Shape Perturbation of Grushin Eigenvalues

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Abstract

We consider the spectral problem for the Grushin Laplacian subject to homogeneous Dirichlet boundary conditions on a bounded open subset of $\mathbb{R}^N$. We prove that the symmetric functions of the eigenvalues depend real analytically upon domain perturbations and we prove an Hadamard-type formula for their shape differential. In the case of perturbations depending on a single scalar parameter, we prove a Rellich–Nagy-type theorem which describes the bifurcation phenomenon of multiple eigenvalues. As corollaries, we characterize the critical shapes under isovolumetric and isoperimetric perturbations in terms of overdetermined problems and we deduce a new proof of the Rellich–Pohozaev identity for the Grushin eigenvalues.

1 Introduction

In this paper, we consider the following degenerate elliptic operator in $\mathbb{R}^N$:

$$\Delta_G := \Delta_x + |x|^{2s} \Delta_y, \qquad s \in \mathbb{N}.$$

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Here and throughout the paper $N \in \mathbb{N}$, $N \geq 2$, $h, k \in \mathbb{N}$, $h + k = N$, $x \in \mathbb{R}^h$, $y \in \mathbb{R}^k$, where $\mathbb{N}$ denotes the set of positive integers. The vector x denotes the first h components of a vector $z \in \mathbb{R}^N$, and similarly y denotes the last k ones, *i.e.* $z = (x, y) \in \mathbb{R}^h \times \mathbb{R}^k = \mathbb{R}^N$. By Δ_x and Δ_y we denote the standard Laplacians with respect to the x and y variables, respectively. The operator Δ_G is nowadays known as the Grushin Laplacian, and has been introduced in a preliminary version by Baouendi [5] and Grushin [27,28]. In [5], Baouendi has studied the regularity of the solutions of a boundary value problem for an elliptic operator, whose coefficients may vanish on the boundary of the open set where the problem is considered. In [27,28], Grushin has considered a class of operators that degenerate on a submanifold. Later on, a more general notion of the Grushin Laplacian has been introduced and studied by Franchi and Lanconelli [19–21]. In recent years, these operators have been studied under several points of view. Here, we mention just a few contributions, without the aim of completeness. For example, inequalities and estimates related to the Grushin operator have been investigated by many authors. D'Ambrosio [11] has studied Hardy inequalities related to Grushin-type operators. Garofalo and Shen [26] have obtained Carleman estimates and unique continuation for the Grushin operator. Symmetry, existence and uniqueness properties of extremal functions for the weighted Sobolev inequality are obtained in Monti [45]. Monticelli et al. [48] have obtained Poincaré inequalities for Sobolev spaces with matrix-valued weights with applications to the existence and uniqueness of solutions to linear elliptic and parabolic degenerate partial differential equations. Furthermore, several authors have investigated issues related to the solutions to problems for degenerate equations. Kogoj and Lanconelli have proved in [33] a Liouville theorem for a class of linear degenerate elliptic operators, whereas in [34] they have obtained some existence, nonexistence and regularity results for boundary value problems for semilinear degenerate equations. Monticelli [46] has obtained a maximum principle for a class of linear degenerate elliptic differential operators of the second order. Thuy and Tri [58,59] and Tri [60,62] have analyzed boundary value problems for linear or semilinear degenerate elliptic differential equations. Other references can be found in the survey of Kogoj and Lanconelli [35], where the authors have discussed linear and semilinear problems involving the Δ_λ–Laplacians, which contain, as a particular case, the operator introduced by Baouendi and Grushin.

In our work, we are interested in the eigenvalue problem

$$- \Delta_G u = \lambda u, \tag{1}$$

with zero Dirichlet boundary conditions on a variable bounded open subset Ω of $\mathbb{R}^N$. It is well-known that problem (1) admits a divergent sequence of domain dependent eigenvalues of finite multiplicity:

$$0 < \lambda_1[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \cdots \to +\infty.$$

Our main aim is to understand the dependence of the eigenvalues $\lambda_n[\Omega]$, both simple and multiple, upon perturbation of the domain Ω. In particular, we plan to extend the results of Lamberti and Lanza de Cristoforis [41] for the Laplacian and of Buoso

and Lamberti [7,8] for polyharmonic operators and systems to the case of the Grushin Laplacian Δ_G.

Shape sensitivity analysis and shape optimization of quantities and functionals related to partial differential equations are vast topics which have been investigated by several authors with different techniques. We mention, for example, the monographs by Bucur and Buttazzo [6], Daners [12], Delfour and Zolésio [14], Henrot [29], Henrot and Pierre [30], Novotny and Sokołowski [51], Novotny et al. [52], Pironneau [53], and Sokołowski and Zolésio [57]. One of the central problems concerns the analysis of the dependence of the eigenvalues of partial differential operators upon domain perturbations. Many authors studied the qualitative behavior of the eigenvalues of various partial differential operators with respect to shape perturbations proving, for example, continuity, smoothness or even analyticity results. In addition to the above monographs, we mention in this direction the works of Arendt and Daners [1], Arrieta [2], Arrieta and Carvalho [3], Buoso and Lamberti [7,8], Buoso and Provenzano [9], El Soufi and Ilias [16], Fall and Weth [18], Lamberti and Lanza de Cristoforis [41], and Prodi [54]. These issues are closely related to the shape optimization of eigenvalues. Indeed, a first step towards the maximization or minimization of an eigenvalue under suitable constraints (such as fixed volume or perimeter) is to study critical shapes. Accordingly, a detailed analysis of the regularity upon shape perturbations and of the shape differential is crucial for this kind of optimization problems. The problem of minimizing the first eigenvalue of the Dirichlet Laplacian has been solved by Faber [17] and Krahn [36], and later on other authors have generalized their result to different operators (see, *e.g.*, Ashbaugh and Benguria [4] and Nadirashvili [49]). However, in general, finding the shapes which optimize a certain eigenvalue is a hard problem which remains open for several well-studied operators, including the Grushin Laplacian.

Another point of view in spectral shape sensitivity analysis is proving quantitative stability estimates for the eigenvalues in terms of some notion of vicinity of sets. For this topic, which is outside the scope of the present work, we refer to the survey of Burenkov, Lamberti and Lanza de Cristoforis [10].

This paper is in the spirit of studying the qualitative behavior of the eigenvalues of the Grushin Laplacian upon shape perturbations. Namely, in contrast with other approaches in the literature which address only continuity and differentiability issues, in Theorem 5.1 we prove that the symmetric functions of the eigenvalues depend real analytically upon shape perturbations. We note that considering the symmetric functions of the eigenvalues, and not the eigenvalues themselves, is a natural choice. Indeed, a perturbation of the domain can split a multiple eigenvalue into different eigenvalues of lower multiplicity and thus the corresponding branches can have a corner at the splitting point. Furthermore, we obtain the Grushin analog of the Hadamard formula for the shape differential (see formula (30) of Theorem 5.3). In the case of perturbations depending real analytically on a single scalar parameter ε, we prove a Rellich–Nagy-type theorem which describes the bifurcation phenomenon of multiple eigenvalues that we mentioned before. More precisely, given an eigenvalue λ of multiplicity m on Ω and a family of perturbations $\{\phi_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ of Ω depending real analytically on ε and such that ϕ_0 is the identity, our result guarantees that all the branches splitting from λ at $\varepsilon = 0$ are described by m real analytic functions of ε. Moreover, the right derivatives at $\varepsilon = 0$ of the branches splitting from λ coincide with the eigenvalues of

the matrix

$$\left(-\int_{\partial\Omega}\left(\frac{d}{d\varepsilon}\phi_\varepsilon\Big|_{\varepsilon=0}\mathbf{n}^t\right)\frac{\partial v_i}{\partial\mathbf{n}}\frac{\partial v_j}{\partial\mathbf{n}}|\mathbf{n}_G|^2\,d\sigma\right)_{i,j=1,\ldots,m},\tag{2}$$

where $\{v_i\}_{i=1,\ldots,m}$ is an orthonormal basis in $L^2(\Omega)$ of the eigenspace corresponding to λ, $\mathbf{n}$ is the outer unit normal field to $\partial\Omega$, and $\mathbf{n}_G := (\mathbf{n}_x, |x|^s\mathbf{n}_y)$ (see formula (32)). We note that formula (2) and the analogous formulas of the paper based on surface integrals are obtained by assuming that the eigenfunctions are of class $W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)$ or at least of class $W_0^{1,2}(U)\cap W^{2,2}(U)$ for some neighborhood U of the support of the perturbation, here $\frac{d}{d\varepsilon}\phi_\varepsilon\Big|_{\varepsilon=0}$. This assumption is clearly automatically guaranteed by classical regularity theory when $\overline{U}$ does not intersect the set $\{x = 0\}$ (see also Remark 5.8). In any case, our formulas are also presented in an alternative form involving only volume integrals, in which case extra regularity assumptions are not required. Although we do not enter in regularity issues for the solutions of Grushin-type equations, we note that the $W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)$ regularity assumption is satisfied for suitable classes of domains (for instance, if the domain is smooth and has no characteristic points). In this regard, we refer to Kohn and Nirenberg [37] and Jerison [32].

Finally, we show two consequences of our analysis. First, motivated by shape optimization problems, we characterize the critical shapes under isovolumetric and isoperimetric perturbations. In Theorem 6.1, we prove that if a domain Ω is a critical set under the volume constraint $\mathrm{Vol}(\Omega) = \mathrm{const.}$ for the symmetric functions of the eigenvalues bifurcating from an eigenvalue λ of multiplicity m, then

$$\sum_{l=1}^m\left(\frac{\partial v_l}{\partial\mathbf{n}}\right)^2|\mathbf{n}_G|^2 = c \qquad \text{on } \partial\Omega\setminus\{x=0\},$$

for some constant c. Next, we consider the same problem under the perimeter constraint $\mathrm{Per}(\Omega) = \mathrm{const.}$ and in Theorem 6.2 we obtain the additional condition

$$\sum_{l=1}^m\left(\frac{\partial v_l}{\partial\mathbf{n}}\right)^2|\mathbf{n}_G|^2 = c\mathcal{H} \qquad \text{on } \partial\Omega\setminus\{x=0\},$$

where $\mathcal{H}$ is the mean curvature of $\partial\Omega$. Under suitable regularity assumptions on the eigenfunctions, the above extra conditions are also sufficient for Ω to be critical. As a second consequence, we obtain a new simple proof of the Rellich–Pohozaev identity for the Grushin eigenvalues, *i.e.*

$$\lambda = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v}{\partial\mathbf{n}}\right)^2|\mathbf{n}_G|^2((x,(1+s)y)\cdot\mathbf{n})\,d\sigma_z,$$

where v is an eigenfunction normalized in $L^2(\Omega)$.

The paper is organized as follows: In Sect. 2 we introduce some notation and preliminaries. Section 3 is devoted to the eigenvalue problem for the Dirichlet Grushin

Laplacian and to some well-known basic results about it. In Sect. 4 we define the set of admissible domain perturbations ϕ's and we prove that the ϕ-pullback is a linear homeomorphism. Section 5 contains our main results, namely, we show that the symmetric functions of the eigenvalues depend real analytically upon shape perturbations and we prove the Hadamard formula and the Rellich–Nagy-type theorem. In Sect. 6 we characterize the critical shapes under isovolumetric and isoperimetric perturbations and we formulate the corresponding overdetermined problems. Finally, in Sect. 7 we provide a new proof of the Rellich–Pohozaev formula for the Grushin eigenvalues.

2 Notation and Preliminaries

To deal with the Grushin Laplacian Δ_G, we need to introduce a well-known class of associated weighted Sobolev spaces. Let U be a bounded open subset of $\mathbb{R}^N$. We denote by $W_G^{1,2}(U)$ the space of real-valued functions in $L^2(U)$ such that $\partial_{x_i} u \in L^2(U)$ for all $i \in \{1, \ldots, h\}$ and $|x|^s \partial_{y_j} u \in L^2(U)$ for all $j \in \{1, \ldots, k\}$. The space $W_G^{1,2}(U)$ can be endowed with the following scalar product:

$$
\langle u, v \rangle_{W_G^{1,2}(U)} := \langle u, v \rangle_{L^2(U)} + \sum_{i=1}^{h} \langle \partial_{x_i} u, \partial_{x_i} v \rangle_{L^2(U)}
$$
$$
+ \sum_{j=1}^{k} \langle |x|^s \partial_{y_j} u, |x|^s \partial_{y_j} v \rangle_{L^2(U)},
$$

for all $u, v \in W_G^{1,2}(U)$. Here, $\langle \cdot, \cdot \rangle_{L^2(U)}$ denotes the standard scalar product in $L^2(U)$. It is well-known that the space $W_G^{1,2}(U)$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{W_G^{1,2}(U)}$ is a Hilbert space. The norm induced by the scalar product $\langle \cdot, \cdot \rangle_{W_G^{1,2}(U)}$ is

$$
\|u\|_{W_G^{1,2}(U)} := \left(\|u\|_{L^2(U)}^2 + \sum_{i=1}^{h} \|\partial_{x_i} u\|_{L^2(U)}^2 + \sum_{j=1}^{k} \||x|^{2s} \partial_{y_j} u\|_{L^2(U)}^2 \right)^{1/2}
$$

for all $u \in W_G^{1,2}(U)$. Throughout the paper we use the following notation:

$$
I_G(z) := \begin{pmatrix} I_{h \times h} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & |x|^s I_{k \times k} \end{pmatrix} \qquad \forall z = (x, y) \in \mathbb{R}^N,
$$

where $I_{h \times h}$ and $I_{k \times k}$ denote the $h \times h$ and $k \times k$ identity matrices, respectively, whereas $\mathbf{0}_{h \times k}$ and $\mathbf{0}_{k \times h}$ denote the $h \times k$ and $k \times h$ null matrices, respectively. Moreover, if $u \in W_G^{1,2}(U)$ we set

$$
\nabla_G u := (\partial_{x_1} u, \ldots, \partial_{x_h} u, |x|^s \partial_{y_1} u, \ldots, |x|^s \partial_{y_m} u) = \nabla u \, I_G, \tag{3}
$$

and we refer to $\nabla_G u$ as the Grushin gradient of u. We note that if u is in $W_G^{1,2}(U)$, in general its gradient ∇u is a distribution. However, if by ∇u we mean the function defined a.e. in U as the distributional gradient of u in $U \setminus \{x = 0\}$, then the last equality of (3) is not only formal but holds almost everywhere. The norm $\| \cdot \|_{W_G^{1,2}(U)}$ is equivalent to the norm $\| \cdot \|'_{W_G^{1,2}(U)}$ defined by

$$\|u\|'_{W_G^{1,2}(U)} \equiv \|u\|_{L^2(U)} + \|\nabla_G u\|_{L^2(U)} \quad \forall u \in W_G^{1,2}(U).$$

Remark 2.1 If $\overline{U} \cap \{x = 0\} = \varnothing$, then the norm in $W_G^{1,2}(U)$ is equivalent to the standard Sobolev norm of $W^{1,2}(U)$.

