Leibniz's Argument Against Infinite Number

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Abstract

This paper deals with Leibniz's well-known *reductio* argument against the infinite number. I will show that while the argument is in itself valid, the assumption that Leibniz reduces to absurdity does not play a relevant role. The last paragraph of the paper reformulates the whole Leibnizian argument in plural terms (i.e. by means of a plural logic) to show that it is possible to derive the contradiction that Leibniz uses in his argument even in the absence of the premise that he refutes.

1. Introduction

It is common to identify the birth of set theory with the work of Georg Cantor (1845–1918) as the official entrance into mathematics of the actual infinite: not an indefinite series without a greatest element (a potential infinite), but rather a set with an amount of infinitely many elements. Cantor managed to build an arithmetic of infinity, i.e. he defined mathematical operations (as sum, product, exponentiation, etc.) between infinite sets. In virtue of this, he was legitimated to extend the concept of number from the standard finite numbers to infinite numbers (i.e. numbers denoting the quantities of elements of infinite collections).

Two centuries before Cantor, G.W. Leibniz also argued for the necessity of the actual infinite; however, he also argued for the non-existence of an infinite number. Contrary to Cantor, who believed that the legitimacy of actual infinite collections was enough to ground the legitimacy of infinite numbers, Leibniz thought that the legitimacy of the former was not enough to ground the legitimacy of the latter. Moreover, he was convinced that the notion of infinite number was a self-contradictory notion. In different places Leibniz developed an argument (a *reductio ad absurdum*) to show that the hypothesis of the existence of an infinite number leads to a contradiction. The present paper aims to advance the discussion of such an argument. In particular, we propose to reformulate the whole Leibnizian argument in plural terms (i.e. by means of a plural logic), and we argue that, thanks to the plural formulation, it is possible to derive the contradiction that Leibniz uses in his argument even in the absence of the premise that he refutes.

The paper is structured as follows: section 2 introduces Leibniz's argument against the infinite number. We shall explain why Leibniz's argument is in itself valid, and that both the Leibnizian and the Cantorian views can be legitimately taken in the face of it. In section 3 and section 4 we present, respectively, a set-theoretical and a mereological reformulation of the argument. The aim of these formulations is to show that we can change the background theories with which we may evaluate the argument, but its structure remains the same: nothing thus depends on the notions that are compared in these formulations, such as the notion of Leibnizian aggregate (or whole), set or mereological fusion.[1] In section 5 we push these lines of thought further, by presenting a plural formulation of the argument. The plural version has the merit of revealing that the argument is based on two different standards of comparison for sizes of collections, the one-to-one correspondence and some version of the part-whole principle. These two standards converge in the finite, but diverge in the infinite, i.e. if we are willing to admit infinite collections, then they give us different and incompatible results. Section 6 concludes.[2]

2. The Leibnizian argument against the infinite number

It is well-known that Leibniz admits the actual infinite in physics. For instance, he holds that the totality of monads does not constitute a potential infinite, and thus it is an actual infinite; it is not the case that we can divide matter indefinitely, but rather matter is actually divided and has infinitely many parts.[3] For such a view, there are actually infinite many monads and created things. When confronted with such a position, one is tempted to ask what prevents Leibniz from concluding that the number of all monads is an infinite number, i.e. that the number of an infinite multiplicity of terms is an infinite number. For instance, Gregory Brown writes:

> absent a sound argument to the effect that infinite number is generally contradictory, we may reasonably say that if the world contains an actual infinity of creatures, as Leibniz does, then the cardinality of the set of creatures is an infinite number. (Brown 2000, 31)

However, Leibniz believes this to be false. In fact, he gives an argument to support the non-existence of infinite number. The argument allows him to reject the implication that leads from 'there is an infinite plurality of elements' to 'the number of this plurality is infinite'. The argument aims to establish the claim that the notion of infinite number is self-contradictory, in the precise sense that once admitted, a contradiction can be derived. Leibniz presents this argument in different places.[4] Perhaps the most famous of these is the following:

> There is no maximum in things, or what is the same thing, the infinite number of all unities is not one whole [*non est unum totum*], but is comparable to nothing. For if the infinite number of all unities, or what is the same thing, the infinite number of all numbers, is a whole, it will follow that one of its parts is equal to it; which is absurd. I will show the force of this consequence as follows. The number of all square numbers is a part of the number of all numbers: but any number is the

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1 We have firmly kept distinct Leibnizian aggregates, i.e. the concept of aggregate or whole which Leibniz works with, from the contemporary notion of mereological fusion or sum, the reason being that such concepts are characterised differently. We shall say something about Leibniz's mereological notions in §5.

