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ORIGINAL RESEARCH

An enhanced model for Rosenkranz's logic of justification

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Abstract

Rosenkranz (2021) devised two bimodal epistemic logics: an idealized one and a realistic one. The former is shown to be sound with respect to a class of neighborhood frames called *i-frames*. Rosenkranz designed a specific i-frame able to invalidate a series of undesired formulas, proving that these are not theorems of the idealized logic. Nonetheless, an unwanted formula and an unwanted rule of inference are not invalidated. Invalidating the former guarantees the distinction between the two modal operators characteristic of the logic, while invalidating the latter is crucial in order to deal with the problem of logical omniscience. In this paper, I present an i-frame able to invalidate all the undesired formulas already invalidated by Rosenkranz, together with the missing formula and rule of inference.

Keywords Being in a position to know · Logical omniscience · Logic of justification · Epistemic logic · Neighborhood semantics

1 Introduction

Rosenkranz (2021) proposed two logics for epistemic justification, one called *idealized* and the other called *realistic*. I am going to focus only on the former, which Rosenkranz showed to be sound with respect to an appropriate class of neighborhood frames, called *idealized frames* (i-frames). We deal with a bimodal propositional logic where $K\varphi$ and $k\varphi$ stands for "one is in a position to know that φ", and "one knows that φ" respectively. The distinction between the two concepts is crucial in Rosenkranz's proposal and constitutes the first motivation behind the present paper, the second being related to the problem of logical omniscience. Before making this point more explicit, some technical background is required. Let us start defining an i-frame.

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Let W be a non-empty set of states and $N, R : W \mapsto \mathscr{P}(\mathscr{P}(W))$ be two neighborhood functions. A neighborhood frame $\mathcal{F} = (W, N, R)$ is an i-frame when it respects the following conditions for all $X, Y \subseteq W$ and all $u, v, w \in W$:

(t_K) if $X \in N(w)$, then $w \in X$
(o) if $X \in R(w)$, then $X \in N(w)$
(l) if $\{v \in W : X \notin N(v)\} \notin N(w)$,
then $\{u \in W : \{v \in W : X \notin N(v)\} \notin N(u)\} \in N(w)$
(z) if $X \in R(w)$,
then $\{u \in W : \{v \in W : X \notin R(v)\} \notin N(u)\} \in N(w)$
(a_0) $X \cap \{v \in W : X \notin R(v)\} \notin N(w)$
(m_K) if $X \subseteq Y$ and $X \in N(w)$, then $Y \in N(w)$
(m_k) if $X \subseteq Y$ and $X \in R(w)$, then $Y \in R(w)$

Rosenkranz (2021, 98–99) associates five different valuations V to the same underlying i-frame $\mathcal{F}$. $V : \mathsf{Prop} \mapsto \mathscr{P}(W)$ is a valuation function, where Prop is a set of countably many propositional variables.[1] In this way, Rosenkranz produces five i-models $\mathcal{M} = (\mathcal{F}, V)$ working as countermodels for a series of undesired formulas.[2] By soundness, this shows that those are not theorems of the idealized logic. The result is even more relevant since it is achieved exploiting the same i-frame. This means that the formulas can all be invalidated in the same i-model. In fact notice that, given different countermodels based on the same underling frame, it is always possible to construct a single countermodel simply having enough propositional variables. Therefore Rosenkranz's countermodels can be easily merged into a single one.

