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RESEARCH ARTICLE

Dynamically consistent objective and subjective rationality

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Abstract

A group of experts, for instance climate scientists, is to advise a decision maker about the choice between two policies f and g. Consider the following decision rule. If all experts agree that the expected utility of f is higher than the expected utility of g, the unanimity rule applies, and f is chosen. Otherwise, the precautionary principle is implemented and the policy yielding the highest minimal expected utility is chosen. This decision rule may lead to time inconsistencies when adding an intermediate period of partial resolution of uncertainty. We show how to coherently reassess the initial set of experts' beliefs so that precautionary choices become dynamically consistent: new beliefs should be added until one obtains the smallest "rectangular set" that contains the original one. Our analysis offers a novel behavioral characterization of rectangularity and a prescriptive way to aggregate opinions in order to avoid sure regret.

Keywords Ambiguity Aversion · Dynamic Consistency · Objective Rationality · Subjective Rationality · Full Bayesian Updating · Rectangularity

1 Introduction

In many applications of decision analysis, a Decision Maker (DM henceforth) will seek the opinions of several experts. Consider a board of Bayesian experts that needs to guide choices of a DM facing alternatives with uncertain outcomes. One can think for instance of a group of climate scientists that should advise the European Union about the best policy in order to fight CO_2 emissions. Another concrete and recent

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example may be the one of epidemiologists advising a prime minister about the best policy to fight the coronavirus outbreak in 2020. In both cases, it is reasonable to think that different experts may assign different probability distributions to possible scenarios.

Consider two policies, f and g. If all experts think that f is better than g (meaning that f has an higher expected utility than g), then it may be reasonable for the DM to prefer policy f rather than g. This decision rule is sometimes referred to as the *unanimity principle*. Note that all experts in the group have veto power: it is sufficient that one expert ranks g above f to break unanimity. Whenever this is the case, and experts disagree, it is not clear which policy the DM should implement. In this situation, especially when uncertainty about different scenarios is high and there are scenarios that can lead to catastrophes, as with pandemics or climate change, several authors suggest to adopt the *precautionary principle*. While there is not an accepted and universal definition of the precautionary principle, one can think of it as saying that a policy should be evaluated through the opinion of the most pessimistic expert.[1]

Gilboa et al. (2010) (GMMS henceforth) offer an axiomatic analysis supporting the use of the Maxmin rule in order to "complete" the unanimity rule when full agreement among experts does not hold. However, if we introduce an intermediary period of partial resolution of uncertainty, i.e. if experts know today that they will have some information tomorrow, this decision rule may violate Dynamic Consistency. This means that decisions taken today may be regretted tomorrow once experts are allowed to update their preferences, no matter which peace of information is learned. Dynamic Consistency is a normative-appealing property and relates to GMMS's concept of subjective rationality. According to GMMS, a DM is subjectively rational if she does not feel "embarrassed" by her preferences. Clearly, making a decision that could cause sure future regret can be a source of embarrassment and can induce a DM to reassess her preferences. This paper provide a refinement of GMMS so that Dynamic Consistency is preserved in a model with a period of partial resolution of uncertainty.

From a decision-theoretical point of view, one can identify each expert with the probability distribution she assigns to possible scenarios. Hence the term expert and probability measure can be used as synonyms and a board of experts can be represented by a set of probability measures. Moreover decision rules can be thought of as preference relations. The unanimity rule corresponds to a Bewley (2002) preference, while the precautionary principle leads to a Maxmin preference of Gilboa and Schmeidler (1989). Throughout the paper we adopt the follwing interpretation: the Bewley and Maxmin preferences represent the preferences of a DM (e.g. a policymaker) who is advised by a group of Bayesian experts.

We model information through an exogenous partition of the state space.[2] The DM knows today that tomorrow she will learn in which element of the partition the

[1] Our approach identifies the precautionary principle with the Maxmin model of Gilboa and Schmeidler (1989), as in Alon and Gayer (2016), Chevé and Congar (2003) and the UNESCO (2005) (written max(i)min, p. 29). See also Gardiner (2006) for a philosophical discussion about the interpretation of the precautionary principle as a Maxmin. Other authors follow different paths as for instance Barrieu and Sinclair-Desgagné (2006), Berger et al. (2016) and Gollier et al. (2000).

[2] See Galanis (2021) for a thorough analysis of the relationship between Dynamic Consistency and the value of information for ambiguity averse preferences.

true state of nature lies. Let $\succsim^*$ be the original Bewley preference relation (unanimity rule), that may lead to violations of Dynamic Consistency once it is completed with a Maxmin preference. Our goal is to modify $\succsim^*$ into a new preference $\succsim^{**}$ in order to avoid those violations. We say that $\succsim^{**}$ is a reassessment of $\succsim^*$ if, whenever f is better than g for $\succsim^{**}$, then f is better than g for $\succsim^*$. It means that there is no reversal of previous rankings, and that no new rankings are formed. It is well known that a reassessment of a Bewley preference is obtained by adding new probabilities (i.e. new experts) to the set of probabilities characterizing the original preference.

We require the new preference relation $\succsim^{**}$ to be coherent with respect to $\succsim^*$ by imposing two axioms, that we call Ex-Post and Ex-Ante Coherence. The intuition behind both axioms is that the new group of experts should agree with the old one whenever there are no problems involving Dynamic Consistency. Ex-Post Coherence says that, once an event E in the information partition is revealed, the resulting new conditional preference $\succsim_E^{**}$ should respect the rankings of the old conditional one $\succsim_E^*$. Ex-Ante Coherence requires that the unconditional new preference $\succsim^{**}$ should agree with the old one $\succsim^*$ when acts f and g are measurable with respect to the information partition.

We say that $\succsim^{**}$ is the *coherent precautionary reassessment* of $\succsim^*$ if it is the most incomplete Bewley preference satisfying Ex-Post and Ex-Ante Coherence. The idea is that we must add as many experts as possible in a coherent way, in order to avoid sure regrets caused by violations of Dynamic Consistency. On the one hand, extending the set of probabilities means that more opinions are taken into account.[3] On the other hand, it implies that it is more likely that two experts disagree on some rankings. Our point here is that the decrease in comparability (i.e. the increase of incompleteness of preferences) is consistent with the use of the precautionary principle, as adding experts means increasing aversion to ambiguity in the sense of Ghirardato et al. (2004).

Our main result, Theorem 1, states that given two Bewley preferences, $\succsim^*$ and $\succsim^{**}$, the relation $\succsim^{**}$ is the coherent precautionary reassessment of $\succsim^*$ if, and only if, it is a Bewley preference represented by the same utility index on consequences, and with a set of priors that is derived through suitable combinations of all conditional beliefs of the original group of experts. Our theorem provides a prescriptive way to aggregate experts' opinions taking into account the information flow and avoiding dynamic inconsistencies that may arise when one completes the unanimity rule with the precautionary principle as in GMMS. This implies that the new set of experts' beliefs is a "rectangularization" *à la* Sarin and Wakker (1998) and Epstein and Schneider (2003) of the set of probabilities characterizing the original preference $\succsim^*$.

Modifying the original Bewley preference has practical relevance in many decision problems. Our leading example, Example 1, which is developed throughout the paper, shows in an Ellsberg (1961) type setting applied to equilibrium climate sensitivity how preferences must change in order to avoid dynamic inconsistencies. We also provide an application to decisions related to pandemics in Sect. 5.1.

The rest of the paper is organized as follows. Section 2 introduces the necessary notations. Section 3 recalls the results of GMMS and gives an example of dynamic

[3] Intuitively, reassessing the original preference expands the set of experts while coherence "adjusts" its size.

inconsistency. Section 4 contains our axioms and main results. Section 5 provides one application and additional results. Section 6 concludes. Proofs are gathered in the Appendix.

2 Framework and notation

Consider a set S of *states of the world*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We say that a function $f : S \to X$ is *simple* if $f(S) := \{f(s) : s \in S\}$ is a finite set. A simple function f is Σ-measurable if $\{s \in S : f(s) = x\} \in \Sigma$ for all $x \in X$. We denote by $\mathcal{F}$ the set of all simple and Σ-measurable functions. A function $f \in \mathcal{F}$ is called *act*.

We assume the set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of outcomes Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as re-stated by Fishburn (1970). Using the linear structure of X, we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1-\alpha)g : S \to X$ by

$$(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s).$$

Also, given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$, we denote by fEg the act delivering the consequences $f(s)$ in E and $g(s)$ in $E^c := S \backslash E$ (the complement of E).

We denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = \sup_{s\in S} |a(s)|$ (called the *supnorm*) and $B(\Sigma)$ will denote the supnorm closure of $B_0(\Sigma)$. In another way, $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm (for more details, see Dunford and Schwartz 1988, Section 5 of Chapter IV). We denote by $ba(\Sigma)$ the Banach space of all finitely additive set functions on Σ endowed with the total variation norm. It is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Note also that the weak* topology $\sigma(ba, B_0)$ of $ba(\Sigma)$ coincides with the eventwise convergence topology. Throughout the paper, we assume that any subset of $ba(\Sigma)$ is endowed with the topology inherited from the weak* topology.

Given a non-constant mapping $u : X \to \mathbb{R}$, function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s))$, for all $s \in S$. We note that $u(f) \in B_0(\Sigma)$ whenever f belongs to $\mathcal{F}$. Let x be a consequence in X, abusing notation we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set of constant acts in $\mathcal{F}$. We say that a function $u : X \to \mathbb{R}$ is *affine* if for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, $u(\alpha f + (1-\alpha)g) = \alpha u(f) + (1-\alpha)u(g)$. Affine functions $u : X \to \mathbb{R}$ are called *utility functions*.

We denote by $\Delta(S, \Sigma) := \Delta$ the set of all (finitely additive) probability measures $P : \Sigma \to [0, 1]$. Given an act $f \in \mathcal{F}$, a utility index u, and a probability measure $P \in \Delta$, the expected utility of f is denoted by $\int u(f)\, dP$. Consider an event $E \in \Sigma$ and a probability $P \in \Delta$ such that $P(E) > 0$. The *Bayesian update* of P with respect to E is $P^E(A) = \frac{P(A\cap E)}{P(E)}$, for all $A \in \Sigma$. Let $\mathcal{C} \subseteq \Delta$ and $E \in \Sigma$ such that $P(E) > 0$

for all $P \in \mathcal{C}$, then the set $\mathcal{C}^E$ denotes the set of *prior-by-prior Bayesian updates* of $\mathcal{C}$ given E, i.e. $\mathcal{C}^E = \{P^E | P \in \mathcal{C}\}$. We also say that $\mathcal{C}$ is updated following the *full Bayesian updating rule*.

A *preference relation* $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ is a binary relation that satisfies reflexivity, transitivity (*preorder*), continuity and non-triviality. *Transitivity* means that for all $f, g, h \in \mathcal{F}$, $f \succsim g$ and $g \succsim h$ imply $f \succsim h$. *Continuity* means that for all $f, g, h \in \mathcal{F}$ the sets $\{\lambda \in [0, 1] | \lambda f + (1 - \lambda) g \succsim (\precsim) h\}$ are closed in $[0, 1]$. *Non-triviality* means that $\succsim$ has a non-empty strict part. As usual, the strict and weak parts of $\succsim$ are denoted $\succ$ and $\sim$ respectively.