2 The main concern of the paper is clearly more theoretical than historical. For a more historical approach to Leibniz's argument see for example Esquisabel and Raffo Quintana 2017.

3 On this point see for instance Antognazza 2015 and Arthur 2015, 2018a.

4 It must be borne in mind that when Leibniz speaks of the infinite number, he has in mind a hypothetical number of all (finite) numbers. It is thus the greatest number. After Cantor, we know that the admission of an infinite number does not imply the admission of a greatest number: in the transfinite hierarchy, there are infinitely many infinite numbers, but none of them is the greatest. Cantor would have perfectly agreed with Leibniz in rejecting the existence of a maximum number. In what follows, we will analyse Leibniz's argument insofar it denies the legitimacy of an infinite number, and not of a greatest number.

root of some square number, for if it is multiplied into itself, it makes a square number. But the same number cannot be the root of different squares, nor can the same square have different roots. Therefore there are as many numbers as there are square numbers, that is, the number of numbers is equal to the number of squares, the whole to the part, which is absurd. (Leibniz, *De Minimo et Maximo*, 98)[5]

Following Van Atten (2011, §1), we can analyse the argument in the following way:[6]

(1) The infinite multitude of numbers forms an aggregate, a whole (Assumption C);

(2) Each square is a number, but not each number is a square (premise);

(3) The multitude of squares is equal to (has the same number of elements of) a part of the multitude of all numbers (from 1 and 2);

(4) There is a one-to-one correspondence between the numbers and the squares (premise);

(5) The multitude of squares is equal to the multitude of number, i.e. the number of squares is the same as the number of all numbers (from 1 and 4);

(6) A part of all the number (or a part of the whole of numbers) is equal (equinumerous) to the whole of numbers (from 3 and 5);

(7) The whole is greater than each of its own proper pars (premise: part-whole principle, from now on PW);

(8) Contradiction (from 6 and 7);

(9) Therefore, the infinite multitude of numbers do not form a whole (¬C).

The argument is a *reductio ad absurdum.* Leibniz supposes that the infinite of numbers forms a whole, and it exploits this assumption to derive a contradiction. It is clear that by reducing to absurdity premise 1, Leibniz is claiming that the notions of part and whole do not apply to the infinite case, and thus we cannot exploit PW to conclude that the whole of natural numbers is bigger than the whole of square numbers. In this way, the infinite is not a counter example to PW, simply because PW cannot be applied to it[7]. PW is a premise of the argument that Leibniz strongly believed to be true[8]. As such, the Leibnizian argument is valid. In effect, Cantor could accept the argument up to line 8, and conclude with the

5 Leibniz's argument exploits what is known in the literature as Galileo's Paradox.

6 The presentation of the argument is not a direct quotation from Van Atten's text.

7 See Levey 2015, p. 178.

8 Leibniz did not take PW as an evident truth, but he tries to prove it. His proof is based on defining when (an entity) B is less than (an entity) A: *B is less than A (or, which is the same, A is greater than B) if and only if B is equal to a proper part of A.* This is the same as defining 'proper part' as a part which is *not equal* to the whole. Given this definition, it is clear that no proper part can result in being numerically equivalent to the whole.

denial of PW instead of Assumption C. At first sight, it seems that both the Leibnizian and the Cantorian options seem legitimate[9].

In the paragraphs below, we shall show that Assumption C that Leibniz reduces to absurdity does not play any relevant role in the argument. On the contrary, we argue that the origin of the contradiction should be traced back to the presence of two different standards of comparison for sizes of collections: the common one-to-one correspondence and PW. Our argumentative strategy proceeds in two steps: first, we shall present a set-theoretical and a mereological reformulation of the argument. The two reformulations will give us two different characterisations of Assumption C. In the first case Assumption C becomes the claim that there exists *the set* of all (finite) numbers, while in the second case it becomes the claim that there exists *the mereological fusion* of all the (finite) numbers. We argue that the resulting arguments share exactly the same structure and nature of the formulation as the Leibnizian one, and thus irrespective of whether one takes the numbers to form a Leibnizian aggregate, a set or a mereological fusion, one still has to reckon with the same contradiction.