Nonetheless, the i-frame Rosenkranz designed faces a couple of limitations. Firstly, it makes K and k collapse into one another for any V. Let us see why. The semantic clauses for the two modal operators are the following:

$$\mathcal{M}, w \vDash K\varphi \text{ iff } [\![\varphi]\!]^{\mathcal{M}} \in N(w)$$
$$\mathcal{M}, w \vDash k\varphi \text{ iff } [\![\varphi]\!]^{\mathcal{M}} \in R(w)$$

Where $[\![\varphi]\!]^{\mathcal{M}} = \{x \in W : \mathcal{M}, x \vDash \varphi\}$. For the sake of simplicity, I drop the superscript and write $[\![\varphi]\!]$ for $[\![\varphi]\!]^{\mathcal{M}}$. The modal semantic clauses can therefore be restated in the following way:

$$[\![K\varphi]\!] = \{w \in W : [\![\varphi]\!] \in N(w)\}$$
$$[\![k\varphi]\!] = \{w \in W : [\![\varphi]\!] \in R(w)\}$$

The non-modal operators are defined in the usual way, assuming classical logic.

[1] Rosenkranz uses a bivalent interpretation function $I : \mathsf{Prop} \times W \mapsto \{1, 0\}$ instead of V. Anyway the resulting models are isomorphic.

[2] For the sake of simplicity, I shall call "i-models" what Rosenkranz calls "target i-models". Moreover, the term "frame" is not used by Rosenkranz, who refers indirectly to frames talking about *classes of models* (Rosenkranz 2021, 97–99).

Let us now describe the i-frame devised by Rosenkranz (2021, 97) in order to construct his countermodels. It is a neighborhood frame $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4\}$ and N, R are such that

$$
\begin{aligned}
N(w_1) = R(w_1) = N(w_2) = R(w_2) &= \{\{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, W\} \\
N(w_3) = R(w_3) = N(w_4) = R(w_4) &= \{\{w_1, w_3, w_4\}, \{w_2, w_3, w_4\}, W\}
\end{aligned}
$$

The i-frame equates N and R, since for each $w \in W$ we have $N(w) = R(w)$. It immediately follows that $[\![K\varphi]\!] = [\![k\varphi]\!]$ for any φ. $[\![K\varphi]\!] = [\![k\varphi]\!]$ holds iff both $[\![K\varphi]\!] \supseteq [\![k\varphi]\!]$ and $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ hold. On the one hand, $[\![K\varphi]\!] \supseteq [\![k\varphi]\!]$ must be the case, corresponding to condition (o). On the other hand, $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ cannot always be the case. In fact this would amount to accepting the formula $K\varphi \rightarrow k\varphi$, which is false every time one is in a position to know a certain proposition φ without knowing φ. Moreover, as anticipated, accepting $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ would make the two modal operators collapse into one another, entailing $[\![K\varphi]\!] = [\![k\varphi]\!]$. The concepts of "being in a position to know" and "knowing" are distinct and in this resides the interest of Rosenkranz's proposal. The devised i-frame is not able to express this distinction though.

The second limitation faced by the devised i-frame is related to the problem of logical omniscience, which is scrupulously taken into consideration by Rosenkranz. Opposing logical omniscience roughly means trying to devise an epistemic logic for agents with bounded computational capabilities. Rosenkranz refuses to take the rule RN_k as part of his logic for this very reason. RN_k says that if φ is a theorem, then $k\varphi$ is likewise a theorem: if $\vdash \varphi$, then $\vdash k\varphi$. This is a strong idealization, requiring that one knows any logical truth, even the most convoluted. Nonetheless, the i-frame Rosenkranz designed validates RN_k since $W \in R(w)$ for all $w \in W$. It is easy to check that RN_k holds iff this is the case. In fact $W = [\![\top]\!]$, where $\top$ is an abbreviation for any theorem of the logic. Notice that this entails that the rule RN_k and the formula $k\top$ are equivalent. Given that all the other undesired schemas are formulas and not rules, for the rest of the paper I shall refer to $k\top$ instead of RN_k for the sake of uniformity.

The aim of this paper is to present an i-frame able to overcome the limitations faced by the one devised by Rosenkranz, so as to invalidate $K\varphi \rightarrow k\varphi$ and $k\top$. The structure is the following. Firstly, I shall construct a neighborhood frame and show that it is indeed an i-frame (Section 2). This amounts to proving that the new frame meets each of the seven conditions listed at the beginning of this Introduction. Then, I will design a countermodel starting from that i-frame given an appropriate valuation V (Section 3). This i-model will invalidate all the formulas already invalidated by Rosenkranz, together with the additional $K\varphi \rightarrow k\varphi$ and $k\top$.