We denote by $W_{G,0}^{1,2}(U)$ the closure of $C_c^\infty(U)$ in $W_G^{1,2}(U)$. In Theorem 2.1 below, we present an analog of the Rellich–Kondrachov embedding theorem that holds for the Sobolev space $W_{G,0}^{1,2}(U)$. For a proof we refer to the works of Franchi and Serapioni [22, Theorem 4.6] and of Kogoj and Lanconelli [34, Proposition 3.2], which consider a more general class of weighted Sobolev spaces.

Theorem 2.1 (Rellich–Kondrachov) *Let U be a bounded open subset of $\mathbb{R}^N$. Then the space $W_{G,0}^{1,2}(U)$ is compactly embedded in $L^2(U)$.*

It is also known that an analog of the Poincaré inequality holds in the space $W_{G,0}^{1,2}(U)$. Namely, the following theorem holds (for a proof, see, *e.g.*, D'Ambrosio [11, Theorem 3.7], Monticelli and Payne [47, Theorem 2.1] and Monticelli et al. [48]).

Theorem 2.2 (Poincaré inequality) *Let U be a bounded open subset of $\mathbb{R}^N$. Then, there exists $C > 0$ such that*

$$\|u\|_{L^2(U)} \le C \|\nabla_G u\|_{L^2(U)} \quad \forall u \in W_{G,0}^{1,2}(U).$$

3 The Eigenvalue Problem

Here, we introduce the precise formulation of the eigenvalue problem. We fix U to be a bounded open subset of $\mathbb{R}^N$. The classical spectral problem is

$$\begin{cases} -\Delta_G u = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4}$$

in the unknowns λ (the eigenvalues) and u (the eigenfunctions). Actually, to reduce the regularity assumptions, we consider the weak formulation of problem (4). Namely,

$$\int_U \nabla_G u \cdot \nabla_G \varphi \, dz = \lambda \int_U u\varphi \, dz \quad \forall \varphi \in W_{G,0}^{1,2}(U), \tag{5}$$

in the unknowns $\lambda \in \mathbb{R}$ and $u \in W_{G,0}^{1,2}(U)$. We use a standard procedure which enables us to reduce the study of the eigenvalues of (5) to an eigenvalue problem for a compact

self-adjoint operator in a Hilbert space. With a slight abuse of notation, we consider the Grushin Laplacian Δ_G as the operator from $W_{G,0}^{1,2}(U)$ to its dual $(W_{G,0}^{1,2}(U))'$ defined by

$$\Delta_G[u][\varphi] := -\int_U \nabla_G u \cdot \nabla_G \varphi \, dz \qquad \forall u, \varphi \in W_{G,0}^{1,2}(U). \tag{6}$$

Next, we define the following bilinear form

$$Q_G[u, v] := -\Delta_G[u][v] \qquad \forall u, v \in W_{G,0}^{1,2}(U).$$

It is easily seen that the bilinear form Q_G is continuous. Moreover, by the Poincaré inequality of Theorem 2.2, we have that

$$Q_G[u, u] = \int_U |\nabla_G u|^2 \, dz \geq c_2 \|u\|_{W_G^{1,2}(U)}^2 \qquad \forall u \in W_{G,0}^{1,2}(U),$$

for some $c_2 > 0$ and thus the bilinear form Q_G is coercive. In other words, Q_G is a scalar product on $W_{G,0}^{1,2}(U)$ which induces a norm equivalent to the standard one. Thus, we can apply the Riesz representation theorem to deduce that Δ_G is a linear homeomorphism from $W_{G,0}^{1,2}(U)$ onto $(W_{G,0}^{1,2}(U))'$. We denote by J the map from $L^2(U)$ to $(W_{G,0}^{1,2}(U))'$ defined by

$$J[u][\psi] := \langle u, \psi \rangle_{L^2(U)} = \int_U u\psi \, dz \qquad \forall u \in L^2(U), \, \forall \psi \in W_{G,0}^{1,2}(U). \tag{7}$$

Clearly J is continuous and injective. Equation (5) can be rewritten as

$$-\Delta_G^{(-1)} \circ J \circ i[u] = \mu u, \tag{8}$$

where $\mu = \lambda^{-1}$ and i is the embedding of $W_{G,0}^{1,2}(U)$ in $L^2(U)$. Accordingly, it is natural to consider the operator T_G from $W_{G,0}^{1,2}(U)$ to itself defined by

$$T_G[u] := -\Delta_G^{(-1)} \circ J \circ i[u] \qquad \forall u \in W_{G,0}^{1,2}(U).$$

Since the embedding i is compact by Theorem 2.1, T_G is compact in $W_{G,0}^{1,2}(U)$. Moreover, T_G is self-adjoint in $W_{G,0}^{1,2}(U)$ endowed with the scalar product Q_G. Indeed,

$$Q_G[T_G u, v] = -\Delta_G[T_G u][v] = \Delta_G\left[\Delta_G^{(-1)} \circ J \circ i[u]\right][v] = \int_U uv \, dz$$

$$\forall u, v \in W_{G,0}^{1,2}(U).$$

Since Q_G is symmetric, we have that $Q_G[T_G u, v] = Q_G[u, T_G v]$. In addition, T_G is injective because it is the composition of injective maps. It follows that the spectrum of T_G is discrete and consists of a sequence of positive eigenvalues $\mu_j[U]$ of finite

multiplicity converging to zero. More precisely, by classical spectral theory, by the min-max principle (see, *e.g.*, Davies [13, Sect. 4.5]), and by the equivalence of the formulations (5) and (8) we have the following.

Theorem 3.1 *The eigenvalues of equation* (5) *have finite multiplicity and can be represented by means of a divergent sequence*

$$0 < \lambda_1[U] \le \lambda_2[U] \le \ldots \le \lambda_j[U] \le \ldots \to +\infty.$$

Moreover, they coincide with the inverse of the eigenvalues $\mu_j[U]$ *of* T_G, *and*

$$\lambda_j[U] = \min_{\substack{E \subseteq W^{1,2}_{G,0}(U) \\ \dim E = j}} \max_{\substack{u \in E \\ u \ne 0}} \frac{\int_U |\nabla_G u|^2 \, dz}{\int_U u^2 \, dz} \quad \forall j \in \mathbb{N}.$$

4 Admissible Domain Perturbations

Since we plan to consider the eigenvalue problem (5) on a variable domain, the first step is to define what we mean by variable domain. Our point of view is to consider a fixed domain and a family of open sets parametrized by a suitable homeomorphism defined on the fixed domain. Accordingly, we fix

a bounded open subset Ω of $\mathbb{R}^N$ and a bounded open subset O of $\mathbb{R}^N$ such that $\varnothing \ne \overline{\Omega} \cap \{x = 0\} \subseteq O$. (9)

From now on if ϕ is a map with values in $\mathbb{R}^N$, we denote by $\phi_x = (\phi_{x_1}, \ldots, \phi_{x_h})$ and by $\phi_y = (\phi_{y_1}, \ldots, \phi_{y_k})$ the first h and the last k components of ϕ, respectively. Moreover, we denote by π_x and by π_y the projections of $\mathbb{R}^N$ to $\mathbb{R}^h$ and $\mathbb{R}^k$ which take $z = (x, y)$ to x and y, respectively. We set

$$L_{\Omega, O} := \Big\{ \phi \in \mathrm{Lip}(\Omega)^N : \exists \, \tilde{\phi}_x \in \mathrm{Lip}(\pi_x(\Omega \cap O))^h, \ \tilde{\phi}_y \in \mathrm{Lip}(\pi_y(\Omega \cap O))^k$$
$$\text{s.t. } \phi = (\tilde{\phi}_x \circ \pi_x, \tilde{\phi}_y \circ \pi_y) \text{ in } \Omega \cap O, \ \tilde{\phi}_x(0) = 0 \Big\}.$$

It is easily seen that $L_{\Omega, O}$ is a closed linear subspace of the Banach space $\mathrm{Lip}(\Omega)^N$, where $\mathrm{Lip}(\Omega)$ denotes the Banach space of Lipschitz functions in Ω endowed with the norm $\sup_{\Omega} |f| + \sup_{\substack{z_1, z_2 \in \Omega \\ z_1 \ne z_2}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}$. Therefore, $L_{\Omega, O}$ is a Banach space itself. We define the space of admissible shape perturbations as

$$\mathcal{A}_{\Omega, O} := \Big\{ \phi \in L_{\Omega, O} : \inf_{\substack{z_1, z_2 \in \Omega \\ z_1 \ne z_2}} \frac{|\phi(z_1) - \phi(z_2)|}{|z_1 - z_2|} > 0 \Big\}.$$
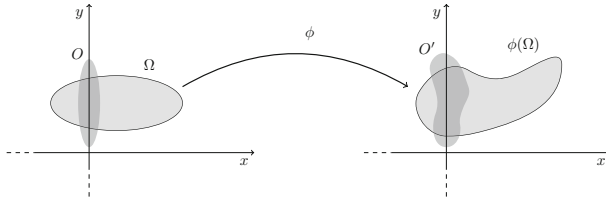
see Fig. 1.

Fig. 1 An example, in dimension $N = 2$, of a homeomorphism $\phi \in \mathcal{A}_{\Omega,O}$ for a possible choice of fixed Ω and O. In the intersection between Ω and O, the homeomorphism ϕ deforms separately the x and y directions

By Lamberti and Lanza de Cristoforis [41, Lemma 3.11], if $\phi \in \mathcal{A}_{\Omega,O}$ then ϕ is injective and $\inf_\Omega |\det D\phi| > 0$. Moreover, $\phi(\Omega)$ is a bounded open set and the inverse map $\phi^{(-1)}$ belongs to $\mathcal{A}_{\phi(\Omega),O'}$ for some open set O' containing $\overline{\phi(\Omega)} \cap \{x = 0\}$.

Remark 4.1 If $\phi \in \mathcal{A}_{\Omega,O}$, then ϕ is a bi-Lipschitz homeomorphism from Ω into his image that near the degenerate set, *i.e.* inside $\Omega \cap O$, deforms separately the x-direction and the y-direction. Moreover, if a point belongs to the degenerate set $\{x = 0\}$, then its image through ϕ has to remain on the degenerate set. Since ϕ is bi-Lipschitz, it is easily seen that there exists $C > 0$ such that $\frac{1}{C}|x| \leq \phi_x(z) \leq C|x|$ for all $z \in \Omega$. Finally, it is worth noting that our setting includes both the case in which the degenerate set $\{x = 0\}$ intersects Ω and the case in which part of the boundary of Ω lies on the degenerate set.

For a transformation $\phi \in \mathcal{A}_{\Omega,O}$, we are able to prove that the ϕ-pushforward (or equivalently the ϕ-pullback), that we will use to transplant the problem to the fixed domain Ω, is a linear homeomorphism.

Lemma 4.1 *Let Ω and O be as in* (9). *Let $\phi \in \mathcal{A}_{\Omega,O}$. Then, the operator $C_{\phi^{(-1)}}$ defined by*

$$C_{\phi^{(-1)}}[u] := u \circ \phi^{(-1)} \qquad \forall u \in L^2(\Omega),$$

is a linear homeomorphism from $L^2(\Omega)$ to $L^2(\phi(\Omega))$ which restricts a linear homeomorphism from $W_G^{1,2}(\Omega)$ onto $W_G^{1,2}(\phi(\Omega))$ and from $W_{G,0}^{1,2}(\Omega)$ onto $W_{G,0}^{1,2}(\phi(\Omega))$. Moreover $C_{\phi^{(-1)}}^{(-1)} = C_\phi$.

Proof Let $u \in L^2(\Omega)$. There exists $c_1 > 0$ such that

$$\|C_{\phi^{(-1)}}[u]\|_{L^2(\phi(\Omega))}^2 = \int_{\phi(\Omega)} (u \circ \phi^{(-1)}(z))^2 \, dz = \int_\Omega (u(z))^2 |\det D\phi| \, dz$$
$$\leq c_1 \int_\Omega (u(z))^2 \, dz = c_1 \|u\|_{L^2(\Omega)}^2.$$

Thus, $C_{\phi^{(-1)}}$ is continuous from $L^2(\Omega)$ to $L^2(\phi(\Omega))$. Since $C_{\phi^{(-1)}}$ is clearly surjective, the Open Mapping Theorem implies that it is a linear homeomorphism from $L^2(\Omega)$ to $L^2(\phi(\Omega))$.