3. A set-theoretical version of the argument

Leibniz's argument is cast out in mereological terms which are not completely clear and are sometimes ambiguous. However, a very natural way for the contemporary reader to interpret the argument is by means of set-theoretical notions. Ambiguous talk of aggregates, wholes or multitudes is to be substituted with talk about sets; in particular, the part-whole relation must receive a set-theoretical interpretation. Talk of square numbers as part of the natural numbers can be translated in the talk of square numbers as forming a subset of the set of the natural numbers. This means that the relation of 'part of' can be interpreted as the subset-relation[10]. We shall assume a very broad conception of set: a set is always a further entity with regard to its elements[11]. The argument becomes as follows:

(1) There is a set of all (finite) numbers (Assumption C-set)

(2) Each square is a number, but not all numbers are square (premise);

(3) The set of squares is a proper subset of the set of all numbers (from 1 and 2);

(4) There is a one-to-one correspondence between the set of all numbers and the set of squares (premise);

(5) The cardinality of the set of squares is equal to the cardinality of the set

9 It is interesting to note the similarity between Leibniz's claim that the multiplicity of numbers does not form a whole, and so no number corresponds to it, and the claim – often made by set-theorists – that the absolute (i.e. the multiplicity of all sets) does not form a set, but rather a proper class. Somewhat anachronistically, one can say that for Leibniz (finite) numbers form a proper class.

10 It is well-known that David Lewis did the same by interpreting the 'subset relation' between sets as the 'part of' relation in his attempt to provide a mereological foundation of set theory (see Lewis 1991).

11 Just for the sake of simplicity, I only assume well-founded sets, i.e. sets that do not contain themselves as elements.

of numbers, i.e. the number of squares is the same as the number of all numbers (from 1 and 4);

(6) A subset of the set of numbers is equinumerous to (i.e. it has the same cardinality of) the set of numbers (from 3 and 5);

(7) Any set is bigger (i.e. more numerous) than any of its proper subsets (premise: PW);

(8) Contradiction (from 6 and 7);

(9) Therefore, the infinite multitude of (finite) numbers does not form a set (¬C-set).

In this set-theoretical formulation, PW becomes the claim that any set is bigger than any of its proper subsets, while the one-to-one correspondence can be defined as usual as a bijective function from a set (the domain) to another set (the co-domain). In this specific case, we can take the set of all (finite) numbers as the domain and the set of all square numbers as the co-domain. Again, both the Leibnizian and the Cantorian options are available: either one considers PW as a truth holding for all sets, and so one denies Assumption C-set, which – in this context – means to deny the existence of infinite sets (their existence would contradict the truth of PW)[12], or one simply denies PW as Cantor did. The present argument has thus the same structure as the previous formulation.

4. A mereological version of the argument

It is well-known that Leibniz defended a mereological conception of number along with the Euclidean idea according to which a number is an aggregate of unities, which are the parts of the aggregate[13]. For this reason, Leibniz speaks of the square numbers as *a part* of the natural numbers (the roots). It is therefore natural, and historically more accurate, to formulate the argument in mereological terms. Here we shall present a reformulation based on *contemporary classical mereology*. We shall assume as the background mereological theory GEM, General Extensional Mereology[14]. GEM embodies a principle of Unrestricted Composition, which we may express with the claim that for an arbitrary condition φ and some objects *a, b, c, …* satisfying φ, there exists the mereological sum (or fusion) of the objects *a, b, c, …*. In the case of (natural) numbers, φ is the property of being a natural number, while *a, b, c, …* are the numbers . 1, 2, 3, …. Unrestricted Composition implies that there is the sum (or fusion) of all (natural) numbers[15]. The argument becomes as follows:

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12 Of course the resulting set theory would be very different from the standard one having its roots in Cantor's work.

13 More on this in section 5.

14 General Extensional Mereology is the classical mereological system of Lesnieski and of Leonard and Goodmann. On mereology and GEM in particular see Varzi 2016.

15 I shall use the terms (mereological) fusion or sum to refer to the contemporary mereological concepts. In contrast, when I speak of aggregate or whole, I refer to the Leibnizian notions.

