ON THE MITRA–WAN FOREST MANAGEMENT PROBLEM IN CONTINUOUS TIME

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Abstract. The paper provides a continuous-time version of the discrete-time Mitra-Wan model of optimal forest management, where trees are harvested to maximize the utility of timber flow over an infinite time horizon. The available trees and the other parameters of the problem vary continuously with respect to both time and age of the trees, so that the system is ruled by a partial differential equation. The behavior of optimal or maximal couples is classified in the cases of linear, concave or strictly concave utility, and positive or null discount rate. All sets of data share the common feature that optimal controls need to be more general than functions, *i.e.* positive measures. Formulas are provided for *golden-rule* configurations (uniform density functions with cutting at the ages that solve a Faustmann problem) and for *Faustmann policies*, and their optimality/maximality is discussed. The results do not always confirm the corresponding ones in discrete time.

Key words: Optimal harvesting, Forest Management, Measure-valued Controls.
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1. Introduction

Although forest economics has a centuries-long history (see, *e.g.* Samuelson, 1995), the first complete formulation of the forest management problem as a Ramsey-like optimal control model in discrete time is contained in two papers by Mitra and Wan (1985, 1986) of the early eighties. The authors there discuss the structure of the

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cutting/replanting strategy that maximizes, over an infinite horizon, the sum of utilities of timber flows obtained by harvesting the trees of a forest. Trees have different ages up to a maximum age, and are grown on a unit piece of land that cannot be transferred to other uses; a productivity function gives the amount of wood obtained harvesting trees of a given age; cutting and replanting costs are zero; new saplings are immediately replanted on the cleared land. The main results are that: 1) the *Faustmann policy* (*i.e.*, cutting trees that reach an age maximizing the present value of bare land subject to an infinite sequence of planting cycles) is optimal when the utility function is linear, and it generates a cycle in the configuration of the forest, 2) optimal paths converge to the golden rule configuration (the uniform forest with maximal sustainable yield) when the utility function is strictly concave and the discount factor is equal to 1, and 3) cycles of the optimal path reappear when the future utility is discounted, even when the utility function is strictly concave. Following this lead, almost the entire theory of optimal forest management has been developed in terms of discrete time (see Tahvonen, 2004, and Khan and Piazza, 2012, for recent lists of the extensions of the model) while, to our knowledge, a consistent continuous time version has never entered the literature.

The technical complexity of continuous time models explains the lack of contributions in optimal forest management. Indeed: 1) the ages of capital goods (*i.e.*, trees) vary continuously, so that the system evolution in time is ruled by a partial differential equation; 2) the control appears also in the boundary condition; 3) unlike in other vintage capital models, optimal controls are not functions but measures (Dirac's Deltas). Then the equation itself cannot be interpreted (pointwise) in $\mathbb{R}$, and calls for an extended formulation to make sense. Moreover, continuous time models are not a straightforward multi-sectoral generalization of the Ramsey model (for a similar conclusion see Khan and Piazza, 2011). Briefly after Samuelson's 1976 survey of the forestry literature (Samuelson, 1995), Kemp and Moore (1979) attacked the problem but concluded that (p.142) "The asymptotic behavior of cutting and planting on individual one-tree plots and of the age configuration of the entire forest [...] remain open questions". For example, Tahvonen and Salo (1999) (see also Tahvonen et al., 2001) studied a model in which time is continuous but trees are indivisible. This implies that harvesting does not provide a continuous flow of timber but a sequence of mass points in connection with the jumps in the state variables. Salo and Tahvonen (2003), on the other hand, maintained the discrete time structure, but let the length of the period go to zero. Finally, Salant (2013) analyzed equilibrium price paths of different vintages of trees in a simple model in which the forest land can be used alternatively, but deforestation is irreversible: that allows to study optimal continuous paths avoiding the complexities of distributed state variables. Independently, Heaps (see Heaps and Neher, 1979; Heaps, 1984, 2014) tried to establish an appropriate maximum principle for a Faustmann model in continuous time. In Heaps (1984), for example, it is assumed that harvesting occurs only for trees of the oldest ages, and a pre-theoretical argument is added to suggest that, in a model where cutting is feasible at all ages, it is not optimal to cut trees of an age s^* leaving trees older than s^* standing (Proposition 1 of Heaps, 1984). Heaps there proves that optimality conditions take the form of a delay differential equation. The same "old-first principle" emerged as a theorem in some specific cases: in Salant (2013), for example, it holds in a strong form (*i.e.*, older trees are harvested strictly before younger trees) while it is found in a weak form in the discrete-time two-age-classes Mitra-Wan model where, depending on the initial configuration, it is optimal either to cut only the old trees or the old *and* part of the young trees (see for example Tahvonen, 2004). In general, however, it seems to require strong assumptions on the productivity function (Heaps always assumes older trees are more productive, concavity of the productivity function in the undiscounted case, and some weakened concavity in the discounted case), satisfied neither in the original Mitra-Wan formulation (*a fortiori*, under weaker assumptions as in Khan and Piazza (2012)) nor in the model discussed in this paper.

The new approach used here is reformulating the control problem for the partial differential equation as an equivalent problem for an evolution equation in an infinite dimensional space, and developing *ad hoc* techniques for its analysis. In this setting, we need neither to reduce the dimensionality of the problem (as in Tahvonen and Salo, 1999, or Salant, 2013), nor to constrain the controls (as in Heaps, 1984). We allow strategies to be measures rather than functions, with the consequence that instantaneous cutting for forests of any given age is possible (the golden rule configuration is of such type), although we require the associated trajectories to be functions, so to avoid mass points. To this extent, it is enough to allow initial configurations of the forest which are square integrable functions and prove that property is preserved by the whole trajectory. We show that:

a) the analogue of golden-rule and *modified* golden-rule configurations is available for the continuous-time model;
b) modified golden rules are optimal stationary solutions for the discounted model, with both optimal cutting age and timber stationary consumption level monotonically not increasing in the rate of discount while, in the undiscounted case, the golden rule is maximal when the utility function is linear, and optimal when the utility is strictly concave, provided it is unique;
c) if the golden-rule configuration is unique, then undiscounted maximal (or optimal) paths exist from any given initial configuration and, for a strictly concave utility, converge in time to the golden rule;
d) the Faustmann policy is optimal when the utility is linear and the discount positive, is maximal (and not optimal) when the utility is linear and the discount null, it is not optimal when the utility is strictly concave and the discount positive, for initial data in any neighborhood of the optimal steady state. In particular this result contradicts the analogue in discrete time.

We show for the undiscounted case (see (c) and (d)) that the conclusions by Brock (1970) on the existence and "average" convergence of maximal paths hold also in our framework, and that the results can be strengthened to existence and asymptotic convergence of optimal paths as in Gale (1967) if the utility function is strictly concave (Mitra and Wan, 1986, and Khan and Piazza, 2010a, have already shown that the same holds in discrete time). In addition, we refine the above results by providing an example in the style of Brock (1970) and Peleg (1973) (see also Khan and Piazza (2010b) for a related example in the case of the Robinson-Solow-Srinivasan model), which proves that optimal paths do not exist in the linear case. On the other hand, the comparative statics results under (b) are specific to the continuous-time setting (hints may be found in Samuelson, 1995, see Figure 2 at p. 133 and the first paragraph at p. 134) and have no counterpart in discrete time forestry literature.[1] It is interesting to note that monotonicity of the modified golden rule consumption may not hold in models with several capital goods like ours, while it holds in the one sector Ramsey discounted model (see for example Mas-Colell et al., 1995, pages 758-9). Regarding the Faustmann Policy in (d) we show that, similarly to what occurs in discrete time (see Mitra and Wan, 1985), discounting does not affect the structure of optimal policies when the utility is linear. However, for a strictly concave utility we show that the periodic solution is not optimal, contrary to what is distinguishing of the discounted discrete-time model (Mitra and Wan, 1985, p. 265; Salo and Tahvonen, 2003, Proposition 1). This fact confirms the intuitions of Salo and Tahvonen (2003), who prove that in discrete-time optimal cycles tend to disappear as the length of the period approaches zero and, arguing that optimality of cycles in discrete time is due to the discrepancy between the discrete measure of time and the continuous measure of space, expect that optimal cycles do not exist in continuous time.

We recall that the technique of rephrasing the problem in a space of functions is well known in functional analysis (see Bensoussan et al., 2007). First introduced in the economic literature by Barucci and Gozzi (1998, 2001), it was later studied in various works, mainly on problems with vintage capitals: under the point of view of theoretical Dynamic Programming (Faggian, 2005, 2008, with finite horizon; Faggian and Gozzi, 2010, with infinite horizon) and that of applications (Faggian and Gozzi, 2004; Barucci and Gozzi, 1999; Faggian and Grosset, 2013). Nevertheless, in none of these works the control space need be a space of measures.

[1]As pointed out to us by a referee, a monotonicy results for the Faustmann cutting age in discrete time is reported in a forthcoming paper (see Piazza and Pagnoncelli, 2014).

In Section 2 we describe the model in continuous time, in Section 3 we build modified golden rules and Faustmann policies, in Section 4, we classify as optimal or maximal Golden Rules and Faustmann policies, in the cases when the utility is linear, concave, or strictly concave, and the discount rate is positive or null. In Section 5 we draw the conclusions and suggest future development of theory and applications. The Appendix contains the formulation of the abstract model, the proofs of the many theorems and all auxiliary results.

2. The continuous time model

2.1. **Notation.** We denote by $[a]$ and $\{a\}$ respectively the integer and the fractional part of the real number a. If X is a Banach space and X' its dual space, we denote by $\langle \cdot, \cdot \rangle_{X',X}$ or by $\langle \cdot, \cdot \rangle$ the duality pairing. If $-\infty \leq \sigma_1 < \sigma_2 \leq \infty$, we denote by $L^p(\sigma_1, \sigma_2; X)$ (or by $L^p(\sigma_1, \sigma_2)$ when $X = \mathbb{R}$) the space of function with integrable p-norm, from $[\sigma_1, \sigma_2]$ (or $[\sigma_1, +\infty)$, when $\sigma_2 = +\infty$) to X. We write $H^1(\sigma_1, \sigma_2)$ for the space of functions of $L^2(\sigma_1, \sigma_2)$ with (weak) derivative in $L^2(\sigma_1, \sigma_2)$. We also denote by $L^2_{loc}(\sigma_1, \sigma_2; X)$ the set of X-valued functions from $[\sigma_1, \sigma_2]$ which are square integrable on every compact interval contained in $[\sigma_1, \sigma_2]$, and with $L^\infty(\sigma_1, \sigma_2; X)$ the set of X-valued functions with bounded essential supremum in $[\sigma_1, \sigma_2]$. If $k \in \mathbb{N} \cup \{\infty\}$, then $C^k([\sigma_1, \sigma_2]; X)$ ($C^k([\sigma_1, \sigma_2])$ when $X = \mathbb{R}$) is the space of functions of class C^k from $[\sigma_1, \sigma_2]$ to X. For $S > 0$ (chosen in the following section), $\mathcal{R}$ is the set of (positive) Radon measures on $[0, S]$, endowed with the norm $|c|_{\mathcal{R}}$ (the finite measure of $[0, S]$ with respect to c), for all $c \in \mathcal{R}$. The support of any measure g is denoted by $supp(g)$. Finally we define the cut-off function $\psi \in C^\infty([0, S]; \mathbb{R}^+)$ such that for fixed $\bar{s}, s_1, s_2$, with $0 < \bar{s} < s_1 < s_2 < S$, $\psi \equiv 1$ on $[0, s_1]$, $\psi \equiv 0$ on $[s_2, S]$, ψ decreasing on $[s_1, s_2]$.

2.2. **From discrete to continuous time.** The discrete time model in Mitra and Wan (1985, 1986) may be formulated equivalently (see also Salo and Tahvonen, 2003 p. 1414-1415) as follows. Denoted by $x_{t,s}$ the land area occupied by trees of age s at the period t and by $c_{s,t}$ the land area with trees of age s harvested at period t (being t and s any natural numbers with s smaller than some maximal age S), the dynamics of the discrete time model is described by

$$x_{t+1,s+1} = x_{t,s} - c_{t,s}, \qquad s \in \{0, .., S\},\ t \in \mathbb{N}, \tag{1}$$

saying that trees $x_{t,s}$ can only grow into $x_{t+1,s+1}$ or being cut ($c_{t,s}$). The equation is coupled with the following *replanting rule* (for normalized land area)

$$x_{t+1,0} = 1 - \sum_{i=0}^{S} x_{t+1,i} = \sum_{i=0}^{S} c_{t,i} \tag{2}$$

saying that all the harvested area is immediately replanted.

The supplementary condition $c_{S,t} = x_{S,t}$ implies no tree older than S exists, so that the system can be described as a point on the $(S+1)$-simplex. In continuous time the evolution of the forest is observed at any instant, considering not only the "cohort year" of the trees but their exact age. The forest composition is then described by a density function $x(t, s)$ (representing the part of the forest covered at time t by trees of age s) so that, at time t, $x(t, s)\,\mathrm{d}s$ is the area on which trees of age between s and $s + \mathrm{d}s$ are standing. Similarly, $c(t, s)$ denotes the density of the cutting rate, so that $c(t, s)\,\mathrm{d}s\,\mathrm{d}t$ is the area with trees of age in $[s, s + \mathrm{d}s]$ cleared in the time interval $[t, t + \mathrm{d}t]$. Chosen $\mathrm{d}t = \mathrm{d}s \equiv \mathrm{d}h$, using (1) with period $\mathrm{d}h$ rather than 1, and dividing by $(dh)^2$, we get

$$\frac{x(t + \mathrm{d}h, s + \mathrm{d}h) - x(t, s)}{\mathrm{d}h} = -c(t, s)$$

that, letting $dh \to 0^+$, gives

$$\frac{\partial x(t, s)}{\partial t} + \frac{\partial x(t, s)}{\partial s} = -c(t, s), \tag{3}$$

meaning that the variation of density $\frac{\partial x}{\partial t}(t, s)$ is due to aging of trees $-\frac{\partial x}{\partial s}(t, s)$, and to harvesting $-c(t, s)$. On the other hand, recalling that $ds = dh = dt$, (2) becomes

$$x(t + \mathrm{d}h, 0)\,\mathrm{d}h = \sum_{i=1}^{S/\,\mathrm{d}h} c(t, i\,\mathrm{d}h)\,\mathrm{d}h\,\mathrm{d}h$$

that, dividing by $\mathrm{d}h$ and letting $dh \to 0^+$, gives the *boundary condition*

$$x(t, 0) = \int_0^S c(t, s)\,\mathrm{d}s, \tag{4}$$

meaning that the quantity $x(t,0)$ of saplings of age zero at time t coincides with the total amount of trees (of different ages) cut at time t. Hence (3) and (4), with the assignation of an initial density

$$x(s,0) = x_0(s), \quad s \in [0,S] \tag{5}$$

give the continuous version of the Mitra-Wan model. In addition, the strategy-trajectory couples (c,x) are required to satisfy the constraints

$$c(t,s) \geq 0, \text{ and } x(t,s) \geq 0, \ \forall t \geq 0, \ 0 \leq s \leq S \tag{6}$$

(only non-negative quantities are cut, and remainders are non-negative at all ages and times). The state equation has a solution which can be written easily by means of the characteristic method, as long as the control is an integrable function:

$$x(t,s) = \begin{cases} x_0\,(s-t) - \int_0^t c(t-\tau, s-\tau)\mathrm{d}\tau & s \geq t \\ \int_0^S c(t-s,r)\mathrm{d}r - \int_0^s c(t-\tau, s-\tau)\,\mathrm{d}\tau & 0 \leq s < t. \end{cases} \tag{7}$$

Note that $x(t,s)$ does not represent a spatial density. As a consequence, it may be imagined that trees grow far from one another, and not reciprocally interfering. Moreover, since the size of the forest is normalized to 1 at the initial time, that is $\int_0^S x_0(s)\,\mathrm{d}s = 1$, $\int_{\sigma_1}^{\sigma_2} x(t,s)\,\mathrm{d}s$ may be interpreted as the fraction of the forest which is covered at time t by trees of age between σ_1 and σ_2. As a consequence of the boundary condition, the surface of the forest is covered in time by the constant amount 1 of trees of different ages (see Proposition A.3), that is

$$\int_0^S x(t,s)\,\mathrm{d}s = \int_0^S x_0(s)\,\mathrm{d}s = 1, \quad \forall t \geq 0.$$

Now let $f(s)$ represent the *productivity* of a tree of age s. We assume

$$f \in H^1(0,S), \ f \geq 0, \text{ and } \ supp(f) \subset (0,S) \tag{8}$$

implying in particular that $f(s) \equiv 0$ both in some incubation interval $[0,\lambda]$, for a $\lambda > 0$, and for s big enough, say $s \geq \bar{s}$ for some $\bar{s} \in (0,S)$ (old trees are considered unproductive). Moreover, f in H^1 implies f more regular than continuous but less than continuously differentiable.