Next, we fix $u \in C^1(\Omega) \cap W_G^{1,2}(\Omega)$. Since ϕ is invertible, we have that $\phi(\Omega \cap O) \cap \phi(\Omega \setminus O) = \varnothing$ and that $\phi(\Omega \cap O) \cup \phi(\Omega \setminus O) = \phi(\Omega)$ and thus

$$
\begin{aligned}
\||\nabla_G C_{\phi^{(-1)}}[u]\||_{L^2(\phi(\Omega))}^2 &= \int_{\phi(\Omega)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2 \, dz \\
&= \int_{\phi(\Omega \cap O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2 \, dz \\
&\quad + \int_{\phi(\Omega \setminus O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2 \, dz.
\end{aligned}
\tag{10}
$$

We now consider the first summand $\int_{\phi(\Omega \cap O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2 \, dz$ in the right hand side of (10). We have

$$
\begin{aligned}
&\int_{\phi(\Omega \cap O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2 \, dz \\
&= \int_{\phi(\Omega \cap O)} |\nabla u(\phi^{(-1)}(z))(D\phi^{(-1)}(z))I_G(z)|^2 \, dz \\
&= \int_{\Omega \cap O} |\nabla u(z)(D\phi(z))^{-1}I_G(\phi(z))|^2 |\det D\phi(z)| \, dz.
\end{aligned}
$$

We now observe that for almost every $z \in \Omega \cap O$ we have

$$
\begin{aligned}
(D\phi(z))^{-1}I_G(\phi(z)) &= \begin{pmatrix} D_x\phi_x(z) & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & D_y\phi_y(z) \end{pmatrix}^{-1} \begin{pmatrix} I_{h \times h} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & |\phi_x(z)|^s I_{k \times k} \end{pmatrix} \\
&= \begin{pmatrix} (D_x\phi_x(z))^{-1} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & (D_y\phi_y(z))^{-1} \end{pmatrix} \begin{pmatrix} I_{h \times h} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & |\phi_x(z)|^s I_{k \times k} \end{pmatrix} \\
&= \begin{pmatrix} I_{h \times h} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & |\phi_x(z)|^s I_{k \times k} \end{pmatrix} \begin{pmatrix} (D_x\phi_x(z))^{-1} & \mathbf{0}_{h \times k} \\ \mathbf{0}_{k \times h} & (D_y\phi_y(z))^{-1} \end{pmatrix} \\
&= I_G(\phi(z))(D\phi(z))^{-1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\Omega \cap O} |\nabla u(z)(D\phi(z))^{-1}I_G(\phi(z))|^2 |\det D\phi(z)| \, dz \\
&= \int_{\Omega \cap O} |\nabla u(z)I_G(\phi(z))(D\phi(z))^{-1}|^2 |\det D\phi(z)| \, dz.
\end{aligned}
$$

Then, we note that

$$
\nabla u(z)I_G(\phi(z)) = (\nabla_x u(z), |\phi_x(z)|^s \nabla_y u(z)) \qquad \text{for almost every } z \in \Omega,
$$

and that there exists a constant $C > 0$ such that $\frac{1}{C}|x| \leq |\phi_x(z)| \leq C|x|$ for all $z \in \Omega$. As a consequence, since $\phi \in \mathrm{Lip}(\Omega)^N$, we deduce the existence of $c_2 > 0$ such that

$$\int_{\Omega \cap O} |\nabla u(z) I_G(\phi(z))(D\phi(z))^{-1}|^2| \det D\phi(z)|\, dz \leq c_2 \int_{\Omega \cap O} |\nabla_G u(z)|^2\, dz,$$

and accordingly that

$$\int_{\phi(\Omega \cap O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2\, dz \leq c_2 \int_{\Omega \cap O} |\nabla_G u(z)|^2\, dz. \tag{11}$$

We now turn to the second summand $\int_{\phi(\Omega \setminus O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2\, dz$ in the right hand side of (10). By Remark 2.1, the $W_G^{1,2}$-norm is equivalent to the standard Sobolev norm of $W^{1,2}$ if we are far from the degenerate set $\{x = 0\}$ and thus there exist $c_3, c_4, c_5 > 0$ such that

$$\begin{aligned}
\int_{\phi(\Omega \setminus O)} |\nabla_G(u \circ \phi^{(-1)})(z)|^2\, dz &\leq c_3 \int_{\phi(\Omega \setminus O)} |\nabla(u \circ \phi^{(-1)})(z)|^2\, dz \\
&= c_3 \int_{\Omega \setminus O} |\nabla u(z)(D\phi(z))^{-1}|^2| \det D\phi(z)|\, dz \\
&\leq c_4 \int_{\Omega \setminus O} |\nabla u(z)|^2\, dz \\
&\leq c_5 \int_{\Omega \setminus O} |\nabla_G u(z)|^2\, dz.
\end{aligned} \tag{12}$$

Thus by (10) and by summing up the inequalities in (11) and (12), there exists $c_6 > 0$ such that

$$\| |\nabla_G C_{\phi^{(-1)}}[u]| \|^2_{L^2(\phi(\Omega))} \leq c_6 \| |\nabla_G u| \|^2_{L^2(\Omega)}.$$

Since $C_{\phi^{(-1)}}$ is continuous from $L^2(\Omega)$ to $L^2(\phi(\Omega))$, since $W_G^{1,2}(\Omega)$ is continuously embedded in $L^2(\Omega)$, and since $C^1(\Omega) \cap W_G^{1,2}(\Omega)$ is dense in $W_G^{1,2}(\Omega)$ (see Franchi et al. [23]), one can realize that $C_{\phi^{(-1)}}$ is continuous from $W_G^{1,2}(\Omega)$ to $W_G^{1,2}(\phi(\Omega))$. To show the surjectivity, we take $v \in W_G^{1,2}(\phi(\Omega))$. Following the same argument as above together with the inequality $\frac{1}{C}|x| \leq |\phi_x(z)|$ one can realize that $v \circ \phi \in W_G^{1,2}(\Omega)$ and, clearly, $C_{\phi^{(-1)}}[v \circ \phi] = v$. By the Open Mapping Theorem $C_{\phi^{(-1)}}$ is a linear homeomorphism from $W_G^{1,2}(\Omega)$ to $W_G^{1,2}(\phi(\Omega))$.

Finally, by a standard mollification argument $C_{\phi^{(-1)}}[u] \in W_{G,0}^{1,2}(\phi(\Omega))$ for all $u \in C_c^\infty(\phi(\Omega))$. Therefore, since $W_{G,0}^{1,2}(\phi(\Omega))$ is a closed subspace of $W_G^{1,2}(\phi(\Omega))$, we have that $C_{\phi^{(-1)}}[u] \in W_{G,0}^{1,2}(\phi(\Omega))$ for all $u \in W_{G,0}^{1,2}(\Omega)$ and thus $C_{\phi^{(-1)}}$ restricts a linear homeomorphism from $W_{G,0}^{1,2}(\Omega)$ onto $W_{G,0}^{1,2}(\phi(\Omega))$. The last part of the statement is obvious. $\qquad \square$

5 Analyticity Results and Hadamard Formula

In this section, we perform the shape sensitivity analysis of the Grushin eigenvalue problem. As in the previous section, we fix Ω and O as in (9) and $\phi \in \mathcal{A}_{\Omega,O}$. We consider

$$\int_{\phi(\Omega)} \nabla_G v \cdot \nabla_G \psi \, dz = \lambda \int_{\phi(\Omega)} v \psi \, dz \quad \forall \psi \in W^{1,2}_{G,0}(\phi(\Omega)), \tag{13}$$

in the unknowns $v \in W^{1,2}_{G,0}(\phi(\Omega))$ and $\lambda \in \mathbb{R}$. By the results of Sect. 3, the eigenvalues of equation (13) have finite multiplicity and can be represented by means of a divergent sequence

$$0 < \lambda_1[\phi] \le \lambda_2[\phi] \le \ldots \le \lambda_j[\phi] \le \ldots \to +\infty,$$

where we have set

$$\lambda_j[\phi] := \lambda_j[\phi(\Omega)] \quad \forall j \in \mathbb{N}.$$

In general, if we want to study the regularity of an eigenvalue upon a parameter, which in our case is ϕ, we face a first problem. Namely, we cannot expect to prove smooth dependence of the eigenvalues themselves upon the parameter, when the eigenvalues are not simple. This is due to bifurcation phenomena of splitting from a multiple eigenvalue to different eigenvalues of lower multiplicity (cf. Rellich [56, p. 37]). Hence, to circumvent this problem, we consider the elementary symmetric functions of the eigenvalues. This is the point of view introduced by Lamberti and Lanza de Cristoforis [41] and later adopted in many other works (see, *e.g.*, [7,8,40,43]). Clearly, when a certain eigenvalue is simple, for example in the case of the first Grushin eigenvalue under the assumption that $\Omega \setminus \{x = 0\}$ is connected (see Monticelli and Payne [47, Theorem 6.4]), our regularity result for the symmetric functions of the eigenvalues implies that the same regularity is valid for the eigenvalue.

To perform this strategy, we need to introduce two subspaces of $\mathcal{A}_{\Omega,O}$. Let $F \subseteq \mathbb{N}$ be a finite set of indexes and we consider the subset of $\mathcal{A}_{\Omega,O}$ of those maps for which the eigenvalues with index in F do not coincide with the eigenvalues with index outside F. That is

$$\mathcal{A}^F_{\Omega,O} := \left\{ \phi \in \mathcal{A}_{\Omega,O} : \lambda_n[\phi] \neq \lambda_m[\phi], \, \forall n \in F, \, \forall m \in \mathbb{N} \setminus F \right\}.$$

We find also convenient to consider the set $\Theta^F_{\Omega,O}$ of those maps in $\mathcal{A}^F_{\Omega,O}$ such that all the eigenvalues with index in F coincide. Namely,

$$\Theta^F_{\Omega,O} := \left\{ \phi \in \mathcal{A}^F_{\Omega,O} : \lambda_n[\phi] = \lambda_m[\phi], \, \forall n, m \in F \right\}.$$

For $\phi \in \mathcal{A}_{\Omega,O}$ we introduce the following two operators.

(i) J_ϕ is the map from $L^2(\Omega)$ to $(W^{1,2}_{G,0}(\Omega))'$ defined by

$$J_\phi[u][v] := \int_\Omega uv |\det D\phi|\, dz \quad \forall u \in L^2(\Omega),\ \forall v \in W^{1,2}_{G,0}(\Omega).$$

(ii) $\Delta_{G,\phi}$ is the map from $W^{1,2}_{G,0}(\Omega)$ to $(W^{1,2}_{G,0}(\Omega))'$ defined by

$$\Delta_{G,\phi}[u][v] := -\int_\Omega \nabla u (D\phi)^{-1} I_G(\phi)(\nabla v (D\phi)^{-1} I_G(\phi))^t |\det D\phi|\, dz$$
$$\forall u, v \in W^{1,2}_{G,0}(\Omega).$$

Remark 5.1 Let $\phi \in \mathcal{A}_{\Omega,O}$. By performing a change of variable, one can readily verify that

$$J_\phi = C^t_{\phi^{(-1)}} \circ J \circ C_{\phi^{(-1)}} \qquad \Delta_{G,\phi} = C^t_{\phi^{(-1)}} \circ \Delta_G \circ C_{\phi^{(-1)}},$$

being J and Δ_G the operators defined in (7) and (6) with $U = \phi(\Omega)$, respectively, and $C_{\phi^{(-1)}}$ the ϕ-pushforward operator introduced in Lemma 4.1.

Since Δ_G is a linear homeomorphism from $W^{1,2}_{G,0}(\phi(\Omega))$ onto $(W^{1,2}_{G,0}(\phi(\Omega)))'$ and since J is a linear and continuous injection from $L^2(\phi(\Omega))$ to $(W^{1,2}_{G,0}(\phi(\Omega)))'$, Lemma 4.1 immediately implies the following.

Corollary 5.1 Let Ω and O be as in (9) and $\phi \in \mathcal{A}_{\Omega,O}$. Then the operator $\Delta_{G,\phi}$ is a linear homeomorphism $W^{1,2}_{G,0}(\Omega)$ onto $(W^{1,2}_{G,0}(\Omega))'$ and the operator J_ϕ is a linear and continuous injection from $L^2(\Omega)$ to $(W^{1,2}_{G,0}(\Omega))'$.

Next, in order to reformulate problem (13) into a spectral problem for a compact self-adjoint operator, we set $T_{G,\phi}$ to be the map from $W^{1,2}_{G,0}(\Omega)$ to itself defined by

$$T_{G,\phi}[u] := -\Delta^{(-1)}_{G,\phi} \circ J_\phi \circ i[u] \quad \forall u \in W^{1,2}_{G,0}(\Omega). \tag{14}$$

Here, i denotes the embedding of $W^{1,2}_{G,0}(\Omega)$ in $L^2(\Omega)$. Clearly, equation (13) is equivalent to

$$T_{G,\phi}[u] = \mu u$$

with $u = v \circ \phi$ and $\mu = \lambda^{-1}$. Furthermore, we set

$$Q_{G,\phi}[u, v] := -\Delta_{G,\phi}[u][v] \quad \forall u, v \in W^{1,2}_{G,0}(\Omega). \tag{15}$$

Adapting the same computations of the proof of Lemma 4.1, it is easily seen that $Q_{G,\phi}$ is a scalar product on $W^{1,2}_{G,0}(\Omega)$ which induces a norm equivalent to the standard one in $W^{1,2}_{G,0}(\Omega)$.

We now consider the operator $T_{G,\phi}$ acting on $\left(W^{1,2}_{G,0}(\Omega), Q_{G,\phi}\right)$ and we prove that it is a compact self-adjoint operator and that it depends real analytically on ϕ. Before doing this, we need some notation. If $\mathcal{X}$, $\mathcal{Y}$ are two Banach spaces, we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of linear and continuous operators from $\mathcal{X}$ to $\mathcal{Y}$, we set $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$ and we denote by $\mathcal{B}_s(\mathcal{X})$ the space of bilinear symmetric forms on $\mathcal{X}$. These spaces are endowed with their standard norms.

Proposition 5.1 *Let Ω and O be as in* (9) *and $\phi \in \mathcal{A}_{\Omega,O}$. Then*

(i) *$T_{G,\phi}$ is a compact self-adjoint operator in $\left(W^{1,2}_{G,0}(\Omega), Q_{G,\phi}\right)$.*

(ii) *The map from $\mathcal{A}_{\Omega,O}$ to $\mathcal{L}\left(W^{1,2}_{G,0}(\Omega)\right) \times \mathcal{B}_s\left(W^{1,2}_{G,0}(\Omega)\right)$ which takes ϕ to $(T_{G,\phi}, Q_{G,\phi})$ is real analytic.*

Proof First we consider statement (i). The compactness of $T_{G,\phi}$ is a consequence of the compactness of the embedding of $W^{1,2}_{G,0}(\Omega)$ in $L^2(\Omega)$. For the self-adjointess we note that

$$Q_{G,\phi}[T_{G,\phi}u, v] = -\Delta_{G,\phi}[T_{G,\phi}u][v] = -\Delta_{G,\phi}\left[-\Delta^{(-1)}_{G,\phi} \circ J_\phi \circ i[u]\right][v]$$

$$= J_\phi[i[u]][v] \qquad \forall u, v \in W^{1,2}_{G,0}(\Omega).$$

Next, we prove statement (ii). It is easily seen that the maps which take ϕ to $\Delta_{G,\phi}$, J_ϕ and $Q_{G,\phi}$ from $\mathcal{A}_{\Omega,O}$ to $\mathcal{L}(W^{1,2}_{G,0}(\Omega), (W^{1,2}_{G,0}(\Omega))')$, $\mathcal{L}(L^2(\Omega), (W^{1,2}_{G,0}(\Omega))')$ and $\mathcal{B}_s(W^{1,2}_{G,0}(\Omega))$, respectively, are real analytic. Then, since the map which takes an invertible operator to its inverse is real analytic we can conclude that $T_{G,\phi}$ depends real analytically on ϕ. □

We are ready to prove that the elementary symmetric functions of the eigenvalues depend real analytically upon the domain's shape ϕ.