3 Objective and subjective rationality and dynamic consistency

This section discusses the interplay between the unanimity rule and the precautionary principle, and the role of Dynamic Consistency. Section 3.1 recalls the axiomatic analysis of GMMS in which it is studied how to complete a Bewley preference with a Maxmin preference. Section 3.2 introduces an intermediate period of partial resolution of uncertainty and provides an Ellsberg-type example in which dynamic inconsistencies arise.

Most of the concepts discussed in Sect. 3 are well known. However they are needed to motivate our main results of Sect. 4. The reader who is familiar with GMMS and with the notion of Dynamic Consistency may skip this section and proceed directly to Sect. 4.

3.1 Completion of a bewley preference by a maxmin preference

In the context of social decisions, the unanimity principle postulates that society should prefer f to g if every individual prefers f to g. Consider a group of individuals (applying unanimity) in which each member has Subjective Expected Utility preferences with possibly different probability distributions and utility functions. Danan et al. (2016) proposed normative (Pareto) principles in order to aggregate individual preferences into a unanimity rule in which individuals' utilities capturing tastes are combined *à la* Harsanyi (1955) into one (social) utility function on consequences. Therefore, it is as if society is represented by one DM with a Bewley (2002) preference in which the set of probability distributions is characterized by the convex hull of the subjective priors of the members.[4]

Formally, let $u : X \to \mathbb{R}$ be a utility function and $\mathcal{C} \subseteq \Delta$ be a nonempty, convex, and $\sigma(ba, B_0)$-compact subset of Δ. We say that $\succsim^*$ is a *Bewley preference* represented by $(u, \mathcal{C})$ if for all $f, g \in \mathcal{F}$,

$$f \succsim^* g \Leftrightarrow \int u(f) dP \geq \int u(g) dP, \forall P \in \mathcal{C}. \tag{1}$$

[4] This can be viewed as an application of Theorem 2 of Danan et al. (2016). Their results are actually more general as each preference of the group members can be itself a Bewley preference.

The criterion (1), axiomatized by Bewley (2002), says that f is preferred to g with respect to the preference $\succsim^*$ if, and only if, the expected utility of f is higher than the expected utility of g according to every probability $P \in \mathcal{C}$. If each probability distribution in $\mathcal{C}$ represents the opinion of one expert, then f is better than g if, and only if, every expert ranks f above g.[5] This justifies the name unanimity rule: experts should all agree. In Sect. 4, we consider $\succsim^*$ as a primitive of our model and we interpret it as a (incomplete) rule to aggregate different opinions.

In general, this decision rule is incomplete, i.e. does not rank every pair $f, g \in \mathcal{F}$. It may happen that there are $P_1, P_2 \in \mathcal{C}$ such that $\int u(f)dP_1 > \int u(g)dP_1$ and $\int u(g)dP_2 > \int u(f)dP_2$. If two acts cannot be compared, but a decision must be taken, then the DM has several options to aggregate experts' opinions. For instance, one can think about the smooth ambiguity model of Klibanoff et al. (2005), the variational model of Maccheroni et al. (2006), the confidence model of Chateauneuf and Faro (2009) etc. An interesting way to obtain an Expected Utility DM by combining geometrically the probability distributions in $\mathcal{C}$ (preserving Dynamic Consistency) was proposed in a recent paper by Dietrich (2021).

We choose to use the precautionary principle, which we identify in this article with the Maxmin model (as we said in the Introduction, see Footnote 1, there are several possible definitions of the precautionary principle). This principle states that f is better than g if, and only if, the minimum expected utility of f is higher that the minimum expected utility of g, where the minimum is considered over all probabilities in a set $\mathcal{C}$. Put formally, if $u : X \to \mathbb{R}$ is a utility function and $\mathcal{C} \subseteq \Delta$ a nonempty, convex, and $\sigma(ba, B_0)$-compact subset of Δ, then $\succsim^\#$ is a *Maxmin preference* represented by $(u, \mathcal{C})$ if for all $f, g \in \mathcal{F}$,

$$f \succsim^\# g \Leftrightarrow \min_{P \in \mathcal{C}} \int u(f)dP \geq \min_{P \in \mathcal{C}} \int u(g)dP. \tag{2}$$

Maxmin was introduced by Wald (1950) in statistical decision theory and it has been axiomatized in our framework by Gilboa and Schmeidler (1989). It is easy to see that the preference relation $\succsim^\#$ represented by (2) is complete, i.e. it allows to compare every pair of acts f and g.

GMMS analyze the interplay between a Bewley preference $\succsim^*$ and a complete preference $\succsim^\#$ (note that in our definition of preference relation in Sect. 2 completeness is not required). In GMMS, $\succsim^*$ represents the *objective rationality* of the DM: the DM can convince others that f is better than g in an uncontroversial way. The preference $\succsim^\#$ represents the *subjective rationality* of the DM: the DM cannot be convinced of being wrong choosing f rather than g (the DM does not feel embarrassed after her choice). With our interpretation in terms of experts, it is also clear that $f \succsim^* g$ implies that f is objectively better than g as there is an unanimous agreement in favor of f. GMMS study under which conditions $\succsim^\#$ is a Maxmin preference that can be used to compare acts that are not comparable with respect to $\succsim^*$. Note that the choice of completing a Bewley preference through a Maxmin preference is, in a sense, a subjective choice,

[5] Note that all probabilities, and hence all experts, have the same importance. One can generalize this decision rule assigning different weights to different experts as in Faro (2015).

and this is why we take as primitive the former preference and not the latter. GMMS justify this completion imposing two axioms on the couple $(\succsim^*, \succsim^\#)$.

Consistency. For all $f, g \in \mathcal{F}$, $f \succsim^* g$ implies $f \succsim^\# g$.

Default to Certainty. For all $f \in \mathcal{F}$ and $x \in X$, if not $f \succsim^* x$ then $x \succ^\# f$.

Consistency says that the decision rule $\succsim^\#$ agrees with $\succsim^*$ whenever acts are comparable. In the spirit of GMMS, if it is objectively rational to prefer f to g, then it is subjectively rational too. Consistency implies that the preference $\succsim^\#$ is a *completion* of $\succsim^*$, meaning that $\succsim^\#$ is complete and $\succsim^* \subseteq \succsim^\#$. Default to Certainty says that if an uncertain act f is not unanimously preferred to a constant $x \in X$, then $\succsim^\#$ should rank x above f. Default to Certainty behaviorally justifies the identification of the Maxmin rule with the precautionary principle.[6]

Theorem 0 [GMMS, THEOREM 4] *Let $\succsim^*$ be a Bewley preference represented by $(u, \mathcal{C})$ and let $\succsim^\#$ be a complete preference relation. Then:*

(i) The pair $(\succsim^*, \succsim^\#)$ jointly satisfies Consistency and Default to Certainty;

(ii) $\succsim^\#$ is a Maxmin preference represented by $(u, \mathcal{C})$.

The following Ellsberg (1961) example will be used throughout the paper in order to illustrate the theoretical concepts that we present. Here we explicit how to complete a Bewley preference through a Maxmin preference. In order to show a possible application of our approach, we apply Ellsberg's example to climate change. One of the main summary statistics in climate change science is the equilibrium climate sensitivity. It denotes the equilibrium increase in global mean temperatures that would occur if the concentration of atmospheric CO_2 were doubled. Uncertainty about estimates of this parameter is high. See Figure 1 in Meinshausen et al. (2009) in which are plotted estimated probability density functions for climate sensitivity from several published studies. This is one possible situation in which it is reasonable to complete the unanimity rule with the precautionary principle.

Example 1 There are three possible future scenarios: low, medium and high climate sensitivity. A group of experts knows that the probability $P(\text{low}) = \frac{1}{3}$, moreover they think that $P(medium)$ is at least $\frac{1}{6}$ and at most $\frac{1}{2}$. Suppose that experts can be identified with the set

$$\mathcal{C} = \left\{ P = \left(\frac{1}{3}, p, \frac{2}{3} - p \right) \in \Delta \middle| p \in \left[\frac{1}{6}, \frac{1}{2} \right] \right\}$$

Note that, given set $\mathcal{C}$, the events "low" and "medium or high" are unambiguous since $P(\text{low}) = \frac{1}{3}$ and $P(\text{medium or high}) = \frac{2}{3}$ for all $P \in \mathcal{C}$.

This example is formally equivalent to an Ellsberg urn containing 90 balls, 30 of which are red, while the remaining 60 are either blue or green with the number of blue balls being in between 15 and 45. Let us denote low, medium and high by R, B and G respectively. Suppose that a government has to choose between the two policies yielding the payoffs summarized in the table below.

[6] A generalization of GMMS is given in Cerreia-Vioglio (2016) and provides other type of completions with uncertainty averse preferences. Other papers studying pairs of binary relations in which one of them is a Bewley-type relation are Giarlotta and Greco (2013), Cerreia-Vioglio et al. (2020), Frick et al. (2022) and Grant et al. (2021).

	Red	Blue	Green
f	10	0	10
g	0	10	10

First, remark that act g is not ambiguous in the sense that $P(g = 10) = \frac{2}{3} = 1 - P(g = 0)$ for all probabilities $P \in \mathcal{C}$. On the contrary, f is ambiguous.

Consider a DM who is advised by this group of experts. Note that f and g are not comparable with respect to a Bewley preference represented by $(u, \mathcal{C})$. For instance let $P_1 = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$, $P_2 = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ and suppose w.l.o.g. $u(0) = 0$. Then $\int u(f)dP_1 = \frac{5}{6}u(10) > \frac{4}{6}u(10) = \int u(g)dP_1$ and $\int u(f)dP_2 = \frac{1}{2}u(10) < \frac{2}{3}u(10) = \int u(g)dP_2$. If the DM uses a Maxmin preference in order to compare the two acts she gets

$$\min_{P \in \mathcal{C}} \int u(f)dP = \frac{1}{2}u(10) < \frac{2}{3}u(10) = \min_{P \in \mathcal{C}} \int u(g)dP$$

i.e. when the Bewley preference $\succsim^*$ represented by $(u, \mathcal{C})$ is completed by a Maxmin preference $\succsim^\#$ represented by $(u, \mathcal{C})$, one obtains $g \succ^\# f$.

3.2 Dynamic (in)Consistency

The framework of Sect. 3.1 is static. It does not take into consideration how a DM would react to new information that could be obtained over time. Let us add an intermediate period of partial resolution of uncertainty.

Let $\mathcal{P} = \{E_1, \ldots, E_n\}$ denote a finite partition of measurable events of S, meaning that E_1, ..., $E_n \in \Sigma$, $S = \cup_{i=1}^n E_i$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. Partition $\mathcal{P}$ models the information structure: the DM knows today that tomorrow she will learn that $s \in E_i$ for some $i = 1, \ldots, n$. Consider a DM with a (unconditional or ex-ante) preference relation $\succsim$ over $\mathcal{F}$. Given $E \in \mathcal{P}$, we call $\succsim_E$ the *conditional (or ex-post) preference*. It is interpreted as the preference of the DM once she knows $s \in E$.

The following axiom called *Consequentialism* says that a DM should not be concerned about the consequences of an act in states that are known not to occur. We denote $f|_E$ the restriction of act f on event E.

