality. To anticipate, regions with diameters = 0 are *unextended*. Hence, the triangle inequality is effectively the *restriction* of the more general condition (iv) to *unextended regions*. It then becomes clear why condition (iv) holds for extended regions as well. To see this consider world $w_4$ again where I now drew the respective diameters of the extended regions.

Figure 8 makes it clear that (*) $d(r_1, r_2) > d(r_1, r_3) + d(r_3, r_2)$—so that $w_4$ violates the triangle inequality. But adding $|r_3|$ on the right side of (*) is enough to get $d(r_1, r_2) \leq d(r_1, r_3) + d(r_3, r_2) +$

$|r_3|$, so that the more general condition (iv) is by contrast satisfied. And exactly for the expected reason: because $r_3$ is extended. Interestingly, equations (1)-(4) can all be proven.[25]

Now define the following:

**Definition 4. Atoms and Points**: Let $\mathscr{R} = (\mathcal{R}, \leq, d, |\ \ |)$ be a pseudo-metric space. A region $r$ is an *atom* iff it is a $\leq$-minimal $\mathcal{R}$-region. A region $r$ is a *point* iff it is an $\mathcal{R}$-region with 0-diameter, i.e. a region $r$ such that $|r| = 0$.

The suggestion is then the following:

**Definition 5. Extension$_{|\ |}$**: A region $r$ is extended$_{|\ |}$ iff it is not a point. Equivalently: a region $r$ is extended$_{|\ |}$ iff it is an $\mathcal{R}$-region such that its diameter $|r| \neq 0$: $E_{|\ |}(r) \equiv_{\text{df}} |r| > 0$.

**Definition 6. Extended$_{|\ |}$ Simple Region**: A region $r$ is an extended$_{|\ |}$ simple region iff it is an atom but not a point. Equivalently: an extended simple region $r$ is a minimal non-point in $\mathcal{R}$: $E_{|\ |}S(r) \equiv_{\text{df}} A(r) \wedge |r| > 0$.

That is the core of the *Diameter* proposal. In a nutshell, according to the proposal an extended simple region $r$ is a simple region whose diameter is strictly greater than 0.[26]

Let me be upfront. First, I don't lay any claim that Extension$_{|\ |}$ is *the one true notion* of extension. In effect, I tend to be pluralist. There are different notions of extensions that are readily definable and I don't see any reason not to use them all in our theorizing.[27] I will rehearse some (but not all!) of them in §6. Indeed, whenever we are dealing with *defined* notions one may think of pluralism as the default position. Second, I don't lay any claim that Extension$_{|\ |}$ captures our pre-theoretical notion of extension. What I do claim is that Extension$_{|\ |}$ is one possible useful notion that we should add to our logical and metaphysical toolbox. In the rest of the section I am going to provide reasons in favor of this last claim. In particular, I aim to argue that, in general, the *Diameter* proposal has the same theoretical benefits of the *Least Distance* proposal, and then some more. In effect, it is easier to start with the cases in which *Diameter* outperforms *Least Distance*. I am going to mention two that strike me as important:

*Classification*: The *Diameter* proposal easily handles all the worlds $w_1$-$w_7$. That is, it correctly classifies extended and unextended regions in all such worlds.

*Intrinsicality*: On the face of it, Extension$_{|\ |}$ is an *intrinsic* property of regions. This, I take it, is desirable. At least if the intrinsic notion of Extension$_{|\ |}$ can match the explanatory power of the exstrinsic notion of Extension$_d$—a claim that I will argue for shortly.[28]

[25] See e.g., Gerla (1990: 208).

[26] It is worth pointing out the "terminological similarity" with other proposals in the literature, such as e.g., Gilmore (2018: 25-26). However, we should not be fooled. "Atoms" and "Point" here are very specific, technical notions, that are not equivalent to the ones used by Gilmore.

[27] I myself suggested different ones in [REDACTED].

[28] I concede that to make a more robust claim in favor of intrinsicness one needs to look at different accounts of it. Clearly, this goes beyond the scope of the paper. Now, the account of intrinsicness according to which a property $P$ is intrinsic to $x$ iff $x$ has $P$ in a possible world in which there is only $x$ and its parts is widely regarded as unsatisfactory. However we still consider the case as a sort of *litmus test* for more sophisticated accounts. And the *Diameter* proposal passes this litmus test.