2 The new i-frame

Let us consider the following neighborhood frame $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and N, R are such that

$$\begin{aligned}
N(w_1) = N(w_2) &= \{W\} \\
R(w_1) = R(w_2) = R(w_3) &= \emptyset \\
N(w_3) = N(w_4) = R(w_4) &= \{\{w_1, w_3, w_4, w_6\}, \{w_2, w_3, w_4, w_6\}, \\
&\quad\ \{w_1, w_2, w_3, w_4, w_6\}, \{w_1, w_3, w_4, w_5, w_6\}, \\
&\quad\ \{w_2, w_3, w_4, w_5, w_6\}, W\} \\
R(w_5) &= \{\{w_1, w_3, w_4, w_5, w_6\}, W\} \\
N(w_5) = N(w_6) = R(w_6) &= \{\{w_1, w_4, w_5, w_6\}, \{w_1, w_2, w_4, w_5, w_6\}, \\
&\quad\ \{w_1, w_3, w_4, w_5, w_6\}, W\}
\end{aligned}$$

Before showing that this is an i-frame, I shall spend a few words on some feature of the frame.

The first consideration concerns the fact that $R(w) = \emptyset$ for some $w \in W$. This can be regarded as an undesirable property, since it corresponds to total ignorance of the agent in state w. However, this is a necessary feature of any i-frame invalidating $k\top$. Let us see why. As already seen, invalidating $k\top$ amounts to having some $w \in W$ such that $W \notin R(w)$. Let us remember that (m_k) holds in every i-frame, i.e., R must be superset-closed. But in case $R(w) \neq \emptyset$, superset-closure immediately entails $W \in R(w)$. We conclude that in order to invalidate $k\top$, we need to have at least one $w \in W$ such that $R(w) = \emptyset$.

The second consideration is twofold and concerns the choice of $R(w_5)$. Notice that both $R(w_1) = R(w_2) = R(w_3) = R(w_5) = \emptyset$ and $R(w_5) = \{W\}$ would have generated perfectly working i-frames with the additional quality of being simpler than the one provided (the interested reader can verify this, by tweaking the proofs in the next paragraph until the end of the paper). Nonetheless, I believe that both constitutes undesirable idealizations, which are not required in order to invalidate neither $K\varphi \to k\varphi$ nor $k\top$.

Let us start considering $R(w_1) = R(w_2) = R(w_3) = R(w_5) = \emptyset$. In this case $K\varphi \to k\varphi$ would be false only in those states with an empty neighborhood for R. In fact in those states, what one is in a position to know trivially exceeds what one knows since $N(w) \neq \emptyset$ for any $w \in W$ such that $R(w) = \emptyset$.[3] But total ignorance is a limit epistemic state, which cannot be the only reason why $K\varphi \to k\varphi$ fails. We need $R(w) = \emptyset$ for some w, in order to invalidate $k\top$, but $K\varphi \to k\varphi$ should not be false only in such w. The designed frame avoids the problem since something is known in w_5 given $R(w_5) \neq \emptyset$.

[3] Notice that in an i-frame no $w \in W$ can be such that $N(w) = \emptyset$. In fact (l) can be restated in the following way: either $\{v \in W : X \notin N(v)\} \in N(w)$ or $\{u \in W : \{v \in W : X \notin N(v)\} \notin N(u)\} \in N(w)$.

Let us consider $R(w_5) = \{W\}$ now. In this case, $K\varphi \to k\varphi$ would be false also in a state with a non-empty neighborhood; nonetheless, the only element of this neighborhood would be W. This corresponds to a state in which only $\top$ is known. In other words, one only knows theorems of the logic and no contingently true propositions. The designed frame avoids this idealization since something that is not $\top$ is known in w_5, i.e., the proposition corresponding to $\{w_1, w_3, w_4, w_5, w_6\}$.