CONSEQUENTIALISM. For all $f, g \in \mathcal{F}$ and $E \in \mathcal{P}$, if $f|_E = g|_E$, then $f \sim_E g$.

The second and fundamental axiom is *Dynamic Consistency*. It relates the unconditional preference $\succsim$ to the conditional preference $\succsim_E$.

DYNAMIC CONSISTENCY. For all $f, g \in \mathcal{F}$ and $E \in \mathcal{P}$, $f \succsim_E g \Leftrightarrow fEg \succsim g$.

The axiom says that f is better than g conditional on E if, and only if, whenever one replaces act g with act f on E, the resulting act fEg is better than g. We refer the reader to Ghirardato (2002) for a detailed interpretation of this property. Violations of Dynamic Consistency can cause embarrass to the DM. Consider the following proposition.

Proposition 1 *Suppose that $\succsim$ and $\succsim_{E_i}$ satisfy Dynamic Consistency and Consequentialism. Then $f \succsim_{E_i} g$ for all $i = 1, \ldots, n$ implies $f \succsim g$.*

Proposition 1 shows that, if Dynamic Consistency is violated, it can happen that today a DM prefers $g \succ f$, but tomorrow, once she updates her preferences to $\succsim_{E_i}$, her choice will be regretted whatever event E_i is realized. Finally, note that if the couple of preferences $(\succsim, \succsim_E)$ satisfies Dynamic Consistency, then $\succsim_E$ satisfies Consequentialism (see Faro and Lefort 2019).

Let $\succsim^*$ be a Bewley preference relation represented by $(u, \mathcal{C})$. Consider an event $E \in \Sigma$ such that $P(E) > 0$ for all $P \in \mathcal{C}$. Then Ghirardato et al. (2008) prove that Dynamic Consistency and Consequentialism are equivalent to $\succsim^*_E$ being a Bewley preference represented by $(u, \mathcal{C}^E)$, where $\mathcal{C}^E$ is the set obtained from $\mathcal{C}$ by the full Bayesian updating rule, i.e. $\mathcal{C}^E = \{P^E | P \in \mathcal{C}\}$ and $P^E(\cdot) = \frac{P(\cdot \cap E)}{P(E)}$. The equivalence between Dynamic Consistency and the prior-by-prior updating rule in Bewley's model was also previously discussed by Bewley (1987) and Epstein and Le Breton (1993).

The conditional Bewley preference $\succsim^*_E$ is in general incomplete. If one is willing to directly use Theorem 0 in order to complete $\succsim^*_E$ with a Maxmin preference $\succsim^\#_E$ represented by $(u, \mathcal{C}^E)$, then possible dynamic inconsistencies may arise. Consider the following dynamic version of Example 1.[7]

Example 1 - cont Consider the setting of Example 1. Let us add an intermediary period of partial resolution of uncertainty. Information is modeled through the partition

$$\mathcal{P} = \{G, RB\}.$$

Note that events, G and RB are ambiguous w.r.t. probabilities in $\mathcal{C}$, i.e. different experts assign different evaluations to those events. Recall from the previous part of this example, page 7, that act g was not ambiguous w.r.t. set $\mathcal{C}$. Now, if the true state happens to be either Red or Blue, act g becomes ambiguous, in the sense that $P(g = 10|RB)$ depends on the probability under consideration.

Suppose that probabilities in $\mathcal{C}$ are updated through the full Bayesian rule. In our example one has $\mathcal{C}^G = \{(0, 0, 1)\}$, i.e. experts know that the true state is G, and $\mathcal{C}^{RB} = \{(q, 1-q, 0) \in \Delta | q \in \left[\frac{2}{5}, \frac{2}{3}\right]\}$.

By Consequentialism we obtain $f \sim^*_G g$. This implies $f \sim^\#_G g$ (by Consistency). Note that f and g are not comparable with respect to the Bewley preference $\succsim^*_{RB}$ represented by $(u, \mathcal{C}^{RB})$ (consider for instance $Q_1 = (\frac{2}{5}, \frac{3}{5}, 0)$ and $Q_2 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$). By considering the precautionary completion $\succsim^*_{RB}$ given by the Maxmin preference $\succsim^\#_{RB}$ represented by $(u, \mathcal{C}^{RB})$. Then

$$\min_{Q \in \mathcal{C}^{RB}} \int u(f) dQ = u(10)\frac{2}{5} > u(10)\frac{1}{3} = \min_{Q \in \mathcal{C}^{RB}} \int u(g) dQ$$

i.e. $f \succ^\#_{RB} g$, and Dynamic Consistency would imply $f = fRBg \succsim^\# g$, which contradicts what we found before, namely $g \succ^\# f$ (see the end of Example 1, p. 7). In general using prior-by-prior updating and then applying Theorem 0 violates Dynamic Consistency.

[7] This example is inspired by Example 1 in Ghirardato et al. (2008). The authors acknowledge that they owe the example to Denis Bouyssou.

An axiomatization of the full Bayesian updating in the model of objective and subjective rationality has been proposed by Faro and Lefort (2019) in a framework without the inclusion of a partition in the primitives (see also Frick et al. 2022). In their work, dynamic inconsistencies are allowed and interpreted as a product of what they call forced choices (decisions that must be made and based only on subjective grounds). In the perspective of this paper, Dynamic Consistency is viewed as a property of preferences fundamentally related to rationality. If a preference relation is not dynamically consistent then we may have $f \succ g$ but $g \succ_E f$ and $g \succ_{E^c} f$. This means that for dynamically inconsistent preferences, decisions taken today, meaning choosing f over g, may be regretted tomorrow, i.e. the conditional preference will rank g above f no matter if $s \in E$ or $s \in E^c$, see Proposition 1. We think therefore that it is reasonable to require Dynamic Consistency for subjectively rational preferences (defined in the spirit of GMMS).

It is well known that a Maxmin preference relation $\succsim^{\#}$ represented by $(u, \mathcal{C})$ is not dynamically consistent in general. Epstein and Schneider (2003) prove that Dynamic Consistency holds if, and only if, the set of priors $\mathcal{C}$ is rectangular.[8] In Sect. 4 we show how the unanimity rule $\succsim^*$ should be revised in order to achieve a rectangular set, so that Dynamic Consistency holds for the derived Maxmin preference relation.

4 Axioms and main results

Our starting point is a Bewley-type DM with preference $\succsim^*$ who must rank acts at time $t = 0$. As put forward in the Introduction, one can think about a policymaker who aggregates the opinions (represented by probability distributions) of several experts. Whenever this DM cannot compare two acts, she uses the precautionary principle (the completion with the precautionary principle is justified by GMMS axiomatic analysis, and more specifically by the axiom Default to Certainty). One can interpret $\succsim^*$ as the preference relation representing objective evidence *before* the introduction of the intermediary period of partial resolution of uncertainty and its related partition $\mathcal{P}$, see Example 1. As shown in Sect. 3.2, if her ex-post preference $\succsim^*_E$ is completed by the corresponding Maxmin preference represented by $(u, \mathcal{C}^E)$, dynamic inconsistencies may arise. Moreover, the introduction of the information partition $\mathcal{P}$ may increase the ambiguity perceived ex-ante by the DM (as in Example 1 and in Sect. 5.1). We study here how $\succsim^*$ should be transformed into a "new" Bewley preference $\succsim^{**}$ in order to avoid dynamic inconsistencies and to take into account the possible increase in ambiguity.

Throughout this section we consider two Bewley preferences, $\succsim^*$ and $\succsim^{**}$, represented respectively by $(u, \mathcal{C})$ and $(\hat{u}, \hat{\mathcal{C}})$. We fix a finite partition $\mathcal{P} = \{E_1, \cdots, E_n\} \subseteq \Sigma$, such that $P(E_i) > 0$ for all $i \in \{1, \cdots, n\}$ and all $P \in \mathcal{C}$ and we denote $\succsim^*_E$ and $\succsim^{**}_E$ the dynamically consistent updates of $\succsim^*$ and $\succsim^{**}$, respectively. Therefore, $\succsim^*_E$

[8] Note that there exists different ways to cope with dynamic inconsistencies. For instance, Siniscalchi (2011) studies consistent-planning for preferences over decision trees. Gul and Pesendorfer (2021) take into account the sequencing of the resolution of uncertainty and weaken the law of iterate expectation. Finally Dietrich (2021) proposes a geometric aggregation of beliefs.

and $\succsim_E^{**}$ are Bewley preferences represented by $(u, \mathcal{C}^E)$ and $(\hat{u}, \hat{\mathcal{C}}^E)$ respectively, see Sect. 3.2.

Since we modify $\succsim^*$ only in order to avoid violations of Dynamic Consistency, we want the new preference $\succsim^{**}$ to satisfy two desiderata. (i) No new information is added, therefore $\succsim^{**}$ cannot be more complete than $\succsim^*$. This means that there is no ranking-reversal (i.e. it cannot happen $f \succsim^* g$, but $g \succ^{**} f$) and no new rankings are formed (i.e. if f and g are incomparable for $\succsim^*$, then they are also incomparable for $\succsim^{**}$). (ii) The new preference $\succsim^{**}$ must coincide with the old one whenever violations of Dynamic Consistency do not occur.

The first desideratum brings us to the definition of a reassessment.

Definition 1 We say that $\succsim^{**}$ is a *reassessment* of $\succsim^*$ if for all $f, g \in \mathcal{F}$, $f \succsim^{**} g \Rightarrow f \succsim^* g$.[9]

If $\succsim^{**}$ is a reassessment of $\succsim^*$, then it is well known by Ghirardato et al. (2004) that $\hat{\mathcal{C}} \supseteq \mathcal{C}$ and w.l.o.g. that $\hat{u} = u$. If we interpret $\mathcal{C}$ and $\hat{\mathcal{C}}$ as sets of (probabilistic) opinions of a group of experts, then reassessing a preference relation means "adding" experts to $\mathcal{C}$ to ensure that more opinions are taken into account. This is in line with a precautionary attitude towards decision making and reflects the possible increase of ambiguity that can result from an information partition that is not in line with previous information.[10] This happens for instance for partition $\mathcal{P} = \{G, RB\}$ in the example at page 9, in which the information partition breaks the non-ambiguity of events R and BG.

In the extreme case in which $\succsim^{**}$ is a Bewley preference represented by (u, Δ), one has that $\succsim^{**}$ is a reassessment of $\succsim^*$. In this case an act f is preferred to g if and only if $u(f(s)) \geq u(g(s))$ for all states $s \in S$. Completing the unanimity-over-states rule with a Maxmin preference represented by (u, Δ) would generate dynamically consistent decisions. However, this decision rule would completely ignore the opinions of the initial group of experts and the information structure given by partition $\mathcal{P}$.

Desideratum (ii) takes into consideration the discussion in the previous paragraph and forces $\succsim^{**}$ to rank acts coherently with respect to $\succsim^*$, whenever dynamic inconsistencies cannot occur. The following two axioms on the pair of preferences $(\succsim^*, \succsim^{**})$ represent the main behavioral novelty of the paper.

EX- POST COHERENCE. For all $f, g \in \mathcal{F}$, for all $E \in \mathcal{P}$, $f \succsim_E^* g \Rightarrow f \succsim_E^{**} g$.

