Ca' Foscari University of Venice

Department of Economics

Working Paper

Diana Barro
Marco Corazza
Martina Nardon

Cumulative Prospect Theory portfolio selection

ISSN: 1827-3580
No. 26/WP/2020

Working Papers
Department of Economics
Ca' Foscari University of Venice
No. 26/WP/2020
ISSN 1827-3580

Cumulative Prospect Theory portfolio selection

Diana Barro
Ca' Foscari University of Venice

Marco Corazza
Ca' Foscari University of Venice

Martina Nardon
Ca' Foscari University of Venice

Abstract
We introduce elements of Cumulative Prospect Theory into the portfolio selection problem and then compare stock portfolios selected under the behavioral approach with those selected according to classical approaches, such as Mean Variance and Mean Absolute Deviation ones. The mathematical programming problem associated to the behavioral portfolio selection is highly non-linear and non-differentiable; for these reasons it is solved using a Particle Swarm Optimization approach. An application to the STOXX Europe 600 equity market is performed.

Keywords
Cumulative Prospect Theory, Portfolio Selection, Particle Swarm Optimization

JEL Codes
G40, G11, C61

Address for correspondence:
Martina Nardon
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
Phone: (++39) 041 2347413
Fax: (++39) 041 2349176
e-mail: mnardon@unive.it

The Working Paper Series
is available only on line
(http://www.unive.it/nqcontent.cfm?a_id=86302)
For editorial correspondence, please contact:
wp.dse@unive.it

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

Cumulative Prospect Theory portfolio selection

Diana Barro, Marco Corazza and Martina Nardon

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, 30121 Venezia, Italy
d.barro@unive.it, corazza@unive.it, mnardon@unive.it

Abstract

We introduce elements of Cumulative Prospect Theory into the portfolio selection problem and then compare stock portfolios selected under the behavioral approach with those selected according to classical approaches, such as Mean Variance and Mean Absolute Deviation ones. The mathematical programming problem associated to the behavioral portfolio selection is highly non-linear and non-differentiable; for these reasons it is solved using a Particle Swarm Optimization approach. An application to the STOXX Europe 600 equity market is performed.

Keywords: Cumulative Prospect Theory, Portfolio Selection, Particle Swarm Optimization.

JEL Classification: G40, G11, C61.

1 Introduction

The literature on portfolio selection models is wide and since the founding work of Markowitz [5] has grown rapidly and has been extended along many different directions. Among them, modeling risk measures and performance evaluation are certainly crucial themes.

Our aim is to suggest a selection model able to provide optimal decisions tailored to individual attitudes to risk and loss aversion, allowing for a larger flexibility in the description of the portfolio problem. In particular, Prospect Theory (PT) [3] provides a framework to effectively represent a wide range of risk attitudes. Within this framework, [6] propose a behavioral portfolio model.

In this contribution, we apply Cumulative Prospect Theory (CPT) [7] to the portfolio selection problem. CPT relies on two key transformations: a value function, which replaces the utility function and models risk and loss aversion in the evaluation of outcomes, and a probability weighting function which models distortion of probabilities of ranked outcomes. Risk attitudes and actual investment decisions of a Prospect Investor (PI) depend on the shapes of these functions as well as their interaction.

First, we collect preliminary evidence on the role of these specifications on the investment choices. Then, we compare compositions and performances of Behavioral Portfolios (BP) defined under CPT, with those of Mean-Variance (MV) and Mean Absolute Deviation (MAD) portfolios.

In order to solve BP optimization problem, which is highly non-linear and non-differentiable, we resort to an evolutionary metaheuristic, Particle Swarm Optimization (PSO).

The remainder of this paper is organized as follows. Section 2 synthesizes the main features of CPT. Section 3 introduces the BP selection model. Section 4 describes the PSO solution approach. Section 5 synthesizes the Mean Variance and Mean Absolute Deviation portfolio models. In Section 6 an application to the European equity market and the comparison with MV and MAD portfolios are discussed. Section 7 concludes.

2 Cumulative Prospect Theory

PT has been proposed in [3] to explain actual behaviors; decision makers do not always take their decisions consistently with the maximization of expected utility: they are risk averse when they evaluate gains and risk seeking with respect to losses, they are more sensitive to losses than gains of comparable magnitude (loss aversion). Investment opportunities are evaluated not in terms of final wealth, but based on potential gains and losses relative to a *reference point*. Moreover, decision makers apply decision weights that are biased with respect to objective probabilities and are more sensitive to changes in the probability of extreme outcomes than mid outcomes; medium and high probabilities tend to be underweighted and low probabilities of extreme outcomes are overweighted. Risk attitude and loss aversion are modeled through a value function v and probabilistic risk perception through a probability weighting function w.