(1) There exists the mereological fusion of all numbers (Assumption C-fusion).

(2) Each square is a number, but not all numbers are squares (premise);

(3) The fusion of squares is a proper part of the fusion of all numbers, i.e. it is a part of the fusion of all numbers and it is not identical with it (from 1 and 2);

(4) There is a one-to-one correspondence between *the members of* the fusion of all numbers and *the members of* the fusion of all squares (premise);

(5) The fusion of the squares has as many members as the fusion of all numbers (from 1 and 4);

(6) A fusion which is a part of the fusion of all numbers is equinumerous to the fusion of all numbers (from 3 and 5);

(7) Any fusion is bigger than any of its proper part (premise: PW);

(8) Contradiction (from 6 and 7);

(9) Therefore, the infinite multitude of numbers does not form a mereological fusion (¬C-fusion).

In standard presentations of mereology, the key mereological concepts are introduced by means of set-theoretical notions. For instance, Unrestricted Composition is sometimes expressed by the claim that any specifiable *non-empty set* of objects has a mereological sum[16]. Or line 4 of the argument above can be understood as claiming that there is a one-to-one correspondence between the set of all numbers and the set of all squares. If this were the only way in which mereological notions can be understood, then the present formulation of the argument would be parasitic to the set-theoretical formulation. But we are not forced to use set theory to express mereology. In fact, we can drop any talk of set, and express mereological concepts by means of plural terms (i.e. terms that refer to many individuals at once), as we actually did in the argument. Unrestricted Composition becomes the claim that *any plurality* of objects has a sum, and the one-to-one correspondence can be defined with regard to *the members of* the fusion of natural and square numbers. In other words, to say that there is a one-to-one correspondence between the fusion of the (finite) numbers and the fusion of the square numbers amounts to the claim that there is a bijective function whose domain is the *plurality* of (finite) numbers and the co-domain is the *plurality* of square numbers[17]. In the next paragraph we shall explain in detail what plurals are. For the time being, we note that even this formulation of

16 See Varzi 2019, §4.4

17 From a formal point of view, this requires implementing the language of GEM (the theory in which such a formulation of the argument is developed) by means of plural resources.

Leibniz's argument presents the same structure as those above: one can accept PW unconditionally and deny that numbers form a mereological fusion (PW applies only to fusions, which means that it makes sense to say that *x is a part of something* only if there is a fusion (a sum) of which *x is a part*). Alternatively, one can drop PW, and follow the traditional Cantorian path.

5. A plural version of the argument[18]

In the previous paragraphs, we saw three different versions of the argument. The first was based on Leibniz's own mereological terminology, the second on set-theoretical terms, while the last on contemporary mereological terms. The fact that we can formulate the argument within these different background theories shows that there is nothing in the concepts of Leibnizian aggregate, set and mereological fusion on which the contradiction depends. In the present paragraph, we push this line of thought further, by developing a strategy to show that we can completely avoid any of these notions in formulating the argument above, which definitely shows that the contradiction in line 8 does not depend on any version of Assumption C. Our strategy makes appeal to *plural logic*, i.e. a logic that admits plural terms denoting more individuals at once.[19] If we use the terms *plural, plurality* or *multiplicity* to indicate *the things* to which a plural term refers, then it is useful to clarify the nature of pluralities by comparing them to sets. Where a set is always a further object with regard to its elements,[20] a plurality is not a different entity from its members; rather, it simply consists in those members taken simultaneously. Here is an example:

There are five children in the garden (the children in the garden are five)

The plural term 'the children' does not refer to a single object, which is distinct from the children and collects them, because this object (the set of children) is one, not five; nor does it refer to the single child taken one by one, because each child is one, not five. Instead, the term 'children' as the predicate 'being five' refers to the children simultaneously (the predicate 'being five' is a collective predicate). This shows the salient features of pluralities in comparison with sets: while a set is an ontological unity distinct from its elements, a plurality is – from an ontological point of view – many: the unity of the objects in plurality is simply given by the *semantic unity* of our referring to many things at once. In other words, the ontological commitment towards a plurality coincides with the commitment towards its members, while the ontological commitment towards the

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18 A similar approach has been anticipated in Levey 2015, which uses the notion of plurality to interpret the Leibnizian claim that natural numbers do not form a whole. At the time of writing this paper, I was not aware of Levey's contribution; I have to thank an anonymous referee at this journal for having brought it to my attention

19 Plural logic has been developed by Boolos in the '80s. Boolos showed that second-order logic is interpretable into plural first-order logic. From this result, he drew the philosophical consequence that it is possible to interpret second-order logic in such a way that it does not commit us to the existence of sets and classes.