EX- ANTE COHERENCE. For all $f, g \in \mathcal{F}$ s.t. f, g are $\mathcal{P}$-measurable, $f \succsim^* g \Rightarrow f \succsim^{**} g$.

Ex-Post Coherence says that the new conditional preference relation $\succsim_E^{**}$ should preserve rankings originally revealed by $\succsim_E^*$ for all $E \in \mathcal{P}$. Note that, while the definition of a reassessment allows an increase of incompleteness of the ex-ante preference, Ex-Post Coherence moves in the opposite way. As a result, it is simple to see that the conjunction of Reassessment and Ex-Post Coherence implies that $\succsim_E^*$ and $\succsim_E^{**}$ coincide (see Lemma 3 in the Appendix). To understand the normative validity of this axiom, fix $f, g \in \mathcal{F}$ and $E \in \mathcal{P}$ and suppose $f \succsim_E^* g$. Since Bewley preferences are dynamically consistent (see Sect. 3.2) we have $f \succsim_E^* g \Leftrightarrow fEg \succsim^* g$. Because

9 Equivalently, this can be written $\succsim^{**} \subseteq \succsim^*$.

10 Note that adding experts means increasing aversion to ambiguity in the sense of Ghirardato et al. (2004).

of Consistency of GMMS, $f \succsim_E^* g \Rightarrow f \succsim_E^\# g$ and $fEg \succsim^* g \Rightarrow fEg \succsim^\# g$. Therefore there is no violation of Dynamic Consistency for acts $f, g \in \mathcal{F}$ and for the event $E \in \mathcal{P}$.

Ex-Ante Coherence simply says that $\succsim^{**}$ should rank $\mathcal{P}$-measurable acts as $\succsim^*$.[11] As for Ex-Post Coherence, Ex-Ante Coherence is normatively appealing since the acts involved in the axiom do not generate violations of Dynamic Consistency. To see this, consider two $\mathcal{P}$-measurable acts f and g and an event $E_i \in \mathcal{P}$. Let f_i and g_i denote the constant value taken by f and g on E_i. Just observe that $fE_ig \succsim^\# g$ if and only if $u(f_i) \geq u(g_i)$ if and only if $f \succsim_{E_i}^\# g$.

We say that $\succsim^{**}$ is a *coherent reassessment* of $\succsim^*$ if it is a reassessment and if the pair $(\succsim^*, \succsim^{**})$ satisfies Ex-Post Coherence and Ex-Ante Coherence. A coherent reassessment satisfies desiderata (i) and (ii), but it does not guarantee Dynamic Consistency as shown in Example 2 in the Appendix. We actually need something more. Consider the next definition.

Definition 2 We say that $\succsim^{**}$ is the *coherent precautionary reassessment* of $\succsim^*$, if $\succsim^{**}$ is the most incomplete coherent reassessment of $\succsim^*$.

Definition 2 says that not only we want $\succsim^{**}$ to be a coherent reassessment of $\succsim^*$, but we want it to be the *most incomplete* coherent reassessment. This is the reason why we call it "precautionary": we want to add as many probabilities as possible to the set $\mathcal{C}$ characterizing the initial preference. A bigger set of probabilities implies more ambiguity aversion in the sense of Ghirardato et al. (2004). Moreover it implies a more pessimistic evaluation of each act, once the Maxmin completion is computed. Finally it is fundamental to note that if we do not require $\succsim^{**}$ to be the most incomplete coherent reassessment, we can still get dynamic inconsistencies.[12]

We can now state our main result.

Theorem 1 *The following assertions are equivalent:*

(i) The preference $\succsim^{**}$ is the coherent precautionary reassessment of $\succsim^*$;
(ii) For all $f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Leftrightarrow$$
$$\sum_{i=1}^{n} P_0(E_i) \int u(f) dP_i^{E_i} \geq \sum_{i=1}^{n} P_0(E_i) \int u(g) dP_i^{E_i}, \forall P_0, P_1, \ldots, P_n \in \mathcal{C}. \tag{3}$$

Expression (3) in Theorem 1 tells how acts are evaluated by the coherent precautionary reassessment. First one should fix $n+1$ probabilities $P_0, P_1, \ldots, P_n \in \mathcal{C}$, i.e. $n+1$ experts should be chosen. Each probability P_i, $i = 1, \ldots, n$ should be assigned

[11] We thank an anonymous referee for pointing out an error in the definition of Ex-Ante Coherence given in a previous version of the paper.

[12] Note that an alternative way to solve both problems of incompleteness and time inconsistency is to pick only one expert out of the group. If there is not an objective procedure to select the "best expert", it seems reasonable for a DM to opt for a plurality of opinions.

to a set in the partition $\mathcal{P}$ (for simplicity we denote P_i the probability assigned to set E_i). Then for act f the quantity $\sum_{i=1}^{n} P_0(E_i) \int u(f) dP_i^{E_i}$ should be computed. Expression $\int u(f) dP_i^{E_i}$ is the expected utility of f calculated through the Bayesian update with respect to E_i of the corresponding probability P_i. Then expected utilities are aggregated through a convex combination in which weights are given by $P_0(E_i)$, $i = 1, \ldots, n$. To summarize:

$$\underbrace{\sum_{i=1}^{n} P_0(E_i)}_{\substack{\text{convex} \\ \text{combination}}} \underbrace{\int u(f) dP_i^{E_i}}_{\substack{\text{EU of } f \text{ w.r.t.} \\ \text{update } P_i^{E_i}}} . \tag{4}$$

The expression in (4) should be calculated for all possible $n + 1$ choices of probabilities $P_0, P_1, \ldots, P_n \in \mathcal{C}$, for both acts f and g. If for all choices of probabilities the value obtained for f is higher than the one obtained for g, then $f \succsim^{**} g$. This result can be viewed as a prescriptive way about how to aggregate opinions. Hence, a Bewley preference $\succsim^{*}$ represented by $(u, \mathcal{C})$ should be revised in the following way: first, compute the Bayesian update for all $P \in \mathcal{C}$ and all events in the partition; second, compute the expected value under these conditional probabilities; and third, take convex combinations using as weights the opinions of the members on likelihood of events in $\mathcal{P}$. The decision criterion (3) is one in which new experts acquire veto power. These new experts are "constructed" from the old ones precisely as we just described. For a numerical illustration see the computations related to Sect. 5.1 (given in the Appendix, p. 25). Therefore a Bewley-type DM who wants to complete her preferences through the precautionary principle avoiding dynamic inconsistencies should add probability measures as prescribed by Theorem 1. Such a DM can be thought of as "sophisticated" in the sense that at $t = 0$ she can construct a reassessment $\succsim^{**}$ for any possible partition of the state space. Finally, note that Theorem 1 does not make reference to the Maxmin completion of $\succsim^{**}$. However, the whole point of computing the precautionary reassessment is motivated by the lack of Dynamic Consistency caused by a direct approach of GMMS.

Given a finite partition $\mathcal{P} = \{E_1, \cdots, E_n\} \subseteq \Sigma$, it is well known that a probability Q can be written as $Q = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}$ for some $P_0, P_1, \ldots, P_n \in \mathcal{C}$ if, and only if, Q is in the rectangular hull of $\mathcal{C}$. This notion is formalized in Definition 3, based on the previous contributions of Sarin and Wakker (1998), Epstein and Schneider (2003) and Ghirardato et al. (2008).

Definition 3 The rectangular hull of a set of priors $\mathcal{C} \subseteq \Delta$ w.r.t. partition $\mathcal{P}$ is given by

$$r_{\mathcal{P}}(\mathcal{C}) := \left\{ \sum_{i=1}^{n} P_0(E_i) \cdot P_i^{E_i} \,\middle|\, P_0, P_1, \cdots, P_n \in \mathcal{C} \right\}.$$

We say that a set $\mathcal{C} \subseteq \Delta$ is rectangular (w.r.t. $\mathcal{P}$) when $\mathcal{C} = r_{\mathcal{P}}(\mathcal{C})$.

The rectangular hull of the set $\mathcal{C}$ for a partition $\mathcal{P}$ is obtained by considering all convex combinations of the conditional probabilities (conditioned using Bayesian updating on events in $\mathcal{P}$) with weights given by the unconditional probabilities. The link with (4) should be evident. Given Definition 3, the following corollary is immediate.

Corollary 1 *Item (ii) of Theorem 1 is equivalent to*

(iii) For all $f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Leftrightarrow \int u(f)dQ \geq \int u(g)dQ, \forall Q \in r_{\mathcal{P}}(\mathcal{C}). \tag{5}$$

Obviously $\mathcal{C} \subseteq r_{\mathcal{P}}(\mathcal{C})$, which reflects the fact that $\succsim^{**}$ is a reassessment of $\succsim^{*}$. Corollary 1 shows that our result offers a novel behavioral foundation of rectangularity based on the interaction of two Bewley preferences. According to Corollary 1, a different way to state Theorem 1 could have been: "A Bewley preference admits a representation with a rectangular set of priors if and only if it is its own coherent precautionary reassessment". However, this would have been a purely mathematical result without any economic justification as Bewley preferences are dynamically consistent and therefore they do not need to be "rectangularized".

On the other end Theorem 1 proves useful when one wants to complete $\succsim^{**}$ maintaining Dynamic Consistency. An application of Theorem 0 yields the following result.

Corollary 2 *Let $\succsim^{**}$ be the coherent precautionary reassessment of $\succsim^{*}$ and assume that $\succsim^{\#\#}$ is a complete preference relation. The following are equivalent:*

(i) The pair $(\succsim^{**}, \succsim^{\#\#})$ jointly satisfies Consistency and Default to Certainty;
(ii) $\succsim^{\#\#}$ is a Maxmin preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$.

Moreover for any $E \in \mathcal{P}$, $r_{\mathcal{P}}(\mathcal{C})^E = \mathcal{C}^E$ and if $\succsim_E^{\#\#}$ is a Maxmin preference represented by $(u, r_{\mathcal{P}}(\mathcal{C})^E)$ then the pair $(\succsim^{\#\#}, \succsim_E^{\#\#})$ satisfy Dynamic Consistency.

The last sentence of Corollary 2 says that Dynamic Consistency is satisfied. This result is derived from Epstein and Schneider (2003), see also Amarante and Siniscalchi Amarante and Siniscalchi (2019). As we argued in Sect. 3.2, Dynamic Consistency is an important property that a subjectively rational preference should satisfy. Corollary 2 together with Theorem 1 give a new behavioral characterization of rectangularity and Dynamic Consistency.

