Cheating at Craps: A Quantitative Analysis

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ABSTRACT

Craps is a simple dice game that is popular in casinos around the world. While the rules for Craps, and its mathematical analysis, are reasonably straightforward, this paper instead focuses on the best ways to cheat at Craps, by using loaded (biased) dice. We use both analytical modeling and simulation modeling to study this intriguing dice game. Our modeling results show that biasing a die away from the value 1 or towards the value 5 lead to the best (and least detectable) cheating strategies, and that modest bias on two loaded dice can increase the winning probability above 50%. Our Monte Carlo simulation results provide validation for our analytical model, and also facilitate the quantitative evaluation of other scenarios, such as heterogeneous or correlated dice.

1. INTRODUCTION

Dice have played a pivotal role in gambling games for over a thousand years [3]. Craps is a dice game that dates back to ancient times well over 500 years ago. It uses two six-sided dice, which are rolled one or more times in succession to determine the outcome of the game (win or lose) based on the sum of the dice values rolled. The name of the game is believed to come from the French word *crapaud* (toad), in reference to the frog-like squatting position of participants when playing the game in the streets.

The rules for Craps are extremely simple. If the opening roll by a player is 7 or 11, then the player wins immediately. If the opening roll is 2, 3, or 12, then the player immediately loses. If any other value occurs during the opening roll, then this value is recorded as the target value, called "point", for subsequent rolls. The goal of the player is then to roll the target point value again, before rolling a 7. If they roll the point value, then they win, but if they roll a 7, they lose. If any other value is rolled, it is ignored, and the game continues with successive rolls until either the point value (win) or a 7 (lose) occurs.

There are several aspects of the Craps game that make it appealing for players and for casinos. First, it is an extremely simple game of chance that is easy to learn and understand, with minimal or no strategy involved. Second, it is well-suited for wagering, both by players themselves (i.e., betting against the "house" that they will win) and by spectators (i.e., betting against the house whether the player will win or lose on the next roll). Third, and perhaps most important, is that the winning probability is approximately 0.493. These odds are much better than for some other casino games, and tantalizingly close to the 50% level at which a player could expect to break even (or win) in the long run. In short, the odds are still in favour of the casino, but not by much.

The latter observation is the primary motivation for our paper. In essence, we are asking how much cheating is required in order to tip the odds in favour of the Craps player, rather than the casino. More specifically, we consider the possibility of loaded dice, which are biased to produce certain roll outcomes slightly more frequently than expected with fair dice. In this work, we ignore how the cheating is achieved (see, e.g., [1, 6] for works on the analysis of randomness in fair dice), and just assume that the outcome probabilities of some faces can be altered, either by the skills of the player or by the manufacturing of the die.

The specific cheating questions that we address are:

- If biased dice are used, which outcomes are preferable?
- Should both dice be biased, or only one?
- If two biased dice are used, should they have homogeneous bias to the same preferred outcome, or should they be different?
- What is the minimal level of cheating (i.e., bias) required in order to achieve a 50% winning probability?

We use analytical and simulation modeling to answer these questions. The analytical model starts with a precise mathematical model of the original Craps game, and then extends this model to include a bias parameter that affects the outcome on the loaded dice. The basic model assumes homogeneous bias levels across the two dice, but possibly heterogeneous preferred values for the dice outcomes (e.g., 3 and 6). Our simulation model uses Monte Carlo simulation to estimate the winning probability for the Craps game in different configurations. We use this model initially to validate the analytical model, and then to explore additional scenarios with heterogeneous bias levels and correlated dice.

The main insight that emerges from our modeling efforts is that a biased dice with a preferred value of 5 is often the best in any of the scenarios considered. This result makes sense since an outcome of 5 on one die increases the odds of winning in the opening roll, while completely eliminating the chance of losing on the opening roll. The effect on subsequent rolls is more complicated, but also analyzable. Our modeling results show that using two biased dice is better than having just one biased die, and that biasing the dice either towards 5 or away from 1 are both good cheating strategies. Bias levels of about 5% on each die suffice to increase the winning probability to 50% or more.

Similar to our approach, the authors in [8] studied the best strategies for dice rolling in Craps. However, in contrast with that work, we assume that the die can be arbitrarily loaded, and we seek the optimal way of doing so. The analysis of gambling games has also been used for pedagogy. For instance, the authors in [7] studied the optimal strategy for the dice game known as *Pig*. Despite the differences from Craps, their work shares several methodological aspects with our own. Specifically, the notions of probabilistic modeling, simulation, constraint optimization, and Monte Carlo simulation are used in elegant ways to solve an easily understandable problem. The analysis of optimal strategies in dice or other games has always attracted the interest of researchers working in applied probability [2, 4, 5]. However, in our case, we study the optimal way of cheating with loaded dice, which makes our contribution novel.

The rest of the paper is organized as follows. Section 2 presents our analytical model, and its numerical results. Section 3 presents our simulation model, along with simulation results. Finally, Section 4 concludes the paper.

2. ANALYTICAL MODEL

In this section, we define and develop a probabilistic model for the game of Craps. While the probability of winning has been known for a long time, we will use our model to determine the least cheating required to raise the winning probability above 50%.

We consider two slightly different cheating scenarios:

- We can arbitrarily change the probability of one face on the die, while the other faces all have the same equal (renormalized) probability of being selected.
- We can arbitrarily change the probabilities for multiple faces on the die. This is the case when the game is implemented computationally as an online game.