Let us now verify whether the designed frame is indeed an i-frame. Since $\emptyset \subset X$ for any set $X \neq \emptyset$, it follows $R(w_1) \subset N(w_1)$, $R(w_2) \subset N(w_2)$ and $R(w_3) \subset N(w_3)$. Additionally, $R(w_4) = N(w_4)$ and $R(w_6) = N(w_6)$. Finally, $R(w_5) \subset N(w_5)$. We conclude $R(w) \subseteq N(w)$ is the case for any $w \in W$, and so (o) holds. Since $w \in X$ for any $w \in W$ and any $X \in N(w)$, (t_K) holds. Since R and N are superset-closed, (m_k) and (m_K) both hold.

In order to show that (l), (z) and (a_0) do hold, some observations, given an arbitrary formula φ, are needed:

- If $[\![\varphi]\!] = W$, then $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$ and $[\![K\varphi]\!] = W$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K\varphi]\!] = \emptyset = [\![K\neg k\varphi]\!] = [\![K\neg K\varphi]\!]$. Hence, $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.
- If $[\![\varphi]\!]$ contains exactly five states, then there are six possible combinations to consider:

  (1) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_5\}$. Then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. So, $[\![\neg K\varphi]\!] = [\![K\neg K\varphi]\!] = W$ and $[\![\neg K\neg K\varphi]\!] = \emptyset$.
  (2) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_6\}$. Then $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2, w_5, w_6\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.
  (3) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_5, w_6\}$. Then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).
  (4) $[\![\varphi]\!] = \{w_1, w_2, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_6\}$ and $[\![K\varphi]\!] = \{w_5, w_6\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3, w_4, w_5\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2, w_3, w_4\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.
  (5) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$ and $[\![K\varphi]\!] = \{w_3, w_4, w_5, w_6\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.
  (6) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. Follow case (2).

- If $[\![\varphi]\!]$ contains exactly four states, we have fifteen possible combinations, but we can gather them in three cases:

  (7) For $[\![\varphi]\!] = \{w_1, w_3, w_4, w_6\}$ and $[\![\varphi]\!] = \{w_2, w_3, w_4, w_6\}$, we have the same result: $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. Follow case (2).
  (8) $[\![\varphi]\!] = \{w_1, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_6\}$ and $[\![K\varphi]\!] = \{w_5, w_6\}$. Follow case (4).
  (9) For the remaining combinations we have the same result: $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).

- If $[\![\varphi]\!]$ contains at most three states, then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).

Considering that, given an arbitrary φ, it is always the case that $[\![\neg K\neg K\varphi]\!] \subseteq [\![K\neg K\neg K\varphi]\!]$ and $[\![k\varphi]\!] \subseteq [\![K\neg K\neg k\varphi]\!]$, we conclude that (l) and (z) hold.

What about (a_0)? There are only nine cases to consider given an arbitrary φ.

(a) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(b) $[\![\varphi]\!] = \{w_1, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_4, w_5\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(c) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_2, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(d) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(e) $[\![\varphi]\!] = \{w_1, w_2, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_4, w_5\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(f) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_3\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(g) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_2, w_3, w_5, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(h) $[\![\varphi]\!] = W$ and $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.
(i) For all other $[\![\varphi]\!]$, $[\![k\varphi]\!] = \emptyset$. So, $[\![\varphi \wedge \neg k\varphi]\!] = [\![\varphi]\!]$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

We conclude that (a_0) holds.