Corollary 2 is linked to the papers of Riedel et al. (2018) and of Ceron and Vergopoulos (2021). In the first paper the authors study the issue of Dynamic Consistency in the model of imprecise probabilistic information of Gajdos et al. (2008). They show that, in order to ensure Dynamic Consistency, a DM first should rectangularize a given set of probabilities, and then select a subset from the resulting rectangular hull. The second paper builds upon the ideas presented in this article about achieving Dynamic Consistency in a model of objective and subjective rationality. Ceron and Vergopoulos (2021) adapt the axioms of Consistency and Default to Certainty of GMMS to a

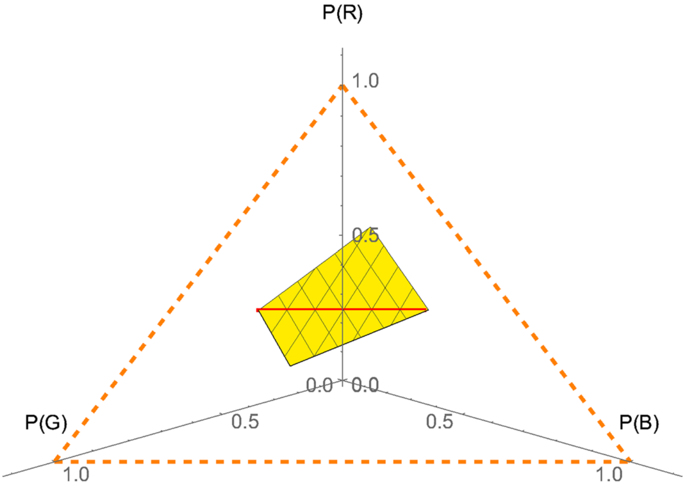

Fig. 1 The set $\mathcal{C}$ (the thick red segment) and its rectangular hull

framework *à la* Epstein and Schneider (2003) and show how to complete the update of a Bewley preference represented by $(u, \mathcal{C})$ through a Maxmin preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$.

Finally, note that Corollary 2 considers prior-by-prior updating. One could also consider the maximum likelihood rule of Gilboa and Schmeidler (1993) as the two rules coincide under rectangularity, see Gilboa and Marinacci (2016), Section 5. We use prior-by-prior updating as a benchmark, as it is equivalent to Dynamic Consistency for a Bewley preference as shown in Ghirardato et al. (2008).

We conclude this section solving the dynamic inconsistencies in Example 1.

Example 1 - cont In the Ellsberg example we get that $\bar{P} \in r_{\mathcal{P}}(\mathcal{C})$ if, and only if, there are $P_0, P_1, P_2 \in \mathcal{C}$ such that

$$\bar{P}(R) = P_0(G)P_1^G(R) + (1 - P_0(G))P_2^{RB}(R) = 0 + (1 - P_0(G))P_2^{RB}(R)$$
$$\bar{P}(B) = P_0(G)P_1^G(B) + (1 - P_0(G))P_2^{RB}(B) = 0 + (1 - P_0(G))P_2^{RB}(B)$$
$$\bar{P}(G) = P_0(G)P_1^G(G) + (1 - P_0(G))P_2^{RB}(G) = P_0(G)$$

Call $P_0(G) = p$ and $P_2^{RB}(R) = q$. Then by Corollary 1, the coherent precautionary reassessment $\succsim^{**}$ is a Bewley preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$ with

$$r_{\mathcal{P}}(\mathcal{C}) = \left\{ (1-p)\,(q, 1-q, 0) + p\,(0, 0, 1) \,\middle|\, p \in \left[\frac{1}{6}, \frac{1}{2}\right], q \in \left[\frac{2}{5}, \frac{2}{3}\right] \right\}.$$

Figure 1 gives a graphical illustration of the set $\mathcal{C}$, represented as a thick red segment, and its rectangular hull $r_{\mathcal{P}}(\mathcal{C})$.

Acts f and g are not comparable with respect to the Bewley preference $\succsim^{**}$. Computing the Maxmin formula obtained in item (ii) of Corollary 2 one gets

$$I(g) = \min_{p \in \left[\frac{1}{6}, \frac{1}{2}\right], q \in \left[\frac{2}{5}, \frac{2}{3}\right]} u(10)[(1-p)(1-q)+p] = \frac{4}{9}u(10)$$

$$I(f) = \min_{p \in \left[\frac{1}{6}, \frac{1}{2}\right], q \in \left[\frac{2}{5}, \frac{2}{3}\right]} u(10)[(1-p)q+p] = \frac{1}{2}u(10) > I(g).$$

Hence $f \succ^{\#\#} g$ and no Dynamic Consistency problem will arise once the set $r_{\mathcal{P}}(\mathcal{C})$ is updated with the prior-by-prior Bayes rule, since $r_{\mathcal{P}}(\mathcal{C})^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$.

5 One application and additional results

5.1 To Lockdown or Not To Lockdown?

A modification of Example 1 shows that the perceived increase in ambiguity resulting from the introduction of the information partition creates possible dynamic inconsistencies even when unanimity holds in the atemporal model. We illustrate this in a thought-example in which we apply our results to government decisions about locking down a country. We show that, even if not-locking down is unanimously preferred ex-ante due to scientific evidence, a government may regret its decision if it does not take into account the information partition and the coherent precautionary reassessment.

Suppose that a potentially unknown virus start to spread. Epidemiologists know that there are three possible states of the wold:

– R: the virus is extremely deadly and spreads quickly;
– B: false alarm (no one dies);
– G: a (known) seasonal flu with very low fatality rate;

but they may disagree about the probability associated to each state. A democratically elected government (the DM) is considering either to lock down the country (policy $lock$) or to leave everything open (policy $\neg lock$). The outcome of a policy is given by the popularity of the government (high / low). Consider the following policies:

	Red	Blue	Green
$lock$	high	low	low
$\neg lock$	low	high	high

It is clear that $lock$ would guarantee high popularity only if the true state turns out to be R, while the opposite is true with $\neg lock$. Suppose that epidemiologists think that set of possible priors is $\mathcal{C} = \left\{P = \left(\frac{1}{3}, p, \frac{2}{3} - p\right) \in \Delta | p \in \left[\frac{1}{90}, \frac{59}{90}\right]\right\}$. Assume further that the government becomes aware that the information partition is given by

$\mathcal{P} = \{G, RB\}$, (i.e. if the virus is a flu it will be recognized and cured, otherwise uncertainty will remain). In this case we are again in a dynamic-type Ellsberg setting as in Example 1. The set $\mathcal{C}$ and the information partition $\mathcal{P}$ represent hard scientific information (of course, in data applications, $\mathcal{C}$ and $\mathcal{P}$ must be chosen in accordance to the medical and epidemiological literature). The set $\mathcal{C}$ is equivalent to an Ellsberg urn containing 90 balls, 30 of which are red, and at least one of the remaining is blue ad another one is green.

In order to simplify notation call $f' = lock$ and $g = \neg lock$. One can show that $g \succ^* f'$, i.e. experts unanimously agree that not locking down is better than locking down, which implies $g \succ^{\#} f'$ (by Consistency). This ranking looks "intuitive" since $\neg lock$ guarantees high popularity with (objective) probability $\frac{2}{3}$, while $lock$ only with (objective) probability $\frac{1}{3}$. However, when preferences are updated through full Bayesian updating, acts f' and g are not comparable with respect to $\succsim^*_{RB}$. In case that $\succsim^*_{RB}$ is completed with the precautionary principle identified by the Maxmin preference $\succsim^{\#}_{RB}$, then $f' \succ^{\#}_{RB} g$. Therefore, if it turns out that the virus is either extremely dangerous or nothing at all, the updated Maxmin preference would recommend the lockdown. If Dynamic Consistency holds, then $f'RBg \succsim^{\#} g$.

However, note that $f'RBg$ (an act that gives low popularity on blue and high otherwise) is equivalent to act f of Example 1 (where "high popularity" is replaced by 10 and "low popularity" by 0). From that example we know that $g \succ^{\#} f'RBg$, a contradiction.

In order to avoid these contradictions, we compute the coherent precautionary reassessment $\succsim^{**}$ of $\succsim^*$. As it turns out, acts f' and g are not comparable with respect to $\succsim^{**}$ (computations are detailed in the Appendix, p. 25). When the completion $\succsim^{\#\#}$ (a Maxmin preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$) is used, we obtain $f' \succ^{\#\#} g$, i.e. lockdown is preferred. Without the intermediary period of partial resolution of uncertainty, the government would choose $\neg lock$ as it guarantees high popularity with higher probability (2/3 vs. 1/3). However, once $\mathcal{P}$ is introduced, the government anticipates that the majority of experts think that the true state of nature will belong to RB, and if the true state turns out to be in RB, $\neg lock$ will be regretted. The increase of ex-ante ambiguity and the willingness to eliminate possible dynamic inconsistencies make the government more ambiguity averse so that it chooses $lock$ over $\neg lock$.[13]

To summarize, we started with two acts, f' and g, that are not ambiguous with respect to the probability set $\mathcal{C}$ (meaning that $\int u(f')dP = \int u(f')dQ$ for all $P, Q \in \mathcal{C}$, and the same holding for g). However, the information partition $\mathcal{P} = \{G, RB\}$ is not aligned with the ex-ante structure of information given by $\mathcal{C}$. The ambiguity perceived ex-ante by the government increases and leads to dynamic inconsistencies (on the other hand note that ex-post ambiguity, once uncertainty is partially resolved, remains the same, as $r_{\mathcal{P}}(\mathcal{C})^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$). Once the coherent precautionary reassessment $\succsim^{**}$ is computed, f' and g are not comparable anymore. Finally the Maxmin completion $\succsim^{\#\#}$ reverses the initial ranking and suggests the more cautious policy of locking down the country.

[13] Note that in this example if the true state is G, then $\neg lock$ will not be regretted. For acts $lock$ and $\neg lock$, violations of Dynamic Consistency can occur only if the true state is in RB.

5.2 Additional results

In this section we study what happens when either Ex-Ante or Ex-Post Coherence is dropped. This allows us to interpret these axioms in terms of the representation they imply, providing further support besides their normative appeal.

We say that $\succsim^1$ is the *ex-ante coherent reassessment* of the Bewley preference $\succsim^*$ if it is a reassessment and it satisfies Ex-Ante Coherence. Preference $\succsim^1$ is the *ex-ante-coherent precautionary reassessment* of $\succsim^*$ if it is the most incomplete ex-ante coherent reassessments. The *ex-post-coherent precautionary reassessment* $\succsim^2$ is defined analogously. Consider the following sets of probabilities.

$$\mathcal{C}_1 = \{Q \in \Delta : \exists P \in \mathcal{C} \text{ s.t. } P(E) = Q(E), \forall E \in \mathcal{P}\}$$

and

$$\begin{aligned}\mathcal{C}_2 &= \{Q \in \Delta | \exists P_1, \ldots, P_n \in \mathcal{C} \text{ s.t. } P_i^{E_i} = Q^{E_i}, \forall i = 1, \ldots, n\} \\ &= \{Q \in \Delta | \forall i = 1, \ldots, n\, \exists P_i \in \mathcal{C} \text{ s.t. } P_i^{E_i} = Q^{E_i}\}\end{aligned}$$

We show now that $\succsim^1$ and $\succsim^2$ are characterized by utility index u and sets $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proposition 2 *$\succsim^1$ is the ex-ante-coherent precautionary reassessment of $\succsim^*$ if and only if $\succsim^1$ is represented by $(u, \mathcal{C}_1)$.*

Proposition 3 *$\succsim^2$ is the ex-post-coherent precautionary reassessment of $\succsim^*$ if and only if $\succsim^2$ is represented by $(u, \mathcal{C}_2)$.*

The link between Ex-Ante and Ex-Post Coherence and sets $\mathcal{C}_1$ and $\mathcal{C}_2$ gives additional justifications to these behavioral axioms. Ex-Ante Coherence implies that the new added experts must agree on the likelihood of events in $\mathcal{P}$ with at least one old expert in $\mathcal{C}$. Ex-Post Coherence implies that once information is revealed to a new expert, her updated beliefs should correspond to the updated belief of at least one old expert in $\mathcal{C}$. Therefore, in term of cardinal representation, Ex-Ante and Ex-Post Coherence require that there is enough agreement between the original group of experts and the newly added experts on elements of the information partition $\mathcal{P}$. Ex-Ante Coherence imposes agreement before the partial revelation of uncertainty while Ex-Post Coherence forces agreement after the realization of event $E_i \in \mathcal{P}$.