Remark 2.1 The assumption of an initially null f is used also in discrete time, e.g. by Mitra and Wan (1985, 1986), as well as by Heaps (2014) in his continuous time model. The condition $f(s) \equiv 0$ for any s big enough (*i.e.*, $s \geq \bar{s}$) is the counterpart of the assumption of a finite number of possible vintages in discrete models, and here reduces the complexity of the problem allowing only densities with support in $[0, \bar{s}]$. The H^1-regularity assumption is linked to the continuous time setting (in discrete time there is no need of continuity). For example, in Heaps (1984), f is C^2. Besides that, we have no assumption on the behavior of f, conversely Mitra and Wan (1985, 1986) assume f increasing until a certain age and then decreasing (and concave for some results) and Salo and Tahvonen (2002, 2003) and Tahvonen (2004) use an increasing f. Concavity is used also in Heaps (1984, 2014). In discrete time, Khan and Piazza (2010a, 2012) are the first making no assumption on f, and any $f\colon \mathbb{N} \to \mathbb{R}$ is fit. □

Summing all wood $f(s)c(t,s)ds$ of different ages s harvested at time t, we obtain the total wood harvested at time t, that is $w(c(t)) = \int_0^S f(s)c(t,s)ds$. The instantaneous utility function is defined as a function u satisfying

(9) $$u \in C^1(\mathbb{R}^+, \mathbb{R}^+) \text{ and concave,}$$

while the overall utility U_T at a finite horizon T, with $0 \leq T \leq +\infty$, is

$$U_T(c) = \int_0^T e^{-\rho t} u\left(w(c(t))\right) \, \mathrm{d}t, \quad \text{with } U(c) \equiv U_\infty(c).$$

The problem is maximizing in a suitable sense the overall utility $U(c)$ with infinite horizon, over a set of admissible strategies, with or without discount ($\rho > 0$ or $\rho = 0$, respectively). Note that when $\rho > 0$ the concavity of u implies the finiteness of $U(c)$, while when $\rho = 0$, $U(c)$ may be infinite valued. We say that a control strategy $\tilde{c}$ *catches up to* a control strategy c if

(10) $$\liminf_{T \to \infty} \big(U_T(c^*) - U_T(c)\big) \geq 0.$$

For a given initial stock x_0, an admissible control strategy c^* is *optimal at* x_0 if it catches up to every control strategy c admissible at the same initial stock x_0. If the utility function is not strictly concave, then optimality proves a too strong requirement (in some cases no control matching the definition is available), so that a weaker property is taken into account. We say that an admissible control strategy c^* is *maximal at* x_0 if, given any other control c admissible at x_0, one has

$$\limsup_{T\to\infty} \big(U_T(c^*) - U_T(c)\big) \geq 0. \tag{11}$$

Optimality implies maximality, but the viceversa is false in general. [2]

A last definition completes the continuous framework, that of a *stationary program*. When the control is a function, a stationary program is defined as a time-independent couple (x, c) satisfying (7) with a null time derivative of x, that is $x(s) = x(s-t) - \int_{s-t}^{s} c(r)\mathrm{d}r$ (for all t) when $s \geq t$, and $\int_s^S c(r)\mathrm{d}r$ when $0 \leq s < t$, uniquely satisfied by

$$x(s) = \int_s^S c(r)\mathrm{d}r. \tag{12}$$

Remark 2.2 Formulas (7) and (12) are defined only when controls are functions of s (for instance in L^2) but, unfortunately, we show that optimal controls are not functions but Dirac's Deltas, so that (7) and (12) need be interpreted in more general sense. This fact has an economic interpretation, as it is profit maximization and competitive arbitrage that lead, at least in the long-run, to concentrate harvesting on a finite set of ages, so that controls that act on single points of the tree configuration have to be allowed (details are in Section 3.1).

2.3. **Admissible controls and initial data.** We denote a trajectory starting at x_0 and driven by a control c by $x(\cdot; x_0, c)$ or by $x(t)$, and make two simplifying assumptions: a) initial densities x_0 are in $L^2(0, S)$ rather than in $\mathcal{R}$, ruling out mass concentrated at certain ages; b) x_0 is compactly supported in $[0, \bar{s}]$, where $\bar{s}$ is the age at which trees become unproductive (see (8) and comments below). Accordingly, we assume admissible controls are positive measures in $\mathcal{R}$, null for $s \geq \bar{s}$ and engendering

[2]Note that the terminology is not consistent across different papers: we follow McKenzie (1986, p. 1286) and recall that our notion of optimal (catching up) controls was formalized by Von Weizsäcker (1965) in continuous time, and by Gale (1967) and McKenzie (2009) in discrete time. Maximality of controls in the above sense was introduced by Brock (1970), while Halkin (1974) adapted the concept to continuous time.

trajectories which are null for $s \geq \bar{s}$. We make use also of the space D' containing $\mathcal{R}$, defined in Appendix A.1. Such requirements are formally expressed as follows.

Initial data. Initial densities x_0 are chosen in the set

$$\Pi := \left\{ x \in L^2(0,S) \ : \ x \geq 0, \ \ \mathrm{supp}(x) \subseteq [0,\bar{s}], \ \ \int_0^{\bar{s}} x(s)\, \mathrm{d}s = 1 \right\}. \tag{13}$$

Admissible control strategies. The set $\mathcal{U}_{x_0}$ of control strategies admissible at x_0 is

$$\mathcal{U}_{x_0} := \left\{ c \in L^2_{loc}(0,+\infty; D') \ : \ \begin{array}{l} supp(c(t)), supp(x(t)) \subseteq [0,\bar{s}] \ \forall t \geq 0 \\ c(t) \text{ and } x(t;x_0,c) \text{ lie in } \mathcal{R}, \ \forall t \geq 0 \end{array} \right\}. \tag{14}$$

Note that the condition "$c(t)$ and $x(t;x_0,c)$ lie in $\mathcal{R}$" in (14) translates the non-negativity constraints (6) in terms of measures. Moreover from (8) optimal controls c^* in $\mathcal{U}_{x_0}$ are expected to satisfy $supp(c^*(t)) \subseteq [\lambda, \bar{s}]$ for almost all $t \geq 0$, for a $\lambda > 0$.

Proposition 2.3 *Consider an initial datum x_0 in Π and a control $c \in \mathcal{U}_{x_0}$. Then there exists a unique solution $x(\cdot; x_0, c)$ of (31) and it belongs to $C^0([0,+\infty); D')$. Moreover, for any $t \in [0,+\infty)$, $x(t;x_0,c)$ belongs to $L^2(0,S)$ (hence it is a function).*

In some cases we consider a restricted class of admissible controls. If $\lambda > 0$, set

$$\mathcal{U}^{\lambda}_{x_0} := \left\{ c \in L^\infty(0,+\infty; \mathcal{R}) : \begin{array}{l} supp(c(t)) \subseteq [\lambda,\bar{s}], supp(x(t)) \subseteq [0,\bar{s}], \\ c(t), x(t;x_0,c) \in \mathcal{R}, \ \forall t \geq 0 \end{array} \right\}, \tag{15}$$

in particular, controls in $\mathcal{U}^{\lambda}_{x_0}$ in addition need be bounded in the $\mathcal{R}$–norm. The set

$$\mathcal{U}^{\lambda,K}_{x_0} := \left\{ c \in L^\infty(0,+\infty; \mathcal{R}) : \begin{array}{l} supp(c(t)) \subseteq [\lambda,\bar{s}], supp(x(t)) \subseteq [0,\bar{s}] \\ c(t), x(t;x_0,c) \in \mathcal{R}, \ |c(t)|_{\mathcal{R}} \leq K, \ \forall t \geq 0 \end{array} \right\} \tag{16}$$

may also be considered (with $\mathcal{R}$–norms of controls bounded by the same constant K).

3. The Faustmann problem and candidate optimal programs

In this section we describe candidate optimal and maximal programs consistently with Mitra and Wan (1985, 1986), only in continuous time. All candidates are characterized by a cutting age obtained solving the *Faustmann problem*, that is identifying critical ages which maximize "the present discounted value of all net cash receipts [...] calculated over the *infinite chain* of cycles of planting on the given acre of land from now until Kingdom Come" (Samuelson, 1995, p. 122). The rule "cutting any tree that reaches the critical age" is called a *Faustmann Policy*, and candidates generated by that policy are cyclical. Stationary candidates are also prices supported, and are called here *golden rules* or *modified golden rules*, depending on the fact that the discount rate is zero or positive. In continuous time and for $\rho > 0$, the Faustmann problem is finding maximizers of

$$g_\rho(s) = \sum_{n=1}^{\infty} e^{-\rho n s} f(s) = \frac{f(s)}{e^{\rho s} - 1}$$

that is, the value of an infinite sequence of planting cycles with harvesting at age s. Since when $\rho > 0$ maximizers of $g_\rho(s)$ and of $g_\rho(s)(1 - e^{-\rho})$ coincide and $\lim_{\rho \to 0^+} g_\rho(s)(1 - e^{-\rho}) = f(s)/s$, the Faustmann problem becomes identifying

$$\mathcal{A}_\rho \equiv \operatorname{argmax}\{G_\rho(s) : s \in [0, \bar{s}]\}, \quad \forall \rho \geq 0,$$

where $G_\rho(s) = \frac{1-e^{-\rho}}{e^{\rho s}-1} f(s)$, when $\rho > 0$ and $G_0(s) = f(s)/s$. Maximizers enjoy some interesting properties, stated below.

Proposition 3.1 *Assume f satisfies (8). Then $\mathcal{A}_\rho \subset (0, \bar{s}]$, and $\mathcal{A}_\rho \neq \emptyset$, for all $\rho \geq 0$. Moreover, if $0 < \rho_B < \rho_A$, then:*

(i) *There exists $\widetilde{s} \in (0, S]$ such that $\mathcal{A}_{\rho_A} \subseteq (0, \widetilde{s}]$ and $\mathcal{A}_{\rho_B} \subseteq [\widetilde{s}, S]$. Moreover, $\mathcal{A}_{\rho_A}$ and $\mathcal{A}_{\rho_B}$ may be non-disjoint only if f is not differentiable at $\widetilde{s}$.*

(ii) *For any chosen $M_\rho \in \mathcal{A}_\rho$, the selections $\rho \mapsto M_\rho$ and $\rho \mapsto \frac{f(M_\rho)}{M_\rho}$ are nonincreasing. Moreover $\mathcal{A}_\rho$ is not a singleton for at most countable set of values of ρ.*

(iii) *For every selection M_ρ of $\mathcal{A}_\rho$, there exists $\lim_{\rho \downarrow 0^+} M_\rho = m_0 = \min \mathcal{A}_0$.*

3.1. **The Golden Rule.** A *modified golden rule* (x_ρ, c_ρ) (or *golden rule*, when $\rho = 0$) is a couple in $\Pi \times \mathcal{R}$ so defined (see Figure 1):

$$x_\rho(s) \equiv \frac{1}{M_\rho} \chi_{[0, M_\rho]}(s), \tag{17}$$

where $M_\rho \in \mathcal{A}_\rho$, meaning that all ages in the range $[0, M_\rho]$ are uniformly distributed and equal to $1/M_\rho$, while those in the range $[M_\rho, S]$ are null;

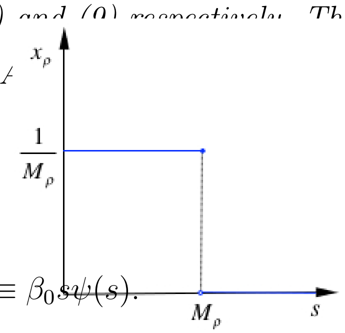
$$c_\rho(t, s) \equiv \frac{1}{M_\rho} \delta_{M_\rho}, \tag{18}$$

where δ_{M_ρ} is the Dirac's Delta at point M_ρ, that is, c_ρ is cutting exactly trees reaching age M_ρ. Note that x_ρ is a function in Π and that c_ρ is not a function of s but a positive measure. By definition of Dirac's deltas, through the control c_ρ the following quantity of wood is harvested

$$w(c_\rho) = \langle c_\rho, f\rangle = \frac{1}{M_\rho}\langle \delta_{M_\rho}, f\rangle = \frac{1}{M_\rho} f(M_\rho). \tag{19}$$

It is not difficult to guess that any golden rule is a stationary couple, as the amount of trees cut at age M_ρ is instantaneously replanted at age 0, preserving the configuration among different ages unaltered, as stated next.

Proposition 3.2 *Let $\rho \geq 0$, and f and u satisfying (8) and (9) respectively. Then (x_ρ, c_ρ) is a stationary couple, in the sense of Definition*



Define now $\beta_\rho \equiv \langle c_\rho, f\rangle = f(M_\rho)/M_\rho$, $\eta_\rho \equiv f(M_\rho)/(e^{\rho M_\rho} - 1)$ and $p_\rho \colon [0, S] \to \mathbb{R}^+$ as

$$p_\rho(s) \equiv \eta_\rho\,(e^{\rho s} - 1)\,\psi(s),\ \ \rho > 0; \quad p_0(s) \equiv \beta_0 s\psi(s). \tag{20}$$

Note that $p_0(s) = \lim_{\rho\to 0^+} p_\rho(s)$ and that, for any $\rho \geq 0$, p_ρ is twice differentiable with $p_\rho(S) = {p_\rho}'(S) = 0$.

FIGURE 1. The golden rule

Remark 3.3 The dual variables in (20) have a direct interpretation as stationary competitive prices associated with a golden rule path (see Cass and Shell, 1976). Indeed, if we interpret $p_\rho(s)$ as the (infinite dimensional) vector of the prices of capital goods (i.e, the prices of the different vintages s of trees) and set $R = \rho\eta_\rho$ the rent rate of the land on which the trees are planted (when $\rho = 0$, define $R = \lim_{\rho\to 0^+} \rho\eta_\rho = \beta_0$), then by definition (20), one has $f(s) \leq p_\rho(s)$, when s is in $[0, \bar{s}]$, and $f(M_\rho) = p_\rho(M_\rho)$. The first inequality means that no cutting process yields a positive profit, while the equality says that the only cutting processes that do not generate losses are those operating at the Faustmann ages. Thus, golden rule policies maximize short run profits. In addition, since for all $s \in [0, \bar{s}]$, $p'_\rho(s) = \rho p_\rho(s) + R$ for $M_\rho \geq s \geq 0$ and $p'_\rho(s) \leq \rho p_\rho(s) + R$ for $s \geq M_\rho$, then the asset-market-clearing conditions holding under competitive arbitrage are satisfied. Clearly, the arbitrage condition in a golden rule takes the form of a "modified Hotelling rule" because a piece of land needs to be rented in order to hold a tree of a given age *in situ*. Since the Faustmann age (when unique) is the only age in $(0, \bar{s}]$ generating no loss, the importance of including measures among admissible controls is confirmed.

3.1.1. *Modified golden rules.* In the following sections we will classify the behavior of candidate optimal or maximal programs, assuming either a positive or null discount, a linear or strictly concave utility function, a singleton or multivalued $\mathcal{A}_\rho$, although when $\rho > 0$, optimality of the golden rule is a general property, as stated below.

Theorem 3.4 *Assume $\rho > 0$, $M_\rho \in \mathcal{A}_\rho$, and f and u satisfying (8) (9) respectively. Then c_ρ is optimal at x_ρ. Moreover, if $\mathcal{A}_\rho = \{M_\rho\}$, then the unique optimal stationary couple is (x_ρ, c_ρ).*

Remark 3.5 It is easy to show (see the Appendix for a formal proof) that the same property holds for any convex linear combination of golden rules, that is, if $\mathcal{A}_\rho = \{M_\rho^1, ..., M_\rho^n\}$, and (x_ρ^i, c_ρ^i) is the golden rule associated to M_ρ^i, then $\tilde{x} = \sum_{i=1}^n \lambda_i x_\rho^i$, $\tilde{c} = \sum_{i=1}^n \lambda_i c_\rho^i$, where $\lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$, is also an optimal stationary program. □

Remark 3.6 Note that by (19) it is $U_T(c_\rho) = \rho^{-1}(1 - e^{-\rho T})u(\beta_\rho)$, when $\rho > 0$ the golden rule is optimal when starting at x_ρ, with maximal overall utility given by

$$\max_{c \in \mathcal{U}_{x_\rho}} U(c) = U(c_\rho) = \lim_{T \to +\infty} U_T(c_\rho) = u(\beta_\rho)\rho^{-1}.$$

The proofs of Theorem 3.4 and of other theorems in the following sections rely on the construction of the *value-loss function*, studied in Corollary A.9:

$$\theta_\rho(c, x) = u(\beta_\rho) - u\left(\langle c, f\rangle\right) + u'\left(\beta_\rho\right)\left[\rho\left\langle x - x_\rho, p_\rho\right\rangle - \left\langle x, A^* p_\rho\right\rangle + \left\langle c, p_\rho\right\rangle\right]. \tag{21}$$

The value $\theta_\rho(c(t), x(t))$, which gives the value-loss of any admissible couple at the steady state competitive prices, is the analogue of the value-loss function commonly used for finite dimensional optimal growth problem (see McKenzie, 1986, for the discrete time case and Magill, 1977, for continuous time). The only aspect that is specific to our infinite dimensional setting is that the unit rental costs function contains an element accounting for the aging process of capital goods.

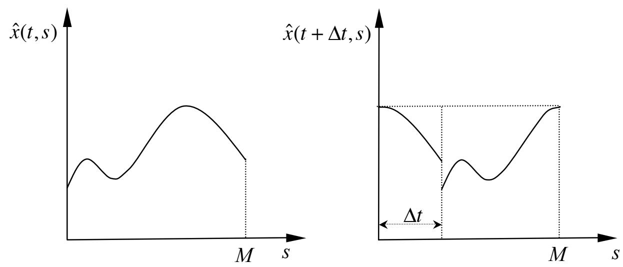


FIGURE 2. The Faustmann Solution.

3.2. **The Faustmann solution.** Besides the golden rule, other controls are candidates to be optimal or maximal when starting at a general initial datum x_0. (Indeed, the golden rule may even be non-admissible at x_0.) Given an initial datum $x_0 \in \Pi$, if M_ρ represents a preferable cutting age providing a maximal harvesting, one may attempt to use the feedback strategy $(\hat{x}, \hat{c})$, where

$$\hat{c}(t) = \hat{x}(t, M_\rho)\delta_{M_\rho}, \quad \forall t \geq 0, \tag{22}$$

that is, $\hat{c}$ cuts existing trees reaching age M_ρ. Such trees vary in time depending on the initial density x_0. We remark that a trajectory of the system starting from an initial datum in Π is defined for almost every s, so that $\hat{x}(t, M_\rho)$ and $\widehat{c}$ may not be well defined. However, in the following lemma we give meaning to both.

Lemma 3.7 *Let $x_0 \in \Pi$, supp$(x_0) \subset [0, M_\rho]$, $\sigma(t) = \left\{\frac{t}{M_\rho}\right\}$, $M_\rho = t - \left[\frac{t}{M_\rho}\right] M_\rho$, and*

$$\widehat{x}(t,s) = \widehat{x}(t)(s) = x_0(s - \sigma(t))\chi_{[\sigma(t), M_\rho]}(s) + x_0(s + M_\rho - \sigma(t))\chi_{[0,\sigma(t)]}(s). \tag{23}$$

Then $\hat{x}(t + M_\rho) = \hat{x}(t)$, for all $t \geq 0$, the control $\hat{c}(t) = \hat{x}(t, M_\rho)\delta_{M_\rho}$ is admissible at x_0 and $\hat{x}$ solves (in mild sense, see (31)) the closed loop equation

$$x'(t) = Ax(t) + Bx(t, M_\rho)\,\delta_{M_\rho}, \;\; x(0) = x_0. \tag{24}$$

Then the *Faustmann solution* or *Faustmann Policy* $(\hat{x}, \hat{c})$ generates M_ρ-periodic trajectories $\hat{x}$ (Figure 2). The golden rule is the Faustmann solution starting at x_ρ.