3 Undesired formulas

Now that we have proved that the designed frame is indeed an i-frame, let us show that it can invalidate the following undesired formulas. Apart from the first two, the others were already invalidated by the i-frame devised in Rosenkranz (2021). I refer to the book for a detailed explanation of why these formulas are undesirable in the idealized logic.

$$
\begin{array}{rl}
(K\text{-}k) & K\varphi \to k\varphi \\
(N_k) & k\top \\
(Agg_K) & K\varphi \wedge K\psi \to K(\varphi \wedge \psi) \\
(Agg_k) & k\varphi \wedge k\psi \to k(\varphi \wedge \psi) \\
(4_K) & K\varphi \to KK\varphi \\
(4_k) & k\varphi \to kk\varphi \\
(5_K) & \neg K\varphi \to K\neg K\varphi \\
(5_k) & \neg k\varphi \to k\neg k\varphi \\
(T_J) & \neg K\neg K\varphi \to \varphi \\
(T_D) & \neg K\neg k\varphi \to \varphi \\
(B_K) & \varphi \to K\neg K\neg\varphi \\
(B_k) & \varphi \to k\neg k\neg\varphi \\
(K_K) & K(\varphi \to \psi) \to (K\varphi \to K\psi)
\end{array}
$$

$$
\begin{array}{ll}
(K_k) & k(\varphi \to \psi) \to (k\varphi \to k\psi) \\
(Agg_J) & \neg K \neg K\varphi \wedge \neg K \neg K\psi \to \neg K \neg K(\varphi \wedge \psi) \\
(Agg_D) & \neg K \neg k\varphi \wedge \neg K \neg k\psi \to \neg K \neg k(\varphi \wedge \psi)
\end{array}
$$

In order to make the proofs easier to follow, I describe the i-frame once again. $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and N, R are such that

$$
\begin{aligned}
N(w_1) = N(w_2) &= \{W\} \\
R(w_1) = R(w_2) = R(w_3) &= \emptyset \\
N(w_3) = N(w_4) = R(w_4) &= \{\{w_1, w_3, w_4, w_6\}, \{w_2, w_3, w_4, w_6\}, \\
&\quad \{w_1, w_2, w_3, w_4, w_6\}, \{w_1, w_3, w_4, w_5, w_6\}, \\
&\quad \{w_2, w_3, w_4, w_5, w_6\}, W\} \\
R(w_5) &= \{\{w_1, w_3, w_4, w_5, w_6\}, W\} \\
N(w_5) = N(w_6) = R(w_6) &= \{\{w_1, w_4, w_5, w_6\}, \{w_1, w_2, w_4, w_5, w_6\}, \\
&\quad \{w_1, w_3, w_4, w_5, w_6\}, W\}
\end{aligned}
$$

The first four formulas are invalidated for any possible valuation, therefore we don't need to assign a particular V.

- (K-k) Since $\emptyset \subset X$ for all $X \neq \emptyset$, we have $R(w_1) = R(w_2) = R(w_3) = \emptyset \subset N(w)$ for any $w \in W$. Moreover $R(w_5) \subset N(w_5)$. In both cases $N(w) \not\subseteq R(w)$ and therefore $[\![K\varphi]\!] \not\subseteq [\![k\varphi]\!]$.
- (N_k) Its failure follows from the fact that $W \notin R(w_1) = R(w_2) = R(w_3)$.
- (Agg_K) Its failure follows from the fact that N is not closed under intersection.[4]
- (Agg_k) Its failure follows from the fact that R is not closed under intersection.

For the remaining undesired formulas, I shall provide a particular countermodel. We need to show that, given the appropriate valuation V, each formula is false in at least one state of the i-model $(\mathcal{F}, V)$. Remember that for some arbitrary φ and ψ the implication $\varphi \to \psi$ is true in each state iff $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Let us assign V to five different propositional variables p_1, p_2, p_3, p_4, p_5.

Let $V(p_1) = \{w_1, w_2, w_3, w_4, w_6\}$.