20 I have in mind the iterative conception of set, which is nowadays acknowledged to be the basis of Zermelo-Fraenkel set theory. According to such a conception, sets are always well-founded, i.e. no set can be an element of itself. See Boolos 1971, in Boolos 1998, 13–29, and Boolos 1989, in Boolos 1998, 88–104.

elements of a set does not imply the commitment towards the set.[21]

The idea of exploiting plural logics simply consists in rewriting the whole argument in plural terms, i.e. we need to paraphrase the argument by means of plural terms. For instance, instead of saying that the collection of squares is a sub-collection (or a subset) of the collection of all finite numbers (which seems to commit us to the existence of special objects as collections or sets), we shall say that the *squared numbers are only some of the finite numbers* (where 'only some' substitutes 'a part of'); or instead of saying the whole of number is bigger than the part of the squares, we shall say *the numbers (roots) are more than the squares*, or *there are more numbers (roots) than squares.* Moreover, following Leibniz, we are assuming that it is not possible to speak of the number of a multiplicity if this multiplicity does not form a whole (whatever this notion means). As a consequence, the argument completely avoids to speak of numbers at all. However, from this we do not conclude – as Leibniz seems to do – that it is not possible to establish relationships of size between multiplicities that do not form a whole in any case. Since it is possible to paraphrase in plural terms both PW and the notion of equality based on the one-to-one correspondence, and because these give us two methods of establishing relationships of size independently from the existence of numerical systems, it is possible to compare different pluralities of objects (without supposing that they form a whole). But then, since both PW and the one-to-one correspondence are present, we are in the position of deriving the contradiction. The plural version of the argument is the following:

(1) There are infinitely many numbers (in the sense of the actual infinite, not the potential).

(2) Every square is a number, but not vice versa (premise).

(3) The squares are as many as some of the numbers (but they are not as many as all the numbers) (from 1 and 2).

(4) There is a one-to-one correspondence between the natural numbers and the squares (premise).

(5) The squares are as many as the numbers (from 1 and 4).

(6) Some numbers (the squares) are equal to (i.e. it has as many members as) all the numbers (from 3 and 5).

(7) All numbers (i.e. the plurality of all numbers) are more than any sub-plurality of numbers (this is a sort of plural version of PW).

(8) Contradiction (from 6 and 7).

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21 To be honest, there is a wide-ranging debate on the ontological innocence of plurals. See Boolos 1985, in Boolos 1998, 54–72 for a defence of the innocence of plurals, and Linnebo 2003, for a critique of Boolos' position. Here we shall assume that plurals are ontologically innocent. If this were not the case, then the same Leibnizian notion of a multiplicity (plural) that does not form a whole would be inconsistent.

Once again, the present argument shares the same structure as the former versions. However, premise 1 only assumes that there are all numbers, without any claim about numbers forming a whole, a set or a fusion. In other words, the plural version derives the 'same' contradiction without exploiting Assumption C. This shows that the assumption that Leibniz reduces to absurdity plays no role in the derivation of the contradiction.

Before proceeding I would like to draw your attention to the plural formulation of PW. This formulation completely avoids any use of the notions of part and whole. This is fundamental to reply to the following objection. One can in fact observe that according to the mereological system within Leibniz works, once the applicability of the notion of the whole has been refuted, it no longer makes any sense to speak of the part[22]. In fact, Leibniz defines a part as something that *is in* the whole (relation of *inesse*), which means that the part implies the existence of the whole of which it is a part. In our present context, the fact that the numbers do not form a whole implies that it makes no sense to speak of parts of them. If PW cannot be dispensed from the notion of part, then we cannot have it in the plural formulation, and so we could note derive the contradiction in line 8.