Lemma 3.8 *Assume supp*$(x_0) \subset [0, M_\rho]$, $\hat{c}$ *the Faustmann policy,* $T \geq 0$, *and* $U_1^\rho := \int_0^{M_\rho} e^{\rho\tau} u\left(f(M_\rho)x_0(\tau)\right) d\tau$, $U_2^\rho(T) := \int_{M_\rho - \sigma(T)}^{M_\rho} e^{\rho\tau}\, u\left(f(M_\rho)x_0(\tau)\right) d\tau$. *Then*

$$U_T(\widehat{c}) = \begin{cases} \frac{1-e^{-\rho n M_\rho}}{e^{\rho M_\rho}-1} \chi_{[M_\rho,\infty)}(T)\, U_1^\rho + e^{-\rho(n+1)M_\rho} U_2^\rho(T), & \rho > 0 \\ nU_1^0 + U_2^0(T), & \rho = 0 \end{cases} \tag{25}$$

Remark 3.9 Note that when $\rho > 0$ the overall utility is finite

$$U(\hat{c}) = \lim_{T\to\infty} U_T(\hat{c}) = U_1^\rho (e^{\rho M_\rho} - 1)^{-1},$$

contrary to the case $\rho = 0$. The formula is consistent with Remark 3.6 when $x_0 = x_\rho$.

3.3. **Null discounts and Good Controls.** Assume $\rho = 0$, $M \in \mathcal{A}_0$ and denote by $(\bar{x}, \bar{c})$ the associated golden rule, that is

$$\bar{x} = \frac{1}{M}\chi_{[0,M]}, \;\; \bar{c} = \frac{1}{M}\delta_M. \tag{26}$$

The case $\rho = 0$ appears immediately more complicated than the case $\rho > 0$. Indeed the utility over a finite horizon T associated to the golden rule is $U_T(\bar{c}) = T\ u(\beta_0)$, diverging when T tends to infinity. A useful notion is that of *good controls*, already introduced in discrete time by Gale (1967).

Definition 3.10 *Assume* $\rho = 0$. *A control* $c \in \mathcal{U}_{x_0}$ *is* good *if there exists* $\theta \in \mathbb{R}$ *s.t.*

$$\inf_{T\geq 0} \big(U_T(c) - U_T(\bar{c})\big) \geq -\theta.$$

Note that a control is defined "good" in comparison to the golden rule, even when $\bar{c}$ is not admissible at x_0. Note also that a control c is good if and only if there exists $\theta \in \mathbb{R}$ such that, for all $T \geq 0$, one has $U_T(c) \geq U_T(\bar{c}) - \theta$, meaning that the utility (over an arbitrary finite horizon T) achieved by means of a good control is dominated by the utility at $\bar{c}$ by at most a finite quantity θ. The following result implies that search of optimal or maximal programs can be restricted to good controls.

Proposition 3.11 *If* $c^* \in \mathcal{U}_{x_0}$ *is maximal (or optimal) then it is good.*

4. Classification of optimal programs

4.1. **Linear utility, positive discount.** In Theorem 3.4 we already established that, when $\rho > 0$, the modified golden rules are optimal in all sets of assumptions. In particular that holds true for u linear. In the following theorem we see that, in the particular case of $\rho > 0$ and u linear, the Faustmann solution $(\hat{x}, \hat{c})$ given by (23) (22) is optimal regardless the initial density x_0, as long as x_0 does not contain trees older than M_ρ. The result is consistent with Theorem 3.4, as $(\hat{x}, \hat{c})$ coincides with the golden rule when the initial datum is x_ρ.

Theorem 4.1 *Assume $\rho > 0$, (8), and $u(r) = ar + b$, $r \geq 0$ (a, b in $\mathbb{R}$). Let $x_0 \in \Pi$, with supp$(x_0) \subseteq [0, M_\rho]$. The Faustmann Solution $(\hat{x}, \hat{c})$ is optimal at x_0.*

Then a modified golden rule is an equilibrium, but not an asymptotic equilibrium (convergence is driven by second order differences, hence to strict concavity of the value function, which in this case is linear).

Remark 4.2 The assumption *supp*$(x_0) \subset [0, M_\rho]$ (used in Theorems 4.1 and 4.3) is technical and related to the continuous time setting. In discrete time Mitra and Wan prove that, with a linear utility function, there are cases in which the optimal policy is cutting at time 0 all trees with age *at least* M_ρ and subsequently those reaching age M_ρ (see Theorem 4.2 in Mitra and Wan, 1985, and Theorem 5.2 in Mitra and Wan, 1986). A corresponding continuous time policy, demanding at time $t = 0$ to cut all trees of age M_ρ and older, would engender a mass of trees of age zero (a Dirac's Delta) immediately after time $t = 0$ by effect of replanting, and the density $x(t)$ would be no longer in L^2. Hence in our framework the described policy would not be admissible.

4.2. **Linear utility, null discount.** In this section we establish that, with null discount and linear u, the Faustmann solution is maximal but not optimal. We show also that optimal programs do not exist. We require

$$\mathcal{A}_0 \quad \text{is singleton, } \mathcal{A}_0 \equiv \{M\}. \tag{27}$$

Although the case of multivalued $\mathcal{A}_0$ is not considered here, multiplicity of maxima is a fragile phenomenon that vanishes under small perturbations of f.

Theorem 4.3 *Let (27) be satisfied, $\rho = 0$, $u(r) = ar + b$ $(a, b \in \mathbb{R})$, $r \geq 0$, and $x_0 \in \Pi$ with supp$(x_0) \subseteq [0, M]$. Then the Faustmann Solution $(\hat{x}, \hat{c})$ is maximal, but not optimal. Indeed no optimal control exists in this set of data.*

The fact applies to the particular case of the golden rule.

Corollary 4.4 *In the assumptions of Theorem 4.3, the golden rule $(\bar{x}, \bar{c})$ is maximal, but not optimal, at $\bar{x}$. Moreover no admissible control at $\bar{x}$ may be optimal.*

As a direct proof of the assertion that neither $\bar{c}$ is optimal, nor an optimal control exists, one may build the following example (proof of Theorem 4.3 is based on a similar construction), where the control $\bar{c}$ is not catching up to c_1 defined by means of (28), admissible at $\bar{x}$. The control c_1 behaves *on average* like $\bar{c}$ but delayed of some initial time interval: the difference in utilities yielded by $\bar{c}$ and c_1 coincide repeatedly with their difference in the initial time interval, precisely because $\rho = 0$ and u is linear.

Example 4.5 Let N be a natural number greater than 1. Define $s_j := jM/N$, for $j = 1, .., N$ and consider a control c_1 and associated trajectory x_1 so defined: when $t \leq M/N$, c_1 cuts the quantity $x_1(t, s_j)$ of available trees of age s_j, subsequently when $t \geq M/N$, c_1 cuts the quantity $x_1(t, M)$ of trees reaching age M, that is

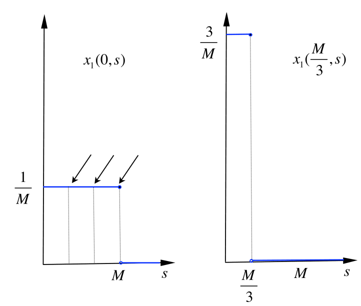


FIGURE 3. The trajectory x_1 for $N = 3$, at time 0 and at time $M/3$.

$$c_1(t) = \begin{cases} \sum_{j=1}^{N} x_1(t, s_j)\delta_{s_j}, & 0 \leq t < \frac{M}{N} \\ x_1(t, M)\delta_M, & t \geq \frac{M}{N} \end{cases} \tag{28}$$

It is easy to check that c_1 is admissible at any x_0 with *supp*$(x_0) \subset [0, M]$. In particular for $x_0 = \bar{x}$ the associated trajectory $x_1(t, s; c_1, \bar{x}) \equiv x_1(t, s)$ is given by

$$x_1(t, s) = (N/M)\chi_{[0,t]}(s) + (1/M) \sum_{j=1}^{N} \chi_{[s_{j-1}+t, s_j]}(s)$$

when $t \in \left[0, \frac{M}{N}\right], s \geq 0$. From $t = M/N$ on, $x_1(t,s)$ is M-periodic, coincides with (29) in all intervals of type $[T_i, T_{i+1}]$ where $T_i = M/N + iM$, $i \in \mathbb{N}$, namely

$$(29) \qquad x_1(t,s) = \begin{cases} \frac{N}{M}\chi_{[t-\frac{M}{N},t]}(s) & t \in \left[\frac{M}{N}, M\right] \\ \frac{N}{M}\left[\chi_{[0,t-M]}(s) + \chi_{[t-\frac{M}{N},M]}(s)\right] & t \in \left[M, M+\frac{M}{N}\right]. \end{cases}$$

In such intervals, c_1 cuts an amount N/M for a time length M/N, while $\bar{c}$ cuts the amount $1/M$ for a time length M (in Figure 3 is represented the case $N = 3$). Then, except on $[0, M/N]$, the utilities yielded by $\bar{c}$ and c_1 on a period length interval are both equal to $f(M)$, and the difference between such utilities is periodically equal to the difference yielded on $[0, M/N]$, that is $\frac{1}{N}\sum_{j=1}^{N-1} f(s_j) > 0$ (provided f is not null everywhere). Then the control $\bar{c}$ does not catch up to c_1. By means of a similar idea one may contradict also the existence of an optimal control. For details we refer the reader to the proof of the general case, Theorem 4.3 in the Appendix. □

4.3. **Strictly concave utility, null discount.** Also in this subsection we assume that (27) is satisfied. Moreover we consider, rather than $\mathcal{U}_{x_0}$, $\mathcal{U}^{\lambda}_{x_0}$ or $\mathcal{U}^{K,\lambda}_{x_0}$ defined in (15) (16) as the set of admissible strategies.

Theorem 4.6 *Let $\rho = 0$, and let (8)(9)(27) be satisfied, with u strictly concave. Let $x_0 \in \Pi$, and $c \in \mathcal{U}^{\lambda}_{x_0}$, with c good. Then the trajectory $x(t; c, x_0)$ converges to the golden rule $\bar{x}$ in $L^2(0,S)$–norm.*

Theorem 4.7 *Assume $\rho = 0$, and that* (8) (9) (27) *hold. Assume moreover that u is strictly concave. Then the golden rule $(\bar{x}, \bar{c})$ is an optimal stationary couple.*

As a consequence of Theorem 4.6, the Faustmann solutions (maximal for linear u) are not maximal anymore for strictly concave u, as they are definitely caught up by the convergent solution.

Corollary 4.8 *In the assumptions of Theorem 4.6, the Faustmann solution is neither optimal nor maximal, except for the particular case of the Golden Rule.*

Theorem 4.9 *Let $x_0 \in \Pi$, $\rho = 0$, and assume (27) is satisfied. Let $\mathcal{U}^{K,\lambda}_{x_0}$ be defined by (16), for a $K > 0$ such that $\mathcal{U}^{K,\lambda}_{x_0} \neq \emptyset$. Then:*

(*i*) *if u is strictly concave, then there exists an optimal control in $\mathcal{U}^{K,\lambda}_{x_0}$;*

(ii) if u is concave, then there exists a maximal control in $\mathcal{U}_{x_0}^{K,\lambda}$.

4.4. **Strictly concave utility, positive discount.** As observed in Corollary 4.8 the Faustmann Policy is not optimal for the case of a strictly concave utility function and null discount. However, that does not preclude the possibility that the Faustmann policy turns out optimal for the discounted model. Indeed, in discrete time and with a strictly concave utility and discounted future utilities, Mitra and Wan (1985) provided examples in which the Faustmann Policy was optimal, and Wan (1994), Salo and Tahvonen (2002, 2003) took the issue further (see also Mitra et al., 1991, for similar results in a different vintage capital model) by showing that optimal Faustmann cycles persist at a neighborhood of the (modified) golden rule when the discount factor approaches unity. In particular, Proposition 1 in Salo and Tahvonen (2003) states that, for discount factors less than one, the Faustmann Policy is optimal for all initial forests sufficiently close to the steady state. On the contrary, for the strictly concave continuous-time discounted model, optimality of cyclical Faustmann solutions is still an open question: neither is a convergence result available, nor a case in which the Faustmann Policy is optimal. However, we establish a partial result by proving that Proposition 1 in Salo and Tahvonen (2003) does not carry over to our continuous-time formulation and, hence, that our model behaves differently from that in discrete-time. Indeed we consider the following simple example. Assume that $M_\rho = 1$ is the unique Faustmann maturity age, and that $f(1) = 1$. Consider, for $0 \leq a \leq \frac{1}{2}$, the initial density depicted in Figure 4 (note that for $a = 0$ one has the golden rule forest)

$$x_a(s) = \chi_{[0,1-2a)}(s) + (1+a)\chi_{[1-2a,1-a)} + (1-a)\chi_{[1-a,1]}.$$

We intend to show that, for any a in a right neighborhood of 0, the Faustmann Policy is not optimal starting at $x_a(s)$. As a consequence, the analogue of Proposition 1 in Salo and Tahvonen (2003) does not hold in continuous time. The cycle induced by the Faustmann Policy $\hat{c}_a(t) = \hat{x}_a(t,1)\delta_1$ comprises an initial phase of length a during which $1-a$ units of timber are harvested, a phase of

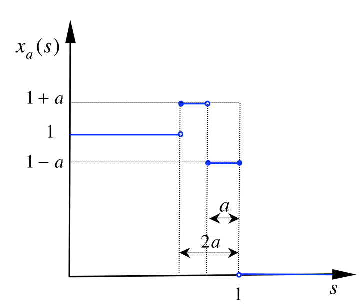


FIGURE 4.

length a during which the harvest is raised to $1+a$, and a final phase of length $1-2a$ with a constant harvest at the golden rule level. On the first cycle, that is for $t \in [0,1]$, one has

$$\langle \hat{c}_a(t), f \rangle = f(1)x_a(t,1) = x_a(t,1) = (1-a)\,\chi_{[0,a]}(t) + (1+a)\,\chi_{[a,2a]}(t) + \chi_{[2a,1]}(t),$$

so that the utility at horizon $T=1$ is given by

$$\begin{aligned} U_1(\hat{c}_a) &= \int_0^a u(1-a)e^{-\rho s}\mathrm{d}s + e^{-\rho a}\int_0^a u(1+a)e^{-\rho s}\mathrm{d}s + e^{-2\rho a}\int_0^{1-2a} u(1)e^{-\rho s}\,\mathrm{d}s \\ &= \left(u(1-a)+u(1+a)e^{-\rho a}\right)\left(1-e^{-\rho a}\right)\rho^{-1} + u(1)(e^{-2\rho a}-e^{-\rho})\rho_{-1}. \end{aligned}$$

Next we note that, starting from any forest $x_a(s)$, a feasible Most Rapid Approach Path to the steady state, namely $c_{mra}(t) = [(1-a)\delta_1 + a\delta_{1-a})]\,\chi_{[0,a]}(t) + \delta_1\chi_{[a,\infty)}(t)$, reaches the golden rule after a units of time by continuously clearing the $1-a$ units of land on which mature trees are planted and the a units of land in excess on which trees of age $1-a$ are grown. The associated utility at horizon $T=1$ is then

$$\begin{aligned} U_1(c_{mra}) &= \int_0^a u\left((1-a)+af(1-a)\right)e^{-\rho s}\mathrm{d}s + e^{-\rho a}\int_0^a u(1)e^{-\rho s}\mathrm{d}s + e^{-2\rho a}\int_0^{1-2a} u(1)e^{-\rho s}\mathrm{d}s \\ &= u\left(1-a+af(1-a)\right)\left(1-e^{-\rho a}\right)\rho^{-1} + u(1)(e^{-\rho a}-e^{-\rho})\rho^{-1}. \end{aligned}$$

Instead, when $T=n\geq 2$, one has

$$U_n(\hat{c}_a) = \sum_{i=0}^{n} e^{-i\rho}U_1(\hat{c}_a) = U_1(\hat{c}_a) + U_1(\hat{c}_a)\frac{e^{-\rho}-e^{-n\rho}}{1-e^{-\rho}}$$

$$U_n(c_{mra}) = U_1(c_{mra}) + \sum_{i=1}^{n} e^{-i\rho}\frac{u(1)}{\rho}\left(1-e^{-\rho}\right) = U_1(c_{mra}) + \frac{u(1)}{\rho}(e^{-\rho}-e^{-n\rho}).$$

As a consequence, $\lim_{n\to\infty}\left(U_n(\hat{c}_a) - U_n(c_{mra})\right) =$

$$= \frac{1-e^{-\rho a}}{\rho}\left[\frac{u(1-a)-u(1)}{1-e^{-\rho}} + \frac{u(1+a)-u(1)}{1-e^{-\rho}}e^{-\rho a} - u(1-a+af(1-a)) + u(1)\right].$$

The sign of the sum in the square brackets, null at $a=0$, for $a>0$ in a neighborhood of 0 is given by the sign of the lowest non-zero derivative order evaluated at 0. Simple calculations show that the first derivative is null, while the second is given by $2(u''(1)-\rho u'(1)) + 2f'(1)u'(1)(1-e^{-\rho})$, which is strictly negative, as $f'(1)(1-e^{-\rho}) = \rho$ by definition of $M_\rho = 1$. Hence the Faustmann Policy is not optimal starting at $x_a(s)$ in a complete neighborhood of the steady state.