- (4_K) Then $[\![Kp_1]\!] = \{w_3, w_4\}$ and $[\![KKp_1]\!] = \emptyset$. So, $[\![Kp_1]\!] \not\subseteq [\![KKp_1]\!]$.
- (4_k) Then $[\![kp_1]\!] = \{w_4\}$ and $[\![kkp_1]\!] = \emptyset$. So, $[\![kp_1]\!] \not\subseteq [\![kkp_1]\!]$.
- (5_K) Then $[\![\neg Kp_1]\!] = \{w_1, w_2, w_5, w_6\}$ and $[\![K\neg Kp_1]\!] = \emptyset$. So, $[\![\neg Kp_1]\!] \not\subseteq [\![K\neg Kp_1]\!]$.

[4]In fact Agg_K is valid in a neighborhood frame iff N is closed under intersection. This is a known result, but I sketch here a proof of the relevant direction of the biconditional, for the sake of clarity. An analogous proof can be carried out for Agg_k. Take a neighborhood frame $\mathcal{F} = (W, N)$ such that Agg_K is valid, i.e., is true at every state for every valuation. Let us suppose, for the sake of contradiction, that for some $X, Y \subseteq W$ we have $X \in N(w)$ and $Y \in N(w)$, but $X \cap Y \notin N(w)$, i.e., N is not closed under intersection. Let us take a valuation V such that $V(p) = X$ and $V(q) = Y$. It follows $[\![p]\!] \in N(w)$ and $[\![q]\!] \in N(w)$, but $[\![p \wedge q]\!] \notin N(w)$. But then Agg_K is not valid. We showed by contradiction that, if Agg_K is valid in a neighborhood frame, then N must be closed under intersection. By contraposition: if N is not closed under intersection, then Agg_K is not valid.

- (5_k) Then $[\![\neg kp_1]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![k\neg kp_1]\!] = \emptyset$. So, $[\![\neg kp_1]\!] \not\subseteq [\![k\neg kp_1]\!]$.
- (T_J) Then $[\![\neg K\neg Kp_1]\!] = W$. So, $[\![\neg K\neg Kp_1]\!] \not\subseteq [\![p_1]\!]$.
- (T_D) Then $[\![K\neg kp_1]\!] = \emptyset$ and $[\![\neg K\neg kp_1]\!] = W$. So, $[\![\neg K\neg kp_1]\!] \not\subseteq [\![p_1]\!]$.

Let $V(p_2) = \{w_5\}$.

- (B_K) Then $[\![\neg p_2]\!] = \{w_1, w_2, w_3, w_4, w_6\}$ and $[\![K\neg p_2]\!] = \{w_3, w_4\}$. Accordingly, $[\![\neg K\neg p_2]\!] = \{w_1, w_2, w_5, w_6\}$ and $[\![K\neg K\neg p_2]\!] = \emptyset$. So, $[\![p_2]\!] \not\subseteq [\![K\neg K\neg p_2]\!]$.
- (B_k) Then $[\![k\neg p_2]\!] = \{w_4\}$. Accordingly, $[\![\neg k\neg p_2]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![k\neg k\neg p_2]\!] = \emptyset$. So, $[\![p_2]\!] \not\subseteq [\![k\neg k\neg p_2]\!]$.

Let $V(p_3) = \{w_1, w_3, w_4, w_5, w_6\}$ and $V(p_4) = \{w_3, w_4, w_5, w_6\}$.

- (K_K) Then $[\![p_3 \to p_4]\!] = \{w_2, w_3, w_4, w_5, w_6\}$ and $[\![K(p_3 \to p_4)]\!] = \{w_3, w_4\}$. Moreover $[\![Kp_3]\!] = \{w_3, w_4, w_5, w_6\}$ and $[\![Kp_4]\!] = \emptyset$ give $[\![Kp_3 \to Kp_4]\!] = \{w_1, w_2\}$. So, $[\![K(p_3 \to p_4)]\!] \not\subseteq [\![Kp_3 \to Kp_4]\!]$.
- (K_k) Then $[\![k(p_3 \to p_4)]\!] = \{w_4\}$. Moreover $[\![kp_3]\!] = \{w_4, w_5, w_6\}$ and $[\![kp_4]\!] = \emptyset$ give $[\![kp_3 \to kp_4]\!] = \{w_1, w_2, w_3\}$. So, $[\![k(p_3 \to p_4)]\!] \not\subseteq [\![kp_3 \to kp_4]\!]$.