Theorem 5.1 *Let Ω and O be as in* (9). *Let F be a finite nonempty subset of $\mathbb{N}$. Let $\tau \in \{1, \ldots, |F|\}$. Then $\mathcal{A}^F_{\Omega,O}$ is open in $L_{\Omega,O}$ and the map $\Lambda_{F,\tau}$ from $\mathcal{A}^F_{\Omega,O}$ to $\mathbb{R}$ defined by*

$$\Lambda_{F,\tau}[\phi] := \sum_{\substack{j_1,\ldots,j_\tau \in F \\ j_1 < \cdots < j_\tau}} \lambda_{j_1}[\phi] \cdots \lambda_{j_\tau}[\phi] \qquad \forall \phi \in \mathcal{A}^F_{\Omega,O}$$

is real analytic.

Proof We denote by $\{\mu_j[\phi]\}_{j \in \mathbb{N} \setminus \{0\}}$ the set of eigenvalues of $T_{G,\phi}$. As we have already pointed out $\mu_j[\phi] = (\lambda_j[\phi])^{-1}$. Hence, the set $\mathcal{A}^F_{\Omega,O}$ coincides with the set

$$\left\{\phi \in \mathcal{A}_{\Omega,O} : \mu_n[\phi] \neq \mu_m[\phi], \ \forall n \in F, \ \forall m \in \mathbb{N} \setminus F\right\}.$$

By Proposition 5.1, $T_{G,\phi}$ is self-adjoint with respect to the scalar product $Q_{G,\phi}$ and, moreover, both $T_{G,\phi}$ and $Q_{G,\phi}$ depend analytically on $\phi \in \mathcal{A}^F_{\Omega,O}$. Thus, Lamberti

and Lanza de Cristoforis [41, Theorem 2.30] implies that $\mathcal{A}_{\Omega,O}^F$ is open in $L_{\Omega,O}$ and the map $M_{F,\tau}$ from $\mathcal{A}_{\Omega,O}^F$ to $\mathbb{R}$ defined by

$$M_{F,\tau}[\phi] := \sum_{\substack{j_1,\ldots,j_\tau \in F \\ j_1 < \cdots < j_\tau}} \mu_{j_1}[\phi] \cdots \mu_{j_\tau}[\phi] \qquad \forall \phi \in \mathcal{A}_{\Omega,O}^F \tag{16}$$

is real analytic. If we set $M_{F,0}[\phi] := 1$ for all $\phi \in \mathcal{A}_{\Omega,O}^F$, one can readily verify that

$$\Lambda_{F,\tau}[\phi] = \frac{M_{F,|F|-\tau}[\phi]}{M_{F,|F|}[\phi]} \qquad \forall \phi \in \mathcal{A}_{\Omega,O}^F. \tag{17}$$

Accordingly the statement follows. □

In view of the applications, once we have considered the regularity of the elementary symmetric functions, it is important to have an explicit formula for their shape differential. Thus, our next step is to prove an Hadamard-type formula for the shape differential of the elementary symmetric functions.

Theorem 5.2 *Let Ω and O be as in (9). Let F be a finite nonempty subset of $\mathbb{N}$. Let $\tau \in \{1,\ldots,|F|\}$. Let $\tilde{\phi} \in \Theta_{\Omega,O}^F$ and let $\lambda_F[\tilde{\phi}]$ be the common value of all the eigenvalues $\{\lambda_j[\tilde{\phi}]\}_{j\in F}$. Let $\{v_l\}_{l\in F}$ be an orthonormal basis in $(W_{G,0}^{1,2}(\tilde{\phi}(\Omega)), Q_G)$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$. Then the Frechét differential of the map $\Lambda_{F,\tau}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d_{|\phi=\tilde{\phi}}(\Lambda_{F,\tau})[\psi]$$
$$= -\lambda_F^\tau[\tilde{\phi}]\binom{|F|-1}{\tau-1}\sum_{l\in F}\left\{\int_{\tilde{\phi}(\Omega)}(\lambda_F[\tilde{\phi}]v_l^2 - |\nabla_G v_l|^2)\mathrm{div}\left(\psi\circ\tilde{\phi}^{(-1)}\right)dz\right.$$
$$+ \int_{\tilde{\phi}(\Omega)}(\nabla v_l)(D(\psi\circ\tilde{\phi}^{(-1)})I_G^2 + I_G^2 D(\psi\circ\tilde{\phi}^{(-1)})^t)(\nabla v_l)^t\,dz$$
$$\left. - \int_{\tilde{\phi}(\Omega)}2s|x|^{2s-2}|\nabla_y v_l|^2 x\cdot(\psi\circ\tilde{\phi}^{(-1)})_x\,dz\right\} \qquad \forall\psi\in L_{\Omega,O}. \tag{18}$$

Proof We set $u_l := v_l\circ\tilde{\phi}$ for all $l\in F$ and we note that $\{u_l\}_{l\in F}$ is an orthonormal basis in $\left(W_{G,0}^{1,2}(\Omega), Q_{G,\tilde{\phi}}\right)$ for the eigenspace corresponding to the eigenvalue $\lambda_F^{-1}[\tilde{\phi}]$ of the operator $T_{G,\tilde{\phi}}$. We recall that $M_{F,\tau}$ is the operator defined in (16). By Lamberti and Lanza de Cristoforis [41, Theorem 2.30] it follows that

$$d_{|\phi=\tilde{\phi}}(M_{F,\tau})[\psi] = \lambda_F^{1-\tau}[\tilde{\phi}]\binom{|F|-1}{\tau-1}\sum_{l\in F}Q_{G,\tilde{\phi}}\left[d_{|\phi=\tilde{\phi}}T_{G,\phi}[\psi][u_l], u_l\right]$$

$$\forall\psi\in L_{\Omega,O}.$$

Thus, exploiting formula (17), we have that

$$
\begin{aligned}
d_{|\phi=\tilde{\phi}}&(\Lambda_{F,\tau})[\psi] \\
&= \left\{ d_{|\phi=\tilde{\phi}} M_{F,|F|-\tau}[\psi] M_{F,|F|}[\tilde{\phi}] - M_{F,|F|-\tau}[\tilde{\phi}] d_{|\phi=\tilde{\phi}} M_{F,|F|}[\psi] \right\} \lambda_F^{2|F|}[\tilde{\phi}] \\
&= \left\{ \lambda_F^{1-2|F|+\tau}[\tilde{\phi}] \binom{|F|-1}{|F|-\tau-1} - \lambda_F^{1-2|F|+\tau}[\tilde{\phi}] \binom{|F|}{\tau} \right\} \\
&\quad \times \lambda_F^{2|F|}[\tilde{\phi}] \sum_{l \in F} Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_l], u_l \right] \\
&= -\lambda_F^{1+\tau}[\tilde{\phi}] \binom{|F|-1}{\tau-1} \sum_{l \in F} Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_l], u_l \right] \qquad \forall \psi \in L_{\Omega,O}.
\end{aligned}
$$

(19)

Thus, we have to compute the term $Q_{G,\tilde{\phi}}[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_l], u_l]$. By standard rules of calculus in Banach spaces, by the definition (15) of $Q_{G,\phi}$, and since every u_l is an eigenfunction corresponding to the eigenvalue $\lambda_F^{-1}[\tilde{\phi}]$, we have that

$$
\begin{aligned}
Q_{G,\tilde{\phi}} &\left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_l], u_l \right] \\
&= Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} (-\Delta_{G,\phi}^{(-1)} \circ J_\phi \circ i)[\psi][u_l], u_l \right] \\
&= Q_{G,\tilde{\phi}} \left[-\Delta_{G,\tilde{\phi}}^{(-1)} \circ \left(d_{|\phi=\tilde{\phi}} (J_\phi \circ i)[\psi] \right) [u_l], u_l \right] \\
&\quad + Q_{G,\tilde{\phi}} \left[\left(d_{|\phi=\tilde{\phi}} (-\Delta_{G,\phi}^{(-1)})[\psi] \right) \circ J_{\tilde{\phi}} \circ i[u_l], u_l \right] \\
&= \left(d_{|\phi=\tilde{\phi}} (J_\phi \circ i)[\psi][u_l] \right) [u_l] \\
&\quad - \Delta_{G,\tilde{\phi}} \left[\Delta_{G,\tilde{\phi}}^{(-1)} \circ \left(d_{|\phi=\tilde{\phi}} (\Delta_{G,\phi})[\psi] \right) \circ \Delta_{G,\tilde{\phi}}^{(-1)} \circ J_{\tilde{\phi}} \circ i[u_l] \right] [u_l] \\
&= \left(d_{|\phi=\tilde{\phi}} (J_\phi \circ i)[\psi][u_l] \right) [u_l] + \lambda_F^{-1}[\tilde{\phi}] \left(d_{|\phi=\tilde{\phi}} (\Delta_{G,\phi})[\psi] \right) [u_l][u_l] \\
&\qquad \forall \psi \in L_{\Omega,O}
\end{aligned}
$$

(cf. Lamberti and Lanza de Cristoforis [41, Lemma 3.26]). Hence, in order to have an explicit representation of the differential, we need to compute the terms

$$
\left(d_{|\phi=\tilde{\phi}} (J_\phi \circ i)[\psi][u_l] \right) [u_l] \qquad \text{and} \qquad \left(d_{|\phi=\tilde{\phi}} (\Delta_{G,\phi})[\psi] \right) [u_l][u_l].
$$

Standard rules of calculus in Banach spaces yield

$$
\left[\left(d_{|\phi=\tilde{\phi}} (\det D\phi)[\psi] \right) \circ \tilde{\phi}^{(-1)} \right] \det D\tilde{\phi}^{(-1)} = \operatorname{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right) \qquad \forall \psi \in L_{\Omega,O}.
$$

(20)

We note that the map from $\{ f \in L^\infty(\Omega) : \operatorname{essinf}_\Omega |f| > 0 \}$ to $L^\infty(\Omega)$ which takes f to $|f|$ is differentiable and its differential at f is the map from $L^\infty(\Omega)$ to itself which

maps h to $\mathrm{sgn}(f)h$. By the above equality (20) and by a change of variable we obtain

$$
\left(d_{|\phi=\tilde{\phi}}(J_\phi \circ i)[\psi][u_l]\right)[u_l]
$$
$$
= \int_\Omega u_l^2 \, d_{|\phi=\tilde{\phi}}(|\det D\phi|)[\psi] \, dz
$$
$$
= \int_{\tilde{\phi}(\Omega)} v_l^2 \, d_{|\phi=\tilde{\phi}}((|\det D\phi|)[\psi]) \circ \tilde{\phi}^{(-1)} |\det D\tilde{\phi}^{(-1)}| \, dz
$$
$$
= \int_{\tilde{\phi}(\Omega)} v_l^2 \, \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dz \qquad \forall \psi \in L_{\Omega,O}.
$$

Next, we turn to consider the shape differential of the term $((\Delta_{G,\phi})[\psi])[u_l][u_l]$. By standard rules of calculus we have

$$
d_{|\phi=\tilde{\phi}}(D\phi)^{-1}[\psi] = -(D\tilde{\phi})^{-1} D\psi (D\tilde{\phi})^{-1} \qquad \forall \psi \in L_{\Omega,O},
$$

and

$$
d_{|\phi=\tilde{\phi}} I_G(\phi)[\psi] = \begin{pmatrix} \mathbf{0}_{h\times h} & \mathbf{0}_{h\times k} \\ \mathbf{0}_{k\times h} & s|\tilde{\phi}_x|^{s-2}\tilde{\phi}_x \cdot \psi_x I_{k\times k} \end{pmatrix} \qquad \forall \psi \in L_{\Omega,O}.
$$

Hence,

$$
\left(d_{|\phi=\tilde{\phi}}(\Delta_{G,\phi})[\psi]\right)[u_l][u_l]
$$
$$
= \int_\Omega \nabla u_l (D\tilde{\phi})^{-1} D\psi (D\tilde{\phi})^{-1} I_G(\tilde{\phi})(\nabla u_l (D\tilde{\phi})^{-1} I_G(\tilde{\phi}))^t |\det D\tilde{\phi}| \, dz
$$
$$
+ \int_\Omega \nabla u_l (D\tilde{\phi})^{-1} I_G(\tilde{\phi})(\nabla u_l (D\tilde{\phi})^{-1} D\psi (D\tilde{\phi})^{-1} I_G(\tilde{\phi}))^t |\det D\tilde{\phi}| \, dz
$$
$$
- \int_\Omega \nabla u_l (D\tilde{\phi})^{-1} I_G(\tilde{\phi})(\nabla u_l (D\tilde{\phi})^{-1} I_G(\tilde{\phi}))^t (d_{|\phi=\tilde{\phi}}|\det D\phi|)[\psi] \, dz
$$
$$
- \int_\Omega \nabla u_l (D\tilde{\phi})^{-1} (d_{|\phi=\tilde{\phi}} I_G(\phi)[\psi])(\nabla u_l (D\tilde{\phi})^{-1} I_G(\tilde{\phi}))^t |\det D\tilde{\phi}| \, dz
$$
$$
- \int_\Omega \nabla u_l (D\tilde{\phi})^{-1} I_G(\phi)(\nabla u_l (D\tilde{\phi})^{-1} (d_{|\phi=\tilde{\phi}} I_G(\phi)[\psi])(\tilde{\phi}))^t |\det D\tilde{\phi}| \, dz
$$
$$
= \int_{\tilde{\phi}(\Omega)} (\nabla v_l)(D(\psi \circ \tilde{\phi}^{(-1)}) I_G)(\nabla_G v_l)^t \, dz
$$
$$
+ \int_{\tilde{\phi}(\Omega)} (\nabla_G v_l)(D(\psi \circ \tilde{\phi}^{(-1)}) I_G)^t (\nabla v_l)^t \, dz
$$
$$
- \int_{\tilde{\phi}(\Omega)} |\nabla_G v_l|^2 \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dz
$$
$$
- 2s \int_{\tilde{\phi}(\Omega)} |x|^{2s-2} |\nabla_y v_l|^2 x \cdot (\psi \circ \tilde{\phi}^{(-1)})_x \, dz
$$

$$
= \int_{\tilde{\phi}(\Omega)} (\nabla v_l) \Big(D(\psi \circ \tilde{\phi}^{(-1)}) I_G^2 + I_G^2 D(\psi \circ \tilde{\phi}^{(-1)})^t \Big) (\nabla v_l)^t \, dz
$$

$$
- \int_{\tilde{\phi}(\Omega)} |\nabla_G v_l|^2 \mathrm{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right) dz
$$

$$
- \int_{\tilde{\phi}(\Omega)} 2s|x|^{2s-2} |\nabla_y v_l|^2 x \cdot (\psi \circ \tilde{\phi}^{(-1)})_x \, dz \qquad \forall \psi \in L_{\Omega,O}.
$$

Accordingly, we have proved that

$$
Q_{G,\tilde{\phi}} \Big[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_l], u_l \Big]
$$

$$
= \int_{\tilde{\phi}(\Omega)} v_l^2 \mathrm{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right) dz
$$

$$
+ \lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla v_l) \Big(D(\psi \circ \tilde{\phi}^{(-1)}) I_G^2 + I_G^2 D(\psi \circ \tilde{\phi}^{(-1)})^t \Big) (\nabla v_l)^t \, dz
$$

$$
- \lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |\nabla_G v_l|^2 \mathrm{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right) dz
$$

$$
- \lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} 2s|x|^{2s-2} |\nabla_y v_l|^2 x \cdot (\psi \circ \tilde{\phi}^{(-1)})_x \, dz \qquad \forall \psi \in L_{\Omega,O}.
$$

Putting together all the above equalities one verifies that formula (18) holds. □

Now, our aim is to rewrite formula (18) in a simpler form and obtain a Grushin analog of the classical Hadamard formula. To achieve this goal, we prove an intermediate technical lemma where we provide a suitable representation formula for $Q_{G,\tilde{\phi}} \Big[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_1], u_2 \Big]$, where u_1, u_2 are two eigenfunctions associated with the same eigenvalue. The following lemma is the analog in the Grushin setting of Lamberti and Lanza de Cristoforis [41, Lemma 3.26] for the standard Laplacian. We note that, although the idea behind the proof is the same, the Grushin case requires a careful and not straightforward analysis of several terms which do not appear in the standard case. For this reason we include a detailed proof.