It remains to be argued that *Diameter* can match the benefits of *Least Distance*, as they were presented in §2. Let me go over them, one by one.

*Parsimony*: Extension$_{|\ |}$, exactly like Extension$_d$, is defined in terms of another primitive, so that it cuts down the primitive notions by one.

*Liberality*: Extension$_{|\ |}$, exactly like Extension$_d$, is liberal in that it allows from some regions to be extended and simple.

*Explanatory Power, Part I*: Extension$_{|\ |}$ explains why points satisfy the triangle inequality. This is because points are regions with diameter = 0. If we plug that in clause (iv) of **Definition** 3—the *generalized triangle inequality*—we obtain exactly the *triangle inequality*.

*Explanatory Power, Part II*: Extension$_{|\ |}$, exactly like Extension$_d$, not only yields that extended simple regions are possible. It also explains what they are like. They are regions $r$ with diameter $|r| > 0$. This also answers the question asked by Goodsell, Duncun and Miller—the part in italics in the relevant quote in §2. On an intrinsic view of extension the difference between a point and an extended simple region is exactly that, in the technical sense defined above, extended simple regions are not *points*—they do not have diameter = 0.

The *Diameter* proposal seems to match the *Least Distance* proposal in all the important respects, and outperforms it (or so it seems) in other important ones. It is also based on solid mathematical ground. As I noted already, it provides a rigorous, broadly metrical intrinsic notion of extension that allows for extended simple regions. I contend that these are reasons enough to consider it as a candidate to satisfactorily and rigorously capture *one* notion of extension.

## 5 | THE *DIAMETER* PROPOSAL: A DEVELOPMENT

I framed the *Diameter* proposal in terms of pseudo-metric spaces. As we saw, these are structures that contain two distinct functions, $d$ and $|\ |$. The latter played the lion's share in §4. Here is a natural question. Can we do it all with just $|\ |$? In particular: can we define $d$ in terms of $|\ |$? It turns out that we can, under certain (admittedly controversial) assumptions. In this section I explore such assumptions, the resulting definition, and discuss some of the consequences.

**Definition 7. Diametric Poset**: [29] A diametric poset is a structure $\mathscr{R} = (\mathcal{R}, \leq, |\ |)$ where $(\mathcal{R}, \leq)$ is a set of regions partially ordered by parthood, and $|\ |$ is a function $|\ | : \mathcal{R} \to \mathbb{R}^{\geq 0}$ that, for every $r_1, r_2$, satisfies:

(i) $r_1 \leq r_2 \to |r_1| \leq |r_2|$
(ii) $r_1 \circ r_2 \to \exists r(r_1 \leq r \wedge r_2 \leq r \wedge |r| \leq |r_1| + |r_2|)$
(iii) $\exists r(|r| < \infty \wedge r_1 \circ r \wedge r_2 \circ r)$

Let us say that a region $r$ is bounded iff its diameter is finite, that is $r < \infty$. In plain English, clause (i) ensures that $|\ |$ is order-preserving, clause (ii) says that for any two overlapping regions

[29] This definition provides only a "starting point", so to speak. The most relevant structure is introduced in **Definition** 9.

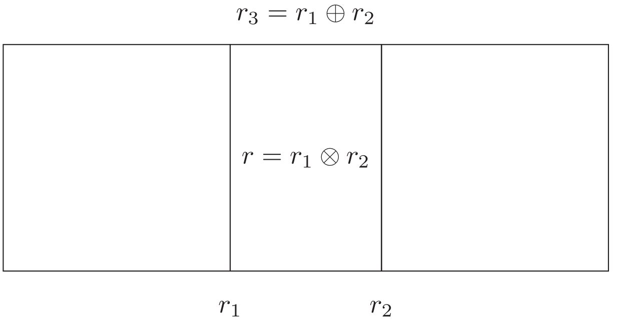


**Figure 9** World $w_8$.

there is another which includes both whose diameter is less or equal to the sum of their diameters, and finally according to (iii) for any two regions there is a bounded region that overlaps both. Then we can define $d$ in terms of $|\ |$ as follows, where $\bigvee x$ is the greatest lower bound of $x$:

**Definition 8. Pseudo-Distance**: $d$ is a function $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{R}^{\geq 0}$ defined as follows: $d(r_1, r_2) = \bigvee \{|r| \ : r_1 \circ r \wedge r_2 \circ r\}$.