Proposition 2 and 3 show that the set of priors characterizing the coherent precautionary reassessment $\succsim^{**}$ is the intersection of sets $\mathcal{C}_1$ and $\mathcal{C}_2$. Figure 2 depicts sets $\mathcal{C}_1$ and $\mathcal{C}_2$ for $\mathcal{C}$ defined as in Example 1. It is readily seen that the rectangular hull of $\mathcal{C}$, shown in Fig. 1 is obtained as $r_{\mathcal{P}}(\mathcal{C}) = \mathcal{C}_1 \cap \mathcal{C}_2$.

6 Conclusion

Consider a group of experts that should advise a DM about the choice between two policies. The unanimity rule says that a policy f is preferred to g if, and only if, every

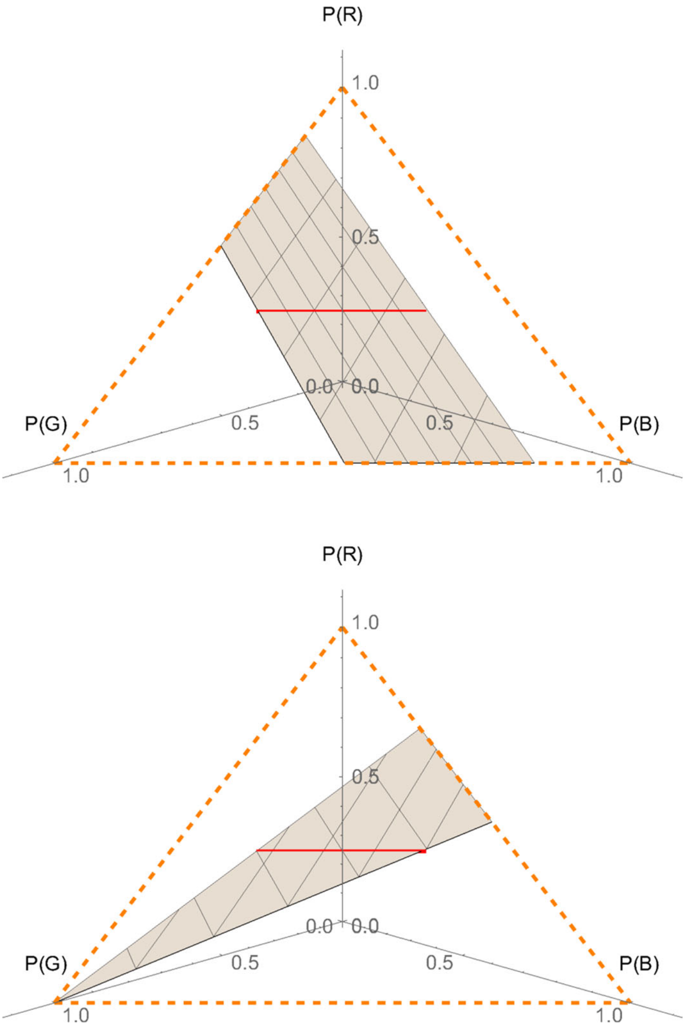

Fig. 2 The sets $\mathcal{C}$ (red line), $\mathcal{C}_1$ (left) and $\mathcal{C}_2$ (right)

expert assigns higher expected utility to f rather than g. If two experts disagree, this rule is unable to tell which policy is better. When a decision must be taken, several authors suggest to compare policies through the precautionary principle: the policy with the highest minimal expected utility should be chosen.

This rule may generate possible dynamic inconsistencies when an intermediary period of partial resolution of uncertainty is added. This implies that choices made today are regretted tomorrow no matter the additional information learned. In order to avoid this problem, we provide axioms that modify the original group of experts. We derive a new unanimity rule called coherent precautionary reassessment. New opinions are formed by taking convex combinations of experts' updated beliefs. This makes the completion of the new unanimity rule dynamically consistent.

Finally, we would like to underline that our paper takes as primitive a Bewley-type DM who completes her preferences through the precautionary principle and who wants to avoid dynamic inconsistencies. One interesting open question is which behavioral axioms on the couple $(\succsim^{\#}, \succsim^{\#\#})$ characterize a (direct) rectangularization of a Maxmin preference.

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Appendix

Throughout this Appendix, $\mathcal{P} = \{E_1 \ldots, E_n\}$ denotes a fixed partition of S. Preferences $\succsim^*$ and $\succsim^{**}$ are Bewley preferences represented by $(u, \mathcal{C})$ and $(\hat{u}, \hat{\mathcal{C}})$, respectively. We assume that for all $E \in \mathcal{P}$, $P(E) > 0$ for all $P \in \mathcal{C}$. Finally, we assume w.l.o.g. that $[-1, 1] \subseteq u(X)$.

Proof of Proposition 1 We prove first that for all $i = 1, \ldots, n-1$, $fE_iE_{i+1}\ldots E_ng \succsim fE_{i+1}\ldots E_ng$ (where $fE_iE_{i+1}\ldots E_ng$ is the act fEg with $E = E_i \cup E_{i+1} \cup \cdots \cup E_n$). By hypothesis $f \succsim_{E_i} g$ and by Consequentialism $f \succsim_{E_i} fE_{i+1}\ldots E_ng$. By Dynamic Consistency $fE_i(fE_{i+1}\ldots E_ng) \succsim fE_{i+1}\ldots E_ng$ and hence $fE_iE_{i+1}\ldots E_ng \succsim fE_{i+1}\ldots E_ng$. Applying transitivity $n-1$ times we can conclude $f \succsim fE_ng$. Since, by hypothesis, $f \succsim_{E_n} g$, by Dynamic Consistency $fE_ng \succsim g$ and again by transitivity $f \succsim g$. □

Lemma 1 *$\succsim^{**}$ is a reassessment of $\succsim^*$ if and only if $\hat{u}$ and u represent the same preference over X and $\hat{\mathcal{C}} \supseteq \mathcal{C}$ (and therefore we can take $\hat{u} = u$).*

Proof This result follows from Ghirardato et al. (2004), Proposition 6. □

Lemma 2 *Let $\succsim^{**}$ be a reassessment of $\succsim^*$. Then $(\succsim^*, \succsim^{**})$ satisfies Ex-Ante Coherence if and only if for all $Q \in \hat{\mathcal{C}}$ there exists $P \in \mathcal{C}$ s.t. $P(E) = Q(E)$, for all $E \in \mathcal{P}$.*

Proof $(\Rightarrow)$ Define the mapping $\varphi : ba(\Sigma) \to \mathbb{R}^n$ by $\varphi(\mu) := (\mu(E_1), \ldots, \mu(E_n))$, $\forall \mu \in ba(\Sigma)$, and denote the image of $\mathcal{C} \subseteq ba(\Sigma)$ as

$$A := \varphi(\mathcal{C}) = \{(P(E_1), \ldots, P(E_n)) : P \in \mathcal{C}\} \subseteq \mathbb{R}^n.$$

First, the function φ is weak* continuous. In fact, given $E \in \Sigma$, each mapping φ_E defined by $\varphi_E(\mu) := \mu(E)$ is linear on $ba(\Sigma)$, which implies that φ_E is $\sigma(ba, B_0)$-continuous and, therefore, φ is weak* continuous. Thus, since $\varphi(\mathcal{C}) = A$ and $\mathcal{C}$ is compact, we have that $A \subseteq \mathbb{R}^n$ is compact for the usual topology. Second, since $\mathcal{C}$ is convex and φ is a linear mapping, it is straightforward to show that $\varphi(\mathcal{C}) = A$ is a convex subset of $\mathbb{R}^n$.

Consider now $Q \in \hat{\mathcal{C}}$ and define $y \in \mathbb{R}^n$ by $y_i = Q(E_i)$ for $i = 1, \ldots, n$. If $y \in A$ then there is $P \in \mathcal{C}$ such that $Q(E_i) = P(E_i)$ for all $i \in \{1, \ldots, n\}$ and we are done. Suppose $y \notin A$. Since A is convex and compact, by the separating hyperplane theorem there exist $v \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle y, v\rangle < \alpha < \langle x, v\rangle, \ \ \forall x \in A.$$

Define $\bar{\alpha} = (\alpha, \ldots, \alpha) \in \mathbb{R}^n$ and $\bar{v} = v - \bar{\alpha}$. Then

$$\begin{aligned}
\langle y, v\rangle - \alpha &< 0 < \langle x, v\rangle - \alpha, \ \ \forall x \in A \\
\langle y, v\rangle - \langle y, \bar{\alpha}\rangle &< 0 < \langle x, v\rangle - \langle x, \bar{\alpha}\rangle, \ \ \forall x \in A \\
\langle y, v - \bar{\alpha}\rangle &< 0 < \langle x, v - \bar{\alpha}\rangle, \ \ \forall x \in A \\
\langle y, \bar{v}\rangle &< 0 < \langle x, \bar{v}\rangle, \ \ \forall x \in A
\end{aligned}$$

Clearly we can suppose w.l.o.g that $\|\bar{v}\| = 1$ (otherwise just consider $\frac{\bar{v}}{\|\bar{v}\|}$) and therefore $\bar{v}_i \in [-1, 1]$ for all $i = 1 \ldots, n$. Consider any two $\mathcal{P}$-measurable acts $f, g \in \mathcal{F}$ such that $u(f(s)) - u(g(s)) = \bar{v}_i$ for all $s \in E_i$. This can be done since w.l.o.g. we assumed $[-1, 1] \subseteq u(X)$. Denote $f_i = u(f(s))$ for $s \in E_i$ and g_i in a similar way. Then $\langle x, \bar{v}\rangle = \sum_i x_i(f_i - g_i) > 0 \ \forall x \in A$ implies

$$\int [u(f(s)) - u(g(s))]\, dP = \sum_{i=1}^{n} (f_i - g_i) P(E_i) > 0, \ \ \forall P \in \mathcal{C}$$

i.e. $f \succsim^* g$. However $\langle y, \bar{v}\rangle < 0$ implies

$$\int [u(f(s)) - u(g(s))]\, dQ = \sum_{i=1}^{n} (f_i - g_i) Q(E_i) < 0$$

and hence $f \not\succsim^{**} g$, contradicting ex-ante coherence. Hence we must have $y \in A$.