Lemma 5.1 *Let Ω and O be as in* (9). *Let $\tilde{\phi} \in \mathcal{A}_{\Omega,O}$. Suppose that $\tilde{\phi}(\Omega)$ is of class C^1. Let $v_1, v_2 \in W_{G,0}^{1,2}(\tilde{\phi}(\Omega))$ be two eigenfunctions corresponding to an eigenvalue $\lambda[\tilde{\phi}]$ of* (5). *Suppose that $v_1, v_2 \in W_0^{1,2}(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))$. Let $u_1 := v_1 \circ \tilde{\phi}$, $u_2 := v_2 \circ \tilde{\phi}$. Then*

$$
Q_{G,\tilde{\phi}} \Big[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_1], u_2 \Big] = \lambda[\tilde{\phi}]^{-1} \int_{\partial \tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{(-1)} \mathbf{n}^t) \frac{\partial v_1}{\partial \mathbf{n}} \frac{\partial v_2}{\partial \mathbf{n}} |\mathbf{n}_G|^2 \, d\sigma
$$

$$
\forall \psi \in L_{\Omega,O}, \quad (21)
$$

where $\mathbf{n}$ denotes the outward unit normal field to $\partial \tilde{\phi}(\Omega)$ and

$$
\mathbf{n}_G := \mathbf{n} \, I_G = (\mathbf{n}_x, |x|^s \mathbf{n}_y) \qquad on \, \partial \tilde{\phi}(\Omega). \quad (22)
$$

Proof We fix $\psi \in L_{\Omega,O}$ and for the sake of simplicity we set $\omega := \psi \circ \tilde{\phi}^{(-1)}$. A minor modification of the proof of Theorem 5.2 shows that

$$
Q_{G,\tilde{\phi}}\left[d_{|\phi=\tilde{\phi}}T_{G,\phi}[\psi][u_1], u_2\right]
$$
$$
= \int_{\tilde{\phi}(\Omega)} v_1 v_2 \mathrm{div}\,(\omega)\,dz
$$
$$
+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla v_1)\left((D\omega)I_G^2 + I_G^2(D\omega)^t\right)(\nabla v_2)^t\,dz
$$
$$
- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla_G v_1)(\nabla_G v_2)^t \mathrm{div}\,(\omega)\,dz
$$
$$
- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} 2s|x|^{2s-2}(\nabla_y v_1)(\nabla_y v_2)^t x \cdot \omega_x\,dz
$$
$$
\forall \psi \in L_{\Omega,O}. \tag{23}
$$

We start by considering the second term in the right hand side of formula (23). A simple computation shows that

$$
\int_{\tilde{\phi}(\Omega)} \nabla v_1\left((D\omega)I_G^2 + I_G^2(D\omega)^t\right)(\nabla v_2)^t\,dz
$$
$$
= \int_{\tilde{\phi}(\Omega)} \nabla_x v_1\left(D_x\omega_x + (D_x\omega_x)^t\right)(\nabla_x v_2)^t\,dz
$$
$$
+ \int_{\tilde{\phi}(\Omega)} |x|^{2s}\nabla_y v_1\left(D_y\omega_y + (D_y\omega_y)^t\right)(\nabla_y v_2)^t\,dz
$$
$$
+ \int_{\tilde{\phi}(\Omega)} \nabla_y v_1\left(D_x\omega_y + |x|^{2s}(D_y\omega_x)^t\right)(\nabla_x v_2)^t\,dz
$$
$$
+ \int_{\tilde{\phi}(\Omega)} \nabla_y v_2\left(D_x\omega_y + |x|^{2s}(D_y\omega_x)^t\right)(\nabla_x v_1)^t\,dz. \tag{24}
$$

We consider the second term in the right hand side of equation (24). By the Divergence Theorem we have that

$$
\int_{\tilde{\phi}(\Omega)} |x|^{2s}\nabla_y v_1\left(D_y\omega_y + (D_y\omega_y)^t\right)(\nabla_y v_2)^t\,dz
$$
$$
= \sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} \left(|x|^{2s}\frac{\partial\omega_{y_i}}{\partial y_j}\frac{\partial v_1}{\partial y_i}\frac{\partial v_2}{\partial y_j} + |x|^{2s}\frac{\partial\omega_{y_j}}{\partial y_i}\frac{\partial v_1}{\partial y_i}\frac{\partial v_2}{\partial y_j}\right)dz
$$
$$
= -\sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} \left(|x|^{2s}\omega_{y_i}\frac{\partial^2 v_1}{\partial y_j\partial y_i}\frac{\partial v_2}{\partial y_j} + |x|^{2s}\omega_{y_i}\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_j^2}\right)dz
$$
$$
- \sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} \left(|x|^{2s}\omega_{y_j}\frac{\partial^2 v_1}{\partial y_i^2}\frac{\partial v_2}{\partial y_j} + |x|^{2s}\omega_{y_j}\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_j}\right)dz
$$

$$+ \sum_{i,j=1}^{k} \int_{\partial \tilde{\phi}(\Omega)} \left(|x|^{2s} \omega_{y_i} \mathbf{n}_{y_j} \frac{\partial v_1}{\partial y_i} \frac{\partial v_2}{\partial y_j} + |x|^{2s} \omega_{y_j} \mathbf{n}_{y_i} \frac{\partial v_1}{\partial y_i} \frac{\partial v_2}{\partial y_j} \right) d\sigma_z.$$

We note that

$$\sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_{y_i} \frac{\partial^2 v_1}{\partial y_j \partial y_i} \frac{\partial v_2}{\partial y_j} \, dz + \sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_{y_j} \frac{\partial v_1}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} \, dz$$

$$= \sum_{i,j=1}^{k} \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_{y_i} \frac{\partial}{\partial y_i} \left(\frac{\partial v_1}{\partial y_j} \frac{\partial v_2}{\partial y_j} \right) dz$$

$$= - \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \mathrm{div}_y \omega_y \, dz$$

$$+ \int_{\partial \tilde{\phi}(\Omega)} |x|^{2s} (\omega_y \mathbf{n}_y^t)((\nabla_y v_1)(\nabla_y v_2)^t) \, d\sigma_z,$$

and

$$\sum_{i,j=1}^{k} \int_{\partial \tilde{\phi}(\Omega)} \left(|x|^{2s} \omega_{y_i} \mathbf{n}_{y_j} \frac{\partial v_1}{\partial y_i} \frac{\partial v_2}{\partial y_j} + |x|^{2s} \omega_{y_j} \mathbf{n}_{y_i} \frac{\partial v_1}{\partial y_i} \frac{\partial v_2}{\partial y_j} \right) d\sigma_z$$

$$= \int_{\partial \tilde{\phi}(\Omega)} \left(|x|^{2s} (\omega_y \nabla_y v_1^t)(\mathbf{n}_y \nabla_y v_2^t) + |x|^{2s} (\omega_y \nabla_y v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \right) d\sigma_z.$$

Accordingly, the second term in the right hand side of equation (24) equals

$$\int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_y v_1 \left(D_y \omega_y + (D_y \omega_y)^t \right) (\nabla_y v_2)^t \, dz$$

$$= - \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\nabla_y v_1) \omega_y^t \, dz$$

$$- \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \mathrm{div}_y \omega_y \, dz$$

$$- \int_{\partial \tilde{\phi}(\Omega)} |x|^{2s} (\omega_y \mathbf{n}_y^t)((\nabla_y v_1)(\nabla_y v_2)^t) \, d\sigma_z$$

$$+ \int_{\partial \tilde{\phi}(\Omega)} \left(|x|^{2s} (\omega_y \nabla_y v_1^t)(\mathbf{n}_y \nabla_y v_2^t) + |x|^{2s} (\omega_y \nabla_y v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \right) d\sigma_z.$$

Similarly, the first term in the right hand side of equation (24) equals

$$\int_{\tilde{\phi}(\Omega)} \nabla_x v_1 \left(D_x \omega_x + (D_x \omega_x)^t \right) (\nabla_x v_2)^t \, dz$$

$$= -\int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\nabla_x v_1) \omega_x^t \, dz$$

$$- \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_x v_2) \omega_x^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} (\nabla_x v_1)(\nabla_x v_2)^t \mathrm{div}_x \omega_x \, dz$$

$$- \int_{\partial\tilde{\phi}(\Omega)} (\omega_x \mathbf{n}_x^t)((\nabla_x v_1)(\nabla_x v_2)^t) \, d\sigma_z$$

$$+ \int_{\partial\tilde{\phi}(\Omega)} \left((\omega_x \nabla_x v_1^t)(\mathbf{n}_x \nabla_x v_2^t) + (\omega_x \nabla_x v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \right) \, d\sigma_z.$$

Since v_1, v_2 are eigenfunctions we have

$$- \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_x v_2) \omega_x^t \, dz - \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$= - \int_{\tilde{\phi}(\Omega)} \Delta_G v_1 (\nabla v_2) \omega^t \, dz + \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_x v_2) \omega_x^t \, dz$$

$$= \lambda[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} v_1 (\nabla v_2) \omega^t \, dz + \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_x v_2) \omega_x^t \, dz,$$

and the same holds interchanging the role of v_1 and v_2. Accordingly,

$$- \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_x v_2) \omega_x^t \, dz - \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$- \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\nabla_x v_1) \omega_x^t \, dz - \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\nabla_y v_1) \omega_y^t \, dz$$

$$= -\lambda[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} v_1 v_2 \mathrm{div}(\omega) \, dz + \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_x v_2) \omega_x^t \, dz + \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\nabla_y v_1) \omega_y^t \, dz$$

$$+ \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\nabla_x v_1) \omega_x^t \, dz.$$

Thus, by substituting all the above relations into equality (23), one can verify that

$$Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_1], u_2 \right]$$

$$= \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\nabla_y v_1) \omega_y^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla_x v_1)(\nabla_x v_2)^t \mathrm{div}_y \omega_y \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_x \mathbf{n}_x^t)(\nabla_x v_1)(\nabla_x v_2)^t \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left((\omega_x \nabla_x v_1^t)(\mathbf{n}_x \nabla_x v_2^t) + (\omega_x \nabla_x v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \right) \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_x v_2) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\nabla_x v_1) \omega_x^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \mathrm{div}_x \omega_x \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s} (\omega_y \mathbf{n}_y^t)(\nabla_y v_1)(\nabla_y v_2)^t \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left(|x|^{2s} (\omega_y \nabla_y v_1^t)(\mathbf{n}_y \nabla_y v_2^t) + |x|^{2s} (\omega_y \nabla_y v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \right) \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_1 \left(D_x \omega_y + |x|^{2s} (D_y \omega_x)^t \right) (\nabla_x v_2)^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_2 \left(D_x \omega_y + |x|^{2s} (D_y \omega_x)^t \right) (\nabla_x v_1)^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} 2s |x|^{2s-2} (\nabla_y v_1)(\nabla_y v_2)^t x \cdot \omega_x \, dz. \tag{25}$$

We now set

$$I_1 := \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\nabla_y v_2) \omega_y^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\nabla_y v_1) \omega_y^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla_x v_1)(\nabla_x v_2)^t \mathrm{div}_y \omega_y \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\nabla_x v_2) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\nabla_x v_1) \omega_x^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \mathrm{div}_x \omega_x \, dz$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\tilde\phi(\Omega)} \nabla_y v_1 \left(D_x \omega_y + |x|^{2s}(D_y \omega_x)^t \right)(\nabla_x v_2)^t \; dz$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\tilde\phi(\Omega)} \nabla_y v_2 \left(D_x \omega_y + |x|^{2s}(D_y \omega_x)^t \right)(\nabla_x v_1)^t \; dz$$

$$- \lambda^{-1}[\tilde\phi] \int_{\tilde\phi(\Omega)} 2s |x|^{2s-2}(\nabla_y v_1)(\nabla_y v_2)^t x \omega_x^t \; dz \,,$$

$$I_2 := - \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega_x \mathbf{n}_x^t)(\nabla_x v_1)(\nabla_x v_2)^t \; d\sigma_z$$

$$- \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} |x|^{2s}(\omega_y \mathbf{n}_y^t)(\nabla_y v_1)(\nabla_y v_2)^t \; d\sigma_z \,,$$

$$I_3 := \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} \left((\omega_x \nabla_x v_1^t)(\mathbf{n}_x \nabla_x v_2^t) + (\omega_x \nabla_x v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \right) d\sigma_z$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} \left(|x|^{2s}(\omega_y \nabla_y v_1^t)(\mathbf{n}_y \nabla_y v_2^t) + |x|^{2s}(\omega_y \nabla_y v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \right) d\sigma_z \,.$$