2.1 Model Overview and Assumptions

Let N be the random variable (RV) denoting the number of dice rolls, and let $X_1^{(i)}$ and $X_2^{(i)}$ be two independent random variables denoting the outcomes of the i-th dice roll, for $i = 1, \ldots, N$. The RVs $X_d^{(i)}$ are independent and have support $\{1, \ldots, 6\}$. The density function for dice 1 and 2 are described by:

$$\mathbf{q}_1 = (q_{11}, \ldots, q_{16}), \quad \mathbf{q}_2 = (q_{21}, \ldots, q_{26}).$$

For a fair game, $q_{dj} = 1/6$ for $d \in \{1, 2\}$ and $j \in \{1, \ldots, 6\}$. By independence, the probability of obtaining an outcome s from a single dice roll is:

$$o_s(\mathbf{q}_1, \mathbf{q}_2) = \sum_{j=\max(1,s-6)}^{\min(6,s-1)} q_{1j} q_{2(s-j)}, \qquad \text{for } s = 2, \ldots, 12.$$

In what follows, we simplify our notation to o_s unless we wish to emphasize the dependence on vectors $\mathbf{q}_1$ and $\mathbf{q}_2$.

2.2 Analytical results

In this section, we formally derive the distribution of N, which is the number of dice rolls required to win or lose a game, as well as the probability of winning.

We start by considering the case in which $s = X_1^{(1)} + X_2^{(1)} \in \{4, 5, 6, 8, 9, 10\}$, so that the game does not end on the very first roll. For brevity, denote $Y^{(i)} = X_1^{(i)} + X_2^{(i)}$, and let us compute

$$Pr\{N = n | Y^{(1)} = s\}.$$

Each dice roll is an independent Bernoulli experiment whose probability of success is $o_7 + o_s$, i.e.:

$$Pr\{N = n | Y^{(1)} = s\} = (o_7 + o_s)(1 - o_7 - o_s)^{(n-2)}.$$

for $n = 2, 3, \ldots$ (recall that, given the conditioning, we have at least 2 dice rolls). The probability of winning is $o_s/(o_s + o_7)$. Therefore, the complete distribution of N is:

$$Pr\{N = n\} = \begin{cases} \sum_{s\in\{2,3,7,11,12\}} Pr\{Y^{(1)} = s\} & \text{if } n = 1 \\ \sum_{s\in\{4,5,6,8,9,10\}} Pr\{N = n | Y^{(1)} = s\} & \\ \quad \cdot Pr\{Y^{(1)} = s\} & \text{if } n \geq 2 \end{cases}$$

From the distribution of N, we can derive the expected number of dice rolls for each game:

$$\sum_{n=1}^{\infty} n Pr\{N = n\} = \frac{557}{165} = 3.37576.$$

The probability of winning can be easily derived by the law of total probabilities, and is expressed by:

$$p(\mathbf{q}_1, \mathbf{q}_2) = o_7(\mathbf{q}_1, \mathbf{q}_2) + o_{11}(\mathbf{q}_1, \mathbf{q}_2) + \sum_{j\in\{4,5,6,8,9,10\}} \frac{o_j(\mathbf{q}_1, \mathbf{q}_2)^2}{o_j(\mathbf{q}_1, \mathbf{q}_2) + o_7(\mathbf{q}_1, \mathbf{q}_2)}.$$

Analogously to what we have done for $o_s(\mathbf{q}_1, \mathbf{q}_2)$, we write only p instead of $p(\mathbf{q}_1, \mathbf{q}_2)$ when there is no ambiguity. For the fair game case,

$$p = \frac{244}{495} \approx 0.492929.$$

2.3 Cheating Strategies

We propose two cheating strategies:

Strategy 1: Single-Face Optimization. We assume that we can increase or decrease the outcome probability of one die face, and the remaining faces are all equally likely. That is, if face f on die d has probability x, then the vector $\mathbf{q}_d^f(x)$ is defined as follows:

$$q_{dj}^f(x) = \begin{cases} x & \text{if } j = f, \\ (1-x)/5 & \text{otherwise.} \end{cases}$$

Strategy 2: Multi-Face Optimization. We assume that the outcome probability for each die face can be set independently of the other faces. That is, the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ can have arbitrary components in $[0, 1]$ that sum to unity.

Our goal is to make the winning probability $p > 0.5$ with minimal modifications to the dice. For both cheating strategies, let us measure the bias of a die $d = 1, 2$ using a sum of squared deviations from a fair die as follows:

$$\Psi(\mathbf{q}_d) = \sum_{j=1}^{6} \left(q_{dj} - \frac{1}{6} \right)^2.$$

2.4 Solution for Single-Face Strategy 1

In Strategy 1, we can change the outcome probability for just one face on each die. All of the remaining faces on each die are equally likely.

Initially, we allow bias levels that are as large as necessary to overcome the house advantage. Denoting by x and y the chosen face probabilities for each die, we solve the following optimization problem:

$$\underset{x,y}{\operatorname{argmin}} \left(x - \frac{1}{6}\right)^2 + \left(y - \frac{1}{6}\right)^2$$

$$\text{s.t.}$$

$$p(\mathbf{q}_1^{f_1}(x), \mathbf{q}_2^{f_2}(y)) \geq \frac{1}{2}\,.$$

Table 1 shows the results from solving this optimization problem. These results provide several high-level insights into cheating at Craps. First, winning is possible for any combination of die face values, if the bias levels are large enough. Second, some dice combinations, such as (1,1), (1,5), (1,6), (2,5), and (5,5), seem more promising than others, since less bias is required. Third, the solution landscape is not always symmetric. That is, the level of bias is not always the same magnitude or in the same direction on each face in a pairing. This observation even applies for the cases for doubles, in which both face values are the same. Specifically, (1,1) through (5,5) have symmetric bias levels, while (6,6) is distinctly asymmetric. In particular, the bias level for the latter needs to be substantially increased for one die and substantially decreased for the other, presumably to avoid a 12 in the opening roll. In general, the bias levels required for winning are larger when a 6 is present at all, suggesting that a 6 is not especially helpful in Craps.