In other words, the distance between $r_1$ and $r_2$ is the greatest lower bound of the diameters of regions that overlap both. It follows that, if $r_1$ and $r_2$ overlap, then $d(r_1, r_2) = 0$, as expected—see equation (4). A more interesting and substantive theorem is the following.

**Theorem 1.** *The function d defined in **Definition** 8 satisfies the condition of a pseudo-metric space as defined in **Definition** 3.*

For the proof, see e.g., Gerla and Paolillo (2010), THEOREM 2.7. In the light of the previous discussions, this is indeed a highly significant result. For now we could cut our primitives by one once more, and make do with just the diameter function $|\ |$.

One important question however needs to be addressed. We now have a different mathematical structure, that of a diametric poset. How are we to be sure that it does not rule out some of the worlds we used to put pressure on *Least Distance*, so to speak? A little more precisely: are all worlds that provided counterexamples to *Least Distance* models of diametric posets as defined in **Definition** 7? The answer is no. In effect, clause (iii) rules out world $w_4$. Under plausible—but not undeniable—assumptions it rules out $w_3$ and $w_5 - w_7$ as well. The former (i.e., $w_3$) can be slightly modified to deliver a model of a diametric poset.[30] Suppose we "add" an extended simple region $r$ that is part of both $r_1$ and $r_2$. That is, the model has one extended simple region, $r$, and mereologically complex regions $r_1$, $r_2$ and $r_3$ such that $r_3 = r_1 \oplus r_2$ and $r = r_1 \otimes r_2$—Fig. 9:

Unfortunately for *Least Distance* even $w_8$ is a counterexample to it. This is because the following holds:

$$d(r, r_1) = d(r, r_2) = d(r, r_3) = d(r_1, r_2) = d(r_1, r_3) = d(r_2, r_3) = 0 \tag{23}$$

And therefore, *Least Distance* misclassifies $r$ as unextended. Naturally, world $w_8$ has an "heterodox" mereological structure—indeed the same heterodox structure exhibited by $w_1$ and $w_2$. It is surely an interesting question whether there are models of diametric posets that e.g., (i) are

[30] Even the others can be modified similarly. It is easier to modify $w_3$ so I'll stick to it.

models of classical mereology and (ii) provide counterexamples to *Least Distance*. I will not press this point because I actually want to make a different suggestion. In what follows I will introduce topological vocabulary and define what I shall call "Weakly Connected Diametric Posets". I will argue that even in that case we can actually define $d$ in terms of $|\ |$. And it turns out that there are indeed well-behaved models of weakly connected diametric posets that provide counterexamples to *Least Distance* but not to *Diameter*.

We already introduced the topological notion of connection. Let us be precise and assume it as another two-place primitive, $C$. We assume that $C$ is reflexive and symmetric—but not transitive. We can then define tangential connection $TC$ as follows. Two regions are tangentially connected ($TC$) iff they are connected but disjoint: $TC(r_1, r_2) \equiv_{\text{df}} C(r_1, r_2) \wedge r_1 \wr r_2$. It can be proven that $TC$ is irreflexive and symmetric but not transitive. We also need to define the notion of a *minimally connected region* of two regions. A region $r$ is a minimally connected region of two regions $r_1$ and $r_2$ iff (i) it is tangentially connected to both, and (ii) for any other region $r'$ that is tangentially connected to both, the diameter of $r$ is less or equal that of $r'$: $MWC(r, r_1, r_2) \equiv_{\text{df}} TC(r, r_1) \wedge TC(r, r_2) \wedge \forall r'((TC(r', r_1) \wedge TC(r', r_2) \rightarrow |r| \leq |r'|))$. Then we can define:

**Definition 9. Weakly Connected Diametric Poset**: A weakly connected diametric poset is a structure $\mathscr{R} = (\mathcal{R}, \leq, C, |\ |)$ where $(\mathcal{R}, \leq)$ is a set of regions partially ordered by parthood, C is the two-place topological notion of connection, and $|\ |$ is a function $|\ | : \mathcal{R} \rightarrow \mathbb{R}^{\geq 0}$ such that, for any $r_1, r_2$ in $\mathcal{R}$, the following hold:

(i) $C$ is reflexive and symmetric
(ii) Parthood and connection are monotonic, i.e., $r_1 \leq r_2 \rightarrow \forall r_3(C(r_1, r_3) \rightarrow C(r_2, r_3)$.
(iii) $|\ |$ is order preserving, i.e., $r_1 \leq r_2 \rightarrow |r_1| \leq |r_2|$.
(iv) The space of regions is weakly globally connected in that for any two disconnected regions $r_1$ and $r_2$ there is a minimally connected region $r$: $\neg C(r_1, r_2) \rightarrow \exists r(MWC(r, r_1, r_2))$.
(v) There is a fusion of tangentially connected regions, i.e., $TC(r_1, r_2) \rightarrow \exists r_3 = r_1 \oplus r_2$.
(vi) The space of regions is "bounded and complete" in that for any two tangentially connected regions their sum has a diameter that is less or equal to the sum of their diameters: $TC(r_1, r_2) \rightarrow \exists r(r = r_1 \oplus r_2 \wedge |r| \leq |r_1| + |r_2|)$.

Clauses (i)-(ii) above detail the relations between parthood and connection. Clause (iii) is the usual order-preserving clause for $|\ |$. Finally, clauses (iv)-(vi) impose certain constraints over the mereology, the connectibility, and the sizes of members of $\mathcal{R}$. I submit that we can now define the function $d$ in terms of the diameter function as follows:

**Definition 10. Pseudo Distance**: $d$ is a function $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{R}^{\geq 0}$ defined as follows:

$$\begin{cases} d(r_1, r_2) = 0 & \text{if } C(r_1, r_2) \\ d(r_1, r_2) = \bigvee\{|r| \ : TC(r_1, r) \wedge TC(r, r_2)\} & \text{otherwise} \end{cases}$$

That is to say that $d$ is defined as follows: it is 0 whenever two regions $r_1$ and $r_2$ are connected, and it is the greatest lower bound of diameters of regions that are tangentially connected to both $r_1$ and $r_2$ otherwise. I take it that this precisely cashes out something in the vicinity of the notion of "least distance" that Goodsell, Duncan and Miller had in mind. Or at least, this is one proposal. This is also why I think of *Diameter* as sharing the same "broadly metric spirit" of *Least Distance*. In any case, the substantive result is that the following can be proven:

**Theorem 2.** *The function d defined in* ***Definition*** *10 satisfies the condition of a pseudo-distance as defined in* ***Definition*** *3.*

A detailed statement of the theorem and a proof is in the **Appendix**. This is important for the discussion at hand. First, it shows that we could define distance in terms of diameters—under suitable conditions. One could not however straightforwardly claim to have cut down on one more primitive. This is because the definition of $d$ crucially depends on $C$—which we did not have before. However, it is widely held that $C$ has to be taken as a primitive anyway, as it cannot be e.g., defined in terms of parthood.[31] If so, this move would also foster some sort of ideological parsimony. Second, particular worlds that spelled trouble for *Least Distance* are now admissible models, namely $w_3$, $w_5 - w_7$. These worlds are well-behaved mereologically. And as we saw all of them provide counterexamples to *Least Distance*. But they do not tell against *Diameter*. Let me then conclude the discussion of this one possible development of *Diameter* by simply noting that this seems a significant advantage of the proposal.

## 6 | EXTENSIONS, EXTENSION OF A REGION, BEING LESS EXTENDED

In this paper I reviewed an extant proposal to define extension in such a way as to allow for extended simple regions. I found that proposal somewhat unsatisfactory in some respects. I then suggested another definition that seems to fare better. As I said, the definition of extension I proposed rigorously captures *one* notion of extension. But there are others. A few examples from the literature include:

*Mereological Extension.* $x$ is extended iff $x$ is not atomic.[32]
*Mereological and Locational Extension*: $x$ is extended iff the exact location of $x$ is not atomic.[33]
*Metrical Extension.* $x$ is extended iff the exact location of $x$ includes two parts at non zero-distance.[34]
*Measure Theoretic Extension.* $x$ is extended iff the exact location of $x$ has non zero-measure.[35]

Clearly, it is a substantive questions how all these notions relate to one another and to Extension$_{|\,|}$.[36] It has been pointed out that both the mereological notions of extension above

[31] For arguments see e.g., Casati and Varzi (1999).