However, line 7 of the plural version of the argument is a plural reformulation of PW such that it does not mention the words 'part' and 'whole' at all. We might agree with Leibniz that once you have dismissed talk of a whole, you should also dismiss talk of parts. But the plural version of the argument does not require such talk, because PW just becomes the claim that all numbers are always more than some numbers (where 'some' must be read as 'some, but not all'). If you find the standard version of PW compelling, then – for exactly the same reasons – you should find this plural version compelling too. In fact, why is PW so intuitive? The reason is that a proper part is a part such that we must add something to obtain the whole, i.e. the whole contains this specific part *and* something more. It is therefore natural to hold that the whole is always bigger than its proper parts. But the same reason also obtains in the case of plural PW: all numbers contain the squares and some *more* numbers (the non-square numbers). Even though plural PW does not mention the words 'part' and 'whole' at all, it seems that if one believes in PW, one should believe in plural PW too.

5.1. Pluralities and wholes

Once the notion of plurality has been introduced, one might ask what relationships there are between pluralities and the Leibnizian notion of 'whole' (*totum*), which is the word Leibniz uses in his argument (section 2). Leibniz's argument is for the claim that numbers do not form a whole, but what exactly is a whole? In the previous discussion, we

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22 Mereology is the study of the part-whole relation. By 'part' Leibniz means something which *is in* (*inesse*) the whole, and it is *homogeneous* to the whole. The definition of part is given relative to the notion of the whole (see De Risi 2007, 192, n. 64). A part is characterised as a *requisitum* for the whole, in the sense that the existence of the whole implies the existence of the part; a part is *diversum* from the whole, which simply means that Leibniz is only considering proper parts; a part is *immediatum* with regard to the whole, in the sense that the part and the whole coexist; finally, the part is *in recto cum correquisitis*, which means that the part is in the whole (*inesse*), and it is not said of the whole. The last feature seems to indicate that what characterises a part of a whole does not characterise the same whole, and therefore the proper features of the part cannot be predicated of the whole. On the contrary, the *inesse* relation seems to indicate that the existence of a part B implies the existence of the whole A of which B is a proper part. This is a key feature of Leibniz's mereological system, because it implies that if we cannot have a whole – as Leibniz claims for the numbers – then we cannot have parts either.

presupposed that the notion of whole indicates something more than the mere presence of the elements that constitute the whole. This may indicate a further entity – as in the set-theoretical interpretation – or this may simply indicate that *the elements* form a sort of unity (as happens with mereological fusions). In this latter setting, which is certainly closer to Leibniz's view than the set-theoretical interpretation, the words 'whole' and 'plurality' are not synonyms; rather, 'whole' would indicate something more than the mere presence of all elements of a certain kind. To say that numbers do not form a whole would mean that while *there are* (note the plural form!) all the numbers, they do not form a true (metaphysical) unity. Just as an aggregate (such as a flock of sheep) is not a true unity (and so is not a true being), because its unity derives from our perception,[23] so the plurality of all numbers does not have a true metaphysical unity, and so it is not one single totality, or – as Leibniz says – it is not a whole (*non est unum totum*).

This interpretation of the notion of whole fits well with some defences, which have recently appeared in the literature, of the idea that Leibniz endorses the actual infinite in the case of numbers too[24]. Such defences claims that the totality of all numbers does not constitute a potential infinite, and thus it is an actual infinite; but this actual infinite is 'syncategorematic' in the precise sense that the fact that numbers are infinitely many simply means that they exceed any finite number. Plural tools are very useful to capture this view: there is the plurality of all numbers, but this plurality does not form a whole (a set, a fusion, etc.).

In such a scenario, Assumption C becomes the claim that the plurality of numbers has a true unity, and so it is a true being. The consequence is that the *plural* reformulation[25] of the argument is *not* based on Assumption C, which Leibniz reduces to absurdity. Moreover, as said above, the plural argument does not make any appeal to the notion of number: for instance, at line 5 it is said that 'the squares are as many as all the numbers'; it is not said that 'the number of the squares is equal to the number of all the numbers'. In this way, the argument does not presuppose the legitimacy of infinite numbers.

The argument presupposes the possibility of comparing different pluralities whenever all their elements are in some sense given, i.e. whenever the pluralities (if infinite) form actual, and not potential, infinite. If the pluralities (or multiplicities) of integers and squares are considered to be actual infinite, then there seems to be no obstacle in comparing them with regard to their extension (without introducing the notion of numbers). One can say that there are as many integers as squares, because for each integer there is a square, and vice versa; or one can say that there are more integers than squares, because each square is an integer, but not vice versa. But then the argument shows that the contradiction in line 8 does not depend on Assumption C, because we can derive the contradiction without assuming C. The Leibnizian strategy of reducing to absurdity the

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23 On this point I refer the reader to the illuminating Lodge 2001.