We next consider a more realistic scenario, in which the bias levels are constrained, so that the cheating is less likely to be detectable. For this purpose, we assume that the modified face must have an outcome probability that deviates no more than $\epsilon = 20\%$ from the original 1/6 probability; specifically, the new value must lie between 2/15 and 1/5 (inclusive). This implies that:

$$\Psi(\mathbf{q}_d) \leq \frac{1}{750}\,,$$

but we also impose a constraint on the maximum bias of the face we are changing.

We formulate the following optimization problem for the face pair f_1 and f_2:

$$\underset{x,y}{\operatorname{argmax}}\, p(\mathbf{q}_1^{f_1}(x), \mathbf{q}_2^{f_2}(y)) \tag{1}$$

$$\text{s.t.}$$

$$\frac{2}{15} \leq x, y \leq \frac{1}{5}\,,$$

Table 2 shows the solution results from this optimization problem. If we are only allowed to decrease the outcome probability of one of the die faces, then the optimal choice is to decrease the probability of face 1 in both dice. If we are only allowed to increase the probability of a certain die face, then the optimal choice is to increase the probability of face 5. Having one die biased against 1, and the other in favour of 5, is also a very good solution, but is not optimal.

Table 1: Minimum cheating required on a single face to reach a winning probability of $1/2$. The first column indicates the faces f_1 and f_2 that are modified on each die. The second and third columns show the bias required on each die to achieve a winning probability of $1/2$.

f_1, f_2	P[f_1] in die 1	P[f_2] in die 2
1, 1	−10.4%	−10.4%
1, 2	−19.2%	−5.16%
1, 3	−19.2%	5.26%
1, 4	−19.1%	5.31%
1, 5	−14.52%	9.52%
1, 6	−20.5%	1.58%
2, 2	−28.6%	−28.6%
2, 3	−35.2%	36.1%
2, 4	−34.4%	35.1%
2, 5	−12.9%	21.6%
2, 6	−53.3%	−20.8%
3, 3	28.9%	28.9%
3, 4	−13.2%	56.7%
3, 5	11.6%	24.8%
3, 6	59.9%	4.60%
4, 4	28.9%	28.9%
4, 5	11.8%	24.5%
4, 6	59.3%	7.80%
5, 5	15.4%	15.4%
5, 6	28.9%	5.96%
6, 6	−59.1%	60.8%

Table 2: Solution of the constrained single-face optimization problem for Strategy 1. The first column indicates the faces f_1 and f_2 that are modified on each die. The second column shows the best winning probability attainable with cheating, while columns 3 and 4 show the outcome probabilities used to attain the maximum winning probability.

f_1, f_2	P[win]	P[f_1] in die 1	P[f_2] in die 2
1, 1	**0.506653**	2/15	2/15
1, 2	0.501677	2/15	2/15
1, 3	0.501694	2/15	1/5
1, 4	0.501716	2/15	1/5
1, 5	0.504172	2/15	1/5
1, 6	0.500405	2/15	1/5
2, 2	0.497644	2/15	2/15
2, 3	0.497022	2/15	1/5
2, 4	0.497080	2/15	1/5
2, 5	0.500695	2/15	1/5
2, 6	0.49557	2/15	2/15
3, 3	0.497667	1/5	1/5
3, 4	0.496433	1/5	1/5
3, 5	0.499808	1/5	1/5
3, 6	0.495316	1/5	2/15
4, 4	0.497667	1/5	1/5
4, 5	0.499865	1/5	1/5
4, 6	0.495294	1/5	2/15
5, 5	**0.502144**	1/5	1/5
5, 6	0.498334	1/5	1/5
6, 6	0.493711	2/15	1/5

Table 3: Probability of not rejecting null hypothesis H_0 for single-face Strategy 1 solutions.

n	Face 1: P[n.r. H_0]	Face 5: P[n.r. H_0]
200	0.8650	0.8690
400	0.7641	0.7653
600	0.6514	0.6570
800	0.5272	0.5422
1,000	0.4252	0.4472
2,000	0.0920	0.1312
5,000	0.00001	0.00003

In general, the structural results in Table 2 align well with those in the earlier Table 1, in terms of the directionality of the bias required to improve the odds of winning at Craps. There are a few exceptions, however, such as the cases for (3,4), (3,6), and (4,6) in the tables. In each of these cases, the constrained search settles on a local optimum at an endpoint of the permitted range, but without finding a winning probability of at least 1/2. In fact, only 8 of the 15 face combinations in Table 2 produce winning strategies under the cheating constraints given. However, dice combinations with $f_2 = 5$ seem very robust, regardless of f_1, in terms of winning or almost breaking even.

We next turn our attention to the detectability of the cheating strategy. There are several approaches that a casino observer could use for this purpose. We consider two approaches: monitoring the outcomes of individual die rolls, and monitoring the distribution for the number of rolls in each Craps game. Other approaches could consider the proportion of games won, or the monetary winnings.