As a consequence, equality (25) reads as

$$Q_{G,\tilde\phi}\left[d_{|\phi=\tilde\phi} T_{G,\phi}[\psi][u_1], u_2 \right] = I_1 + I_2 + I_3 \,. \tag{26}$$

Now, we rewrite the terms I_2 and I_3. We first consider I_2.

$$I_2 = - \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega_x \mathbf{n}_x^t)(\nabla_G v_1)(\nabla_G v_2)^t \; d\sigma_z$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} |x|^{2s}(\omega_x \mathbf{n}_x^t)(\nabla_y v_1)(\nabla_y v_2)^t \; d\sigma_z$$

$$- \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega_y \mathbf{n}_y^t)(\nabla_G v_1)(\nabla_G v_2)^t \; d\sigma_z$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega_y \mathbf{n}_y^t)(\nabla_x v_1)(\nabla_x v_2)^t \; d\sigma_z$$

$$= - \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega \mathbf{n}^t)(\nabla_G v_1)(\nabla_G v_2)^t \; d\sigma_z$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} |x|^{2s}(\omega_x \mathbf{n}_x^t)(\nabla_y v_1)(\nabla_y v_2)^t \; d\sigma_z$$

$$+ \lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} (\omega_y \mathbf{n}_y^t)(\nabla_x v_1)(\nabla_x v_2)^t \; d\sigma_z.$$

We now consider the last two summands of the right hand side of the previous equality. We have:

$$\lambda^{-1}[\tilde\phi] \int_{\partial\tilde\phi(\Omega)} |x|^{2s}(\omega_x \mathbf{n}_x^t)(\nabla_y v_1)(\nabla_y v_2)^t \; d\sigma_z$$

$$= \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \operatorname{div}_x \left(|x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \omega_x \right) dz$$

$$= \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \operatorname{div}_x \omega_x \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_y v_1 \left(D_x \nabla_y v_2 \right) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_y v_2 \left(D_x \nabla_y v_1 \right) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} 2s |x|^{2s-2} x \cdot \omega_x (\nabla_y v_1)(\nabla_y v_2)^t \, dz,$$

and, similarly,

$$\lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \mathbf{n}_y^t)(\nabla_x v_1)(\nabla_x v_2)^t \, d\sigma_z$$

$$= \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla_x v_1)(\nabla_x v_2)^t \operatorname{div}_y \omega_y \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_x v_1 \left(D_y \nabla_x v_2 \right) \omega_y^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_x v_2 \left(D_y \nabla_x v_1 \right) \omega_y^t \, dz.$$

Therefore, we deduce the following expression for I_2

$$I_2 = - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega \mathbf{n}^t)(\nabla_G v_1)(\nabla_G v_2)^t \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} (\nabla_y v_1)(\nabla_y v_2)^t \operatorname{div}_x \omega_x \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_y v_1 \left(D_x \nabla_y v_2 \right) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_y v_2 \left(D_x \nabla_y v_1 \right) \omega_x^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} 2s |x|^{2s-2} x \omega_x^t (\nabla_y v_1)(\nabla_y v_2)^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} (\nabla_x v_1)(\nabla_x v_2)^t \operatorname{div}_y \omega_y \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_x v_1 \left(D_y \nabla_x v_2 \right) \omega_y^t \, dz$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_x v_2 \left(D_y \nabla_x v_1 \right) \omega_y^t \, dz. \tag{27}$$

Next, we turn to consider I_3. We recall that $\mathbf{n}_G$ is the outward field to $\partial\Omega$ defined in (22). Therefore

$$
\begin{aligned}
I_3 &= \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left((\omega_x \nabla_x v_1^t)(\mathbf{n}_x \nabla_x v_2^t) + (\omega_x \nabla_x v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \right) d\sigma_z \\
&\quad + \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left(|x|^{2s}(\omega_y \nabla_y v_1^t)(\mathbf{n}_y \nabla_y v_2^t) + |x|^{2s}(\omega_y \nabla_y v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \right) d\sigma_z \\
&= \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left((\omega_x \nabla_x v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega_x \nabla_x v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \right) d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_1^t)(\mathbf{n}_y \nabla_y v_2^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \, d\sigma_z \\
&\quad + \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left((\omega_y \nabla_y v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega_y \nabla_y v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \right) d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_1^t)(\mathbf{n}_x \nabla_x v_2^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \, d\sigma_z \\
&= \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega \nabla v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega \nabla v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_1^t)(\mathbf{n}_y \nabla_y v_2^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_1^t)(\mathbf{n}_x \nabla_x v_2^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \, d\sigma_z.
\end{aligned}
\tag{28}
$$

We consider the last four terms in the right hand side of the previous equation:

$$
\begin{aligned}
&- \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_1^t)(\mathbf{n}_y \nabla_y v_2^t) \, d\sigma_z \\
&\quad - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} |x|^{2s}(\omega_x \nabla_x v_2^t)(\mathbf{n}_y \nabla_y v_1^t) \, d\sigma_z \\
&= -\lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \mathrm{div}_y \left(|x|^{2s}(\omega_x \nabla_x v_1^t)\nabla_y v_2 + |x|^{2s}(\omega_x \nabla_x v_2^t)\nabla_y v_1 \right) dz \\
&= -\lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\omega_x \nabla_x v_1^t) \, dz
\end{aligned}
$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\omega_x \nabla_x v_2^t) \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_x (D_y \nabla_x v_1) \nabla_y v_2^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_x (D_y \nabla_x v_2) \nabla_y v_1^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_x v_1 (D_y \omega_x) \nabla_y v_2^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_x v_2 (D_y \omega_x) \nabla_y v_1^t \, dz,$$

and similarly

$$- \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_1^t)(\mathbf{n}_x \nabla_x v_2^t) \, d\sigma_z - \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega_y \nabla_y v_2^t)(\mathbf{n}_x \nabla_x v_1^t) \, d\sigma_z$$

$$= -\lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\omega_y \nabla_y v_1^t) \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\omega_y \nabla_y v_2^t) \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \omega_y (D_x \nabla_y v_1) \nabla_x v_2^t \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \omega_y (D_x \nabla_y v_2) \nabla_x v_1^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_1 (D_x \omega_y) \nabla_x v_2^t \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_2 (D_x \omega_y) \nabla_x v_1^t \, dz.$$

We can now rewrite the right hand side of equation (28) and deduce that

$$I_3 = \lambda^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (\omega \nabla v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega \nabla v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \, d\sigma_z$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_2 (\omega_x \nabla_x v_1^t) \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \Delta_y v_1 (\omega_x \nabla_x v_2^t) \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_x (D_y \nabla_x v_1) \nabla_y v_2^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \omega_x (D_y \nabla_x v_2) \nabla_y v_1^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_x v_1 (D_y \omega_x) \nabla_y v_2^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} |x|^{2s} \nabla_x v_2 (D_y \omega_x) \nabla_y v_1^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_2 (\omega_y \nabla_y v_1^t) \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \Delta_x v_1 (\omega_y \nabla_y v_2^t) \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \omega_y (D_x \nabla_y v_1) \nabla_x v_2^t \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \omega_y (D_x \nabla_y v_2) \nabla_x v_1^t \, dz$$

$$- \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_1 (D_x \omega_y) \nabla_x v_2^t \, dz - \lambda^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla_y v_2 (D_x \omega_y) \nabla_x v_1^t \, dz. \tag{29}$$

By equalities (25), (26), (27), and (29) and by noting that $(D_x \nabla_y v_l)^t = D_y \nabla_x v_l$, we deduce that

$$Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_1], u_2 \right]$$

$$= -\lambda^{-1}[\tilde{\phi}] \int_{\partial \tilde{\phi}(\Omega)} (\omega \mathbf{n}^t)(\nabla_G v_1)(\nabla_G v_2)^t \, d\sigma_z$$

$$+ \lambda^{-1}[\tilde{\phi}] \int_{\partial \tilde{\phi}(\Omega)} (\omega \nabla v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega \nabla v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \, d\sigma_z.$$

We note that, since $v_1, v_2 \in W_0^{1,2}(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))$, the gradients $\nabla v_1, \nabla v_2$ are parallel to $\mathbf{n}$ on $\partial \tilde{\phi}(\Omega)$ and $\nabla_G v_1, \nabla_G v_2$ are parallel to $\mathbf{n}_G$ on $\partial \tilde{\phi}(\Omega)$. Accordingly,

$$\nabla_G v_i = \nabla v_i \, I_G = (\nabla v_i \, \mathbf{n}^t)\mathbf{n} \, I_G = \frac{\partial v_i}{\partial \mathbf{n}} \mathbf{n}_G, \qquad \text{a.e. on } \partial \tilde{\phi}(\Omega) \quad \text{for } i \in \{1, 2\}.$$

Then

$$\int_{\partial \tilde{\phi}(\Omega)} (\omega \mathbf{n}^t)(\nabla_G v_1)(\nabla_G v_2)^t \, d\sigma_z = \int_{\partial \tilde{\phi}(\Omega)} (\omega \mathbf{n}^t) \frac{\partial v_1}{\partial \mathbf{n}} \frac{\partial v_2}{\partial \mathbf{n}} |\mathbf{n}_G|^2 \, d\sigma_z \,,$$

and

$$\int_{\partial \tilde{\phi}(\Omega)} (\omega \nabla v_1^t)(\mathbf{n}_G \nabla_G v_2^t) + (\omega \nabla v_2^t)(\mathbf{n}_G \nabla_G v_1^t) \, d\sigma_z$$

$$= 2 \int_{\partial \tilde{\phi}(\Omega)} (\omega \mathbf{n}^t) \frac{\partial v_1}{\partial \mathbf{n}} \frac{\partial v_2}{\partial \mathbf{n}} |\mathbf{n}_G|^2 \, d\sigma_z.$$

We can finally conclude that

$$Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\psi][u_1], u_2 \right] = \lambda^{-1}[\tilde{\phi}] \int_{\partial \tilde{\phi}(\Omega)} (\omega \mathbf{n}^t) \frac{\partial v_1}{\partial \mathbf{n}} \frac{\partial v_2}{\partial \mathbf{n}} |\mathbf{n}_G|^2 \, d\sigma_z.$$

$$\square$$

Now, combining Theorem 5.2, formula (19), and Lemma 5.1 we are able to deduce our main result regarding the Hadamard-type formula for the shape differential of the symmetric functions of the eigenvalues.

Theorem 5.3 *Let Ω and O be as in* (9). *Let F be a finite nonempty subset of $\mathbb{N}$. Let $\tau \in \{1, \ldots, |F|\}$. Let $\tilde{\phi} \in \Theta_{\Omega,O}^F$ and let $\lambda_F[\tilde{\phi}]$ be the common value of all the*

eigenvalues $\{\lambda_j[\tilde\phi]\}_{j\in F}$. *Let* $\{v_l\}_{l\in F}$ *be an orthonormal basis in* $(W_{G,0}^{1,2}(\tilde\phi(\Omega)), Q_G)$ *of the eigenspace associated with* $\lambda_F[\tilde\phi]$. *Suppose that* $\tilde\phi(\Omega)$ *is of class* C^1 *and* $\{v_l\}_{l\in F} \subseteq W_0^{1,2}(\tilde\phi(\Omega)) \cap W^{2,2}(\tilde\phi(\Omega))$. *Then the Frechét differential of the map* $\Lambda_{F,\tau}$ *at the point* $\tilde\phi$ *is delivered by the formula*

$$d_{|\phi=\tilde\phi}(\Lambda_{F,\tau})[\psi] = -\lambda_F^\tau[\tilde\phi]\binom{|F|-1}{\tau-1}\sum_{l\in F}\int_{\partial\tilde\phi(\Omega)} (\psi\circ\tilde\phi^{(-1)}\mathbf{n}^t)\left(\frac{\partial v_l}{\partial\mathbf{n}}\right)^2 |\mathbf{n}_G|^2\, d\sigma$$

$$\forall\psi\in L_{\Omega,O}. \tag{30}$$

Remark 5.2 In order to prove formula (21) and subsequently the Hadamard formula (30) we had to assume some extra regularity for the eigenfunctions to avoid regularity problems near $\partial\tilde\phi(\Omega) \cap \{x = 0\}$. However, if $\psi \in L_{\Omega,O}$ is such that $\psi_{|\Omega\cap O} = 0$, then, since any problem around the degenerate set is canceled by ψ, formulas (21) and (30) hold without requiring that the eigenfunctions are of class $W_0^{1,2} \cap W^{2,2}$.

Next, we consider the case of a family of domain perturbations depending real analytically upon one scalar parameter. In this case it is possible to prove a Rellich–Nagy-type theorem and describe all the branches splitting from a multiple eigenvalue of multiplicity m by means of m real analytic functions of the scalar parameter. Namely, we have the following.

Theorem 5.4 *Let* Ω *and* O *be as in* (9). *Let* $\tilde\phi \in \mathcal{A}_{\Omega,O}$ *and* $\{\phi_\varepsilon\}_{\varepsilon\in\mathbb{R}} \subseteq \mathcal{A}_{\Omega,O}$ *be a family depending real analytically on* ε *and such that* $\phi_0 = \tilde\phi$. *Let* λ *be a Dirichlet Grushin eigenvalue on* $\tilde\phi(\Omega)$ *of multiplicity* m. *Let* $\lambda = \lambda_n[\tilde\phi] = \cdots = \lambda_{n+m-1}[\tilde\phi]$ *for some* $n \in \mathbb{N}$. *Let* $v_1, \ldots, v_m$ *be an orthonormal basis in* $(W_{G,0}^{1,2}(\tilde\phi(\Omega)), Q_G)$ *of the eigenspace associated with* λ. *Suppose that* $\tilde\phi(\Omega)$ *is of class* C^1 *and that* $v_1, \ldots, v_m \in W_0^{1,2}(\tilde\phi(\Omega)) \cap W^{2,2}(\tilde\phi(\Omega))$.