5. Conclusions

In this paper we developed and analyzed a continuous time version of the Mitra and Wan (1985) model of optimal forest management. One of our main purposes was to isolate the set of phenomena that are independent of the way time is modeled. Table 1 gives an overview of the results we have established in continuous-time. It turned out that many of the results in discrete time carry over to continuous time, with an important exception: cyclical optimal solutions, proper of the discounted strictly-concave discrete model, disappear in continuous time. Unlike in discrete

	$\rho = \mathbf{0}$	$\rho > \mathbf{0}$
u linear	• If $\mathcal{A}_0$ is singleton, GR maximal, but not optimal • If $\mathcal{A}_0$ is singleton, FS maximal at any admissible x_0 with $supp(x_0) \subset [0, M_\rho]$ • There do not exist optimal controls	• Any MGR is optimal • FS is optimal at any admissible x_0 satisfying $supp(x_0) \subset [0, M_\rho]$
u strictly conc.	If $\mathcal{A}_0$ singleton: • GR is the unique optimal stationary couple • There exists an optimal control • Any optimal trajectory converges to the GR	• Any MGR is optimal • It is not true (as in discrete time) that FS is optimal for all initial forests close to the MGR.
u conc.	If $\mathcal{A}_0$ singleton: • GR is the unique maximal stationary couple • There exists a maximal (admissible) control	• Any MGR is optimal

Table 1. Results at one glance: FS stands for Faustmann Solution, GR for Golden Rule, MGR for modified golden rule.

time, modeling timber production in continuous time required a quantum leap from the received vintage capital theory. Indeed, in the typical vintage capital model in continuous time, (*e.g.* for optimal investment Feichtinger et al., 2003, 2006) optimal investments are spread over a continuum of ages, so that controls are functions. In forest management in continuous time, however, timber production cannot be modeled this way, because the Faustmann condition implies that generically it is optimal to fell down only trees of a single age. Therefore, we had to develop an entirely new class of vintage models in which measure-valued controls are allowed. Since this is the first attempt to formulate the Mitra-Wan model in continuous time, we have been concentrating on the basic features of the model, without attempting to use minimal assumptions and without taking into account recent refinements of the theory (Khan and Piazza, 2012). We expect that our results and methods of analysis will find useful applications in studying other models with non-homogeneous natural resources, as age distributed fisheries (see Tahvonen, 2009) or the so called orchard model (Mitra et al., 1991), vintage capital models, as the continuous-time Ramsey version of the clay-clay vintage capital model of Solow et al. (1966) developed by Boucekkine et al. (1997) (see also Boucekkine et al., 1998), and more in general, all kind of models comprising age distributed state variables (*e.g.*, demographic models).

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Appendix A. The Abstract Problem and Proofs

We advise that a full understanding of the content of the Appendix requires a good knowledge of functional analysis. We refer the reader to Engel and Nagel (1999) or Pazy (1983) for the general theory of strongly continuous semigroups and evolution equations, to Bensoussan et al. (2007) for control in infinite dimension.

A.1. **The extended framework.** We start by formulating an intermediate abstract problem in $L^2(0,S)$, using the translation semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(0,S)$, namely the linear operators $T(t) : L^2(0,S) \to L^2(0,S)$ such that $[T(t)f](s) = f(s-t)$, if $s \in [t,S]$, and $[T(t)f](s) = 0$ otherwise. The generator of $T(t)$ is the operator $A : D(A) \to L^2(0,S)$ where $D(A) = \{f \in H^1(0,S) \ : \ f(0) = 0\}$, given by $[Af](s) = -\partial f(s)/\partial s$. The adjoint of A is then $A^* : D(A^*) \to L^2(0,S)$ with $D(A^*) = \{f \in H^1(0,S) \ : \ f(S) = 0\}$ defined by $[A^*f](s) = \partial f(s)/\partial s$, generating itself a translation semigroup $T^*(t) : L^2(0,S) \to L^2(0,S)$ given by $T^*(t)f(s) = f(s+t)$, if $s \in [0, S-t]$, and $T^*(t)f(s) = 0$ otherwise. Then we generalize of all previous notions to a wider space. We set $D \equiv D(A^*)$, and $D' \equiv D(A^*)'$, and assume D' *is both the control space and the state space of the abstract problem.* Indeed by standard arguments, in particular by replacing the scalar product in L^2 with the duality pairing $\langle \phi, \psi \rangle_{D',D}$ with $\phi \in D'$, $\psi \in D$ (coinciding with the scalar product in L^2 when $\phi \in L^2$ – we use the notation $\langle \cdot, \cdot \rangle$ in both cases, unless it is ambiguous); the semigroup $\{T(t)\}_{t\geq 0}$ can be extended to a strongly continuous semigroup on D', while $T^*(t)$ can be restricted to a strongly continuous semigroup on D; their generators are respectively an extension and a restriction of those in $L^2(0,S)$.[3] For simplicity we keep denoting such operators by

[3]For details we refer the reader to Faggian (2005) and Faggian and Gozzi (2010).

$T(t)$, $T^*(t)$, A, and A^*. The role of $L^2(0,S)$ is that of pivot space between D and D', namely $D \subset L^2 \subset D'$, with continuous inclusions. It is very important to say that *such formulation enables the possibility of choosing controls which are positive measures rather than functions.* More precisely, we use the subset $\mathcal{R} \subset D'$ (with continuous inclusion) containing all Dirac's measures δ_{s_0}, with $s_0 \in [0,\bar{s}]$. The cut-off function ψ defined in Section 2.1 is in $D^2 \equiv D([A^*]^2) = \{g \in H^2(0,S) : g(S) = g'(S) = 0\}$, namely, the domain of the generator of the adjoint semigroup $T^*(t)$ restricted to D. When c has support in $[0,\bar{s}]$, by means of the linear bounded functional $D' \to \mathbb{R}$, $c \mapsto \langle c, \psi \rangle$ we write the boundary condition as $x(t,0) = \langle c(t), \psi \rangle$, and enclose it in the definition of the control operator $B\colon D' \to D'$, with $Bc := -c + \langle c, \psi \rangle \delta_0$, where δ_0 is the Dirac's Delta at 0 (see Barucci and Gozzi (2001), pp. 25-26 for a detailed explanation).[4] The adjoint operator of B is given by $B^*\colon D \to D$, with $B^* v := -v + \langle \delta_0, v \rangle \, \psi$. Then (3)(4)(5) become the *state equation*

$$\begin{cases} x'(t) = Ax(t) + Bc(t), \quad t > 0 \\ x(0) = x_0 \end{cases} \tag{30}$$

and rewritten in mild form (see *e.g.* Bensoussan et al., 2007 Section 3.II.1) as

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bc(\tau)\,\mathrm{d}\tau. \tag{31}$$

The total wood harvested at t may be written as $w(c(t)) = \langle c(t), f \rangle$, coinciding with $\int_0^S f(s)c(t,s)ds$ when $c(t)$ is in $L^2(0,S)$. Then the *objective functional* is written as

$$U_T(c) = \int_0^T e^{-\rho t} u\left(\langle c(t), f \rangle\right) \mathrm{d}t, \quad 0 \le T \le +\infty. \tag{32}$$

When $c \in \mathcal{R}$, (12) may be interpreted in the following abstract way.

[4]The lift of the boundary condition in the state equation can be understood as follows: as the abstract equation aggregates quantities with respect to s, it accounts for both the distributed control $-c$ (cutting) and the boundary control $\langle c, \psi \rangle \delta_0$ (replanting), the latter with effect concentrating (by means of δ_0) on the subdomain $\{0\} \times [0,+\infty)$ of trees of age 0. It is a case of unbounded boundary control operators (Bensoussan et al., 2007, Sections II-2-1.1.2 page 177 and II-2-2.1 page 188).

Definition A.1 *We say that* $(\tilde{x}, \tilde{c}) \in \Pi \times \mathcal{R}$ *is a* stationary couple *if, for all* $t \geq 0$,

$$\tilde{x} = T(t)\tilde{x} + \int_0^t T(t-s)B\tilde{c}\,\mathrm{d}s. \tag{33}$$

A stationary couple $(\tilde{x}, \tilde{c})$ *is* optimal *if* $c(t) \equiv \tilde{c}$ *is optimal at* $\tilde{x}$.

A.2. **Proofs for Section 2.3.**

Proposition A.2 *For any* $T > 0$, *the operator* $\mathcal{S}\colon D' \times L^2(0,T;D') \to C([0,T];D')$ *defined by* $\mathcal{S}(x_0, c)(t) := T(t)x_0 + \int_0^t e^{(t-s)A}Bc(s)\,\mathrm{d}s$ *is continuous. In particular, for any* $x_0 \in D'$, *and for any* $c \in \mathcal{U}_{x_0}$, *the function* $[0,T] \to D'$, $t \mapsto \mathcal{S}(x_0,c)(t)$ *is also continuous.*

Proof. See *e.g.* Bensoussan et al. (2007) Section II.1.3. □

Proof of Proposition 2.3. The first statement follows from Proposition A.2. To prove the second, note that (31) implies

$$x(t) = T(t)x_0 + \int_0^t \langle c(\tau), \psi\rangle\, T(t-\tau)\delta_0\,\mathrm{d}\tau - \int_0^t T(t-\tau)c(\tau)\,\mathrm{d}\tau. \tag{34}$$

It is enough to show that the right hand side defines a measure with null singular part. Indeed, by definition, the first two addenda are both positive functions in $L^2(0,S)$; in particular, the second lies in $L^2(0,S)$ as a consequence of Proposition 3.1 in Bensoussan et al. (2007) p. 212, as Hypothesis 3.1 there contained holds true: if $B_1\colon D' \to D'$, $B_1\phi := \langle \phi, \psi\rangle\,\delta_0$ then $B_1^*\colon D \to D$ is given by $B_1^* h := \langle \delta_0, h\rangle \psi$, and $|B_1^* T^*(\tau)h|_D \leq |h(\tau)|_{\mathbb{R}}\,|\psi|_D$, so that

$$\int_0^T |B_1^* T^*(\tau)h|_D^2\,\mathrm{d}\tau \leq |\psi|_D^2 \int_0^S |h(\tau)|_{\mathbb{R}}^2\,\mathrm{d}\tau = |\psi|_D^2 |h|_{L^2(0,S)}^2.$$

In addition, the term $-\int_0^t T(t-\tau)c(s)\,\mathrm{d}\tau$ is a negative distribution in D', it is a negative measure on $[0,S]$ by Proposition 2.3 page 270 of Hirsch and Lacombe (1999), and then the sum of an absolutely continuous and a singular part (w.r.t. the Lebesgue measure on $[0,S]$), both negative. Hence the singular part of right side of (34) is negative, whereas that on the left hand side is positive, hence the singular part on both is null. □

Proposition A.3 *Assume $x_0 \in \Pi$. Then the trajectory $x = x(\cdot; x_0, c)$ of* (31) *satisfies $\langle x(t), \psi\rangle \equiv \langle x_0, \psi\rangle$ for all $t \geq 0$, and for all $c \in \mathcal{U}_{x_0}$.*

Proof. Let first $c \in L^2_{loc}(0, +\infty; L^2(0, S))$, so that (7) implies

$$\langle x(t), \psi\rangle = \langle x_0, \psi\rangle - \int_0^{\bar{s}} \int_0^{t\wedge s} c(t-\tau, s-\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^{t\wedge s} \int_0^{\bar{s}} c(t-s, \tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

By a change of variables, the second and the third addenda are opposites, so that $\langle x(t), \psi\rangle = \langle x_0, \psi\rangle$. The claim for a general c in $\mathcal{U}_{x_0}$ follows by density and by continuity of the operator $\mathcal{S}$ defined in Proposition A.2. □

Proposition A.4 *Let x be the solution to (31) when $x_0 \in \Pi$, $c \in \mathcal{U}_{x_0}$. Let also $T > 0$ and p in D^2. Then $x(t)$ is a* weak solution *(31), that is*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \langle x(t), p\rangle = \langle x(t), A^* p\rangle - \langle c(t), p\rangle + \langle \delta_0, p\rangle \langle \psi, c(t)\rangle, & \forall t \in (0, T] \\ \langle x(0), p\rangle = \langle x_0, p\rangle = 1. \end{cases}$$

Proof. As suggested at p.204 in Bensoussan et al., 2007 Section II.3.1, it is enough to repeat the construction of the weak solution at p. 203, with $k \in D^2$. □

Remark A.5 The function p_ρ defined in (20) is in D^2.

A.3. **Proofs for Section 3.** For a selection M_ρ in $\mathcal{A}_\rho$, we define the *support function*

(35) $$h_\rho(s) = g_\rho(M_\rho)(e^{\rho s} - 1), \quad \text{when } \rho > 0, \quad h_0(s) = f(M_0){M_0}^{-1}.$$

Note that since $h_\rho(s) \geq f(s)$, for all s in $(0, S]$ and $h_\rho(M_\rho) = f(M_\rho)$, one has

(36) $$\mathcal{A}_\rho = \{s \ \in (0, S] : \ h_\rho(s) = f(s)\}.$$

Proof of Proposition 3.1. Note that $g_\rho(\cdot)$ continuous with compact support in $[\lambda, \bar{s}]$ implies $\mathcal{A}_\rho \neq \emptyset$, while (8) implies $0 \notin \mathcal{A}_\rho$ for any $\rho \geq 0$. Now let $M_{\rho_A} \in \mathcal{A}_{\rho_A}$ and $M_{\rho_B} \in \mathcal{A}_{\rho_B}$. A simple analysis shows that $\rho_B < \rho_A$ implies h_{ρ_A}definitively greater than h_{ρ_B}, and there exists $\widetilde{s} \in (0, +\infty)$ such that $h_{\rho_A}(\widetilde{s}) = h_{\rho_B}(\widetilde{s})$, $h_{\rho_A}(s) < h_{\rho_B}(s)$ for all $s \in (0, \widetilde{s})$, and $h_{\rho_A}(s) > h_{\rho_B}(s)$ for all $s \in (\widetilde{s}, +\infty)$. Indeed $(0, \tilde{s}] \ni M_{\rho_A}$, as g_{ρ_B} maximal at M_{ρ_B} implies $h_{\rho_A}(M_{\rho_A}) \leq h_{\rho_B}(M_{\rho_A})$. Moreover, by maximality of g_{ρ_A} at M_{ρ_A}, one derives $h_{\rho_A}(M_{\rho_B}) \geq h_{\rho_B}(M_{\rho_B})$, which implies $\widetilde{s} \leq S$ and $M_{\rho_B} \in [\widetilde{s}, S]$, and the first assertion in (i) is proved. If in addition f is differentiable at $\widetilde{s}$ and, by contradiction, $\mathcal{A}_{\rho_A} \cap \mathcal{A}_{\rho_B} = \{\widetilde{s}\}$, then (36) implies $f(\widetilde{s}) = h_{\rho_A}(\widetilde{s}) = h_{\rho_B}(\widetilde{s})$, and $h'_{\rho_A}(\widetilde{s}) > h'_{\rho_B}(\widetilde{s})$, against the fact that the graph of f lies underneath the graph of both support functions.

Next we prove (ii). The selection $\rho \mapsto M_\rho$ is nonincreasing as a direct consequence of (i). From $M_{\rho_A} \leq M_{\rho_B}$ and the convexity of $h_{\rho_B}(s)$ it follows $h_{\rho_B}(M_{\rho_B})/M_{\rho_B} \geq h_{\rho_B}(M_{\rho_A})/M_{\rho_A}$. Then (36) implies $f(M_{\rho_B}) = h_{\rho_B}(M_{\rho_B})$ and $h_{\rho_B}(M_{\rho_A}) \geq f(M_{\rho_A})$, and then $f(M_{\rho_B})/M_{\rho_B} \geq f(M_{\rho_A})/M_{\rho_A}$. The last claim in (ii) is a consequence of (i) and of countability of the discontinuities of a decreasing function. The limit m_0 in (iii) exists and is in $[0, S]$, as any selection M_ρ in $\mathcal{A}_\rho$ is nonincreasing and there contained. By continuity of f any $\mathcal{A}_\rho$ has a positive minimum, and $\mathcal{A}_\rho \cap \mathcal{A}_0$ contains at most one element, which implies $m_0 \leq \min \mathcal{A}_0$. Suppose by contradiction that $m_0 < \min \mathcal{A}_0$, then $h_0(s) - f(s)$ is always strictly positive on $[\lambda, m_0]$. Moreover, if we define $k_\rho(s) = f(\min \mathcal{A}_0)(e^{\rho s} - 1)/(e^{\rho \min \mathcal{A}_0} - 1)$, we may observe that: (i) $h_\rho(s) \geq k_\rho(s)$, and (ii) $k_\rho(s) - h_0(s)$ converges uniformly to 0 on $[\lambda, m_0]$, when $\rho \to 0$. Hence there exists $\hat{\rho}$ small enough such that, $h_{\hat{\rho}}(s) \geq k_{\hat{\rho}}(s) > f(s)$, for any $s \in [\lambda, m_0]$. This implies $\mathcal{A}_{\hat{\rho}} \subset (m_0, \min \mathcal{A}_0]$, a contradiction. □

Lemma A.6 *Let $a, b \in [0, S]$, $a \leq b$. Then*

$$T(t)\chi_{[a,b]} = \chi_{[a+t,(b+t)\wedge S]}, \quad \textit{and} \quad \int_0^t T(\tau)\delta_b \, \mathrm{d}\tau = \chi_{[b,(b+t)\wedge S]}. \tag{37}$$

Proof. The first assertion follows from

$$T(t)\chi_{[a,b]}(s) = \chi_{[a,b]}(s-t)\chi_{[t,S]}(s) = \chi_{[a+t,b+t]}(s)\chi_{[t,S]}(s) = \chi_{[a+t,(b+t)\wedge S]}(s).$$

For the second, note that, if ϕ is any test function in D, one has $\langle \delta_b, T^*(\tau)\phi \rangle = \phi(b+\tau)$ if $b+\tau \leq S$, and 0 otherwise, then by changing the variable in the integral one derives

$$\left\langle \int_0^t T(\tau)\delta_b \, \mathrm{d}\tau, \phi \right\rangle = \int_0^t \langle \delta_b, T^*(\tau)\phi \rangle \, \mathrm{d}\tau = \int_b^{(b+t)\wedge S} \phi(\sigma) \, \mathrm{d}\sigma = \langle \chi_{[b,(b+t)\wedge S]}, \phi \rangle. \quad \square$$