24 See for instance Arthur 2001a, 2001b, 2018a, 2018b who calls this view 'actual and syncategorematic infinite', and Levey 1998, 2015. Levey 2015, 178 writes: 'When Leibniz denies that infinity is one or a whole he is not saying that there is no such thing as infinity, but rather he is denying that an infinity of things forms a unity or single whole. […] Leibniz's position here is subtle. There are actually infinitely many natural numbers, on his view, but they do not form a totality'.

25 Pedantically at least, what I have presented is not a reformulation of the same Leibnizian argument, but rather a different argument, since once one drops an assumption in an argument, what one obtains – if it is still an argument – is a different argument. In the main text, I have spoken of reformulation for matter of simplicity, and because I think that there is no danger of confusion here.

premise C is no longer available in the plural version of the argument.

At this point, the problem is how to deal with the contradiction in line 8. If we accept 1 as valid, i.e. we accept that there is an actual infinite of numbers, then there are only two options available. Either we can follow the orthodox Cantorian solution, and declare 7 – the plural version of PW – false, or we can follow the theory of *numerosities*[26], which denies the validity of the one-to-one correspondence to measure (infinite) sets, and accepts PW (this implies the denial of passage from 1 and 4 to 5). Of course, avoiding the contradiction by accepting only one of the standards, while rejecting the other, exactly presupposes the recognition that the contradiction comes from the combination of *different standards* of sizes for collections of objects. However, I think that neither solution would have any appeal for Leibniz. Since his definition of equality (see n. 7), PW turns out to be a logical truth, and so the Cantorian approach is not available to him. But also the alternative solution – to dismiss the one-to-one correspondence – is not available to Leibniz, who needs to maintain the one-to-one correspondence also in the infinite case, since his definition of the infinite requires it. Leibniz's definition is that there are infinitely many terms when there are *more* terms than any number: the 'more' in this very last sentence must be interpreted as saying that there is no one-to-one correspondence between an infinite plurality and any (particular) number. Here Leibniz cannot appeal to PW, because he has dismissed talk of part and whole in the infinite case. At this point, the only possibility available is to deny premise 1 and abandon the idea of an actual infinite. Of course, this would undermine the idea of the 'actual and syncategorematic infinite', i.e. the idea that the multiplicity of numbers forms an actual infinite that exceeds any finite number.

5.2. What if wholes are pluralities?

However, one might question the claim that wholes are more than the mere presence of their constituents. Only substances – which are simple, i.e. they have no parts – are true metaphysical unities for Leibniz, and thus one might claim that the notion of whole is very close to (if not the same as) our notion of plurality. If this is right, to say that numbers do not form a whole means that numbers do not form an actual infinite, i.e. there is *no plurality* of all integers. In this context, Assumption C becomes the claim that numbers form an actual infinite (a plurality), and a reformulation in plural terms of Leibniz's argument would simply show that the integers form a potential infinite. In this sense, the argument would really show that Assumption C is false. This goes well with some interpretations of Leibniz, according to which he admitted the actual infinite only in physics, while in mathematics he only admitted the potential infinite[27].

If the word 'whole' is a sort of synonym for the term 'plurality', then premise 1 of the plural version of the argument can be read as a more rigorous translation of Assumption C, and the argument can be exploited to show that numbers cannot form an actual infinite. The idea being that to say that the squares are *equal* to the natural numbers or that the natural numbers are more than the squares, both the squares and the natural numbers

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26 The theory of *numerosities* (see Vieri & Di Nasso 2003; Vieri, et al. 2007; Mancosu 2009) is a non-Cantorian set theory that assumes PW and rejects the idea that the one-to-one correspondence between sets indicates that those sets have the same numbers of elements. Contrary to Leibniz, such a theory admits infinite numbers.