Suppose that an observer wants to use a χ-square goodness-of-fit test to check if the dice are fair. The null hypothesis H_0 is that the outcome of a dice roll has the same 1/6 probability for all the faces. A significance level of 0.05 is assumed. The observer focuses on one die, and performs n rolls. We determine the probability that the hypothesis H_0 is rejected when the die has some outcome distribution $\hat{\mathbf{q}}_d$. In order to determine this probability, we use a Monte Carlo simulation that considers 10,000 independent experiments, each with n rolls, and we count how many of these experiments lead the observer to reject the null hypothesis.

In Table 3, we show the results of Monte Carlo simulation experiments to assess the effectiveness of detecting Strategy 1. We have two columns: one in which the probability of a 1 is decreased to 2/15, and the other in which the probability of a 5 is increased to 1/5. This cheating strategy, with 20% bias, seems eminently detectable. Indeed, with only 1,000 rolls, the observer rejects the null hypothesis more than 50% of the time.

In Figure 1, we show the difference in the probability of observing a certain number of dice rolls between the fair and cheated games for the two optimal solutions (in **bold**) from Table 2. We can see that, although the manipulation of the outcome probability of face 1 gives a higher probability of winning, it also changes the distribution of the number of dice rolls in a more prominent way (i.e., fewer games end after the opening roll). In particular, it is much easier to detect the (1,1) cheating strategy than it is to detect the (5,5) cheating strategy. As such, the (5,5) strategy might be preferable, based on this detection metric.

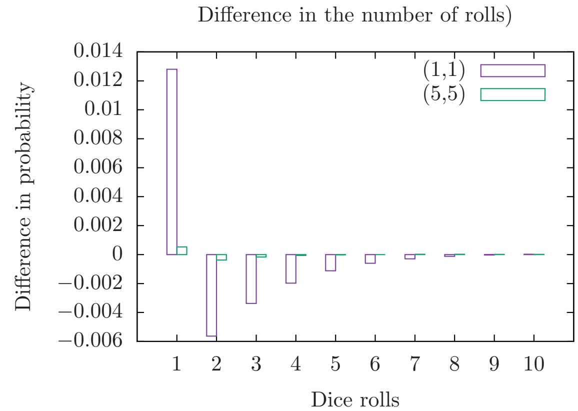


Figure 1: Difference in the probability of observing a certain number of dice rolls in the fair game and when cheating according to single-face Strategy 1.

2.5 Solution for Multi-Face Strategy 2

We next consider the multi-face Strategy 2, which seems inherently stronger than the single-face Strategy 1. In Strategy 2, the probabilities for all six faces can be manipulated, as long as they sum to unity.

Let Ψ_{max} be the maximum level of bias allowed in the dice. Then, for Strategy 2, we solve the following optimization problem using the variables $q_{11}, \ldots, q_{16}, q_{21}, \ldots, q_{26}$:

$$\underset{\mathbf{q}_1,\mathbf{q}_2}{\operatorname{argmax}}\, p(\mathbf{q}_1, \mathbf{q}_2) \tag{2}$$

$$\text{s.t.}$$

$$\Psi(\mathbf{q}_1) \le \Psi_{max}\,, \Psi(\mathbf{q}_2) \le \Psi_{max}\,,$$

$$\forall d \in \{1,2\} : \left(\forall i \in \{1,\ldots,6\} :\; 0 \le q_{di} \le 1, \sum_{i=1}^{6} q_{dij} = 1 \right).$$

Note that $p(\mathbf{q}_1, \mathbf{q}_2)$ is a rational function in the variables q_{dj}, whose numerator and denominator degrees are 4 and 2, respectively. We use the Interior Point Method to solve the optimization problem, where the first and second order derivatives are computed symbolically.

For this optimization, we use a uniform distribution for both dice (i.e., fair dice) as the initial starting point. Although the Interior Point Method may converge to a suboptimal solution, the choice of this starting point is intuitive for finding the minimum bias required on the dice. We have tried the optimization starting from other randomly chosen starting points, and have observed the same convergence point. This empirical observation suggests that the local optimum is also a global optimum, though we do not yet have a mathematical proof of this property.

For $\Psi_{max} = 0.00025$, we obtain the local maximum $p = 0.50002$ corresponding to:

$$\hat{\mathbf{q}_1} = \hat{\mathbf{q}}_2 = (0.1548, 0.1627, 0.1701, 0.1703, 0.1749, 0.1672)\,. \tag{3}$$

This result is quite insightful. Both dice are modified in exactly the same way: face 5 receives the highest probability (about 5% bias), while face 1 receives the lowest probability (about -7% bias). Furthermore, the probability for the number 6 is perturbed the least, reflecting its minor influence on

Table 4: Probability of not rejecting null hypothesis H_0 for multi-face Strategy 2 solution.

n	P[not rejecting H_0]
200	0.9359
400	0.9206
600	0.9085
800	0.8911
1,000	0.8998
2,000	0.7712
5,000	0.4679

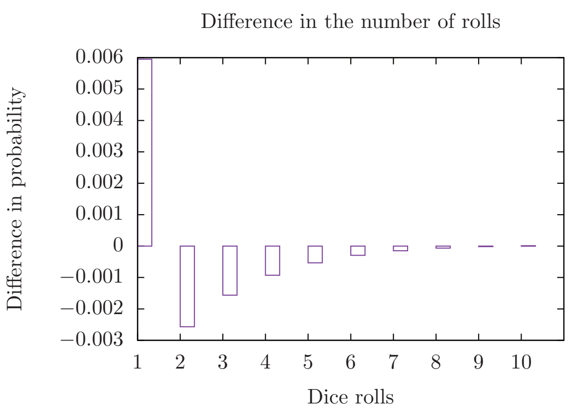


Figure 2: Difference in the probability of observing a certain number of dice rolls in the fair game and when cheating according to multi-face Strategy 2.

the game's outcome. These small modifications suffice to overcome the house advantage in Craps.