Lemma A.7 *Let $g \in L^2_{loc}(0, +\infty; \mathbb{R})$, and $a \in [0, S)$.*

$$\left(\int_0^t g(\tau) T(t-\tau)\delta_a \, \mathrm{d}\tau \right)(s) = \chi_{[a,t+a]}(s) g(t+a-s), \;\; \forall t \in [0, S-a], \; \forall s \in [0, S]. \tag{38}$$

Proof. The proof is follow by Lemma A.6 and by density of step-functions approximating g and satisfying (38). $\square$

Lemma A.8 *Assume* (8) (9)*, and let* $\rho \geq 0$*, and* x_ρ*,* c_ρ p_ρ *be defined by means of* (17) (18) (20)*. Let* $x_0 \in \Pi$*,* $c \in \mathcal{U}_{x_0}$*, and* $x(t) = x(t; x_0, c)$ *a solution of* (30)*. Then*

$$\langle c(t) - c_\rho, f - p\rangle \leq \rho \langle x(t) - x_\rho, p_\rho\rangle - \langle x(t) - x_\rho, A^* p_\rho\rangle, \quad \forall t \geq 0. \tag{39}$$

Proof. Proposition 3.1 implies there exists $M_\rho \in \mathcal{A}_\rho \neq \emptyset$, with $f(s) \leq p_\rho(s)$ for all $s \in [0, \bar{s}]$, and $f(M_\rho) = p_\rho(M_\rho)$. This implies $\langle f, c\rangle \leq \langle p_\rho, c\rangle$ for all positive $c \in D'$, with equality holding at $c = \gamma\delta_{M_\rho}$, with any $\gamma \geq 0$. In particular, for $\gamma = 1$, one has

$$\langle c(t), f - p_\rho\rangle \leq 0 = \langle c_\rho, f - p_\rho\rangle, \quad \forall t \geq 0 \tag{40}$$

By means of Proposition 2.3, $x(t)$ lies in $L^2(0, S)$ and $A^* p_\rho = {p_\rho}'$ so that

$$-\rho \langle x(t), p_\rho\rangle + \langle x(t), A^* p_\rho\rangle = -\rho \int_0^{\bar{s}} p_\rho(s) x(t, s)\, \mathrm{d}s + \int_0^{\bar{s}} {p_\rho}'(s) x(t, s)\, \mathrm{d}s =: \Delta(\rho). \tag{41}$$

If $\rho > 0$, then $supp(x(t)) \subset [0, \bar{s}]$, ${p_\rho}'(s) = \rho\eta_\rho e^{\rho s}$ on $[0, \bar{s}]$. Proposition A.3 gives

$$\Delta(\rho) = \rho\eta_\rho \int_0^{\bar{s}} x(t, s)\, \mathrm{d}s = \rho\eta_\rho \int_0^{\bar{s}} x_0(s)\, \mathrm{d}s = \rho\eta_\rho \tag{42}$$

that is, the quantity on the left of (41) hand side is constant for all admissible trajectories, and in particular for $x = x_\rho$. Then (40), (41) and (42) imply the claim. Similarly for $\rho = 0$, $p_0'(s) = \beta_0$ on $[0, \bar{s}]$, so that

$$\Delta(0) = \int_0^{\bar{s}} p_0'(s) x(t, s)\, \mathrm{d}s = \beta_0 \int_0^{\bar{s}} x(t, s)\, \mathrm{d}s = \beta_0,$$

which leads to the same conclusion. $\square$

Corollary A.9 *In the assumption of Lemma A.8, set* $\beta_\rho := f(M_\rho)/M_\rho$*, and* $\alpha_\rho := u'(\beta_\rho)$*. The value-loss function* (21) *satisfies* $\theta_\rho(c(t), x(t)) \geq 0$*, for all* $t \geq 0$*.*

Proof For all c in D', set $h\colon D' \to \mathbb{R}$, $h(c) := u(\langle c, f\rangle)$, so that $h(c_\rho) = u(\langle c_\rho, f\rangle) = u(\beta_\rho)$; h is differentiable with $h'(c) = u'(\langle c, f\rangle) f \in D$, $h'(c_\rho) = u'(\beta_\rho) f = \alpha_\rho f$. Since h is concave, $u(\langle c, f\rangle) - u(\beta_\rho) \leq \alpha_\rho \langle c - c_\rho, f\rangle$, for all $c \in D'$, and Lemma A.8 gives

$$u(\langle c(t), f\rangle) \leq u(\beta_\rho) + \alpha_\rho \big[\, \langle c(t) - c_\rho, p_\rho\rangle + \rho \langle x(t) - x_\rho, p_\rho\rangle - \langle x(t) - x_\rho, A^* p_\rho\rangle \,\big].$$

To complete the proof, we need to show that $-\langle c_\rho, p_\rho\rangle + \langle x_\rho, A^* p_\rho\rangle = 0$. For $\rho > 0$

$$-\langle c_\rho, p_\rho\rangle + \langle x_\rho, A^* p_\rho\rangle = -\frac{\eta_\rho}{M_\rho}\left(e^{\rho M_\rho} - 1\right) + \frac{\rho \eta_\rho}{M_\rho}\int_0^{M_\rho} e^{\rho s}\,\mathrm{d}s = 0,$$

while, for $\rho = 0$ and with $\bar{x}, \bar{c}$ given by (26), one has

$$\langle \bar{x}, A^* p_0\rangle = \beta_0 = \langle \bar{c}, p_0\rangle. \qquad \square \tag{43}$$

Remark A.10 The previous results remain true when c_ρ is replaced by any positive multiple $\gamma\delta_{M_\rho}$ of the Dirac's Delta at M_ρ. If moreover $\rho = 0$, (39) holds true for a general initial datum x_0 in place of x_ρ, as $\langle x, A^* p_0\rangle = \beta_0$ for all $x \in \Pi$. We summarize these facts in the following generalized version of Lemma A.8 for the case $\rho = 0$. $\square$

Corollary A.11 *Let $\rho = 0$, and $x(t) = x(t; x_0, c)$, with $x_0 \in \Pi$, $c \in \mathcal{U}_{x_0}$. Then*

$$\langle c(t) - \gamma\delta_{M_0}, f - p_0\rangle \leq -\langle x(t) - x_0, A^* p_0\rangle, \ \forall t \geq 0, \ \forall \gamma \geq 0.$$

Corollary A.12 *In the assumption of Corollary A.9*

$$U_T(c_\rho) - U_T(c) \geq \alpha_\rho \left(\langle x_\rho - x_0, p_\rho\rangle - e^{-\rho T}\langle x_\rho - x(T), p_\rho\rangle\right), \tag{44}$$

Proof. From Corollary A.9 $u(\beta_\rho) - u(\langle c(t), f\rangle)$ is greater than

$$\alpha_\rho\big[\rho\langle x_\rho - x(t), p_\rho\rangle - \langle c(t), p_\rho\rangle + \langle x(t), A^* p_\rho\rangle\big] = e^{\rho t}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x(t) - x_\rho, e^{-\rho t} p_\rho\right\rangle.$$

which promptly implies the thesis. $\square$

Lemma A.13 *Let $\rho > 0$, x_ρ, p_ρ defined by* (17) (20), *$x(t) = x(t; x_\rho, c)$, with $c \in \mathcal{U}_{x_\rho}$. Then*

$$\lim_{T\to+\infty}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left[\left\langle x_\rho - x(t), e^{-\rho t} p\right\rangle\right]\,\mathrm{d}t = 0.$$

Proof. Note that

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left[\left\langle x_\rho - x(t), e^{-\rho t} p\right\rangle\right]\,\mathrm{d}t = e^{-\rho T}\langle x_\rho - x(T), p_\rho\rangle.$$

Since $x(T)$ and x_ρ are supported in $[0, \bar{s}]$, and Proposition A.3 holds

$$\left|e^{-\rho T}\langle x_\rho - x(T), p_\rho\rangle\right| \leq e^{-\rho T}\int_0^S |x_\rho(s) - x(T, s)||p_\rho(s)|ds \leq 2e^{-\rho T}\eta_\rho\left(e^{\rho S} - 1\right).$$

The last expression converges to 0 as $T \to +\infty$ and then we get the claim. $\square$

Proof of Proposition 3.2. By definition of B one has

$$Bc_\rho = -\frac{1}{M_\rho}\delta_{M_\rho} + \langle \frac{1}{M_\rho}\delta_{M_\rho}, \psi\rangle \delta_0 = \frac{1}{M_\rho}\left(\delta_0 - \delta_{M_\rho}\right)$$

so that, by making use of (37) one obtains

$$T(t)x_\rho + \int_0^t T(t-\tau)Bc_\rho \,\mathrm{d}\tau = \frac{1}{M_\rho}\chi_{[t,(t+M_\rho)\wedge S]} + \frac{1}{M_\rho}\left(\chi_{[0,t\wedge S]} - \chi_{[M_\rho,(M_\rho+t)\wedge S]}\right) = x_\rho$$

which implies that (c_ρ, x_ρ) satisfies Definition A.1. □

Proof of Theorem 3.4. Let $c \in \mathcal{U}_{x_\rho}$, $x(t) = x(t; x_\rho, c)$, and U_T be given by (32).

$$(45) \quad U_T(c_\rho) - U_T(c) + \alpha_\rho \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left[\langle x_\rho - x(t), e^{-\rho t}p_\rho\rangle\right] \mathrm{d}t =$$

$$= \int_0^T e^{-\rho t}\left[u(\beta_\rho) - u\left(\langle c(t), f\rangle\right) + \alpha_\rho\left(\rho\,\langle x_\rho - x(t), p_\rho\rangle - \langle x(t), A^* p_\rho\rangle + \langle c(t), p_\rho\rangle\right)\right] \mathrm{d}t.$$

where the last equality follows from Proposition A.4 with $p = p_\rho$ and from $\langle p_\rho, \delta_0\rangle = 0$. By means of Corollary A.9 the right hand side is positive, so that Lemma A.13 implies

$$\liminf_{T\to+\infty} \left(U_T(c_\rho) - U_T(c)\right) \geq 0.$$

Now we assume $\mathcal{A}_\rho$ singleton and $(\tilde{x}, \tilde{c})$ optimal stationary couple, and we show that it necessarily coincides with (x_ρ, c_ρ). First we show that $supp(\tilde{c}) = \{M_\rho\}$, by separately proving $supp(\tilde{c}) \cap [0, M_\rho) = \emptyset$ and $supp(\tilde{c}) \cap (M_\rho, \bar{s}] = \emptyset$. Assume by contradiction that $supp(\tilde{c}) \cap [0, M_\rho) \neq \emptyset$ and define, for $\epsilon > 0$, the control c_ϵ, admissible at $\tilde{x}$, as

$$(46) \qquad c_\epsilon(t) \equiv (1-\epsilon)\chi_{[0,M_\rho)}\ \tilde{c} + \chi_{(M_\rho,\bar{s}]}\ \tilde{c} + \epsilon\delta_{M_\rho}\int_{(M_\rho-t)\vee 0}^{M_\rho} \tilde{c}(s)\,\mathrm{d}s$$

coinciding with $\tilde{c}$ when $\epsilon = 0$[5]. If we show that $U(c_\epsilon)$ is strictly increasing at $\epsilon = 0$, then $U(c_\epsilon) > U(\tilde{c})$ for a small ϵ, and $\tilde{c}$ is not optimal. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}U(c_\epsilon)\right|_{\epsilon=0} = u'\left(\langle \tilde{c}, f\rangle\right)\left(\frac{1}{\rho}\left\langle -\tilde{c}\chi_{[0,M_\rho]}, f\right\rangle + f(M_\rho)\int_0^{+\infty} e^{-\rho t}\int_{(M_\rho-t)\vee 0}^{M_\rho}\tilde{c}(s)ds\mathrm{d}t\right)$$

[5] Recall that $\tilde{x}$ is decreasing, then $\tilde{c} = -\partial x$ a positive Radon measure. Then those in (46) need be interpreted as Lebesgue-Stieltjes integrals (see *e.g.* Ash, 2000 Section 1.5 page 35), more precisely $\int_0^{M_\rho}\tilde{c}(s)\,\mathrm{d}s = \int_{M_\rho-t}^{M_\rho}\partial\tilde{x}(s)$ and $\int_0^t \tilde{c}(M_\rho - s)\,\mathrm{d}s = \int_0^{M_\rho}\partial\tilde{x}(s)$.

$$= \rho^{-1} u'\left(\langle \tilde{c}, f\rangle\right)\left(-\left\langle f, \tilde{c}\chi_{[0,M_\rho]}\right\rangle + f(M_\rho)e^{-\rho M_\rho}\int_0^{M_\rho} e^{\rho s}\tilde{c}(s)\,\mathrm{d}s\right)$$

$$= \rho^{-1} u'(\langle \tilde{c}, f\rangle)\left(\left\langle p_\rho - f, \tilde{c}\chi_{[0,M_\rho]}\right\rangle + e^{-\rho M_\rho}\left\langle \tilde{c}\chi_{[0,M_\rho]}, f(M_\rho)\psi - p_\rho\right\rangle\right).$$

where the first equality is obtained by separately integrating over $[0, M_\rho)$ and $[M_\rho, +\infty)$ in t and exchanging the order of integration, while the second by observing that $e^{\rho s} = (p_\rho(s)/\eta_\rho + 1)$ on $[0, M_\rho]$ and (20) holds. The last expression is strictly positive, in fact $u' > 0$, the first addendum in brackets is positive as $p_\rho \geq f$, and the second is strictly positive as $p_\rho(s) < f(M_\rho)\psi$ for all $s \in [0, M_\rho)$ and $supp(\tilde{c}) \cap [0, M_\rho) \neq \emptyset$. By a similar argument one may prove that, if $supp(\tilde{c}) \cap (M_\rho, \bar{s}] \neq \emptyset$, then $(\tilde{x}, \tilde{c})$ cannot be optimal as well, so that necessarily $supp(\tilde{c}) = \{M_\rho\}$. Since the only measures whose support is $\{M_\rho\}$ are of type $\tilde{c} = \gamma\delta_{M_\rho}$, for some real $\gamma \geq 0$, then $\tilde{x} \in \Pi$ if and only if $\gamma = 1/M_\rho$. Then $\tilde{c} = c_\rho$ and $\tilde{x} = x_\rho$. □

Proof of Remark 3.5. By linearity, $\tilde{x}$ satisfies Definition A.1 and hence is a stationary program. What is left to show is that $(\tilde{c}, \tilde{x})$ is optimal. Let $c \in \mathcal{U}_{\tilde{x}}$ and let $x(t) = x(t; \tilde{x}; c)$. Then by optimality of c_ρ^i, equation (45), and concavity of U_T

$$\liminf_{T\to+\infty} \left(U_T(\tilde{c}) - U_T(c)\right) \geq \sum_{i=1}^{n} \lambda_i \liminf_{T\to+\infty} \left(U(c_\rho^i) - U_T(c)\right) \geq 0.$$

Proof of Lemma 3.7. It is straightforward from definition that $\hat{x}$ is M_ρ-periodic. Then it is enough to show that $\hat{x}$ solves (24) for $t \in (0, M_\rho)$, more precisely

$$\langle \hat{x}(t) - T(t)x_0, \varphi\rangle = \left\langle \int_0^t T(t-\tau)B\hat{x}(\tau, M_\rho)\,\delta_{M_\rho}\mathrm{d}\tau, \varphi\right\rangle, \quad \forall \varphi \in D. \tag{47}$$

Note that $\langle T(t)x_0, \varphi\rangle = \int_t^S x_0(s-t)\varphi(s)\,\mathrm{d}s$ while

$$\langle \hat{x}(t), \varphi\rangle = \int_0^S \hat{x}(t,s)\ \varphi(s)\mathrm{d}s = \int_t^M x_0(s-t)\varphi(s)\,\mathrm{d}s + \int_0^t x_0(s+M-t)\varphi(s)\,\mathrm{d}s$$

so that the left hand side in (47) may be rewritten as follows

$$\langle \hat{x}(t) - T(t)x_0, \varphi\rangle = -\int_M^S x_0(s-t)\varphi(s)\,\mathrm{d}s + \int_0^t x_0(M-\tau)\varphi(t-\tau)\,\mathrm{d}\tau \tag{48}$$

On the other hand $\widehat{x}(t, M) = x_0(M-t)$, $\hat{c}(t) = x_0(M-t)\delta_{M_\rho}$, and $\left\langle \delta_{M_\rho}, \psi\right\rangle = \psi(M) = 1$ so that $B\hat{x}(\tau, M_\rho)\,\delta_{M_\rho} = \hat{x}(\tau, M_\rho)\,\delta_{M_\rho} + \left\langle \hat{x}(\tau, M_\rho)\,\delta_{M_\rho}, \psi\right\rangle \delta_0 = x_0(M-t)(\delta_0 - \delta_{M_\rho})$, and the right hand side in (47) is

$$\int_0^t \left\langle\, B\hat{x}(\tau, M_\rho)\,\delta_{M_\rho}, T^*(t-\tau)\varphi\right\rangle \mathrm{d}\tau = \int_0^t \; x_0(M-t)\left\langle(\delta_0 - \delta_{M_\rho}), T^*(t-\tau)\varphi\right\rangle \mathrm{d}\tau$$
$$= \int_0^t \; x_0(M-t)\left(\left[T^*(t-\tau)\varphi\right](0) - \left[T^*(t-\tau)\varphi\right](M_\rho)\right)\mathrm{d}\tau$$
$$= \int_0^t x_0(M-\tau)\varphi(t-\tau)\,\mathrm{d}\tau - \int_{M+t-S}^t x_0(M-\tau)\varphi(t-\tau+M)\,\mathrm{d}\tau$$

which is equal, by means of a change of variables, to the right hand side in (48). □

Proof of Lemma 3.8. Since, by Lemma 3.7, $\widehat{x}$ is M_ρ–periodic with $\widehat{x}(t, M_\rho) = x_0(M_\rho - \sigma(t))$, then $\langle \widehat{c}(t), f\rangle = f(M_\rho)x_0(M_\rho - \sigma(t))$. Hence

$$U_T(\widehat{c}) = \sum_{i=0}^{n-1} e^{-\rho i M_\rho} \int_0^{M_\rho} e^{-\rho t} u\left(f(M_\rho)x_0(M_\rho - t)\right) dt +$$
$$+ e^{-\rho n M_\rho} \int_0^{T-nM_\rho} e^{-\rho t}\, u\left(f(M_\rho)x_0(M_\rho - t)\right) dt \;=\; \frac{1-e^{-n\rho M_\rho}}{e^{\rho M_\rho}-1} U_1^\rho + e^{-\rho(n+1)M_\rho} U_2^\rho(T)$$

where $n \equiv [T/M_\rho]$. When $\rho = 0$, similarly, $U_T(\widehat{c})$ is given by

$$\sum_{i=0}^{n-1} \int_0^{M_\rho} u\left(\, f(M_\rho)x_0(M_\rho - t)\right) dt + \int_0^{T-nM_\rho} u\left(f(M_\rho)x_0(M_\rho - t)\right) dt = nU_1^0 + e^{-\rho M_\rho} U_2^0(T)$$

□

A.3.1. *Optimality and good controls.* In order to prove Proposition 3.11, we need some preliminary results, contained in the following lemmata.