Let $V(p_5) = \{w_2, w_3, w_4, w_5, w_6\}$.

- (Agg_J) Then $[\![Kp_3]\!] = \{w_3, w_4, w_5, w_6\}$ and $[\![Kp_5]\!] = \{w_3, w_4\}$. Hence, $[\![\neg Kp_3]\!] = \{w_1, w_2\}$ and $[\![\neg Kp_5]\!] = \{w_1, w_2, w_5, w_6\}$. Accordingly, $[\![K\neg Kp_3]\!] = [\![K\neg Kp_5]\!] = \emptyset$ and then $[\![\neg K\neg Kp_3]\!] = [\![\neg K\neg Kp_5]\!] = W$. From which, $[\![\neg K\neg Kp_3]\!] \cap [\![\neg K\neg Kp_5]\!] = W$ and therefore $[\![\neg K\neg Kp_3 \wedge \neg K\neg Kp_5]\!] = W$. Moreover $[\![p_3 \wedge p_5]\!] = \{w_3, w_4, w_5, w_6\}$ and therefore $[\![K(p_3 \wedge p_5)]\!] = \emptyset$. It follows that $[\![\neg K(p_3 \wedge p_5)]\!] = W = [\![K\neg K(p_3 \wedge p_5)]\!]$. Accordingly $[\![\neg K\neg K(p_3 \wedge p_5)]\!] = \emptyset$. We conclude that $[\![\neg K\neg Kp_3 \wedge \neg K\neg Kp_5]\!] \not\subseteq [\![\neg K\neg K(p_3 \wedge p_5)]\!]$.
- (Agg_D) Then $[\![kp_3]\!] = \{w_4, w_5, w_6\}$ and $[\![kp_5]\!] = \{w_4\}$. Hence, $[\![\neg kp_3]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg kp_5]\!] = \{w_1, w_2, w_3, w_5, w_6\}$. Accordingly, $[\![K\neg kp_3]\!] = [\![K\neg kp_5]\!] = \emptyset$. Following the analogous steps of the previous case, we obtain $[\![\neg K\neg kp_3 \wedge \neg K\neg kp_5]\!] = W$. Moreover $[\![k(p_3 \wedge p_5)]\!] = \emptyset$. Following again the analogous steps of the previous case, we obtain $[\![\neg K\neg k(p_3 \wedge p_5)]\!] = \emptyset$. We conclude that $[\![\neg K\neg kp_3 \wedge \neg K\neg kp_5]\!] \not\subseteq [\![\neg K\neg k(p_3 \wedge p_5)]\!]$.

4 Conclusion

I showed that it is possible to build an i-model invalidating all the formulas that Rosenkranz (2021) considers undesirable for his idealized logic: all the ones he has already invalidated, with the addition of $K\varphi \to k\varphi$ and $k\top$ (equivalent to RN_k). The collapse of K and k into one another and an unwelcome idealization related to logical omniscience are thus avoided. Constructing a series of countermodels would have been sufficient in order to show that those formulas are not theorems of the logic. Nonetheless, having provided a single countermodel shows a stronger result, namely that they can all be invalidated at once: we don't need to assume one to invalidate another. An additional positive feature of the new i-frame is that it avoids two idealized solutions discussed in Section 2, i.e., invalidating $K\varphi \rightarrow k\varphi$ only because the formula is false in some $w \in W$ such that $R(w) = \emptyset$ or $R(w) = \{W\}$. While the former corresponds to total ignorance in w, the latter corresponds to the circumstance in which only theorems are known in w. Both solutions, albeit available, were avoided in order to provide an i-frame corresponding to a more realistic epistemic scenario.

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Code Availability Not applicable

Declarations

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