In Table 4, we show the results regarding the detectability of Strategy 2. Again, this approach uses Monte Carlo simulations and χ-square goodness-of-fit tests, with the null hypothesis H_0 being that the dice are fair. With only $1,000$ die rolls, the observer is unlikely to reject the null hypothesis. With $5,000$ die rolls, however, the observer has a 53% probability of detecting the cheating.

Finally, we study the change in the distribution of the number of dice rolls caused by the cheating. Figure 2 shows the difference in the probability of observing a certain number of dice rolls between the fair and cheated games. The cheated version has a slightly *lower* probability (-1.8%) of an instant win or loss on the opening roll, while the probability of winning or losing with two dice rolls is slightly higher.

Comparing the multi-face Strategy 2 results in Figure 2 with those for the single-face Strategy 1 in Figure 1, and noting the difference in scales, reveals that Strategy 2 is less detectable than the (1,1) configuration of Strategy 1, but more detectable than the (5,5) configuration in Strategy 1. Thus the multi-face Strategy 2 is not always superior to Strategy 1 with regard to detectability. In fact, neither Strategy 1 nor Strategy 2 strictly dominates the other. This is a somewhat surprising insight from our modeling results.

3. SIMULATION MODEL AND RESULTS

In this section, we use Monte Carlo simulation to explore several aspects of the Craps dice game. The simulation results provide validation for our analytical model, and some insights into intermediate states of the Craps game. Furthermore, the simulation model facilitates explorations beyond the assumptions used in the analytical model.

3.1 Simulation Model

We used a Monte Carlo simulation to model the Craps dice game. The simulator is written in C (about 200 lines of code), and compiled and run on a Linux system. Command-line parameters are used to specify the number of games to play (default is 10 million), as well as the bias level (if any) and preferred value for each of the two dice being rolled. We focus only on positive bias here, since the results for negative bias are qualitatively similar.

The simulator uses the `random()` function from the C library for pseudo-random number generation, from which Uniform(0,1) values are used to generate the results for each roll. In the unbiased case (Fair Dice), an EquiLikely(1,6) function is used to generate values for each of the two dice. In the biased cases, a NotQuiteEquiLikely(1,6) function is used to increase the probability for the preferred die value. Unless specified otherwise, the two dice values are generated independently, and summed to produce each roll outcome.

The simulator models the state evolution of each Craps game, from the opening roll until the final outcome is determined. Statistics are recorded regarding wins, losses, point (target) values, and total dice rolls. Summary statistics are saved from each simulation run, and post-processed to produce graphical results and additional statistical analyses.

3.2 Simulation Verification

Figure 3 shows the simulation results from a simple case with full bias (i.e., probability 1) for the face value 5. This scenario is considered with 0, 1, or 2 Biased-5 dice.

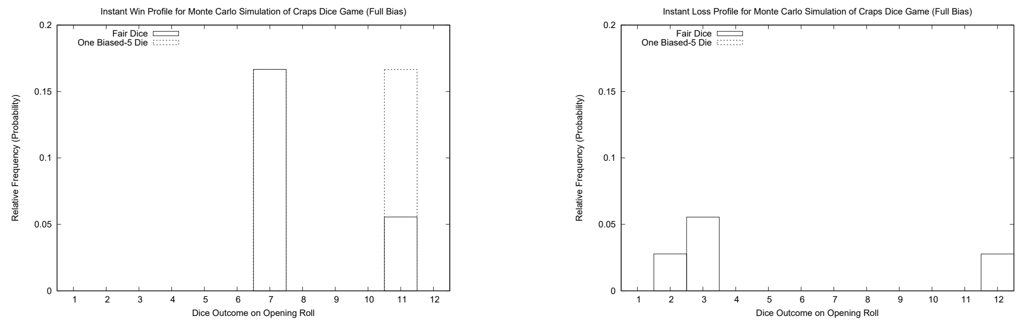
When Fair Dice are used, the winning probabilities from the opening roll match those expected. That is, a 7 is rolled with probability 6/36, and an 11 is rolled with probability 2/36. Similarly, the losing outcomes on the opening roll for 2, 3, and 12 match the expected values of 1/36, 2/36, and 1/36, respectively.

When only one (deterministic) Biased-5 die is used, the opening roll always contains at least one 5, and it is no longer possible to lose in this stage, since the sum already exceeds 3, and cannot possibly reach 12. However, there is only a 1/3 probability of winning in this stage (i.e., 1/6 for a sum of 7, and 1/6 for a sum of 11). All other cases proceed into a subsequent round with one of four possible point values: 6, 8, 9, or 10. At this stage, there is a 50-50 chance of winning. This makes sense since getting a 7 and getting the desired point value are equally likely (1/6 each); all the other possible outcomes for the dice merely prolong the game. So the overall winning probability with such a fully biased die is 2/3.

Playing Craps with two fully-biased Biased-5 dice is trivial, and not shown in the graph. That is, the opening roll is always a 10, which neither wins nor loses, but the subsequent roll of 10 matches the target point value, and wins.

3.3 Results for Medium Bias

Clearly, using fully-biased dice would be easily detected by the casino, so Figure 4 presents a more realistic scenario with only 25% bias in favour of 5. In the Fair Dice



(a) Winning on opening roll

(b) Losing on opening roll

Figure 3: Effect of a Biased-5 die on opening roll in Craps (full bias).

case, the win/loss probabilities in the opening roll match the known values for the Craps game. With a single Biased-5 die, the win probability remains the same for a 7, and increases slightly for an 11. Meanwhile, the loss probabilities for the opening roll decline on all three cases. These trends are consistent with those in the previous example, for the same reasons. Furthermore, these effects are further amplified when both dice are biased in the same way.