Lemma A.14 *Assume* (8) *and* (9)*. Let* $T > 0$*. Then there exists a constant* $B_T > 0$ *(independent of* c *and* x_0*), such that* $U_{T_0}(c) \leq B_T$*, for all* $T_0 \leq T$*,* $x_0 \in \Pi$*,* $c \in \mathcal{U}_{x_0}$*.*

Proof. We prove the assertion for $\rho = 0$, as for $\rho > 0$ it holds *a fortiori*. Let $x_0 \in \Pi$, $c \in \mathcal{U}_{x_0}$ and $x(t) = x(t; x_0, c)$. Set $\varepsilon := \min\{\lambda, S - s_1\}$, where s_1 is that in the definition of cut-off function (see Subsection 2.1), and consider $\phi \in C^1([0,S];\mathbb{R})$ such that $\phi(s) = 1$ for all $s \in [0, \varepsilon/2]$, $\phi(s) = 0$ for all $s \in [\varepsilon, S]$, and $\phi'(s) \leq 0$ for all $s \in [0, S]$. Now $\psi \geq \phi$ and $\phi \in D$, so that Proposition A.3 and (31) imply

$$\langle T(\varepsilon/2)x_0, \phi\rangle + \int_0^{\varepsilon/2} \langle c(\tau), B^*T^*(\varepsilon/2 - \tau)\phi\rangle \;\mathrm{d}\tau = \langle x(\varepsilon/2), \phi\rangle \leq \langle x(\varepsilon/2), \psi\rangle = 1. \tag{49}$$

One defines $\phi_\tau(s) \equiv [T^*(\varepsilon/2-\tau)\phi](s) = \phi(s+\varepsilon/2-\tau)\chi_{[0,S-(\varepsilon/2-\tau)]}(s)$ for all $\tau \in [0,\varepsilon/2]$, so that $B^*T^*(\varepsilon/2-\tau)\phi = -\phi_\tau + \langle\delta_0,\phi_\tau\rangle\psi = -\phi_\tau+\psi$, which implies

$$\langle c(\tau), B^*T^*(\varepsilon/2-\tau)\phi\rangle = \langle c(\tau),\psi-\phi_\tau\rangle \geq \langle c(\tau),\psi-\phi\rangle, \quad \forall\tau\leq\varepsilon/2.$$

Since $\langle T(\varepsilon/2)x_0,\phi\rangle \geq 0$, the latter and (49) give $\int_0^{\varepsilon/2}\langle c(\tau),\psi-\phi\rangle \,\mathrm{d}\tau \leq 1$. Now the argument is iterated. A consequence of (31) is

$$x(t) = T(t-r)x(r) + \int_r^t T(t-\tau)Bc(\tau)\,\mathrm{d}\tau, \quad 0\leq r\leq\tau$$

which yields

$$\int_{n\varepsilon/2}^{(n+1)\varepsilon/2} \langle c(\tau),\psi-\phi\rangle \,\mathrm{d}\tau \leq 1$$

when applied with $t = r+\frac{\varepsilon}{2}$, $r = n\frac{\varepsilon}{2}$ and $n\in\{0,1,..,[2T/\varepsilon]\}$, so that

$$\int_0^T \langle c(t),\psi-\phi\rangle \,\mathrm{d}t \leq \sum_{n=0}^{[2T/\varepsilon]}\int_{n\frac{\varepsilon}{2}}^{(n+1)\frac{\varepsilon}{2}} \left\langle c(\tau),\chi_{[\lambda,\bar{s}]}\right\rangle \,\mathrm{d}\tau \leq \frac{2T}{\varepsilon}+1. \tag{50}$$

Since u is concave, there exist a and b in $\mathbb{R}$ such that $u(q)\leq a+bq$ for all $q\in\mathbb{R}^+$. Moreover chosen $b_1 \geq \max_{s\in[\lambda,\bar{s}]} f(s)$, on has $b_1(\psi-\phi)\geq f$ and

$$u(\langle c(t),f\rangle) \leq a+b\,\langle c(t),f\rangle \leq a+b_1 b\,\langle c(t),\psi-\phi\rangle$$

and hence by (50) there exists $B_T>0$ independent of (x_0,c) such that

$$U_T(c) = \int_0^T u(\langle c(t),f\rangle)\,\mathrm{d}t \leq aT + b_1 b\left(1+\frac{2T}{\varepsilon}\right) =: B_T.$$

Since (9) implies $U_T(c)$ increasing in T, one has $U_{T_0}(c)\leq U_T(c)\leq B_T$. □

Remark A.15 Note that for $\rho=0$ the value-loss function (21) satisfies

$$\theta(c) \equiv \theta_0(c,x) = u\left(\beta_0\right) - u\left(\langle c,f\rangle\right) + \alpha_0\left\langle c-\bar{c},p_0\right\rangle. \tag{51}$$

The concavity of u implies $\theta(c)\geq 0$, with $\theta(\bar{c})=0$. Although evaluated in (21) along the trajectories of the system, θ is a well defined real function on D'. □

Remark A.16 We say that the continuous function $\omega\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *local modulus* if for all $b > 0$, $\lim_{a \to 0^+} \omega(a; b) = 0$. Throughout this section we denote by $\omega(\cdot;\cdot)$ any local modulus, or by $\omega(\cdot)$, if there is no explicit dependence from a parameter b. Since (43) and $x \in \Pi$, imply $\alpha_0 \langle x, A^* p_0 \rangle = \alpha_0 \beta_0$, then (51) implies

$$\theta(c(t)) = u(\beta_0) - u(\langle c(t), f \rangle) + \alpha_0 \langle c(t), p_0 \rangle - \alpha_0 \langle x(t), A^* p_0 \rangle . \tag{52}$$

for $x_0 \in \Pi$ and $c \in \mathcal{U}_{x_0}$, and $x(t)$ the associated trajectory. It is also straightforward that $\theta\colon D' \to \mathbb{R}$ is a continuous function, indeed

$$|\theta(c) - \theta(c_1)| \le |u(\langle c, f \rangle) - u(\langle c_1, f \rangle)| + \alpha_0 |c - c_1|_{D'} |p_o|_D \le \omega_\theta(|c - c_1|_{D'})$$

for some modulus ω_θ, and for $c, c_1 \in D'$ (u is a uniformly continuous function). □

Lemma A.17 *Given a good control $c(\cdot) \in \mathcal{U}_{x_0}$ then it exists and is finite the limit*

$$L_c := \lim_{T \to \infty} \int_0^T \theta(c(t))\, \mathrm{d}t \in [0, +\infty).$$

Proof. As $\theta \ge 0$ implies $T \mapsto \Delta_T := \int_0^T \theta(c(t))\, \mathrm{d}t$ positive and increasing, then L_c exists in $[0, +\infty]$. Now $\frac{\mathrm{d}}{\mathrm{d}t}[\langle x(t), p_0 \rangle] = \langle c(t), p_0 \rangle - \langle x(t), A^* p_0 \rangle$, Proposition A.4 and (52) imply $\Delta_T = U_T(\bar{c}) - U_T(c) - \alpha_0 \langle x(T) - x_0, p_0 \rangle$. As c is good, there exists $\eta \in \mathbb{R}$ such that $U_T(\bar{c}) - U_T(c) \le \eta$ for all T . Moreover, $x(T) - x_0$ is in $L^2(0, S)$, and, since p_0 lies in $L^\infty(0, S)$ with $|p_0|_{L^\infty} \le \beta_0 S$, Hölder inequality implies $|\langle x(T) - x_0, p_0 \rangle| \le |p_0|_{L^\infty}(|x(T)|_{L^1} + |x_0|_{L^1}) = 2|p_0|_{L^\infty}$, so that $\Delta_T \le \eta + 2\alpha_0 \beta_0 S$. □

Remark A.18 As a consequence Lemma A.17, for any fixed $A \in (0, +\infty)$, we have

$$\int_{t-A}^{t} \theta(c(\tau))\, \mathrm{d}\tau \le \omega(1/t)$$

for a suitable modulus ω, that is, the integral is infinitesimal as t tends to $+\infty$. □

Lemma A.19 *For any given x_0 and x_1 in Π, there exists a control $\check{c}(\cdot) = \check{c}(\cdot; x_0, x_1, \bar{s})$ in $\mathcal{U}_{x_0}$, driving the system from x_0 to x_1 in a time length at most $\bar{s}$.*

Proof. We define $d^+(s) := (x_0(s) - x_1(s)) \vee 0$, and $d^-(s) := (x_1(s) - x_0(s)) \vee 0$, for $s \in [0, S]$, so that $d^+(s)$ (respectively, $d^-(s)$) is strictly positive at points where x_0 is strictly more (respectively, less) than x_1. Since $\int_0^S x_0(s)\,\mathrm{d}s = \int_0^S x_1(s)\,\mathrm{d}s = 1$, then

$$J := \int_0^S d^+(s)\,\mathrm{d}s = \int_0^S d^-(s)\,\mathrm{d}s = \int_0^{\bar{s}} d^+(s)\,\mathrm{d}s = \int_0^{\bar{s}} d^-(s)\,\mathrm{d}s.$$

If $J = 0$, then $x_0 = x_1$ and there is nothing to prove. Now assume $J > 0$. We define

$$e^-(t) := \int_{\bar{s}-t}^{\bar{s}} d^-(\tau)\,\mathrm{d}\tau, \quad t \in [0, \bar{s}]. \tag{53}$$

Note that e^- is increasing, with $e^-(0) = 0$, $e^-(\bar{s}) = J$. Set also

$$D^+(t,s) := d^+(s-t)\chi_{[t,\bar{s}]}(s) + d^+(s-t+\bar{s})_{[0,t)}(s), \quad t \in [0,\bar{s}], s \in [0,S]. \tag{54}$$

We show that the claim of the lemma is satisfied by the control $\check{c}$ in a time length $\bar{s}$

$$\check{c}(t,\cdot) = \left(x_0(\bar{s}-t) - \frac{d^+(\bar{s}-t)e^-(t)}{J} \right) \delta_{\bar{s}} + d^-(\bar{s}-t)\frac{D^+(t,\cdot)}{J}, \qquad t \in [0,\bar{s}].$$

Note that $x_0, x_1 \in L^2(0,S)$ imply $d^+, d^- \in L^2(0,S)$, $D^+(t) \in L^2(0,S)$ for any $t \in [0,\bar{s}]$, and $t \mapsto D(T)$ belongs to $C(0,\bar{s}; L^2(0,S))$. Moreover $e^- \in C(0,\bar{s})$, implies $\check{c} \in L^2(0,\bar{s}, D')$. Set $\check{x}(t) = \check{x}(t; x_0, \check{c})$ we need to show $\check{x}(\bar{s}, s) = x_1(s)$ for all s:

$$\check{x}(t) = I_1(t) + I_2(t) + I_3(t) \equiv \left[T(t)x_0 + \int_0^t T(t-\tau)Bx_0(t-\tau)\delta_{\bar{s}}\,\mathrm{d}\tau \right]$$
$$- \frac{1}{J}\left[\int_0^t d^+(\bar{s}-\tau)e^-(\tau)T(t-\tau)B\delta_{\bar{s}}\,\mathrm{d}\tau \right] + \frac{1}{J}\left[\int_0^t d^-(\bar{s}-\tau)T(t-\tau)BD^+(\tau)\,\mathrm{d}\tau \right].$$

From (23), one has $I_1(\bar{s}) = x_0$. Regarding $I_3(t)$, note that

$$T(t-\tau)BD^+(\tau) = T(t-\tau)\left\langle D^+(\tau), \psi \right\rangle \delta_0 - T(t-\tau)D^+(\tau) = J\delta_0 - T(t-\tau)D^+(\tau)$$

so that $I_3(t) = I_{31}(t) + I_{32}(t)$ with

$$I_{31}(t,s) = \int_0^t d^-(\bar{s}-\tau)T(t-\tau)\delta_0\,\mathrm{d}\tau = d^-(\bar{s}-t+s)\chi_{[0,t]}(s)$$

where the last equality is derived by means of (38), while

$$I_{32}(t) = -\int_0^t d^-(\bar{s}-\tau)T(t-\tau)\frac{D^+(\tau)}{J}\,\mathrm{d}\tau.$$

Now note that $T(t-\tau)D^+(\tau)(s) = D^+(\tau)(s-t+\tau)$ if $s-t+\tau \geq 0$ and 0 if $s-t+\tau < 0$, so that the last expression, evaluated at s, gives

$$\frac{1}{J}\int_{(t-s)\vee 0}^{t} d^-(\bar{s}-\tau)D^+(\tau)(s-t+\tau)\,\mathrm{d}\tau = -\frac{1}{J}\int_{(t-s)\vee 0}^{t\wedge(\bar{s}+t-s)} d^-(\bar{s}-\tau)D^+(\tau)(s-t+\tau)\,\mathrm{d}\tau.$$

By means of (53) (54) respectively, the latter is explicited and we find the following expression of $I_{32}(t)(s)$

$$\begin{cases} -\frac{1}{J}\int_{t-s}^{t} d^-(\bar{s}-\tau)d^+(s-t+\bar{s})\,\mathrm{d}\tau = -\frac{1}{J}d^+(s-t+\bar{s})\left(e^-(t)-e^-(t-s)\right),\ s\in[0,t) \\ -\frac{1}{J}\int_0^t d^-(\bar{s}-\tau)d^+(s-t)\,\mathrm{d}\tau = -\frac{1}{J}d^+(s-t)e^-(t),\ \ s\in(t,\bar{s}] \\ -\frac{1}{J}\int_0^{(\bar{s}+t-s)} d^-(\bar{s}-\tau)d^+(s-t)\,\mathrm{d}\tau = -\frac{1}{J}d^+(s-t)e^-(\bar{s}+t-s),\ \ s>\bar{s} \end{cases}$$

Regarding $I_2(t)$, since $-B\delta_{\bar{s}} = \delta_{\bar{s}} - \delta_0$, we may apply again (38) and derive

$$I_2(t,s) = J^{-1}\left[d^+(s-t)e^-(t+\bar{s}-s)\chi_{[\bar{s},\bar{s}+t)}(s) - d^+(\bar{s}+s-t)e^-(t-s)\chi_{[0,t)}(s)\right].$$

As a whole

$$\check{x}(t,s) = \begin{cases} x_0(s)+d^-(\bar{s}-t+s)-d^+(\bar{s}-t+s)e^-(t)J^{-1} & \text{if } \ s\in[0,t) \\ x_0(s)-d^+(s-t)e^-(t)J^{-1} & \text{if } \ s\in[t,\bar{s}] \\ 0 & \text{if } \ s>\bar{s}. \end{cases}$$

so that at all $t \geq 0$ one has $supp(x(t)) \in [0,\bar{s}]$. Moreover $e^-(\bar{s}) = J$ at $t = \bar{s}$ implies $\check{x}(\bar{s},s) = [x_0(s)+d^-(s)-d^+(s)]\,\chi_{[0,\bar{s}]}(s) = x_1(s)$. □

Proof of Proposition 3.11 Assume by contradiction that the maximal control c^* is not good, and denote by x^* the associated trajectory. Then, for any $\theta \in \mathbb{R}$, there exists $T_\theta \geq 0$ with $U_{T_\theta}(c^*) - U_{T_\theta}(\bar{c}) < -\theta$. Next we show that T_θ may be chosen arbitrarily large, for instance $T_\theta > 2\bar{s}$, if θ is chosen sufficiently large. Indeed by means of Lemma A.14 one has $\sup_{t\in[0,2\bar{s}]}|U_t(c^*)-U_t(\bar{c})| \leq B_{2\bar{s}}$ so that for $\theta > B_{2\bar{s}}$, we have $U_T(c^*) - U_T(\bar{c}) < -\theta$ only for values of T which are greater than $2\bar{s}$. We select $\theta > 2B_{\bar{s}} > B_{2\bar{s}}$ and $T_\theta > 2\bar{s}$ and define, with the notation of the previous lemma, $c_1(t) = c(t;x_0,\bar{x})$ and $c_2(t) = c(t;\bar{x},x^*(T_\theta))$, stirring respectively the system from x_0 to $\bar{x}$ and from $\bar{x}$ to $x^*(T)$ in time $\bar{s}$, moreover

$$\tilde{c}(t) = c_1(t)\chi_{[0,\bar{s})}(s) + \bar{c}\chi_{[\bar{s},T_\theta-\bar{s})}(s) + c_2(t)\chi_{[T_\theta-\bar{s},T_\theta)}(s) + c^*(t)\chi_{[T_\theta,+\infty)}(s)$$

We show that $\tilde{c}$ catches up to c^*, so that c^* cannot be maximal, yielding a contradiction. To do so it is enough to observe that, for any $T \geq T_\theta$

$$U_{T_\theta}(\tilde{c}) - U_{T_\theta}(c^*) = U_{\bar{s}}(\tilde{c}) - U_{\bar{s}}(\bar{c}) + U_{T_\theta}(\bar{c}) - U_{T_\theta}(c^*) + \int_{T-\bar{s}}^{T} [u(\langle \tilde{c}(t), f\rangle) - u(\langle \bar{c}, f\rangle)]\,\mathrm{d}t$$
$$\geq U_{\bar{s}}(\tilde{c}) - U_{\bar{s}}(\bar{c}) + \theta + U_{\bar{s}}(\tilde{c}(\cdot + T - \bar{s})) - U_{\bar{s}}(\langle \bar{c}, f\rangle) \geq \theta - 2B_{\bar{s}} > 0. \quad \square$$