In PT individuals maximize the following value

$$V = \sum_{i=-m}^{n} \pi_i \cdot v(z_i)\,, \tag{1}$$

where z_i denotes negative outcomes for $-m \leq i < 0$, and positive outcomes for $0 < i \leq n$, with $z_i \leq z_j$ for $i < j$, considering outcomes interpreted as deviations from a reference point.

According to [3], the value function is concave for gains and convex and steeper for losses. In the application, we adopt the following function

$$v(z) = \begin{cases} v^+(z) = z^a & z \geq 0 \\ v^-(z) = -\lambda(-z)^b & z < 0, \end{cases} \tag{2}$$

with positive parameters that control risk attitude, $0 < a \leq 1$ and $0 < b \leq 1$, and loss aversion, $\lambda \geq 1$. Function (2) is widely used in the literature, it is continuous, strictly increasing, has zero as reference point. In the numerical experiments, we use the parameters estimated by Tversky and Kahneman [7]: $\lambda = 2.25$ and $a = b = 0.88$ (later referred to as 'TK sentiment').

A weighting function w is a strictly increasing function which maps the probability interval $[0, 1]$ into $[0, 1]$, with $w(0) = 0$ and $w(1) = 1$. Empirical evidence suggests a typical *inverse-S shape*: the function is initially concave (probabilistic risk seeking or optimism) for probabilities in the interval $(0, p^*)$, and then convex (probabilistic risk aversion or pessimism) in the interval $(p^*, 1)$, for a certain value of p^*. A linear weighting function describes probabilistic risk neutrality or objective sensitivity towards probabilities, which characterizes Expected Utility. Empirical findings indicate that the intersection (elevation) between the weighting function and the 45 degrees line, $w(p) = p$, is for p in the interval $(0.3, 0.4)$.

In the cumulative version of PT [7], the subjective value (1) depends also on the rank of the outcomes and the decision weights π_i are differences in transformed counter-cumulative probabilities of gains and cumulative probabilities of losses:

$$\pi_i = \begin{cases} w^-(p_{-m}) & i = -m \\ w^-\left(\sum_{j=-m}^{i} p_j\right) - w^-\left(\sum_{j=-m}^{i-1} p_j\right) & i = -m+1, \ldots, -1 \\ w^+\left(\sum_{j=i}^{n} p_j\right) - w^+\left(\sum_{j=i+1}^{n} p_j\right) & i = 0, \ldots, n-1 \\ w^+(p_n) & i = n, \end{cases} \tag{3}$$

where w^- and w^+ denote the weighting function for probabilities of losses and gains, respectively.

Single parameter and two (or more) parameter weighting functions have been suggested; a commonly applied weighting function is

$$w(p) = \frac{p^\gamma}{\left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}}, \tag{4}$$

with $w(0) = 0$ and $w(1) = 1$, and $\gamma > 0$ (with some constraint in order to have an increasing function). When $\gamma < 1$, one obtains the inverse-S shape. In the applications, we use the parameters estimated by Tversky and Kahneman [7]: $\gamma^+ = 0.61$ and $\gamma^- = 0.69$, for w^+ and w^-, respectively. Figure 1 shows some examples of the weighting function used in [7].

3 Behavioral Portfolio selection: a CPT approach

We assume that a Prospect Investor selects the portfolio weights in order to maximize her prospect value (1) subject to the usual budget constraint

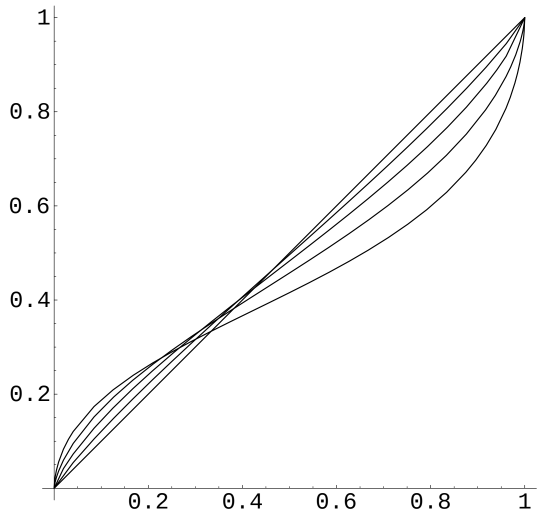

Figure 1: Weighting function $w(p) = \frac{p^\gamma}{(p^\gamma+(1-p)^\gamma)^{1/\gamma}}$ for different values of the parameter $\gamma < 1$. As γ approaches the value 1, w tends to the identity function

and short selling restrictions. Let $\mathbf{x} = (x_1, \dots, x_n)$ be the vector of portfolio weights, such that $x_j \geq 0$ $(j = 1, 2, \dots, n)$ and $\sum_{j=1}^n x_j = 1$. Let us consider m possible scenarios, with r_{ij} the return of equity j in scenario i, and p_i be the probability of each i. In this work we considered equally probable scenarios.