The bottom two graphs in Figure 4 take a closer look at the wins and losses in subsequent rounds. These results are broken down by the different point values possible, with wins in Figure 4(c), and losses in Figure 4(d). These graphs show a lateral shift of winning probabilities from low point values to high point values, and a corresponding decline in loss probabilities as well. At this bias level, the effect on winning probabilities is more pronounced than the effect on loss probabilities. For this particular example, the chances of winning the Craps game are slightly better than 50%, consistent with the analytical model. Any bias stronger than 15.4% achieves this goal when two Biased-5 dice are used.

3.4 Heterogeneous Dice

We next used our simulator to explore dice configurations that are heterogeneous, both in face values and bias levels.

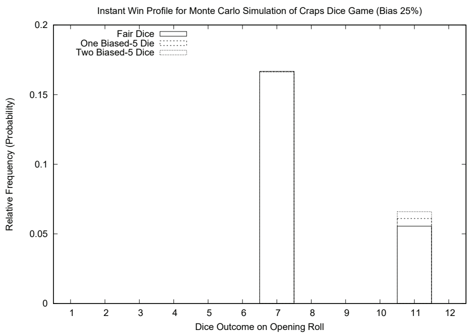
Figure 5 shows the simulation results from these experiments. Each graph shows the winning probability for the Craps game when the first die is biased by 10% toward the value indicated (i.e., towards 1 in Figure 5(a), and towards 6 in Figure 5(f)). Upon each graph is a dotted horizontal line showing the winning probability (0.493) in the baseline configuration of Fair Dice, as well as a solid horizontal line showing the cheating goal of exceeding a 50% winning probability. The remaining six lines show the results for different preferred values (1 through 6) on the second die, as a function of the bias level. We consider bias levels ranging from 0% to 25%, in steps of 5%.

Figure 5(a) shows that having one die biased toward the value 1 is a bad idea. This setting makes the winning probability worse than the baseline case, even when the second die is fair. When the second die is biased, a preferred value of 1 is clearly the worst, since it increases the chances of losing in the opening roll (i.e., sum of 2 or 3). A preferred value of 2 is also a poor choice, for similar reasons. A value of 6 has an intermediate effect: it increases the chance of winning in the opening roll, but also increases the chances of losing on subsequent rolls. The line for this value is close to horizontal over the parameter range considered. The best choice for a preferred value on the second die is 5, as indicated previously by our mathematical model. With a Biased-5 die, one can re-attain the baseline winning probability with a bias of about 15%, but it is not possible to attain the goal of a 50% winning probability, even with 25% bias. Using a Biased-3 or Biased-4 die is less effective, with 25% bias needed just to regain the baseline winning probability. These two lines are very similar, and are almost indistinghishable on the graph.

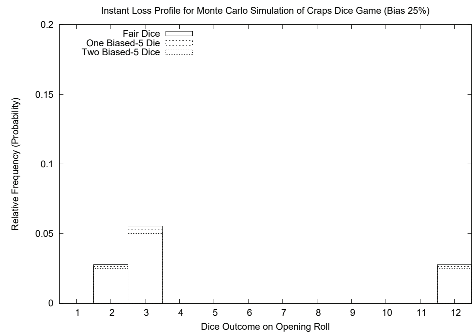
Similar observations apply for Figure 5(b), in which the first die is biased towards the value 2. The initial starting point on the left edge of the graph is slightly better than for the Biased-1 die, since one of the ways to lose in the opening roll (i.e., a sum of 2) is now less likely. However, the winning probability is still worse than the baseline of Fair Dice. Furthermore, the structure of the six lines (i.e., slopes and relative ordering) for different preferred values on the second die remain the same as seen previously, albeit at slightly higher winning probabilities. None of these configurations achieve the desired goal of 50% over the range of bias probabilities considered here.

The simulation results for a Biased-3 die (in Figure 5(c)) and a Biased-4 die (in Figure 5(d)) are very similar. Both have an initial starting point above the Fair Dice baseline, since the losing combinations in the opening roll are less likely. The graphs are otherwise structurally similar to the previous graphs. The only new observations here are that the lines for Biased-3 and Biased-4 on the second die now disambiguate themselves, and that the relative ordering of these two lines depends on the preferred value of the first biased die. Specifically, the results indicate that it is slightly better to roll two 3's or two 4's than it is to roll a 3 and a 4 together (i.e., sum of 7). The intuition is that a 7 hurts the winning probability more across (possibly many) subsequent rolls than it helps in the (single) opening roll.

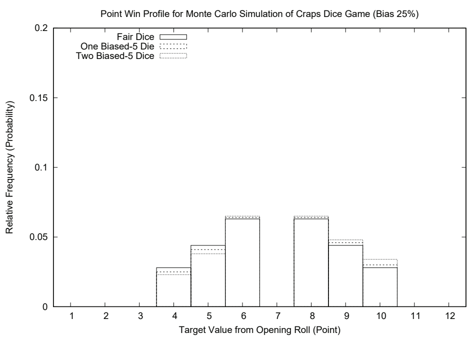
Figure 5(e) shows the simulation results for a Biased-5 die, which has the most pronounced positive impact on the winning probability, as noted earlier. The initial starting point with the first Biased-5 die is distinctly above the baseline for Fair Dice. The effects of the second biased die are structurally similar to the previous results, with Biased-1 being the worst choice, and Biased-5 being the best. Note that Biased-6 as the second die is now more favourable than ob-