27 For such an interpretation, see for instance Antognazza 2015 and Breger 1986.

must have a definite extension, i.e. they must form a plurality. If we deny such a premise, the upshot is that squares and naturals form a potential infinite, i.e. an indefinite sequence that can always be extended. This allows us to argue that, even though there is a one-to-one correspondence between the natural numbers and the squares (premise 4) and all squares are as many as some natural numbers (premise 3), we cannot derive the contradiction because the sizes of the squares and the natural numbers are indefinite, and so it does not make sense to say that they are equal or that one is bigger than the other. The indefiniteness of the sequences does not allow us to compare their sizes, since these sizes are not determined. In this regard, one can stress the fact that Leibniz talked of infinitely many things only distributively, not collectively. Numbers forming a potential infinite would explain why no collection of them (no set, no whole, no fusion) exists and why collective predication fails. We could not say anything concerning the size-relation between all squares and all numbers, because to say that there are fewer squares than numbers seems to presuppose that we can treat squares and numbers collectively.

An important thing to note here is that even though in this scenario the argument can really be exploited to show that Assumption C is false, the plural formulation remains interesting, since plural resources allow for a clearer and more rigorous version of the argument. Moreover, the notion of plurality can be used by a defender of such a position to provide a clear interpretation of the ambiguous notion of (Leibnizian) whole.

However, even if we use the argument against Assumption C, it should be clear that this is possible precisely because the one-to-one correspondence and PW are two different standards of comparison for sizes of collections. In fact, it is the claim that numbers for a potential sequence (i.e. an indefinite sequence) that allows one to coherently accept both principles. More precisely, one can simultaneously accept them, because one has denied the existence of actual infinite pluralities (i.e. because one has denied Assumption C). The argument exactly shows that if all numbers had formed a determined actual infinite plurality, i.e. a plurality whose extension can be compared in size with that of other pluralities, then we would have faced the contradiction. If it were possible to compare in sizes such pluralities, the one-to-one correspondence and PW would have produced two different verdicts on their sizes. This clearly shows that, when infinite pluralities are admitted, PW and the one-to-one correspondence represent two different standards of comparison for sizes of collections. The plural formulation has the merit of bringing out this difference with great clarity.

5.3. Infinite numbers

As we saw above, the argument avoids any mention of numbers. The contradiction can be derived without any appeal to the concept of number, and as such it shows that numbers are not essential to the argument. However, the argument has effect on numbers, since Leibniz's own conception of number is a mereological one, where numbers are mereological aggregates (wholes) whose parts are the unities[28]. Therefore, if there were an infinite number, there would be an infinite whole. But the argument shows that infinite wholes cannot exist, and so infinite numbers too. Whatever one takes a whole to be (a set, a fusion or simply a plurality), an infinite whole presupposes that its members form an

28 More on the Leibnizian concept of number can be found in Sereda 2015.

actual infinite plurality. But then we are in a position to develop the plural version of the argument. At the end of section 5.1. we argued that, due to his definition of equality, Leibniz's reply to such a version of the argument would have probably consisted in the denial of numbers forming an actual infinite, which clearly leaves no space for infinite numbers. As such, the argument has a direct bearing on the Leibnizian conception of number, even though numbers do not play any essential role in it. Of course, the fact that ultimately the argument is based on two different standards for comparison of numbers allows us to accept infinite pluralities, to dismiss one of the two standards, and to modify the concept of numbers so as to allow for infinite numbers (Cantorian transfinite numbers are just an example).

6. Conclusion

In this paper, we have offered a plural reformulation of Leibniz's argument against the infinite number, which is valuable both to those who believe that Leibniz only accepted the potential infinite in mathematics, and to those who believe that Leibniz accepted that the integers form an actual infinite. To the former, the notion of plurality can give them a more rigorous, clear, and direct formulation of the argument, and a way of interpreting the notion of whole. To the latter, the argument shows something important: Assumption C that Leibniz reduces to absurdity does not play any role in the derivation of the contradiction. Generally speaking, Leibniz seems to miss its target, because it reduces to absurdity a premise – namely, that a multiplicity has a number only if it forms a whole – that plays no effective role in the derivation of the contradiction. The contradiction stems from using two different standard of comparison for sizes of collection. In front of this situation only three scenarios are possible: follow Cantor in abandoning PW; or abandon the validity of the one-to-one correspondence in the infinite case (with the necessity of abandoning Leibniz's own definition of an infinite multiplicity); or assume the definition of 'less than' given by Leibniz, and therefore claim that no plurality can constitute an actual infinite.

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