Then there exist an open interval $I \subseteq \mathbb{R}$ *containing zero and* m *real analytic functions* $g_1, \ldots, g_m$ *from* I *to* $\mathbb{R}$ *such that* $\{\lambda_n[\phi_\varepsilon], \ldots, \lambda_{n+m-1}[\phi_\varepsilon]\} = \{g_1(\varepsilon), \ldots, g_m(\varepsilon)\}$ *for all* $\varepsilon \in I$. *Moreover, the derivatives* $g_1'(0), \ldots, g_m'(0)$ *of the functions* $g_1, \ldots, g_m$ *at zero coincide with the eigenvalues of the matrix*

$$\left(-\lambda\int_{\partial\tilde\phi(\Omega)} (\dot\phi_0\circ\tilde\phi^{(-1)}\mathbf{n}^t)\frac{\partial v_i}{\partial\mathbf{n}}\frac{\partial v_j}{\partial\mathbf{n}}|\mathbf{n}_G|^2\, d\sigma\right)_{i,j=1,\ldots,m}, \tag{31}$$

where $\dot\phi_0$ *denotes the derivative at* $\varepsilon = 0$ *of the map* $\varepsilon \mapsto \phi_\varepsilon$. *The same conclusion holds dropping the regularity assumption on the eigenfunctions and requiring that* $\phi_{\varepsilon|\Omega\cap O}$ *is the identity map for all* $\varepsilon \in \mathbb{R}$.

Proof The proof follows by the abstract Rellich–Nagy-type theorem of Lamberti and Lanza de Cristoforis [41, Theorem 2.27, Corollary 2.28] applied to the family of operators $(T_{G,\phi_\varepsilon})_{\varepsilon\in\mathbb{R}}$ defined in (14), which guarantees that there exist an open interval I containing zero and m real analytic functions $f_1, \ldots, f_m$ from I to $\mathbb{R}$ such that $\{\lambda_n[\tilde\phi]^{-1}, \ldots, \lambda_{n+m-1}[\tilde\phi]^{-1}\} = \{f_1(\varepsilon), \ldots, f_m(\varepsilon)\}$ for all $\varepsilon \in I$ and that, if we set

$u_i = v_i \circ \tilde{\phi}$ for all $i = 1, \ldots, m$, the set $\{f_1'(0), \ldots, f_m'(0)\}$ coincides with the set of eigenvalues of the matrix

$$\left(Q_{G,\tilde{\phi}} \left[d_{|\phi=\tilde{\phi}} T_{G,\phi}[\dot{\phi}_0][u_i], u_j \right] \right)_{i,j=1,\ldots,m}.$$

Then we can conclude by setting $g_i = f_i^{-1}$ for all $i = 1 \ldots, m$ and exploiting Lemma 5.1. The last part of the statement follows by the same arguments together with Remark 5.2. $\qquad\square$

We conclude this section with the following remark on the scalar product used.

Remark 5.3 The above formulas are obtained assuming that the orthogonality of the eigenfunctions is taken in $(W_{G,0}^{1,2}(\tilde{\phi}(\Omega)), Q_G)$. If instead one prefers to consider $\{v_l\}_{l \in F}$ to be an orthonormal basis in $L^2(\tilde{\phi}(\Omega))$ endowed with its standard scalar product, then formula (30) of Theorem 5.3 can be rewritten as

$$d_{|\phi=\tilde{\phi}}(\Lambda_{F,\tau})[\psi] = -\lambda_F^{\tau-1}[\tilde{\phi}] \binom{|F|-1}{\tau-1} \sum_{l \in F} \int_{\partial\tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{(-1)} \mathbf{n}^t) \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 \, d\sigma$$

$$\forall \psi \in L_{\Omega,O}.$$

Similarly, in Theorem 5.4 we can choose $v_1, \ldots, v_m$ to be an orthonormal basis in $L^2(\tilde{\phi}(\Omega))$ and in this case the matrix (31) becomes

$$\left(-\int_{\partial\tilde{\phi}(\Omega)} (\dot{\phi}_0 \circ \tilde{\phi}^{(-1)} \mathbf{n}^t) \frac{\partial v_i}{\partial \mathbf{n}} \frac{\partial v_j}{\partial \mathbf{n}} |\mathbf{n}_G|^2 \, d\sigma \right)_{i,j=1,\ldots,m}. \tag{32}$$

6 Critical Shapes and Overdetermined Problems

In this section we consider the problem of studying the critical shapes for the symmetric functions of the Grushin eigenvalues under isovolumetric and isoperimetric perturbations. Let Ω be a bounded open subset of $\mathbb{R}^N$. Let F be a finite nonempty subset of $\mathbb{N}$. We set

$$\mathcal{A}_\Omega^F := \left\{ \phi \in \mathrm{Lip}(\Omega)^N : \exists\, O \text{ as in (9) such that } \phi \in \mathcal{A}_{\Omega,O}^F \right\},$$
$$\Theta_\Omega^F := \left\{ \phi \in \mathcal{A}_\Omega^F : \lambda_n[\phi] = \lambda_m[\phi], \ \forall n, m \in F \right\}.$$

If $\phi \in \mathcal{A}_\Omega^F$, we denote by $\mathcal{V}[\phi]$ and $\mathcal{P}[\phi]$ the volume and the perimeter of the set $\phi(\Omega)$, respectively. Let $\tau \in \{1, \ldots, |F|\}$. Our interest in critical shapes mainly comes from possible applications to shape optimization problems. To the best of our knowledge, the problem of finding optimal shapes for the Grushin eigenvalues under isovolumetric or isoperimetric constraint is a difficult problem which is open at the moment, even in the case of the first eigenvalue. A question related to the minimization of the first Grushin eigenvalue is of course the search for optimal constants for the appropriate

Poincaré inequality, for which only bounds are currently known (cf., *e.g.*, D'Ambrosio [11, Theorem 3.7] for the Poincaré inequality, Garofalo and Nhieu [25, Corollary 1.6] and Milman [44, Theorem 1.1] for analogous inequalities of Poincaré–Wirtinger-type). However, we note that a partial step towards the solution of shape optimization problems is typically represented by the identification of the condition that characterizes the critical shapes under volume and perimeter constraint, respectively. This allows to state the corresponding overdetermined problems.

6.1 The Isovolumetric Problem

First, we consider the problem of finding critical shapes under isovolumetric perturbations. Let Ω and O be as in (9). The volume functional $\mathcal{V}[\cdot]$ is the map from $\mathcal{A}_\Omega^F$ to $\mathbb{R}$ defined by

$$\mathcal{V}[\phi] := \int_{\phi(\Omega)} 1 \, dz = \int_\Omega |\det D\phi| \, dz \quad \forall \phi \in \mathcal{A}_\Omega^F.$$

It is easily seen that $\mathcal{V}_{|\mathcal{A}_{\Omega,O}^F}$ is real analytic and that, by standard rules of calculus in Banach spaces, its differential at the point $\tilde{\phi} \in \mathcal{A}_{\Omega,O}^F$ is delivered by

$$d_{|\phi=\tilde{\phi}}\mathcal{V}_{|\mathcal{A}_{\Omega,O}^F}[\psi] = \int_{\tilde{\phi}(\Omega)} \operatorname{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dz \quad \forall \psi \in L_{\Omega,O}.$$

Under the assumption that $\tilde{\phi}(\Omega)$ is of class C^1, the above differential can be rewritten as

$$d_{|\phi=\tilde{\phi}}\mathcal{V}_{|\mathcal{A}_{\Omega,O}^F}[\psi] = \int_{\partial\tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)}\right) \cdot \mathbf{n} \, d\sigma \quad \forall \psi \in L_{\Omega,O}. \tag{33}$$

For $M \in \,]0, +\infty[$ we set

$$V[M] := \left\{\phi \in \mathcal{A}_\Omega^F : \mathcal{V}[\phi] = M\right\}.$$

Suppose now that a shape $\tilde{\phi} \in \mathcal{A}_{\Omega,O}^F$ is a maximizer (or a minimizer) for the symmetric function $\Lambda_{F,\tau}[\phi]$ under the volume constraint $\phi \in V[M]$ among all the shapes ϕ in $\mathcal{A}_\Omega^F$. Then, for all open sets $O' \subseteq \mathbb{R}^N$ such that

$$O' \subseteq O \text{ and } \overline{\Omega} \cap \{x = 0\} \subseteq O', \tag{34}$$

$\tilde{\phi}$ is a maximizer (or a minimizer) under the volume constraint $\phi \in V[M]$ for all the shapes ϕ in $\mathcal{A}_{\Omega,O'}^F$. Accordingly, for all $O' \subseteq O$ such that $\overline{\Omega} \cap \{x = 0\} \subseteq O'$, $\tilde{\phi}$ is a critical point for $\Lambda_{F,\tau|\mathcal{A}_{\Omega,O'}^F}$ under the volume constraint $\phi \in V[M]$, in other words:

$$\ker d_{\phi=\tilde{\phi}}\mathcal{V}_{|\mathcal{A}_{\Omega,O'}^F} \subseteq \ker d_{\phi=\tilde{\phi}}\Lambda_{F,\tau|\mathcal{A}_{\Omega,O'}^F} \quad \forall O' \subseteq \mathbb{R}^N \text{ as in (34)}.$$

By the Lagrange multipliers theorem, the above condition is equivalent to the fact that for all open sets $O' \subseteq \mathbb{R}^N$ as in (34), there exists a constant $c_{O'} \in \mathbb{R}$ (a Lagrange multiplier) such that

$$d_{\phi=\tilde{\phi}} \Lambda_{F,\tau \mid \mathcal{A}^F_{\Omega,O'}}[\psi] + c_{O'} d_{\phi=\tilde{\phi}} \mathcal{V}_{\mid \mathcal{A}^F_{\Omega,O'}}[\psi] = 0 \qquad \forall \psi \in L_{\Omega,O'}. \tag{35}$$

Inspired by the above discussion, we introduce the following definition.

Definition 6.1 Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $M \in \,]0, +\infty[$. Let F be a finite nonempty subset of $\mathbb{N}$. Let $\tau \in \{1, \ldots, |F|\}$. Let $\tilde{\phi} \in \mathcal{A}^F_\Omega \cap V[M]$. We say that $\tilde{\phi}$ is critical for $\Lambda_{F,\tau}$ under the volume constraint $\phi \in V[M]$ if there exists a bounded open subset O of $\mathbb{R}^N$ with $\overline{\Omega} \cap \{x = 0\} \subseteq O$ such that $\tilde{\phi} \in \mathcal{A}^F_{\Omega,O}$ and such that for all open sets $O' \subseteq \mathbb{R}^N$ as in (34) there exists $c_{O'} \in \mathbb{R}$ such that (35) holds.

In the following proposition we prove a necessary condition for the criticality of shapes under isovolumetric perturbations. We point out that an analogous condition has been obtained in [38,42] for the standard Laplacian in the Euclidean setting and in El Soufi and Ilias [16] for the Laplace–Beltrami operator on a Riemannian manifold. Moreover, for a similar condition in a different context (the eigenvalues of the Laplacian on a compact smooth surface with a one-parametric family of Riemannian metrics) we mention Nadirashvili [50].

Theorem 6.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let F be a finite nonempty subset of $\mathbb{N}$ and $\tau \in \{1, \ldots, |F|\}$. Let $M \in \,]0, +\infty[$. Let $\tilde{\phi} \in \Theta^F_\Omega \cap V[M]$ and let $\lambda_F[\tilde{\phi}]$ be the common value of all the eigenvalues $\{\lambda_j[\tilde{\phi}]\}_{j \in F}$. Assume that $\tilde{\phi}(\Omega)$ is of class C^1. Let $\{v_l\}_{l \in F}$ be an orthonormal basis in $(W^{1,2}_{G,0}(\tilde{\phi}(\Omega)), Q_G)$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$. If $\tilde{\phi}$ is a critical shape for $\Lambda_{F,\tau}$ under the volume constraint $\phi \in V[M]$, then there exists a constant $c_1 \in \mathbb{R}$ such that*

$$\sum_{l \in F} \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 = c_1 \qquad a.e. \text{ on } \partial\tilde{\phi}(\Omega) \setminus \{x = 0\}. \tag{36}$$

Proof Let $\tilde{\phi}$ be a critical shape for $\Lambda_{F,\tau}$ under the volume constraint $\tilde{\phi} \in V[M]$ and let $O \subseteq \mathbb{R}^N$ be as in Definition 6.1. For $O' \subseteq \mathbb{R}^N$ as in (34) we set

$$L^0_{\Omega,O'} := \{\psi \in L_{\Omega,O'} : \psi_{\mid \Omega \cap O'} = 0\}.$$

Then, by Theorem 5.3 and Remark 5.2, we have that

$$d_{\mid \phi=\tilde{\phi}}(\Lambda_{F,\tau})[\psi] = -\lambda^\tau_F[\tilde{\phi}] \binom{|F|-1}{\tau-1} \sum_{l \in F} \int_{\partial\tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{(-1)} \mathbf{n}^t) \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 \, d\sigma$$

$$\forall \psi \in L^0_{\Omega,O'}. \tag{37}$$

Thus, formula (37), formula (33) for the differential of the volume functional, and Definition 6.1 imply that for all open sets $O' \subseteq \mathbb{R}^N$ as in (34) there exists a constant

$c_{O'} \in \mathbb{R}$ such that

$$\sum_{l \in F} \int_{\partial \tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)} \right) \cdot \mathbf{n} \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 \, d\sigma + c_{O'} \int_{\partial \tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)} \right) \cdot \mathbf{n} \, d\sigma = 0$$
$$\forall \psi \in L^0_{\Omega, O'} .$$

On the other hand, if $O' \subseteq \mathbb{R}^N$ is as in (34), then $L^0_{\Omega, O} \subseteq L^0_{\Omega, O'}$. Hence, $c_{O'} = c_O$. That is

$$\sum_{l \in F} \int_{\partial \tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)} \right) \cdot \mathbf{n} \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 \, d\sigma + c_O \int_{\partial \tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)} \right) \cdot \mathbf{n} \, d\sigma = 0$$
$$\forall \psi \in L^0_{\Omega, O'} . \tag{38}$$

By the Fundamental Lemma of Calculus of Variations one can realize that (38) implies that there exists a constant $c_1 \in \mathbb{R}$ such that (36) holds. □

Remark 6.1 If one assumes that the eigenfunctions are of class $W_0^{1,2}(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))$, then it is easily seen that condition (36) becomes also sufficient for the criticality of shapes under isovolumetric perturbations. Moreover, if the $(N-1)$-dimensional measure of $\partial \tilde{\phi}(\Omega) \cap \{x = 0\}$ is zero, *i.e.* $\left| \partial \tilde{\phi}(\Omega) \cap \{x = 0\} \right|_{N-1} = 0$, then (36) can be rewritten as

$$\sum_{l \in F} \left(\frac{\partial v_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 = c_1 \qquad \text{a.e. on } \partial \tilde{\phi}(\Omega).$$

We note that $\left| \partial \tilde{\phi}(\Omega) \cap \{x = 0\} \right|_{N-1} = 0$ is always verified when $\dim\{x = 0\} = k < N - 1$.