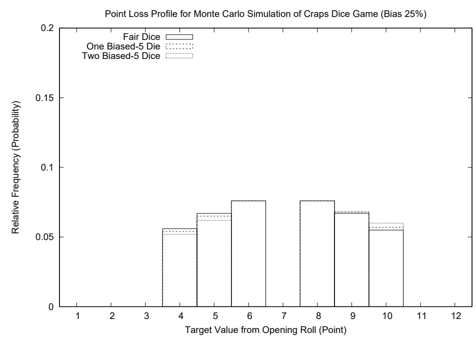
(a) Winning on opening roll



(b) Losing on opening roll



(c) Winning on subsequent rolls



(d) Losing on subsequent rolls

Figure 4: Effect of Biased-5 dice on Craps game outcome when relative bias is 25%.

served previously, with a positive slope across the range of bias values considered. The intuition is the increased probability of an 11 on the opening roll. Note also that the lines for Biased-3 and Biased-4 become indistinguishable once again. The most important observation from Figure 5(e) is that it is now possible to "beat the casino", with a winning probability above 50%. In this example, this can be achieved if the second Biased-5 die has at least 20% bias. This result matches closely with the analytical model, for which a bias of about 15% on each of two Biased-5 dice was sufficient to achieve the goal. The larger bias indicated in Figure 5(e) is because the first die has only 10% bias.

Finally, Figure 5(f) shows the results for a Biased-6 first die. The initial starting point for this graph is close to the baseline value, since the contributions to winning (i.e., sum of 11) and losing (i.e., sum of 12) in the opening roll are comparable, and offset each other. The rest of the graph is similar to those presented earlier. Biased-5 for the second die is again the best choice, since it increases the chances of winning in the opening roll with an 11, and decreases the chances of losing there with a 12. A Biased-6 second die is slightly worse than the baseline with Fair Dice, because of the increased chance of a 12 in the opening roll.

3.5 Correlated Dice

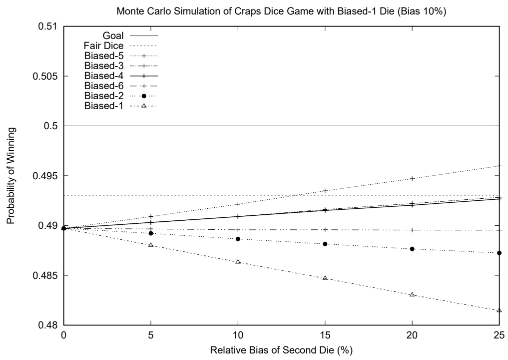
As a final set of simulation experiments, we consider the possibility of correlated dice, which are not reflected in our current analytical model.

We consider two rather simple correlation scenarios. In the positive correlation scenario, there is a tendency for the second die to achieve the same numerical value as the first die (i.e., doubles are achieved more often than normal by the roll). In the negative correlation scenario, the dice tend towards different values, so that doubles are less likely than normal. The level of cross-correlation is 5% in these tests.

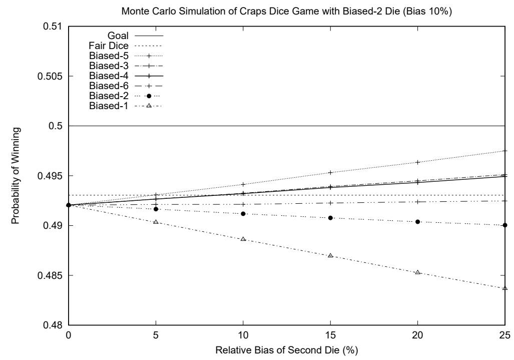
Figure 6 shows the results from this simulation experiment, using a format analogous to that from Figure 4.

The most obvious impact from positively correlated dice is a decline in the win probability for the opening roll. This makes sense since odd-numbered values for the sum are less likely. In the extreme case of 100% positive correlation (not shown here), only doubles are possible as the outcome from the dice roll. In such a scenario, it is not possible to win on the opening roll, but it is possible to lose then, with probability 1/3, if the sum is 2 or 12. In subsequent rounds, 100% positive correlation means a guaranteed win, since it is not possible to roll a 7, so the overall winning probability becomes 2/3. With the 5% positive correlation, the trend is the same, though the effect is less dramatic. Doubles are slightly more likely, which increases the loss probability in the opening roll. For subsequent rounds, there is an increased probability of doubles, and a slight increase in the overall win probability, since 7 is less likely.

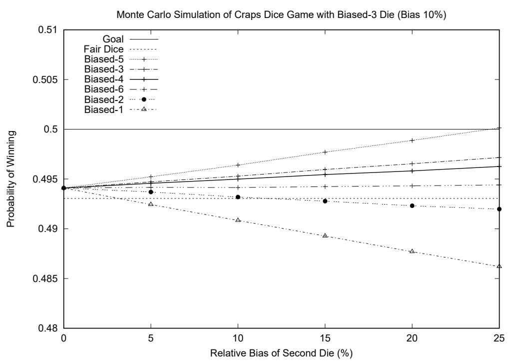
Negatively correlated dice have very different effects. In our simulation, we use a Uniform(0,1) random variate u to generate the value for the first die, and $1 - u$ to generate