($\Leftarrow$) Consider two $\mathcal{P}$-measurable acts $f, g \in \mathcal{F}$ s.t. $f \succsim^* g$. Denote $f_i = u(f(s))$ for $s \in E_i$ and g_i in a similar way. Fix $Q \in \hat{\mathcal{C}}$ and let $P_Q \in \mathcal{C}$ be such that $P_Q(E) = Q(E)$, for all $E \in \mathcal{P}$. Then

$$\begin{aligned}
g_1 Q(E_1) + \cdots + g_n Q(E_n) &= g_1 P_Q(E_1) + \cdots + g_n P_Q(E_n) \\
&\le f_1 P_Q(E_1) + \cdots + f_n P_Q(E_n) = f_1 Q(E_1) + \cdots + f_n Q(E_n).
\end{aligned}$$

Since $Q \in \hat{\mathcal{C}}$ was arbitrarily chosen, the above inequality holds for all $Q \in \hat{\mathcal{C}}$. Hence $f \succsim^{**} g$. □

Lemma 3 *Let $\succsim^{**}$ be a reassessment of $\succsim^*$. Then $(\succsim^*, \succsim^{**})$ satisfies Ex-Post Coherence if and only if for all $E \in \mathcal{P}$, $\mathcal{C}^E = \hat{\mathcal{C}}^E$.*

Proof ($\Rightarrow$) Ghirardato et al. (2004), Proposition 6, implies $\mathcal{C}^E \supseteq \hat{\mathcal{C}}^E$ for all $E \in \mathcal{P}$. Since $\succsim^{**}$ is a reassessment of $\succsim^*$, $\hat{\mathcal{C}} \supseteq \mathcal{C}$. Therefore $\mathcal{C}^E = \hat{\mathcal{C}}^E$ for all $E \in \mathcal{P}$.

($\Leftarrow$) Since for all $E \in \mathcal{P}$, $\mathcal{C}^E = \hat{\mathcal{C}}^E$ we have $\succsim^{**}_E = \succsim^*_E$. Hence the result. □

Recall that $\succsim^{**}$ is a *coherent reassessment* of $\succsim^{*}$ if and only if $\succsim^{**}$ is a reassessment of $\succsim^{*}$ and $(\succsim^{*}, \succsim^{**})$ satisfies Ex-Post Coherence and Ex-Ante Coherence. Lemma 1, 2, 3 can be combined in the following Proposition.

Proposition 4 *$\succsim^{**}$ is a coherent reassessment of $\succsim^{*}$ if and only if*

(i) The affine utility functions $\hat{u}$ and u represent the same preference over X;
(ii) $\hat{\mathcal{C}} \supseteq \mathcal{C}$;
(iii) For all $E \in \mathcal{P}$, $\mathcal{C}^E = \hat{\mathcal{C}}^E$;
(iv) For all $Q \in \hat{\mathcal{C}}$ there exists $P \in \mathcal{C}$ s.t. $P(E) = Q(E)$, for all $E \in \mathcal{P}$.

The next Lemma characterizes the rectangular hull of a set of probabilities $\mathcal{C}$.

Lemma 4 *$r_{\mathcal{P}}(\mathcal{C})$ is the maximal set such that*

(i) $r_{\mathcal{P}}(\mathcal{C})^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$;
(ii) $\forall Q \in r_{\mathcal{P}}(\mathcal{C})$, $\exists P \in \mathcal{C}$ such that $P(E) = Q(E)$, $\forall E \in \mathcal{P}$.

Proof We first prove that $r_{\mathcal{P}}(\mathcal{C})$ satisfies conditions (i) and (ii) and then we show that it is the maximal set satisfying these conditions.

The set $r_{\mathcal{P}}(\mathcal{C})$ satisfies (i):

Fix $E_j \in \mathcal{P}$. We have $P^{E_j} \in r_{\mathcal{P}}(\mathcal{C})^{E_j}$ if, and only if, there exists $P \in r_{\mathcal{P}}(\mathcal{C})$ such that $P^{E_j}(A) = \frac{P(A\cap E_j)}{P(E_j)}$, for all $A \in \Sigma$. The last assertion holds if, and only if, there exists $P_0, P_1, \ldots, P_n \in \mathcal{C}$ such that

$$P^{E_j}(A) = \frac{P(A \cap E_j)}{P(E_j)} = \frac{\sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(A \cap E_j)}{\sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(E_j)} = P_j^{E_j}(A \cap E_j) = P_j^{E_j}(A) \tag{6}$$

for all $A \in \Sigma$. This implies that $P^{E_j} \in \mathcal{C}^{E_j}$.

On the other hand if $P_j^{E_j} \in \mathcal{C}^{E_j}$ then, choosing $n+1$ probabilities $P_0, P_1, \ldots, P_n \in \mathcal{C}$, (6) shows that $P_j^{E_j} \in r_{\mathcal{P}}(\mathcal{C})^{E_j}$.

The set $r_{\mathcal{P}}(\mathcal{C})$ satisfies condition (ii):

Let $Q \in r_{\mathcal{P}}(\mathcal{C})$. Then there are probabilities $P_0, P_1, \ldots, P_n \in \mathcal{C}$ such that for all $E_j \in \mathcal{P}$

$$Q(E_j) = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(E_j) = P_0(E_j) P_j^{E_j}(E_j) = P_0(E_j).$$

Hence P_0 satisfies condition (ii).

$r_{\mathcal{P}}(\mathcal{C})$ is the maximal set satisfying conditions (i) and (ii):

Let Q be a probability over the measurable space (S, Σ) such that $Q^E \in \mathcal{C}^E$ for all $E \in \mathcal{P}$ and such that there exists $P \in \mathcal{C}$ such that $P(E) = Q(E)$ for all $E \in \mathcal{P}$. For all $A \in \Sigma$, by the law of total probability we have $Q(A) = \sum_{i=1}^{n} Q(E_i) Q^{E_i}(A)$. Since $Q^{E_i} \in \mathcal{C}^{E_i}$ by (i) there is $P_i \in \mathcal{C}$ such that $Q^{E_i} = P_i^{E_i}$. Moreover by condition (ii) there is $P_0 \in \mathcal{C}$ such that $Q(E_i) = P_0(E_i)$. This implies that $Q(A) = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(A)$ and hence $Q \in r_{\mathcal{P}}(\mathcal{C})$.

□

Next, we prove our main result, Theorem 1. Recall that $\succsim^{**}$ is the *coherent precautionary reassessment* of $\succsim^*$ if it is the most incomplete coherent reassessment of $\succsim^*$.

Proof of Theorem 1 -($\Rightarrow$) Let $\succsim^{**}$ be a Bewley preference represented by $(\hat{u}, \hat{\mathcal{C}})$. Since $\succsim^{**}$ is the coherent precautionary reassessment of $\succsim^*$, Proposition 4 implies that $\hat{u} = u$ and $\hat{\mathcal{C}}$ satisfies properties (ii), (iii) and (iv) of Proposition 4. By Lemma 4, $r_{\mathcal{P}}(\mathcal{C})$ is the maximal set satisfying properties (iii) and (iv) of Proposition 4. By hypothesis, $\succsim^{**}$ is the most incomplete Bewley preference such that the pair $(\succsim^*, \succsim^{**})$ satisfies Ex-Post and Ex-Ante Coherence, hence $\hat{\mathcal{C}} \supseteq r_{\mathcal{P}}(\mathcal{C})$ and therefore $\hat{\mathcal{C}} = r_{\mathcal{P}}(\mathcal{C})$. This implies the result using Corollary 1.

($\Leftarrow$) By Corollary 1 the representation of $\succsim^{**}$ implies that $\forall f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Leftrightarrow \int u(f)dP \geq \int u(g)dP, \ \forall P \in r_{\mathcal{P}}(\mathcal{C}).$$

It is obvious that properties (i) and (ii) of Proposition 4 are satisfied. By Lemma 4, $r_{\mathcal{P}}(\mathcal{C})$ satisfies properties (iii) and (iv) of Proposition 4. Since moreover $r_{\mathcal{P}}(\mathcal{C})$ is the maximal set satisfying these two properties, $\succsim^{**}$ is the most incomplete Bewley preference such that the pair $(\succsim^*, \succsim^{**})$ satisfies Ex-Post and Ex-Ante Coherence.

□

Example 2 We provide here an example of two Bewley preferences, $\succsim^{\sim}$ and $\succsim^{\wedge}$, which are coherent reassessments of $\succsim^*$, but they are not the most incomplete. We show that in this case, violations of Dynamic Consistency arise. First, let us rewrite sets $\mathcal{C}$ and $r_{\mathcal{P}}(\mathcal{C})$ of Example 1 as

$$\mathcal{C} = co\left\{\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)\right\}$$
$$r_{\mathcal{P}}(\mathcal{C}) = co\left\{\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{5}, \frac{3}{10}, \frac{1}{2}\right), \left(\frac{5}{9}, \frac{5}{18}, \frac{1}{6}\right)\right\}$$

where $co\{\cdot\}$ denotes the convex hull operator, i.e. the set of all convex combinations. Second, we consider the following sets:

$$\tilde{\mathcal{C}} = co\left\{\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{5}, \frac{3}{10}, \frac{1}{2}\right)\right\}$$
$$\hat{\mathcal{C}} = co\left\{\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right), \left(\frac{5}{9}, \frac{5}{18}, \frac{1}{6}\right)\right\}$$

We can note that $\tilde{\mathcal{C}}^{RB} = \hat{\mathcal{C}}^{RB} = \{(q, 1-q, 0) \in \Delta | q \in \left[\frac{2}{5}, \frac{2}{3}\right]\} = \mathcal{C}^{RB}$. Moreover $\tilde{\mathcal{C}}^{G} = \hat{\mathcal{C}}^{G} = \{(0, 0, 1)\} = \mathcal{C}^{G}$. Therefore since $\tilde{\mathcal{C}} \supseteq \mathcal{C}$ and $\hat{\mathcal{C}} \supseteq \mathcal{C}$, Lemma 2 and

Lemma 3 show that the Bewley preferences $\succsim^{\sim}$ and $\succsim^{\wedge}$ represented, respectively, by $(u, \tilde{\mathcal{C}})$ and $(u, \hat{\mathcal{C}})$ are coherent reassessments of $\succsim^{*}$.

Consider now acts f and g of Example 1. It is readily seen that these acts cannot be compared through $\succsim^{\sim}$ or $\succsim^{\wedge}$. If the corresponding Maxmin preference represented by $(u, \tilde{\mathcal{C}})$ and $(u, \hat{\mathcal{C}})$, denoted $\succsim^{\sim\#}$ and $\succsim^{\wedge\#}$, are computed, then $g \succ^{\sim\#} f$ and $g \succ^{\wedge\#} f$, respectively. However, since $\tilde{\mathcal{C}}^{RB} = \hat{\mathcal{C}}^{RB} = \mathcal{C}^{RB}$ and $\tilde{\mathcal{C}}^{G} = \hat{\mathcal{C}}^{G} = \mathcal{C}^{G}$ we have $\succsim^{\sim\#}_{RB} = \succsim^{\wedge\#}_{RB} = \succsim^{\#}_{RB}$ and $\succsim^{\sim\#}_{G} = \succsim^{\wedge\#}_{G} = \succsim^{\#}_{G}$. Therefore, as shown in Example 1, page 9, $f \succsim^{\sim\#}_{RB} g$ and $f \succsim^{\wedge\#}_{RB} g$, a violation of Dynamic Consistency.