The portfolio returns, measured relative to a fixed reference point r_0, are the results subjectively evaluated and weighted through the distorted probabilities computed as in (3). Formally, the BP selection model is defined as:

$$\begin{aligned} &\max_{\mathbf{x}} \sum_{i=1}^m \pi_i \cdot v\left(\sum_{j=1}^n (x_j r_{ij} - r_0)\right) \\ &\text{s.t.} \quad \sum_{j=1}^n x_j = 1 \\ &\qquad x_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \tag{5}$$

The resulting optimization problem is highly non-linear and non-differentiable, so it cannot be solved applying traditional optimization techniques. In

Section 4 an efficient solution approach based on metaheuristics is described.

4 Particle Swarm Optimization and its implementation

Particle Swarm Optimization (PSO) is an iterative bio-inspired population-based metaheuristic for the solution of unconstrained global optimization problems. Instead, our optimization problem is a constrained global one. Because of it, in this section, first we introduce the basics of standard PSO, then we present the implementation performed in order to take into account the presence of constraints.

The basic idea of PSO is to replicate the social behaviour of shoals of fish or flocks of birds cooperating in the pursuit of a given goal. To this purpose, each member – namely, a *particle* – of the shoal/flock – namely, the *swarm* – explores the search area keeping memory of its best position reached so far, and it exchanges this information with the neighbours in the swarm. Thus, generally, the whole swarm tends to converge towards the best global position reached by the particles.

4.1 Basics of PSO

The aforementioned idea may be formalized in the following way. Let us consider the unconstrained global optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}),$$

where $f : \mathbb{R}^d \mapsto \mathbb{R}$ is the objective function. Suppose we apply PSO for its solution, in which M particles are considered. Notice that every particle of the swarm represents a possible solution of such a problem. Initially, each of them is assigned to a random position, $\mathbf{x}_j^0$, and to a random velocity, $\mathbf{v}_j^0$, with $j = 1, \ldots, M$.

At the k-th iteration of the algorithm, with $k = 0, \ldots, K$, three vectors are associated to the j-th particle, with $j = 1, \ldots, M$: $\mathbf{x}_j^k \in \mathbb{R}^d$, which is its current position; $\mathbf{v}_j^k \in \mathbb{R}^d$, which is its current velocity; $\mathbf{p}_j \in \mathbb{R}^d$, which is its best position visited so far.

Furthermore, $pbest_j = f(\mathbf{p}_j)$ is the value of the objective function in the best position visited by the j-th particle, $\mathbf{p}_j$, and $gbest = f(\mathbf{p}_g)$ is the value of the objective function in the best position visited by the swarm, $\mathbf{p}_g$.

The algorithm, in the version with inertia weights, which is the one we use in this work, is reported in the following:

1. Set $pbest_j = +\infty$ for $j = 1, \ldots, M$, $gbest = +\infty$ and $k = 0$. Evaluate $f(\mathbf{x}_j^k)$ for $j = 1, \ldots, M$.

2. If $f(\mathbf{x}_j^k) < pbest_j$ then set $\mathbf{p}_j = \mathbf{x}_j^k$ and $pbest_j = f(\mathbf{x}_j^k)$. If $f(\mathbf{x}_j^k) < gbest$ then set $\mathbf{p}_g = \mathbf{x}_j^k$ and $gbest = f(\mathbf{x}_j^k)$.

3. Update position and velocity of the j-th particle, with $j = 1, \ldots, M$, as
$$\begin{cases} \mathbf{v}_j^{k+1} = w^{k+1}\mathbf{v}_j^k + c_1(\mathbf{p}_j - \mathbf{x}_j^k) + c_2(\mathbf{p}_g - \mathbf{x}_j^k) \\ \mathbf{x}_j^{k+1} = \mathbf{x}_j^k + \mathbf{v}_j^{k+1} \end{cases}$$
where w^{k+1}, c_1 and c_2 are appropriate quantities.

4. If a pre-established convergence criterion is not satisfied then set $k = k + 1$ and go to step 2.

Notice that the values of c_1 and of c_2 affect the strength of the attractive forces towards $\mathbf{p}_j$ and $\mathbf{p}_g$, respectively. In order to get the convergence of the swarm, they have to be set carefully in accordance with the value of the inertia weight w^k, which is generally linearly decreasing with respect to k.