The previous theorem suggests considering the overdetermined system

$$\begin{cases} -\Delta_G u_l = \lambda_j u_l & \text{in } \Omega, \, \forall l \in \{1, \ldots, m\}, \\ u_l = 0 & \text{on } \partial \Omega, \, \forall l \in \{1, \ldots, m\}, \\ \sum_{l=1}^m \left(\frac{\partial u_l}{\partial \mathbf{n}} \right)^2 |\mathbf{n}_G|^2 = \text{const.} & \text{on } \partial \Omega. \end{cases} \tag{39}$$

Here, λ_j is the j-th eigenvalue which has multiplicity $m \in \mathbb{N}$ and $u_1, \ldots, u_m$ is a corresponding orthonormal basis of eigenvalues in $W^{1,2}_{G,0}(\Omega)$ such that the last condition of system (39) makes sense (for example $\{u_l\}_{l=1,\ldots,m} \subseteq W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ when Ω is of class C^1). System (39) is the Grushin analog of the well-known overdetermined

system for the Laplace operator:

$$
\begin{cases}
-\Delta u_l = \lambda_j u_l & \text{in } \Omega, \ \forall l \in \{1, \ldots, m\}, \\
u_l = 0 & \text{on } \partial\Omega, \ \forall l \in \{1, \ldots, m\}, \\
\sum_{l=1}^m \left(\frac{\partial u_l}{\partial \mathbf{n}}\right)^2 = \text{const.} & \text{on } \partial\Omega.
\end{cases}
\tag{40}
$$

It is known that system (40) is satisfied when Ω is a ball (see Lamberti and Lanza de Cristoforis [42]). Moreover, if Ω is connected and $j = 1$ (and then $m = 1$), problem (40) is satisfied if and only if Ω is a ball (see Henry [31]).

It would be of great interest characterizing those bounded domains such that system (39) is satisfied or, at least, find some shapes for which it is satisfied. These problems, to the best of the authors' knowledge, are open.

6.2 The Isoperimetric Problem

Next, we switch to consider the isoperimetric problem. In this section we assume that Ω is a bounded open subset of $\mathbb{R}^N$ of class C^2. Let O be as in (9). We set

$$
L^*_{\Omega,O} := L_{\Omega,O} \cap C^2\left(\overline{\Omega}, \mathbb{R}^N\right) \qquad \mathcal{A}^{*F}_{\Omega,O} := \mathcal{A}^F_{\Omega,O} \cap C^2\left(\overline{\Omega}, \mathbb{R}^N\right)
$$
$$
\mathcal{A}^{*F}_{\Omega} := \mathcal{A}^F_{\Omega} \cap C^2\left(\overline{\Omega}, \mathbb{R}^N\right) \qquad \Theta^{*F}_{\Omega} := \Theta^F_{\Omega} \cap C^2\left(\overline{\Omega}, \mathbb{R}^N\right).
$$

The set $L^*_{\Omega,O}$ is a Banach subspace of $C^2\left(\overline{\Omega}, \mathbb{R}^N\right)$ and $\mathcal{A}^{*F}_{\Omega,O}$ is open in $L^*_{\Omega,O}$. The perimeter functional $\mathcal{P}[\cdot]$ is the map from $\mathcal{A}^{*F}_{\Omega}$ to $\mathbb{R}$ defined by

$$
\mathcal{P}[\phi] := \int_{\partial\phi(\Omega)} 1 \, d\sigma = \int_{\partial\Omega} \left|\mathbf{n}(D\phi)^{-1}\right| |\det D\phi| \, d\sigma \qquad \forall \phi \in \mathcal{A}^{*F}_{\Omega}.
$$

The map $\mathcal{P}_{|\mathcal{A}^{*F}_{\Omega,O}}$ is real analytic and its differential at a point $\tilde{\phi} \in \mathcal{A}^{*F}_{\Omega,O}$ is delivered by

$$
d_{|\phi=\tilde{\phi}}\mathcal{P}_{|\mathcal{A}^{*F}_{\Omega,O}}[\psi] = \int_{\partial\tilde{\phi}(\Omega)} \left(\psi \circ \tilde{\phi}^{(-1)}\right) \cdot \mathbf{n}\,\mathcal{H}\,d\sigma \qquad \forall \psi \in L^*_{\Omega,O},
$$

where $\mathcal{H} = \text{div } \mathbf{n}$ denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$ (see [39]). For $M \in \,]0, +\infty[$ we set

$$
P[M] := \left\{\phi \in \mathcal{A}^{*F}_{\Omega} : \mathcal{P}[\phi] = M\right\}.
$$

Following the lines of the previous section and motivated by isoperimetric optimization problems, we introduce the following definition.

Definition 6.2 Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^2. Let $M \in \,]0, +\infty[$. Let F be a finite nonempty subset of $\mathbb{N}$. Let $\tau \in \{1, \ldots, |F|\}$. Let $\tilde{\phi} \in \mathcal{A}^{*F}_{\Omega} \cap P[M]$.

We say that ϕ is critical for $\Lambda_{F,\tau}$ under the perimeter constraint $\phi \in P[M]$ if there exists a bounded open subset O of $\mathbb{R}^N$ with $\overline{\Omega} \cap \{x = 0\} \subseteq O$ such that $\tilde{\phi} \in \mathcal{A}^{*F}_{\Omega,O}$ and such that for all open sets $O' \subseteq \mathbb{R}^N$ as in (34) there exists $c_{O'} \in \mathbb{R}$ such that

$$d_{\phi=\tilde{\phi}}\Lambda_{F,\tau |\mathcal{A}^{*F}_{\Omega,O'}}[\psi] + c_{O'}d_{\phi=\tilde{\phi}}\mathcal{P}_{|\mathcal{A}^{*F}_{\Omega,O'}}[\psi] = 0 \qquad \forall \psi \in L^*_{\Omega,O'}.$$

Following the lines of the previous section, we are able to prove the following necessary condition for critical shapes under isoperimetric perturbations.

Theorem 6.2 *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^2. Let F be a finite nonempty subset of $\mathbb{N}$ and $\tau \in \{1, \ldots, |F|\}$. Let $M \in]0, +\infty[$. Let $\tilde{\phi} \in \Theta^{*F}_{\Omega} \cap P[M]$ and let $\lambda_F[\tilde{\phi}]$ be the common value of all the eigenvalues $\{\lambda_j[\tilde{\phi}]\}_{j \in F}$. Let $\{v_l\}_{l \in F}$ be an orthonormal basis in $(W^{1,2}_{G,0}(\tilde{\phi}(\Omega)), Q_G)$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$. If $\tilde{\phi}$ is a critical shape for $\Lambda_{F,\tau}$ under the perimeter constraint $\phi \in P[M]$, then there exists a constant $c_2 \in \mathbb{R}$ such that*

$$\sum_{l \in F}\left(\frac{\partial v_l}{\partial \mathbf{n}}\right)^2 |\mathbf{n}_G|^2 = c_2\mathcal{H} \qquad \text{a.e. on } \partial\tilde{\phi}(\Omega) \setminus \{x = 0\}. \tag{41}$$

Remark 6.2 As in Remark 6.1, if the eigenfunctions are of class $W^{1,2}_0(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))$, then condition (41) becomes also sufficient for the criticality of shapes under isoperimetric perturbations. Also, when $\left|\partial\tilde{\phi}(\Omega) \cap \{x = 0\}\right|_{N-1} = 0$, the same considerations as in Remark 6.1 can be done.

As before, Theorem 6.2 suggests that it would be of interest to study the overdetermined system

$$\begin{cases} -\Delta_G u_l = \lambda_j u_l & \text{in } \Omega, \ \forall l \in \{1, \ldots, m\}, \\ u_l = 0 & \text{on } \partial\Omega, \ \forall l \in \{1, \ldots, m\}, \\ \sum_{l=1}^m \left(\frac{\partial u_l}{\partial \mathbf{n}}\right)^2 |\mathbf{n}_G|^2 = c_2\mathcal{H} & \text{on } \partial\Omega, \end{cases}$$

for some constant $c_2 \in \mathbb{R}$.

7 The Rellich–Pohozaev Identity for the Grushin Laplacian

The aim of this section is to collect a new proof of the Rellich–Pohozaev identity for the Grushin Laplacian. Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1. Let λ be an eigenvalue of the Dirichlet Grushin Laplacian, *i.e.* an eigenvalue of (5). Let u be an eigenfunction in $W^{1,2}_{G,0}(\Omega)$ corresponding to λ normalized with $\|u\|_{L^2(\Omega)} = 1$. Suppose that $u \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$. Then the Rellich-Pohozaev identity reads

$$\lambda = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 |\mathbf{n}_G|^2((x, (1+s)y) \cdot \mathbf{n})\, d\sigma_z. \tag{42}$$

This identity is a consequence of a more general class of Pohozev-type identities (see, e.g., Tri [60,61] for $N = 2$, Kogoj and Lanconelli [34, Sect. 2] for arbitrary N). We also mention Garofalo and Lanconelli [24] for a Pohozaev-type identity for the Heisenberg Laplacian. A proof of (42) can be done following the classical argument that Rellich used for the standard Laplacian in [55]. This argument is rather elementary being based only on integration by parts, but requires some lengthy computations. Instead, our proof is a straightforward application of the Hadamard-type formula that we have proved. More precisely, our strategy is first to differentiate the eigenvalue with respect to the natural dilation in the Grushin setting, and then to match this derivative with the one computed by (30). The same strategy was exploited by di Blasio and Lamberti [15, Theorem 5.1] for the Finsler p-Laplacian.

The natural dilation in the Grushin setting is:

$$\delta_t(z) := (tx, t^{1+s}y) \qquad \forall z = (x, y) \in \mathbb{R}^N, \forall t > 0.$$

We fix Ω to be a bounded open subset of $\mathbb{R}^N$ of class C^1. We set

$$\Omega_t := \delta_t(\Omega) \qquad \forall t > 0.$$

Let $\lambda = \lambda_n[\Omega] = \cdots = \lambda_{n+m-1}[\Omega]$ be a Dirichlet Grushin eigenvalue on Ω of multiplicity m. It is easily seen that

$$t^2\lambda_{n+l}[\Omega_t] = \lambda \qquad \forall t > 0, \forall l = 0, \ldots, m - 1. \tag{43}$$

This can be deduced from the fact that if $l = 0, \ldots, m - 1$ and u is an eigenfunction in $W^{1,2}_{G,0}(\Omega_t)$ corresponding to $\lambda_{n+l}[\Omega_t]$, then we have

$$\int_\Omega \nabla_G(u(\delta_t)) \cdot \nabla_G(\varphi(\delta_t)) \, dz$$

$$= \int_\Omega (\nabla(u(\delta_t))I_G(z)) \cdot (\nabla(\varphi(\delta_t))I_G(z)) \, dz$$

$$= \int_\Omega \nabla_x(u(\delta_t)) \cdot \nabla_x(\varphi(\delta_t)) \, dz + \int_\Omega |x|^{2s}\nabla_y(u(\delta_t)) \cdot \nabla_y(\varphi(\delta_t)) \, dz$$

$$= \int_\Omega t^2\nabla_x u(\delta_t) \cdot \nabla_x\varphi(\delta_t) \, dz + \int_\Omega |x|^{2s}t^{2+2s}\nabla_y u(\delta_t) \cdot \nabla_y\varphi(\delta_t) \, dz$$

$$= \int_{\Omega_t} t^2\nabla_x u \cdot \nabla_x\varphi t^{-h-(1+s)k} \, dz + \int_{\Omega_t} |x|^{2s}t^2\nabla_y u \cdot \nabla_y\varphi t^{-h-(1+s)k} \, dz$$

$$= \int_{\Omega_t} t^2\nabla_G u \cdot \nabla_G\varphi t^{-h-(1+s)k} \, dz$$

$$= t^2\lambda_{n+l}[\Omega_t] \int_{\Omega_t} u\varphi t^{-h-(1+s)k} \, dz$$

$$= t^2\lambda_{n+l}[\Omega_t] \int_\Omega u(\delta_t)\varphi(\delta_t) \, dz \qquad \forall\varphi \in W^{1,2}_{G,0}(\Omega_t).$$

Accordingly, by (43) with $t = 1 + \varepsilon$, we have

$$\lambda_{n+l}[\Omega_{1+\varepsilon}] = (1 + \varepsilon)^{-2}\lambda \qquad \forall \varepsilon > -1, \ \forall l = 0, \ldots, m - 1 \, .$$

Therefore, $\varepsilon \mapsto \lambda_n[\Omega_{1+\varepsilon}]$ is differentiable in $]-1, +\infty[$ and we have

$$\frac{d}{d\varepsilon}\lambda_n[\Omega_{1+\varepsilon}]\bigg|_{\varepsilon=0} = -2\lambda.$$

On the other side, we can also compute the derivative $\frac{d}{d\varepsilon}\lambda_n[\Omega_{1+\varepsilon}]\big|_{\varepsilon=0}$ exploiting our results. Let u be an eigenfunction corresponding to λ normalized such that $\|u\|_{L^2(\Omega)} = 1$, and assume that $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. We note that if O is any bounded open subset of $\mathbb{R}^N$ containing $\overline{\Omega} \cap \{x = 0\}$, then $\delta_{1+\varepsilon} \in \mathcal{A}_{\Omega, O}$ for all $\varepsilon > -1$. We can apply Theorem 5.4 and Remark 5.3 to the family $\{\delta_{1+\varepsilon}\}$ and obtain that the eigenvalues of the matrix (32) are the derivatives at $\varepsilon = 0$ of the branches splitting from λ. As we have already seen above, the domain perturbation $\delta_{1+\varepsilon}$ preserves the multiplicity of the eigenvalue λ and accordingly the matrix (32) is actually a scalar matrix. Thus

$$\frac{d}{d\varepsilon}\lambda_n[\Omega_{1+\varepsilon}]\bigg|_{\varepsilon=0} = -\int_{\partial\Omega}\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 |\mathbf{n}_G|^2((x, (1+s)y) \cdot \mathbf{n}) \, d\sigma_z.$$

By the above two expressions of the derivative of the the eigenvalue we get the Rellich-Pohozaev identity:

$$\lambda = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 |\mathbf{n}_G|^2((x, (1+s)y) \cdot \mathbf{n}) \, d\sigma_z.$$

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