Computations for Sect. 5.1 Recall that the set of states of the world is $S = \{R, B, G\}$, the original set of prior is $\mathcal{C} = \left\{P = \left(\frac{1}{3}, p, \frac{2}{3} - p\right) \in \Delta | p \in \left[\frac{1}{90}, \frac{59}{90}\right]\right\}$ and the information partition is given by $\mathcal{P} = \{G, RB\}$. We assume w.l.o.g. $u(low) = 0$. Act f' gives a $high$ payoff on R and a low one otherwise, while for act g the opposite is true. We have that $g \succ^{*} f'$, but if the true state is either R or B the DM will regret its choices. We compute therefore the coherent precautionary reassessment $\succsim^{**}$ of $\succsim^{*}$ following the steps summarized in equation (4):

1. Compute the Bayesian update of all $P \in \mathcal{C}$ and for all the events in the partition $\mathcal{P}$. This gives $\mathcal{C}^{RB} = \left\{(q, 1-q, 0) \in \Delta | q \in \left[\frac{30}{89}, \frac{30}{31}\right]\right\}$ and $\mathcal{C}^{G} = \{(0, 0, 1)\}$.
2. Compute the expected value under these conditional probabilities: $\int u(f')dP^{RB} = u(high)q$, $\int u(f')dP^{G} = 0$, $\int u(g)dP^{RB} = u(high)(1-q)$ and $\int u(g)dP^{G} = u(high)$.
3. Take convex combinations using as weights the opinions of all members $P \in \mathcal{C}$ on likelihood of events in $\mathcal{P}$: denote $P^{RB}(R) = q \in \left[\frac{30}{89}, \frac{30}{31}\right]$ and $P(G) = p \in [\frac{1}{90}, \frac{59}{90}]$,

$$V(f') = P(RB) \int u(f')dP^{RB} + P(G) \int u(f')dP^{G} = u(high)\,(1-p)\,q$$

$$V(g) = P(RB) \int u(g)dP^{RB} + P(G) \int u(g)dP^{G} = u(high)\,[1 - q\,(1-p)]\,.$$

The coherent precautionary reassessment $\succsim^{**}$ is a Bewley preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$ with

$$r_{\mathcal{P}}(\mathcal{C}) = \left\{(1-p)\,(q, 1-q, 0) + p\,(0, 0, 1)\,\middle|\, p \in \left[\frac{1}{90}, \frac{59}{90}\right], q \in \left[\frac{30}{89}, \frac{30}{31}\right]\right\}.$$

Acts f' and g are not comparable with respect to $\succsim^{**}$. In fact $V(g) > V(f')$ for the probability $(\frac{1}{3}, \frac{1}{90}, \frac{59}{90}) \in r_{\mathcal{P}}(\mathcal{C})$ (take $p = \frac{59}{90}$ and $q = \frac{30}{31}$) and $V(f') > V(g)$ for the probability $(\frac{89}{93}, \frac{89}{2790}, \frac{1}{90}) \in r_{\mathcal{P}}(\mathcal{C})$ (take $p = \frac{1}{90}$ and $q = \frac{30}{31}$). The prior $(\frac{89}{93}, \frac{89}{2790}, \frac{1}{90}) \in r_{\mathcal{P}}(\mathcal{C})$ that was not in $\mathcal{C}$ now has veto power. $\square$

Recall the definitions of sets $\mathcal{C}_1$ and $\mathcal{C}_2$ given in Sect. 5.2.

$$\mathcal{C}_1 := \{Q \in \Delta : \exists P \in \mathcal{C} \text{ s.t. } P(E) = Q(E), \forall E \in \mathcal{P}\}$$

and

$$\begin{aligned}\mathcal{C}_2 &= \{Q \in \Delta | \exists P_1, \ldots, P_n \in \mathcal{C} \text{ s.t. } P_i^{E_i} = Q^{E_i}, \forall i = 1, \ldots, n\} \\ &= \{Q \in \Delta | \forall i = 1, \ldots, n \, \exists P_i \in \mathcal{C} \text{ s.t. } P_i^{E_i} = Q^{E_i}\}\end{aligned}$$

Lemma 5 *The sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are rectangular with respect to $\mathcal{P}$.*

Proof Given $Q \in r_{\mathcal{P}}(\mathcal{C}_1)$, we have that

$$Q = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}$$

for some $P_0, P_1, \ldots, P_n \in \mathcal{C}_1$. Thus, given $i_0 \in \{1, \ldots, n\}$

$$Q(E_{i_0}) = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(E_{i_0}) = P_0(E_{i_0}).$$

Since $P_0 \in \mathcal{C}_1, \exists P \in \mathcal{C}$ s.t. $P_0(E) = P(E), \forall E \in \mathcal{P}$.

Thus, $\exists P \in \mathcal{C}$ s.t. for any $i_0 \in \{1, \ldots, n\}$, $Q(E_{i_0}) = P_0(E_{i_0}) = P(E_{i_0})$, i.e., $Q \in \mathcal{C}_1$ and we conclude that $\mathcal{C}_1$ is rectangular w.r.t. $\mathcal{P}$.

Now, consider $Q \in r_{\mathcal{P}}(\mathcal{C}_2)$. Thus,

$$Q = \sum_{i=1}^{n} P_0(E_i) P_i^{E_i}$$

for some $P_0, P_1, \ldots, P_n \in \mathcal{C}_2$. Take $i_0 \in \{1, \ldots, n\}$ and consider an arbitrary $F \in \Sigma$. We note that

$$\begin{aligned}Q^{E_{i_0}}(F) &= \frac{Q(E_{i_0} \cap F)}{Q(E_{i_0})} = \frac{\sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(E_{i_0} \cap F)}{\sum_{i=1}^{n} P_0(E_i) P_i^{E_i}(E_{i_0})} \\ &= \frac{P_0(E_{i_0}) P_{i_0}^{E_{i_0}}(E_{i_0} \cap F)}{P_0(E_{i_0})} = P_{i_0}^{E_{i_0}}(F)\end{aligned}$$

Since $P_{i_0} \in \mathcal{C}_2, \exists \bar{P}_1, \ldots, \bar{P}_n \in \mathcal{C}$ s.t. $\bar{P}_j^{E_j} = P_{i_0}^{E_j}, \forall j = 1, \ldots, n$. Thus, $\exists \bar{P}_{i_0} \in \mathcal{C}$ such that $\bar{P}_{i_0}^{E_{i_0}} = Q^{E_{i_0}}$. Since i_0 was arbitrarily chosen, $Q \in \mathcal{C}_2$ and therefore $\mathcal{C}_2$ is rectangular w.r.t. $\mathcal{P}$. □

Let $\succsim_i, i = 1, 2$, be Bewley a preference represented by $(\hat{u}_i, \hat{\mathcal{C}}_i)$.

Proof of Proposition 2 ($\Rightarrow$) By Lemma 1 $\hat{u}_1 = u$. By Lemma 2, if $Q \in \hat{\mathcal{C}}_1$ then there exists $P \in \mathcal{C}$ s.t. $P(E) = Q(E)$, for all $E \in \mathcal{P}$. Therefore $Q \in \mathcal{C}_1$ and $\hat{\mathcal{C}}_1 \subseteq \mathcal{C}_1$. Since $\succsim^1$ is the most incomplete preference such that $(\succsim^*, \succsim^1)$ satisfies Ex-Ante Coherence, we must have $\hat{\mathcal{C}}_1 = \mathcal{C}_1$

($\Leftarrow$) By Lemma 2, $(\succsim^*, \succsim^1)$ satisfies Ex-Ante Coherence. Let $\succsim^0$ be a Bewley preference represented by $(\hat{u}_0, \hat{\mathcal{C}}_0)$. Then if $\succsim^0$ is an ex-ante coherent reassessment of $\succsim^*$, by Lemma 1 we have $\hat{u}_0 = u$ and by Lemma 2 and $\hat{\mathcal{C}}_0 \subseteq \mathcal{C}_1$, i.e. $\succsim^1 \subseteq \succsim^0$. Therefore $\succsim^1$ is the most incomplete preference.

$\square$

Proof of Proposition 3 We need the following additional claim.
Claim. $\mathcal{C}_2^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$.

Proof of the Claim Since $\mathcal{C}_2 \supseteq \mathcal{C}$, we have $\mathcal{C}_2^E \supseteq \mathcal{C}^E$ for all $E \in \mathcal{P}$. It is left to show the reverse inclusion. Fix $i_0 \in \{1, ..., n\}$ and let $P \in \mathcal{C}_2^{E_{i_0}}$. Then there is $Q \in \mathcal{C}_2$ such that $Q^{E_{i_0}} = P$. Since $Q \in \mathcal{C}_2$, there exist $\bar{Q}_1, \ldots, \bar{Q}_n \in \mathcal{C}$ such that $Q^{E_i} = \bar{Q}_i^{E_i}$ for all $i = 1, \ldots, n$. Therefore $\exists \bar{Q}_{i_0} \in \mathcal{C}$ such that $Q_{i_0}^{E_{i_0}} = P$, hence $P \in \mathcal{C}^{E_{i_0}}$. Since i_0 was arbitrary, the proof of the claim follows. $\square$

($\Rightarrow$) By Lemma 1 $\hat{u}_2 = u$ and by Lemma 3, $\hat{\mathcal{C}}_2^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$. By the Claim above and Lemma 3, $\hat{\mathcal{C}}_2 \supseteq \mathcal{C}_2$ (because a preference represented by $(u, \mathcal{C}_2)$ would be ex-post coherent w.r.t. $\succsim^*$ and $\succsim^2$ is more incomplete than such a preference). It is left to show that $\hat{\mathcal{C}}_2 \subseteq \mathcal{C}_2$. Note first that $\hat{\mathcal{C}}_2^E \supseteq \mathcal{C}_2^E$ for all $E \in \mathcal{P}$ and since $\mathcal{C}_2 \supseteq \mathcal{C}$, we can conclude $\mathcal{C}_2^E \supseteq \mathcal{C}^E = \hat{\mathcal{C}}_2^E$ for all $E \in \mathcal{P}$. Thus $\hat{\mathcal{C}}_2^E = \mathcal{C}_2^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$. Take now $P \in \hat{\mathcal{C}}_2$. Then $P^{E_i} \in \hat{\mathcal{C}}_2^{E_i} = \mathcal{C}^{E_i}$ for all $i = 1, \ldots, n$. Therefore for all $i = 1, \ldots, n$ there exists $P_i \in \mathcal{C}$ such that $P_i^{E_i} = P^{E_i}$. Hence $P \in \mathcal{C}_2$.

($\Leftarrow$) By the Claim and Lemma 3, $(\succsim^*, \succsim^2)$ satisfies Ex-Post Coherence. It is left to show that $\succsim^2$ is the most incomplete preference with this property. Let $\succsim^0$ be a Bewley preference represented by $(\hat{u}_0, \hat{\mathcal{C}}_0)$. Then if $\succsim^0$ is an ex-post coherent reassessment of $\succsim^*$, by Lemma 1 we have $\hat{u}_0 = u$ and $\hat{\mathcal{C}}_0 \supseteq \mathcal{C}$. Moreover by Lemma 3 $(\hat{\mathcal{C}}_0)^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$, and therefore if $P_0 \in \hat{\mathcal{C}}_0$, then $\forall i = 1, \ldots, n\ \exists P_i \in \mathcal{C}$ s.t. $P_i^{E_i} = P_0^{E_i}$. Hence $P_0 \in \mathcal{C}_2$. Therefore $\succsim^2 \subseteq \succsim^0$ and therefore $\succsim^2$ is the most incomplete preference.

$\square$

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