4.2 The unconstrained BP optimization problem

As stated before, our optimization problem is a constrained global one. For dealing with the presence of constraints, different strategies are proposed in the literature to ensure that feasible positions are generated at any iterations of PSO. However, in this paper we use PSO accordingly to the original intent, that is as a tool for the solution of unconstrained global optimization problems. To this purpose, we reformulate our constrained problem into an unconstrained one using the nondifferentiable ℓ_1 penalty function method described by [2] and already applied in the financial context (see, for instance,

[1]). Such an approach is known as *exact penalty method*, where the term "exact" refers to the correspondence between the minimizers of the original constrained problem and the minimizers of the unconstrained (penalized) one.

The reformulated version of the BP optimization problem (5) is:

$$\max_{\mathbf{x}} \sum_{i=1}^{m} \pi_i \cdot v\left(\sum_{j=1}^{n}(x_j r_{ij} - r_0)\right) - \frac{1}{\epsilon}\left[\left|\sum_{j=1}^{n} x_j - 1\right| + \sum_{j=1}^{n} \max(0, -x_j)\right], \quad (6)$$

where ϵ is the so-called *penalty parameter*.

Note that the correct setting of ϵ ensures the correspondence between the solutions of the original constrained problem and of the reformulated unconstrained one (6).

5 Mean Variance and Mean Absolute Deviation portfolios

In order to test the resulting CPT portfolios obtained from (5), in its reformulated versione (6), against alternative selection models, we consider two very well known and widely used in practice. The first is Markowitz Mean Variance (MV) portfolio selection model [5] and the second is Konno and Yamazaki Mean Absolute Deviation (MAD) model [4].

The classical portfolio selection problem dates back to [5] and is based on the solution of a quadratic programming problem with the goal of minimizing variance and maximizing return. We denote with $\mu = (\mu_1, \ldots, \mu_n)$ the vector of expected returns and with Σ the variance-covariance matrix, given the vector of portfolio weights, $\mathbf{x} = (x_1, \ldots, x_n)$, the portfolio return and variance are, respectively, $\mu(x) = x^T\mu$ and $\sigma^2 = x^T\Sigma x$. In a trade off formulation of the objective function, the resulting quadratic portfolio optimization problem can be casted as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}x^T\Sigma x - \gamma x^T\mu \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \quad (7)$$

where γ is a risk tolerance parameter. The portfolio optimization model by Markowitz is a quadratic programming problem and requires calculation of means and variance-covariance matrix for assets' returns.

The MAD model by Konno and Yamazaki [4] proposes a different risk measure, the Mean Absolute Deviation (MAD) measure, that allows to recast the problem as a linear programming model.

We recall that r_{ij} denotes the return of equity j in scenario i, and p_i is the corresponding probability.

The MAD model can be written as

$$\begin{aligned} \min_{\mathbf{x}} \ & \sum_{i=1}^{m} p_i \sum_{j=1}^{n} |x_j r_{ij} - \bar{r}_0| \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{8}$$

where $\bar{r}_0$ is a target return set by the decision maker, deviations from which are symmetrically penalized.

6 Application to the European equity market

Our main purpose is the comparison between the investment decisions of a rational agent and those of a Prospect Investor (PI). To this aim, we consider the equity market represented by the 10 sectorial indices in the STOXX 600 Europe index, and use weekly data. Then, we solve the reformulated BP selection problem (6) and compare it with the MV [5] and the MAD [4] ones. The overall testing period goes from July, 2018 to June, 2019.

The out-of-sample analysis performed over this period is carried out as follows: a 1-year in-sample period is used to select the various optimal portfolios (BP, MV and MAD), then these portfolios are applied at the realized weekly market returns for a 3-month out-of-sample period (e.g.: first in-sample period, from July, 2017 to June, 2018; first out-of-sample period, from July, 2018 to September, 2018). This scheme is then further applied in a 3-month rolling window approach until the entire 1-year out-of-sample testing period