A.4. **Proofs for Section 4.**

A.4.1. *Linear utility, positive discount.* **Proof of Theorem 4.1**. Assume $u(r) = r$ (the proof may be easily adapted to the case $u(r) = ar + b$). Note that $\widehat{c}(t)$ coincides with c_ρ when $x_0 = x_\rho$, so that (25) applies also with (x_ρ, c_ρ) in place of $(x^*, \widehat{c})$.
If $n = [T/M_\rho]$, $\sigma(T) = \{T/M_\rho\}M_\rho$, then $U_T(\widehat{c}) - U_T(c_\rho) =$

$$= \eta_\rho(1-e^{-\rho n M_\rho}) \int_0^{M_\rho} e^{\rho\tau}\,(x_0(\tau) - 1/M_\rho)\,\mathrm{d}\tau + e^{-\rho(n+1)M_\rho} \int_{M_\rho-\sigma(T)}^{M_\rho} e^{\rho\tau}\,(x_0(\tau) - 1/M_\rho)\,\mathrm{d}\tau.$$

Hence when $T \to +\infty$, and once set $\phi(t) = e^{\rho t}$, we derive

$$U(\widehat{c}) - U(c_\rho) = \lim_{T\to+\infty} \big(U_T(\widehat{c}) - U_T(c_\rho)\big) = \eta_\rho \langle x_0 - x_\rho, \phi\rangle = \langle x_0 - x_\rho, p_\rho\rangle. \tag{55}$$

Now let $c \in \mathcal{U}_{x_0}$, with $x(t) = x(t; c, x_0)$. Let $T > 0$, and note that Corollary A.12 implies ($\alpha_\rho = 1$ for $u(r) = r$) $U_T(c_\rho) - U_T(c) \geq -e^{-\rho T}\langle x_\rho - x(T), p_\rho\rangle + \langle x_\rho - x_0, p_\rho\rangle$. Coupling the previous relation with (55) we derive $U_T(\hat{c}) - U_T(c) \geq \omega(T)$ for a suitable function ω, $\omega(T) \to 0$ as $T \to +\infty$, which implies the thesis. $\square$

A.4.2. *Linear Utility, Null discount.* **Average of a trajectory**. Assume $x_0 \in \Pi$ and $c \in \mathcal{U}_{x_0}$. The *average* $x^A(t)$ of the trajectory $x(s; x_0, c)$ over a time interval $[0, t]$ is

$$x^A(t) := \frac{1}{t}\int_0^t x(s; x_0, c)\,\mathrm{d}s. \tag{56}$$

Lemma A.20 *Let $\rho = 0$, $c \in \mathcal{U}_{x_0} \cap L^\infty(0, +\infty; D')$ a good control, $x_0 \in \Pi$. Then*

$$\left\langle x^A(t), h\right\rangle \to \langle \bar{x}, h\rangle\,, \text{ as } t \to +\infty \qquad \text{for all } h \in D. \tag{57}$$

Proof. Apply Theorem 9.1.3 in Zaslavski (2006) to the modified objective functional $\tilde{U}_T(c) = \int_0^T u\left(\langle c(t), f\rangle\right) - u(\langle c_\rho, f\rangle)\,\mathrm{d}t$. $\square$

Proof of Theorem 4.3. We divide the long proof into several steps.

Claim 1: $\hat{c}$ is a maximal control. Let $\hat{x} = x(\cdot; x_0, \hat{c})$, and let $T > 0$. The control $\hat{c}$ is good, indeed by Lemma 3.8 with $\rho = 0$ and $u(r) = r$ applied both to $\hat{c}$ and $\bar{c}$ one has

$$U_T(\widehat{c}) - U_T(\overline{c}) = f(M) \int_{M-\sigma(T)}^{M} (x_0(\tau) - 1/M)\, d\tau \geq -f(M).$$

By contradiction, if $\hat{c}$ is not maximal there exists $\tilde{c}$ in $\mathcal{U}_{x_0}$ and $\hat{T}, a > 0$ such that

$$U_T(\hat{c}) - U_T(\tilde{c}) < -a, \quad \forall T \geq \hat{T}. \tag{58}$$

Now we assume $R \geq 6\hat{T}$, integrate on $[0, R]$ and divide by R, obtaining

$$\frac{1}{R} \int_0^{\hat{T}} (U_T(\hat{c}) - U_T(\tilde{c}))\ \mathrm{d}T + \frac{1}{R} \int_{\hat{T}}^{R} (U_T(\hat{c}) - U_T(\tilde{c}))\ \mathrm{d}T$$

and the first addendum converges to 0 for $R \to \infty$ while, for R large enough, the second is smaller than $-\frac{5}{6}a$ as (58) holds. Then for R big enough one has

$$\frac{1}{R} \int_0^{R} (U_T(\hat{c}) - U_T(\tilde{c}))\ \mathrm{d}T < -\frac{2}{3}a. \tag{59}$$

On the other hand, if $\tilde{x}(t) = x(t; x_0, \tilde{c})$, and $\tilde{x}^A(t)$ its average, Corollary A.12 implies

$$U_T(\hat{c}) - U_T(\tilde{c}) \geq \int_0^{T} \frac{\mathrm{d}}{\mathrm{d}t} \langle \tilde{x}(t) - \hat{x}(t), p_0 \rangle\ \mathrm{d}t = \langle \tilde{x}(T) - \hat{x}(T), p_0 \rangle \tag{60}$$

Integrating on $[0, R]$ and dividing by R one gets, for a sufficiently large R,

$$\frac{1}{R} \int_0^{R} (U_T(\hat{c}) - U_T(\tilde{c}))\ \mathrm{d}T \geq \left\langle \tilde{x}^A(R) - \hat{x}^A(R), p_0 \right\rangle \geq -\frac{1}{3}a$$

as $\left\langle \hat{x}^A(R) - \tilde{x}^A(R), p_0 \right\rangle \to 0$, as $R \to +\infty$, in view of (57): a contradiction.

Claim 2: the control $\hat{c}$ is not optimal. Assume c_1 defined in (28), and $x_1(t, s) \equiv x_1(t, s; c_1, x_0)$. Since f is continuous and $f(M) > 0$, we may choose N big enough so that $f(s_{N-1}) > 0$. Assume also $x_0 \in \Pi$ satisfies

$$\int_{s_{N-2}}^{s_{N-1}} x_0(r) dr > 0. \tag{61}$$

To prove $\hat{c}$ not optimal, it is sufficient to show that there exists $a > 0$ such that $U_{T_n}(\widehat{c}) - U_{T_n}(c_1) = -a$, $\forall T_n = M/N + nM$, with $n \in \mathbb{N}$. Note that for t in $[0, M/N]$

$$x_1(t,s) = \sum_{j=0}^{N} x_0(s_j + s - t)\chi_{[0,t]}(s) + x_0(s-t)\sum_{j=1}^{N} \chi_{[s_j, s_j+t]}(s)$$

while for $t > M/N$ the solution becomes M-periodic and repeatedly equal to

$$x_1(t,s) = \begin{cases} \chi_{\left[t-\frac{M}{N},t\right]}(s)\sum_{j=1}^{N} x_0(s_{j-1} + s + \frac{M}{N} - t) & t \in \left[\frac{M}{N}, M\right] \\ \chi_{[0,t-M]}(s)\sum_{j=1}^{N} x_0(s_{j-1} + s + \frac{N+1}{N}M - t) + & \\ \quad +\chi_{\left[t-\frac{M}{N},M\right]}(s)\sum_{j=1}^{N} x_0(s_{j-1} + s + \frac{M}{N} - t) & t \in \left[M, M + \frac{M}{N}\right] \end{cases}$$

(the general formula is obtained by replacing t with $\xi(t) = t - [M/N]\, M - M/N$ in the right hand side). Since $t \in [0, M/N]$ implies $x_1(t, s_j) = x_0(s_j - t)$, and $\langle \delta_{s_j}, f \rangle = f(s_j)$,

$$U_{\frac{M}{N}}(c_1) = \sum_{j=1}^{N} f(s_j)\int_0^{\frac{M}{N}} x_0(s_j - t)dt = \sum_{j=1}^{N} f(s_j)\int_{s_{j-1}}^{s_j} x_0(r)dr.$$

Moreover, the facts that x_1 M-periodic for $t \geq M/N$, $x_1(t, M) = 0$ for $M/N < t < M$, and $x_1(t, M) = \sum_{j=1}^{N} x_0(s_{j-1} + M + \frac{M}{N} - t)$ for $M < t < M + M/N$ imply

$$U_{T_n}(c_1) - U_{\frac{M}{N}}(c_1) = n\, f(M)\int_{s_{j-1}}^{s_j} x_0(r)dr = nf(M).$$

Hence if $\widehat{c}$ is the Faustmann policy

$$U_{T_n}(\widehat{c}) - U_{T_n}(c_1) = -\sum_{j=1}^{N-1} f(s_j)\int_{s_{j-1}}^{s_j} x_0(r)dr =: -a \tag{62}$$

Note that (61) implies $a > 0$. As a consequence, $\widehat{c}$ is not optimal. The proof for the case when (61) is not satisfied is easily obtained by applying a control c_2 in place of c_1 shaped as follows: if $m := \max\{j : 1 \leq j \leq N-1,\ \int_{s_{j-1}}^{s_j} x_0(r)dr > 0\}$ (a maximum exists as the forest has positive density and extension 1), and $\tau := \frac{M}{N}(N-1-m)$, then $c_2(t) := \bar{c}\, \chi_{[0,\tau]}(t) + c_1(t-\tau)\chi_{[\tau,+\infty)}(t)$, that is, c_2 coincides with $\bar{c}$ until the associated trajectory x_2 yields a positive integral on $[s_{N-2}, s_{N-1}]$, and with c_1 afterwards.

Claim 3: an optimal control does not exist. We assume by contradiction that $\tilde{c}(t)$ in U_{x_0} is optimal. Then in particular, given any $\varepsilon > 0$, there exists T_ε such that

$$U_T(\tilde{c}) - U_T(\widehat{c}) \geq -\varepsilon \text{ and } U_T(\tilde{c}) - U_T(c_1) \geq -\varepsilon \qquad \forall T \geq T_\varepsilon. \tag{63}$$

On the other hand (62) implies , for a sufficiently small $\nu \in [0, M]$ not depending on n, that $U_T(c_1) - U_T(\widehat{c}) \geq \frac{a}{2}$ for all $T \in [T_n, T_n + \nu]$ from which, if $n_\varepsilon \in N$ is such that $T_n > T_\varepsilon$ for all $n \geq n_\varepsilon$, we derive also

$$U_T(\tilde{c}) - U_T(\widehat{c}) \geq \frac{a}{2} - \varepsilon, \quad \forall T \in [T_n, T_n + \nu], \ \forall n \geq n_\varepsilon. \tag{64}$$

We show first that

$$\liminf_{n \to +\infty} \frac{1}{T_n + \nu} \int_0^{T_n + \nu} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT \geq \frac{\nu a}{4}. \tag{65}$$

Set $\int_0^{T_n+\nu} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT \equiv A + B_n + C_n$, where $A \equiv \int_0^{T_{n_\varepsilon}} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT$,

$$B_n \equiv \sum_{i=n_\varepsilon}^{n-1} \int_{T_i+\nu}^{T_{i+1}} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT \geq -\varepsilon\, \nu\, (n - n_\varepsilon)$$

in view of (63), while (64) implies

$$C_n \equiv \sum_{i=n_\varepsilon}^{n} \int_{T_i}^{T_i+\nu} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT \geq \left(\frac{a}{2} - \varepsilon\right) \nu\, (n - n_\varepsilon + 1)$$

so that (recall that $T_n = nM + M/N$), if $\omega(1/n)$ is infinitesimal as $n \to \infty$, we have

$$\frac{1}{T_n + \nu} \int_0^{T_n+\nu} \left(U_T(\tilde{c}) - U_T(\hat{c})\right) dT \geq \frac{\nu}{M} \left(\frac{a}{2} - 2\varepsilon\right) + \omega(1/n).$$

Choosing $\varepsilon \leq a(2 - M/2)$, and passing to limits we obtain (65). On the other hand, Lemma A.11 with $\gamma = \widehat{x}(t, M)$, Proposition A.4, and Remark A.10 imply

$$\langle \tilde{c}(t) - \widehat{c}(t), f \rangle \leq \langle \tilde{c}(t) - \widehat{c}(t), p_0 \rangle - \langle \tilde{x}(t) - x_0, A^* p_0 \rangle = -\frac{d}{dt} \langle \tilde{x}(t) - \widehat{x}(t), p_0 \rangle$$

Then, for all T, integrating on $[0, T]$ one derives

$$U_T(\widetilde{c}) - U_T(\widehat{c}) \leq -\int_0^T \frac{d}{dt} \langle \widehat{x}(t) - \tilde{x}(t), p_0 \rangle\, dt = \langle \tilde{x}(T) - \widehat{x}(T), p_0 \rangle$$

If $x^A(t)$ is defined by (56), we integrate in $[0, S]$ both sides and divide by S obtaining

$$\frac{1}{S} \int_0^S \left(U_T(\widetilde{c}) - U_T(\widehat{c})\right) dT \leq \left\langle \tilde{x}^A(S) - \widehat{x}^A(S), p_0 \right\rangle \xrightarrow{S \to \infty} 0 \tag{66}$$

as $\widehat{c}$ and $\tilde{c}$ are good and Lemma A.20 holds. That and (66), contradict (65). $\square$

A.4.3. *Strictly concave utility, null discount.*

Proof of Theorem 4.6. Set $H := L^2(0,S)$, so that $D \hookrightarrow H$ (with continuous inclusion). Since $c(t) \in L^\infty(0,+\infty;\mathcal{R})$ we have that $K := \sup |x(t)|_{\mathcal{R}} < +\infty$. Let $\varepsilon > 0$ be fixed. We have to prove that there exists $t(\varepsilon) > 0$, such that

$$i(t) := |x(t) - \bar{x}|_H \leq \varepsilon, \quad \text{for all } t \geq t(\varepsilon). \tag{67}$$

Any $c \in \mathcal{R}$ may be decomposed as $c = c_n + c_f$, with c_n, c_f defined as follows. Since M maximum of $f(s)/s$ implies $p_0(s) - f(s) \geq 0$ for all $s \in [0,S]$ with equality holding at $s = 0$ and $s = M$, and f continuous, then for a sufficiently small $\xi > 0$, there exists a smallest $\zeta(\xi) > 0$, with $\lambda < M - \zeta(\xi)$, such that $|s - M| \geq \zeta(\xi)$ implies $p(s) - f(s) \geq \xi$. Note that $\zeta(\xi) \xrightarrow{\xi \to 0} 0$. Then, since c is a positive measure with $supp(c) \subseteq [\lambda, S]$, one may set $c_n = c\nu_\xi$ and $c_f = c(1 - \nu_\xi)$, where ν_ξ is a $[0,1]$-valued smooth cut-off function with $\nu_\xi(s) \equiv 1$ for $|s - M| \leq \zeta(\xi)/2$ and $\nu_\xi(s) \equiv 0$ when $|s - M| \geq \zeta(\xi)$. Now, we may assume $t > S$. Hence, in (31) we have $T(t)x_0(s) = 0$ for all $s \in [0,S]$, and $T(t-\tau)Bc(\tau) = 0$ for all $\tau \leq t - S$, so that in (67)

$$x(t) - \bar{x} = \int_{t-S}^{t} T(t-\tau)B\,(c(\tau) - \bar{c})\;\mathrm{d}\tau = I_1(t,\xi) + I_2(t,\xi) + I_3(t,\xi), \tag{68}$$

where $I_1(t,\xi) := \int_{t-S}^t T(t-\tau)Bc_f(\tau)\,\mathrm{d}\tau$, $I_2(t,\xi) := \int_{t-S}^t T(t-\tau)B(c_n(\tau) - |c_n(\tau)|_{\mathcal{R}}\delta_M)\,\mathrm{d}\tau$, and $I_3(t,\xi) := \int_{t-S}^t T(t-\tau)B(|c_n(\tau)|_{\mathcal{R}}\delta_M - \bar{c})\,\mathrm{d}\tau$. We estimate H-norms of I_1, I_2 and I_3.

Step 1: A preliminary estimate. We first estimate $|c_f|_{D'}$. Given $x \in \Pi$, and $c \in \mathcal{R}$, we use Remark A.16, and $u(\langle \bar{c}, f\rangle) - u(\langle c, f\rangle) \geq -\alpha_0 \langle c - \bar{c}, f\rangle$ to derive

$$\theta(c) = \theta(c) - \theta(\bar{c}) \geq \alpha_0 \langle c - \bar{c}, p_0 - f\rangle = \alpha_0 \langle c, p_0 - f\rangle \geq \alpha_0 \xi |c_f|_{\mathcal{R}},$$

as c is positive and $\langle \bar{c}, p_0 - f\rangle = 0$. Observe that $D \hookrightarrow C^0([0,S])$ with continuous inclusion so that $\mathcal{R} \hookrightarrow D'$, in particular $|c_f|_{D'} \leq C|c_f|_{\mathcal{R}}$ for a constant $C > 0$. Then

$$|c_f|_{D'} \leq C|c_f|_{\mathcal{R}} \leq \frac{C}{\alpha_0 \xi}\theta(c),\ \forall x \in \Pi,\ \forall c \in \mathcal{R}. \tag{69}$$

Step 2: Estimate on $I_1(t,\xi)$. Note that $\|T(t)\|_{L(D')} \le 1$, so that (69) implies

$$|I_1(t,\xi)|_H \le \|B\|_{L(D')} \int_{t-S}^{t} |c_f(\tau)|_{D'}\,\mathrm{d}\tau \le \frac{C\|B\|_{L(D')}}{\alpha_0 \xi} \int_{t-S}^{t} \theta(c(\tau))\,\mathrm{d}\tau \le \omega_1(1/t;\xi)$$

for a modulus ω_1 (see Remark A.18).