[32] This was arguably the background of many modern disputes in e.g., Locke and Hume. See Pasnau(2011) for a discussion.

[33] See e.g., Pickup (2016), Eagle (2019),Calosi (2023).

[34] This is close to e.g., what Baron and LeBihan (2022) discuss.

[35] See Calosi (2023).

[36] Spencer (2008) provides another characterization of "extension" (explicitly tailored to regions) in terms of *displacement relations*. Let me call the account *Displacement Extension* ($E_D$) for short. He claims that if a region $r$ is displaced from itself by an amount $n \neq 0$ in a given direction, then $r$ is extended$_D$. He is not explicit about the "only if" part. Thus, strictly speaking, this falls short to provide a definition of extension. Spencer does not provide too many details on displacement relations but the ones he provides are interesting. He writes:

suffer from some limitations.[37] For example, it is unclear that we can actually provide a satisfactory account of the extension of a region, or that we can even define the relation of "being less extended than" simply using mereological and locational notions. It is worth noting that Extension$_{|\ |}$ does not (seem to) suffer from the same limitations. Here are two natural suggestions:

**Definition 11. Extension$_{|\ |}$ of a Region**: The extension$_{|\ |}$ of a region $r$ is its diameter $|r|$.

**Definition 12. Less Extended$_{|\ |}$ Than**: A region $r_1$ is less extended$_{|\ |}$ than region $r_2$ ($<_{E_{|\ |}}$) iff the diameter of the former is less than the diameter if the latter: $r_1 <_{E_{|\ |}} r_2 \equiv_{\text{df}} |r_1| < |r_2|$.

This is already enough to show that the e.g., measure theoretic and the diameter notion of extension might come apart. This is because, as Calosi (2023) suggests, one may take the extension of a region to be the measure of that region—given a somewhat appropriate measure. One of the leading contenders in this respect is the Lebesgue measure. There are clear examples in which the Lebesgue measure of a region and its diameter do not coincide. We do not need to look that far. Region $r_3$ in Fig. 4 and Fig. 8 is one such example. This entails that these notions are interestingly different. The question of their relations naturally arises. This extends beyond the diameter of the present paper.

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> A displacement is a vector quantity that can be thought of as a distance measured in a particular direction (Spencer 2008: 27-28).

To put some flesh on these bones, let $D(x, y, d, n)$ be a four-place relation of displacement where $x$ and $y$ range over regions, $d$ over directions, and $n \in \mathbb{R}^{>0}$. Then, Spencer requirement translates into $D(r, r, d, n) \rightarrow E_D(r)$. Consider now a region $r$ with non-zero diameter $|r|$. Arguably, $r$ can be displaced from itself along a direction $d$ by $|r|$. That is to say that $D(r, r, d, |r|)$ holds, and therefore that extended$_{|\ |}$ regions are extended$_D$ regions—which seems like a substantive result. Later on Spencer writes that

> It is probably more natural to take displacement relations as fundamental and analyze distances in terms of displacements. We might, for example, say that something is a distance from another thing just in case it is displaced in some direction from that other thing (Spencer, 2008: 33).

This seems exactly in line with the construction in §5 where distances are defined in terms of diameters. This points to relevant similarities between diameters and displacements, an account of which deserves an independent scrutiny. Thanks to an anonymous referee here.

[37] See e.g., Calosi (2023).

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## APPENDIX: PROOF OF THEOREM 2

**Theorem** 2. Let $\mathscr{R} = (\mathcal{R}, \leq, C, |\ \ |)$ be a weakly connected diametric poset. And let $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}^{\geq 0}$ be defined as follows:

$$\begin{cases} d(r_1, r_2) = 0 & \text{if } C(r_1, r_2) \\ d(r_1, r_2) = \bigvee\{|r| \ : TC(r_1, r) \wedge TC(r, r_2)\} & \text{otherwise} \end{cases}$$

Then $d$ satisfies the requirements of a pseudo-distance, that is:

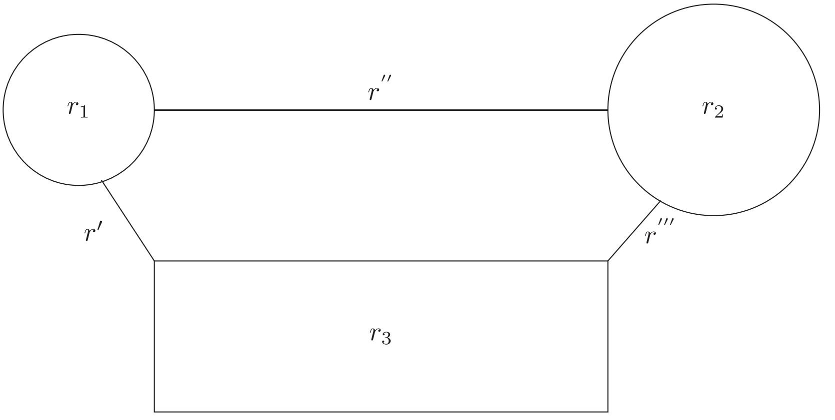


**Figure A1** A Weakly Connected Diameteric Poset.

(i) $d(r_1, r_2) = d(r_2, r_1)$
(ii) $d(r, r) = 0$
(iii) $d(r_1, r_2) \leq d(r_1, r_3) + d(r_2, r_3) + |r_3|$

**Proof of Theorem** 2

Clause (i) is immediate. Clause (ii) follows from the first clause of the definition of $d$ and reflexivity of $C$. For clause (iii) consider any three regions $r_1$, $r_2$, and $r_3$. We will constantly refer to Fig. A1 below.

We first prove the following lemma:

**Lemma 1**. Let $\mathscr{R} = (\mathcal{R}, \leq, C, |\ \ |)$ be a connected diameter poset. Then, for regions $r_1, \ldots, r_n$ in $\mathcal{R}$ such that $TC(r_1, r_2) \wedge TC(r_2, r_3) \wedge \cdots \wedge TC(r_{n-1}, r_n)$, their sum $r$ exists and is such that $|r| \leq |r_1| + |r_2| + \cdots + |r_n|$.

**Proof of Lemma 1**

We do it by induction. Consider the single-region case. Without loss of generality let it be $r_1$. Then, $r = r_1$ and clearly $C(r, r_1)$ and $|r| \leq |r_1|$. Suppose it holds for $n - 1$. Then we show it holds for $n$. By repetition of clause (vi) of weakly connected diametric posets there is a region $r^*$ such that $r^* = r_1 \oplus r_2 \oplus \ldots \oplus r_{n-1}$. Note that the following holds $TC(r^*, r_n)$. By clause (v), their sum $r$ is such that $|r| \leq |r^*| + |r_n|$. This proves **Lemma 1**.

Now we can prove **Theorem** 2.

**Proof of Theorem** 2

Let $r_1$, $r_2$ and $r_3$ be disconnected regions. By (iv) there are regions $r'$, $r''$ and $r'''$ such that $MWC(r', r_1, r_3)$, $MWC(r'', r_1, r_2)$, and $MWC(r''', r_2, r_3)$—as in Fig A1 above. Now, without loss of generality consider $r'$, $r_3$ and $r'''$. They are pairwise $TC$-connected, so by **Lemma 1** their sum $r$ exists and is such that the following holds: (*) $|r| \leq |r'| + |r'''| + |r_3|$. Recall that $MWC(r', r_1, r_3)$. The crucial thing to note is that, given the definition of $d$ above, this is exactly th diameter of the $MWC$-region. Hence, $d(r_1, r_3) = |r'|$. The same argument gives us $d(r_3, r_2) = |r'''|$. Substituting into (*) we get (**) $|r| \leq d(r_1, r_3) + d(r_3, r_2) + |r_3|$. Consider now $r''$. We know that $MWC(r'', r_1, r_2)$. Hence, by definition of $MWC$-region we know that $|r''| \leq |r|$. Repeating the argument we used before we have that $|r''| = d(r_1, r_2)$. Then, by Transitivity of $\leq$ and (**) we get (***) $d(r_1, r_2) \leq d(r_1, r_3) + d(r_3, r_2) + |r_3|$. □