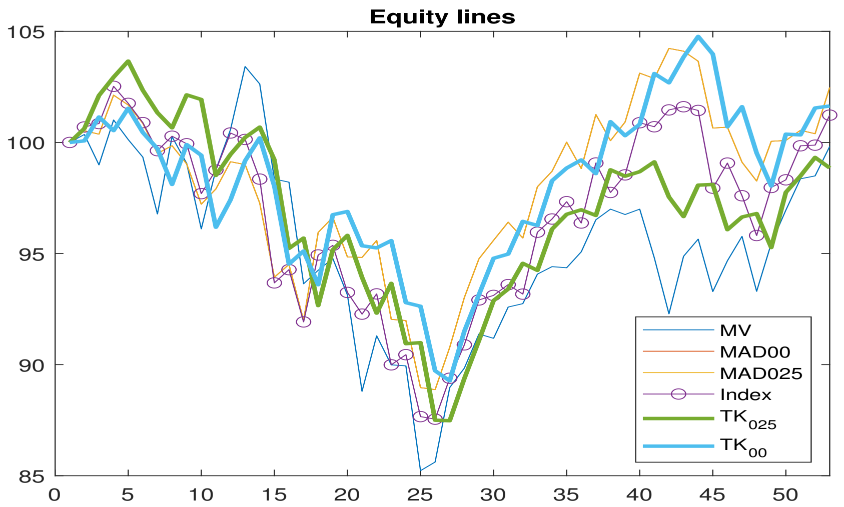

Figure 2: Out-of-sample equity lines for the optimal portfolios and for the Index

Table 1: Statistics for the out-of-sample returns of the optimal portfolios and of other strategies

	MV	MAD_{00}	MAD_{025}	$Index$	TK_{025}	TK_{00}
Mean	-0.0000	0.0005	0.0005	0.0002	-0.0002	0.0003
Standard dev.	0.0216	0.0167	0.0167	0.0176	0.0164	0.0165
Skewness	-0.6472	-0.2375	-0.2375	-0.5630	-0.7246	-0.4072
Kurtosis	2.9549	3.0358	3.0358	3.0943	3.0391	2.6183

is covered. For the BP two different reference points are considered, $r_0 = 0\,\%$ and $r_0 = 2.5\,\%$, and the same for the MAD portfolio deviations.

In Figure 2, we present the out-of-sample equity lines for the optimal portfolios and for the STOXX 600 Europe index (Index), using as starting capital $C = 100$. In Table 1, we report the main statistics for the out-of-sample returns for the portfolios.

In general, the up- and down-trends of the market strongly affect the out-of-sample performances of the optimal portfolios, which behave similarly. The BPs respond differently to various market phases in correspondence of the two reference points. In the case of the lower one, $r_0 = 0\%$, the portfolio performs better in the up-trend market and reduces the losses in the down-trend market (see Figure 2 and column 7 of Table 1). The second reference point, $r_0 = 2.5\%$, rather demanding given the overall market conditions in the considered testing periods, leads to less diversified compositions of the BP that is less performing in terms both of upside capture and downside protection (see Figure 2 and column 6 of Table 1). Similar considerations hold for the MV and MAD portfolios. In particular, the MAD portfolios that produce the minimum deviations for the two reference points are perfectly coincident (see Figure 2 and columns 3 and 4 of Table 1).

Finally, from preliminary findings, we note that BPs display a wider diversification with respect to other selected portfolios in all the 3-months testing periods. The determinants of such a result will be investigated in future experiments.

7 Concluding remarks

In this contribution we investigate a formulation for the portfolio selection problem in a Cumulative Prospect Theory framework. The resulting optimization problem is non-linear and non-differentiable and it is solved through a Particle Swarm Optimization metaheuristic. We considered an application to the 10 sectorial indices in the STOXX 600 Europe Index and test the behaviour of the resulting optimal portfolio compositions over 1-year out-of sample period. We discuss the obtained preliminary results with respect to the Mean-Variance and the MAD portfolios. Further investigations will be carried out, in particular, to the aim of performing sensitivity analysis on the parameters involved both in the reformulated optimization problem (6) and in the used metaheuristic.

References

[1] Corazza, M., Fasano, G., Gusso, R.: Particle Swarm Optimization with no-smooth penalty reformulation, for a complex portfolio selection problem. Appl. Math. Comput. **224**, 611–624 (2013)

[2] Fletcher, R.: Practical Methods of Optimization. John Wiley & Sons, Glichester (1991)

[3] Kahneman, D., Tversky, A.: Prospect theory: An analysis of decision under risk. Econometrica **47**, 263–291 (1979)

[4] Konno, H., Yamazaki, H.: Mean-absolute deviation portfolio optimization model and its applicationsto Tokyo stock market. Manag. Sci. **37**, 519–531 (1991)

[5] Markowitz, H.: Portfolio selection. J. Fin. **7**, 77–91 (1952)

[6] Shefrin, H., Statman, M.: Behavioral portfolio theory. J. Fin. Quant. Analysis **35**, 127–151 (2000)

[7] Tversky, A., Kahneman, D.: Advances in prospect theory: cumulative representation of the uncertainty. J. Risk Uncertain. **5**, 297–323 (1992)