Step 3: Estimate on I_2. Given $\phi \in D$ one has

$$\max_{|s-M|\le\zeta(\xi)} |\phi(s)-\phi(M)| \le \int_{M-\zeta(\xi)}^{M+\zeta(\xi)} |\phi'(s)|\,\mathrm{d}s \le \sqrt{2\zeta(\xi)}\,|\phi'|_H \le \sqrt{2\zeta(\xi)}\,|\phi|_D.$$

Then for $c \in \mathcal{R}$ one has $|\langle c - |c|_{\mathcal{R}}\delta_M, \phi\rangle| \le \sqrt{2\zeta(\xi)}\,|c|_{\mathcal{R}}|\phi|_D$ which implies

$$|c_n(\tau) - |c_n(\tau)|_{\mathcal{R}}\delta_M|_{D'} \le \sqrt{2\zeta(\xi)}\,|c_n(\tau)|_{\mathcal{R}} \le \sqrt{2\zeta(\xi)}\,K, \quad \tau\in[t-S,t] \tag{70}$$

and then $|c_n(\cdot) - |c_n(\cdot)|\delta_M|_{L^2(t-S,t;D')} \le \sqrt{S}\sqrt{2\zeta(\xi)}\,K$. By means of Proposition 3.1 p. 212 in Bensoussan et al. (2007), there exists $C>0$ independent from c such that

$$|I_2(t,\xi)|_H \le C|c_n(\cdot) - |c_n(\cdot)|_R\delta_M|_{L_2(tS,t;D')} \le C\sqrt{2\zeta(\xi)}\,\sqrt{S}K \le \omega_2(\xi)$$

for a modulus ω_2, with $\omega_2(\xi) \xrightarrow{\xi\to 0} 0$.

Step 4: Estimate on $I_3(t,\xi)$. Since u is strictly concave and differentiable, u' is strictly decreasing, $\alpha_0 = u'(\langle \bar{c}, p_0\rangle)$, $\beta_0 = \langle \bar{c}, p_0\rangle$, one defines $\beta_\eta = (1+\eta)\langle \bar{c}, p_0\rangle = (1+\eta)\beta_0$. Then there exist $\gamma>0$ and $0<\eta<1$ such that $\gamma = u(\beta_0) - u(\beta_\eta) + \alpha_0\eta\beta_0 > 0$, and $\Delta = -[u'(\beta_\eta) - \alpha_0] < 0$. Note that γ as a function of η is strictly increasing and attains the value zero at $\eta=0$, so that its inverse $\eta(\gamma)$ is well defined and enjoys the same property, in particular $\eta(\gamma) \xrightarrow{\gamma\to 0} 0$. As a consequence, Δ may itself be regarded as a function of γ, with $\Delta(\gamma) \xrightarrow{\gamma\to 0} 0$. We set $I_3 \equiv I_{31} + I_{32} + I_{33} + I_{34}$ with

$$I_{31}(t,\xi) = \int_{t-S}^{t} \left(|c_n(\tau)|_{\mathcal{R}} - \frac{1}{M} - \eta - \frac{\theta(|c_n(\tau)|_{\mathcal{R}}\delta_M)}{\Delta} \right) T(t-\tau)B\delta_M\,\mathrm{d}\tau$$

$$I_{32}(t,\xi) = \int_{t-S}^{t} \frac{\theta(|c_n(\tau)|_{\mathcal{R}}\delta_M) - \theta(c(\tau))}{\Delta} T(t-\tau)B\delta_M\,\mathrm{d}\tau$$

$$I_{33}(t,\xi) = \int_{t-S}^{t} \frac{\theta(c(\tau))}{\Delta} T(t-\tau)B\delta_M\,\mathrm{d}\tau, \quad I_{34}(t,\xi) = \eta\int_{t-S}^{t} T(t-\tau)B\delta_M\,\mathrm{d}\tau.$$

To estimate $I_{34}(t,\xi)$ it suffices to observe that for every fixed γ,

$$|I_{34}(t,\xi)|_H \le \|B\||\delta_M|_{D'}S\eta =: \omega_{34}(\gamma;\xi) \tag{71}$$

where ω_{34} is a local modulus. Next, Remark A.18 implies

$$|I_{33}(t,\xi)|_H \le \|B\||\delta_M|_{D'} \int_{t-S}^{t} \left|\frac{\theta(c(\tau))}{\Delta}\right| \mathrm{d}\tau \le \omega_{33}(1/t;\gamma,\xi) \tag{72}$$

for some local modulus ω_{33}. Then, Remark A.16 implies

$$|\theta(c(\tau)) - \theta(c_n(\tau))| \le \omega_\theta(|c_f|_{D'}) \le \omega_\theta\left((C/\alpha_0\xi)\theta(c(\tau))\right).$$

Thus, by Remark Remark A.18

$$\int_{t-S}^{t} \frac{|\theta(c(\tau)) - \theta(c_n(\tau))|}{\Delta} \mathrm{d}\tau \le \hat{\omega}(1/t;\gamma,\xi)$$

for some local modulus $\hat{\omega}$. Moreover, in view of (70), one has

$$|\theta(|c_n(\tau)|_{\mathcal{R}}\delta_M) - \theta(c_n(\tau))| \le \omega_\theta(||c_n(\tau)|_{\mathcal{R}}\delta_M - c_n(\tau)|_{D'}) \le \omega_\theta(\sqrt{2\zeta(\xi)}K)$$

so that

$$\int_{t-S}^{t} \frac{|\theta(|c_n(\tau)|_{\mathcal{R}}\delta_M) - \theta(c_n(\tau))|}{\Delta} \mathrm{d}\tau \le \check{\omega}(\gamma;\xi),$$

for some modulus $\check{\omega}$. Hence, once set $\omega_{32} = \|B\||\delta_M|_{D'}(\hat{\omega} + \check{\omega})$, one derives

$$I_{32}(t,\xi) \le \omega_{32}(1/t;\gamma,\xi). \tag{73}$$

Next we estimate I_{31}. By definition of $\theta(c)$ and concavity of u

$$-\theta(c) + \alpha_0 \langle c - \bar{c}, p_0\rangle + \gamma - \alpha_0\eta \langle \bar{c}, p_0\rangle = u(\beta_0) - u(\beta_\eta) \le u'(\beta_\eta) \langle c - (1+\eta)\bar{c}, f\rangle.$$

Recalling that $\langle \bar{c}, p_0\rangle = \langle \bar{c}, f\rangle = \beta_0$, that implies

$$\theta(c) \ge \gamma + \alpha_0 \langle c, p_0\rangle - \alpha_0\beta_0 - \alpha_0\eta\beta_0 - u'(\beta_\eta)\langle f, c\rangle + u'(\beta_\eta)(1+\eta)\beta_0$$

and then $\theta(c) \ge \gamma - \eta\alpha_0\Delta + \langle c - \bar{c}, \alpha_0 p - u'(\beta_\eta) f\rangle$. Then for $c \equiv |c_n(t)|_{\mathcal{R}}\delta_M$

$$\varphi(\tau) := |c_n(t)|_{\mathcal{R}} - \frac{1}{M} - \eta - \frac{\theta(|c_n(t)|_{\mathcal{R}}\delta_M)}{\Delta} \le 0.$$

Now note that, as a consequence of (68), step 2 and step 3, (71) (72) (73), Hölder inequality and the fact that $\langle x(t) - \bar{x}, \psi\rangle = 0$, one has $|\langle x(t) - \bar{x} - I_{31}(t,\xi), \psi\rangle| =$

$$= \left|\int_{t-S}^{t} \varphi(\tau) \langle T(t-\tau)B\delta_M, \psi\rangle \mathrm{d}s\right| + |\langle I_1 + I_2 + I_{32} + I_{33} + I_{34}, \psi\rangle| \le \omega_{31}(1/t;\gamma,\xi),$$

with $\omega_{31} = \sqrt{S}(\omega_1 + \omega_2 + \omega_{32} + \omega_{33} + \omega_{34})$. Now since

$$\langle T(t-\tau)B\delta_M, \psi\rangle = \langle \delta_0 - \delta_M, T^*(t-\tau)\psi\rangle = \psi(t-\tau) - \psi(t-\tau+M)$$

the previous estimate may be rewritten as

$$\left| \int_{t-S}^{t} \varphi(\tau)(\psi(t-\tau) - \psi(t-\tau+M))\, \mathrm{d}\tau \right| \le \omega_4(1/t; \gamma, \xi)$$

for a modulus ω_4. By definition of ψ we have $0 \le \psi(t-\tau) - \psi(t-\tau+M) \le 1$, and $\varphi(\tau) \le 0$, so that the integrand of the last equation is negative. Since by definition of ψ one may show there exists $[t_1, t_2] \subseteq [0, S]$ such that $\psi(t-\tau) - \psi(t-\tau+M) \ge c$, for a suitable $c > 0$, then

$$c \int_{t_1}^{t_2} |\varphi(t-\sigma)|\, \mathrm{d}\sigma \le \int_{t_1}^{t_2} |\varphi(t-\sigma)|\, |(\psi(\sigma) - \psi(\sigma+M))|\, \mathrm{d}\sigma \le \omega_4(1/t; \gamma, \xi). \tag{74}$$

Since (74) is true for all t, we iterate the argument $[S/(t_2 - t_1)] + 1$ times to obtain

$$\int_0^S |\varphi(t-\sigma)|\, \mathrm{d}\sigma \le \frac{1}{c}\frac{S}{(t_2 - t_1)}\omega_4(1/t; \gamma, \xi) = \omega_5(1/t; \gamma, \xi).$$

Thus $|I_{31}(t, \xi)|_H \le \|B\||\delta_M|_{D'}\omega_4(1/t; \gamma, \xi)$. To draw the conclusion, one uses (68), steps 2, 3 and 4, and chooses in order, ξ, γ sufficiently small and $t(\varepsilon)$ sufficiently large, so to derive that $t \ge t(\varepsilon)$ implies $i(t) \le \varepsilon$. □

Proof of Theorem 4.7. For any good control c, and for $x(t) = x(t; \bar{x}, c)$

$$\lim_{T\to+\infty} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} [\langle \bar{x} - x(t), \alpha_0 p_0\rangle]\, \mathrm{d}t = \lim_{T\to+\infty} \langle \bar{x} - x(T), \alpha_0 p_0\rangle = 0.$$

as a consequence of Theorem 4.6, so that Corollary A.9 implies

$$\liminf_{T\to\infty} \big(U_T(\bar{c}) - U_T(c)\big) = \liminf_{T\to\infty} \int_0^T \Big(U_T(\bar{c}) - U_T(c) + \frac{\mathrm{d}}{\mathrm{d}t} [\langle \bar{x} - x_{\bar{x},c}(t), \alpha_0 p_0\rangle]\Big)\, \mathrm{d}t$$

$$= \liminf_{T\to\infty} \int_0^T [u(\langle f, \bar{c}\rangle) - u(\langle f, c(t)\rangle) - \alpha_0 \langle x(t), A^* p_0\rangle + \langle c(t), p_0\rangle]\, u dt \ge 0$$

Then the proof is complete in view of Proposition 3.11. □

Proof of Theorem 4.9 We build the candidate optimal control $\tilde{c}$ as limit of a suitable sequence. Set

$$S \equiv \sup_{c\in\mathcal{U}_{x_0}^{K,\lambda}} (\lim \sup_{T\to+\infty} [U_T(c) - U_T(\bar{c})]), \quad (S \text{ possibly equałto } +\infty).$$

Let $\{c_n\}$ be a maximizing sequence in $\mathcal{U}_{x_0}^{K,\lambda}$, and let θ be defined by (51). Then we can express $U_T(c_n) - U_T(\bar{c})$ as

$$(75) \quad -\int_0^T \left(\theta(c_n(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_0\langle x_n(t), p_0\rangle\right)\mathrm{d}t = -\int_0^T \theta(c_n(t))\,\mathrm{d}t - \alpha_0\langle x_n(T) - x_0, p_0\rangle$$

for all $T > 0$. Since $|p_0|_\infty < +\infty$ and $|x_n(t)|_{L^1} = |x_0|_{L^1} = 1$, then $|\alpha_0\,\langle x_n(t) - x_0, p_0\rangle\,| \le 2\alpha_0|p_0|_\infty$, so that, being $\theta(c_n(t))$ positive for all t, it may happen either (a) $\lim_{T\to+\infty}(U_T(c_n) - U_T(\bar{c})) = -\infty$, ruled out as $\{c_n\}$ is a maximizing sequence, or (b) $\liminf_{T\to+\infty}(U_T(c_n) - U_T(\bar{c})) > -\infty$, the latter implying c_n is a good control. From (75) and the positivity of θ follows also $U_T(c_n) - U_T(\bar{c}) \le 2\alpha_0|p_0|_\infty$, implying $S < +\infty$. Hence with no loss of generality, we may assume that c_n are good controls. For any good control $c \in \mathcal{U}_{x_0}^{K,\lambda}$, Lemma A.17 and Theorem 4.6 imply the following exists and is finite

$$\lim_{T\to+\infty}(U_T(c) - U_T(\bar{c})) = \lim_{T\to+\infty}\int_0^T - \left(\theta(c(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_0\,\langle x(t), p_0\rangle\right)\mathrm{d}t = -L_c - \alpha_0\langle\bar{x} - x_0, p_0\rangle,$$

so that $S = \lim_{n\to\infty}\lim_{T\to+\infty}[U_T(c_n) - U_T(\bar{c})]$. Now, set $h > 0$ and $L_h^2([0,+\infty); D')$ the Hilbert space of all functions $\phi\colon [0,+\infty) \to D'$ such that the norm $\int_0^{+\infty} e^{-ht}|\phi(t)|_{D'}^2\,\mathrm{d}t < +\infty$. It may be shown $\mathcal{U}_{x_0}^{K,\lambda}$ is a sequentially weakly compact subset of $L_h^2([0,+\infty); D')$, then $\{c_n(\cdot)\}$ has a subsequence weakly converging to some $\tilde{c}(\cdot) \in L_h^2([0,+\infty); D')$ and $\tilde{c}(\cdot) \in \mathcal{U}_{x_0}^{K,\lambda}$. Since

$$(76) \qquad \liminf_{T\to+\infty}(U_T(\tilde{c}) - U_T(c)) \ge \liminf_{T\to+\infty}[U_T(\tilde{c}) - U_T(\bar{c})] - S.$$

$\tilde{c}$ is optimal if the right hand side in (76) is positive (or null). We first prove that $\limsup_{T\to+\infty}[U_T(\tilde{c}) - U_T(\bar{c})] = S$. Indeed by definition of S, the inequality $\ge$ holds, while the reverse is obtained by observing that $c \mapsto \limsup_{T\to+\infty}[U_T(c) - U_T(\bar{c})]$ is a concave functional on the convex subset $\mathcal{U}_{x_0}^{K,\lambda}$ of $L_h^2([0,+\infty); D')$, so that passing to limits one obtains $\lim_n \lim_{T\to+\infty}[U_T(c_n) - U_T(\bar{c})] \le S$. Now note that $\{c_n\}$ maximizing for the limsup, implies $\{c_n\}$ is maximizing also for the liminf, more precisely

$$\sup_{c\in\mathcal{U}_{x_0}^{K,\lambda}}\left[\liminf_{T\to+\infty}(U_T(c) - U_T(\bar{c}))\right] = \sup_{c\in\mathcal{U}_{x_0}^{K,\lambda}}\left[\limsup_{T\to+\infty}(U_T(c) - U_T(\bar{c}))\right] = S.$$

Arguing as before about concavity of $c \mapsto \liminf_{T\to+\infty}[U_T(c) - U_T(\bar{c})]$, one derives $\liminf_{T\to+\infty} [U_T(\tilde{c}) - U_T(\bar{c})] = S$.

Next we prove (ii). Let $c \in \mathcal{U}_{x_0}^{K,\lambda}$ be good, $x(t) = x(t, x_0, c)$, and $R > 0$. Then

$$\frac{1}{R}\int_0^R (U_T(c) - U_T(\bar{c}))\, \mathrm{d}T = \frac{1}{R}\int_0^R \int_0^T \theta(c(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_0 \langle x(t), p_0\rangle\, \mathrm{d}t\, \mathrm{d}T$$
$$= \frac{1}{R}\int_0^R \int_0^T \theta(c(t))\,\mathrm{d}t\,\mathrm{d}T + \frac{\alpha_0}{R}\int_0^R [\langle x(T) - \bar{x}, p_0\rangle - \langle x_0 - \bar{x}, p_0\rangle]\, \mathrm{d}T.$$

On one hand, Lemma A.17 implies the first addendum converges to L_c when $R \to \infty$, on the other hand, as a consequence of Theorem 9.1.3 p. 260 in Zaslavski (2006)

$$\frac{1}{R}\int_0^R \langle x(T) - \bar{x}, p_0\rangle\, \mathrm{d}T = \left\langle \frac{1}{R}\int_0^R x(T)\,\mathrm{d}T - \bar{x}, p_0 \right\rangle \to 0, \quad R \to \infty.$$

Hence $\lim_{R\to+\infty} \frac{1}{R}\int_0^R (U_T(c) - U_T(\bar{c}))\, \mathrm{d}T = L_c + \alpha_0 \langle \bar{x} - x_0, p_0\rangle$, so that the limit exists and is finite. Now, given c in $\mathcal{U}_{x_0}^{K,\lambda}$, there exists $\tilde{c}$ maximizing, in $\mathcal{U}_{x_0}^{K,\lambda}$, $\limsup_{R\to+\infty} \frac{1}{R}\int_0^R (U_T(c) - U_T(\bar{c}))\, \mathrm{d}T$ so that, arguing as in (i)

$$\limsup_{R\to+\infty} \frac{1}{R}\int_0^R (U_T(\tilde{c}) - U_T(c))\, \mathrm{d}T \geq \limsup_{R\to+\infty} \frac{1}{R}\int_0^R (U_T(\tilde{c}) - U_T(\bar{c}))\, \mathrm{d}T +$$
$$- \limsup_{R\to+\infty} \frac{1}{R}\int_0^R (U_T(c) - U_T(\bar{c}))\, \mathrm{d}T \geq 0$$

which implies $\limsup_{T\to\infty} \big(U_T(\tilde{c}) - U_T(c)\big) \geq 0$, for all c in $\mathcal{U}_{x_0}^{K,\lambda}$, and the thesis. □

Remark A.21 The control computed above as maximal is the one minimizing $\lim_{T\to\infty} \int_0^T \theta(c(t))\, \mathrm{d}t$. □

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