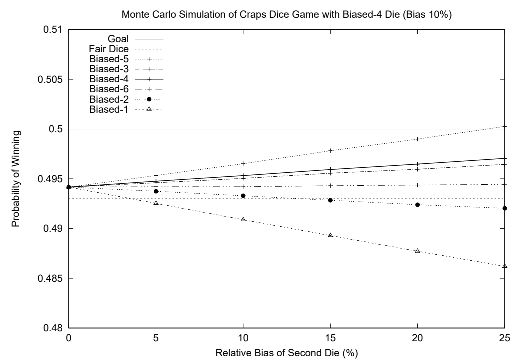
(a) Biased-1 with 10% relative bias



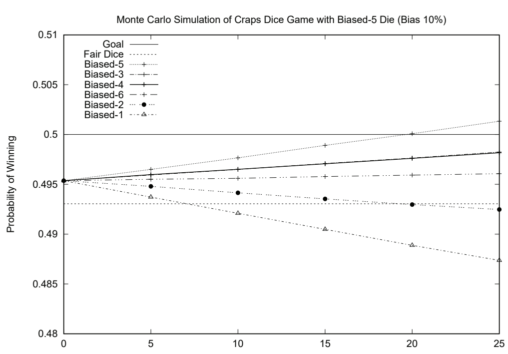
(b) Biased-2 with 10% relative bias



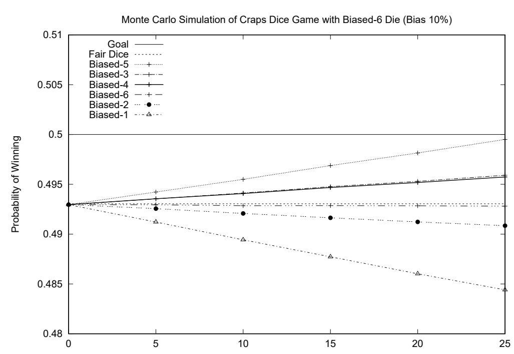
(c) Biased-3 with 10% relative bias



(d) Biased-4 with 10% relative bias

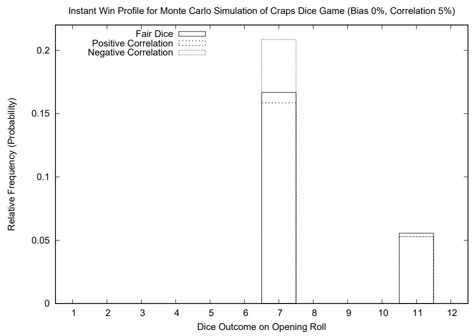


(e) Biased-5 with 10% relative bias

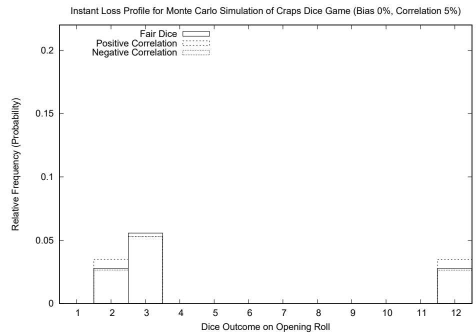


(f) Biased-6 with 10% relative bias

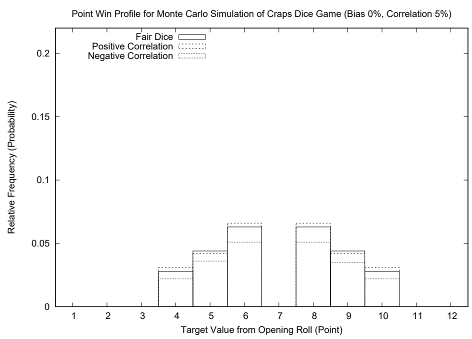
Figure 5: Monte Carlo simulation results for heterogeneous biased dice.



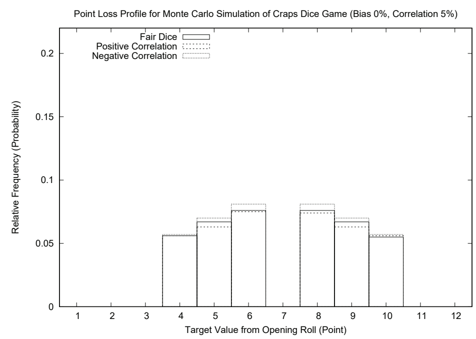
(a) Winning on opening roll



(b) Losing on opening roll



(c) Winning on subsequent rolls



(d) Losing on subsequent rolls

Figure 6: Effect of correlated dice on Craps game outcome (bias 0%; correlation 5%).

the value for the second die. In the 100% correlation scenario, this means that a 7 is always rolled in the opening, and wins. At the 5% correlation level in Figure 6, this effect is evident, but less pronounced. In particular, there is a noticable increase in winning the opening round with a 7, and a noticable decline in the winning probability in subsequent rounds, since rolling a 7 will lead to a loss. The overall result is a decline in the winning probability for the Craps game.

In summary, these simulation experiments show that correlated dice are inferior to two independent biased dice.

4. CONCLUSIONS

In this paper, we used mathematical analysis and simulation to study the Craps casino game with loaded dice.

The main conclusions from our work are the following. First, a die outcome of 5 is extremely powerful in Craps, since it boosts the odds of winning in the opening roll, and eliminates the chances of losing in the opening roll. Second, the most effective cheating strategies for Craps are to have either two Biased-1 dice (with about -10% bias), two Biased-5 dice (with about 15% bias), or one Biased-1 die (-7% bias) and one Biased-5 die (about 5% bias). These perturbations suffice to tip the odds in favour of the Craps player, instead of the casino. Finally, the side effects of these biased dice on the empirically observed number of dice rolls per game are small, and the (5,5) strategy is especially difficult to detect.

Determining whether such biased dice are practically realizable, or detectable by a casino, remains for future work.

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