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Assessing the extent of democratic failures: A t%-Condorcet Jury Theorem

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Abstract The paper quantifies the amount of information aggregated by large elections under qualified majority rules. It shows show that, even when the Condorcet Jury Theorem does not hold, there still can be meaningful information aggregation. In particular, we study the case of information aggregation under rational ignorance and with poorly informed voters.

Keywords Condorcet Jury theorem · probability of success · poor information · rational ignorance

1 Introduction

The Condorcet Jury Theorem considers a committee deciding among two alternatives under majority rules, in a situation where one of the two decisions is the correct one, members vote independently and they are more likely to have correct information than incorrect one. This results consists of three main statements. The first one states that the committee is more likely to select the correct decision than any individual member. The second part states that the probability that the jury chooses the correct decision is monotonically increasing with the size of the jury. Finally, the third part states that the probability that the jury chooses the correct decision tends to one when the jury size tends to infinity.[1] This last part is the so called asymptotic version of the theorem and provides an epistemic justification for a democratic form of government (see Landemore 2012 for discussion and detailed references).

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[1] See Grofman et al. (1983); Miller (1986); Ben-Yashar and Paroush (2000); Berend and Sapir (2005) for precise statements of the hypothesis under which the results hold and for proofs.

This paper addresses an issue regarding the asymptotic form of the theorem that has not been noted (to the author's knowledge) in the literature. While many authors have studied the conditions under which the theorem holds in its different versions (see, for instance, Paroush 1998; Berend and Paroush 1998), nobody has attempted to understand what happens when the asymptotic version of Condorcet Jury Theorem does not hold. It is not clear why a situation where an electorate chooses the best decision 99% of the times, should be classified a "democratic failure". On the other hand, we ought to be concerned if elections yielded the correct decision less than half of the times. Current formulations of the Condorcet Jury Theorem do not allows us to distinguish between the two cases. The paper tackles this issue and goes beyond classical formulations of the Condorcet Jury theorem.

We consider a standard jury model with a fixed number of jurors and two alternatives. The jury has to decide which alternative is selected using a qualified majority rule.[2] We assume that one of the two alternatives is correct. Before voting, jurors do not know which alternative is correct, but have some private and independent information.[3] Jurors are not strategic, but vote according to their private information.[4] The competence of voters (i.e., the probability that a voter casts a correct vote) is heterogenous. We relax the standard assumption that voters are more likely to vote for the right alternative than for the wrong one and allow competence to be less than one half. Thus, our model includes the case where some of the voters follow unreliable information (Mandler 2012 presents strategic foundations for this hypothesis).

Our main result determines the probability that a large electorate selects the correct decision, given a qualified majority rule. Thus, it provides a measure of the amount of information aggregated by large elections. In particular, it generalizes the main result in Berend and Paroush (1998) and extends it to supermajority rules.[5]

Our results are particularly helpful in understanding the effect of rational ignorance, of unreliable information and of outright irrational voters. The rational ignorance hypothesis (Schumpeter 1950; Downs 1957) states that electors have little information since information acquisition is costly and each elector has little probability of being decisive. This is consistent with empirical evidence (see, for instance, Delli Carpini and Keeter 1996; Nannestad and Paldam 2000). The literature casts doubts about the ability of voters to select the correct information (see Hayes et al. 2015) and about the rationality of voters (see Caplan 2007). These facts could have important implications for the quality of democratic deliberations. According to the rational ignorance hypothesis, we should expect individual competence to be low and to decrease with the number of voters. Our main result implies that the Condorcet Jury Theorem holds in majoritarian elections as long as the average competence of the electorate approaches one half at a rate that is slower than one over the square root of the number of electors (see also Paroush 1998). However, there is meaningful (up

[2] We exclude from our analysis the unanimity rule. Other papers dealing with qualified majority rules are Nitzan and Paroush (1984); Fey (2003).

[3] For analysis of the Condorcet Jury Theorem with correlated votes see Ladha (1992); Berg (1993); Peleg and Zamir (2012).

[4] The assumption is not without costs (see Austin-Smith and Banks 1996), but it is a standard one in the literature about the Condorcet Jury Theorem.

[5] See also Giuliano-Antonini (2005).

to "almost complete") information aggregation also when the average competence of the electorate approaches one half at the same rate as the square root of the number of electors. Our results complement the ones by Ben-Yashar and Zaavi (2011) and Martinelli (2006) on the rational ignorance hypothesis. Our findings imply that even a rational ignorant electorate could be able to make the correct decision with high probability. Furthermore, we prove that there is meaningful information aggregation even when some voters base their decision on incorrect information (or are irrational) as long as their number is not too large relative to the committee size (see Mandler 2012 and Hayes et al. 2015). Our characterization is complete, so slight departures from the hypothesis we employ can have dramatic effects.

The structure of the paper is the following: Section **??** presents the model, Section **??** presents the main results, Section **??** presents the applications to poorly informed electorates and Section **??** concludes.

2 The model

There is a committee of n voters. They have to decide between two alternatives, A and B. Without loss of generality, we assume that A is the correct alternative. Let $\beta \in [0,1]$ be the fraction of the electorate required to choose an alternative. That is, at least $\lceil \beta n \rceil$ electors are required to elect alternative A, where $\lceil x \rceil$ denotes the smallest integer greater or equal to x: formally $\lceil x \rceil = \min\{n \in \mathbb{N} \mid n \geq x\}$. If only $\lceil \beta n \rceil - 1$ electors vote for A, A is selected with probability $\alpha \in [0,1)$.[6] Probability α is a *tie-breaking rule*. If less than $\lceil \beta n \rceil - 1$ electors vote for A, then alternative B is selected. Such a qualified majority rule is known as a β*-rule*. For instance, the cases where $\beta = 0$ and $\beta = 1$, correspond to unanimity rules and the case where $\beta^n = \frac{1}{2} + \frac{1}{n}$ corresponds to majority rule. Notice that the qualified majority rule can depend on the size of the committee. For simplicity, we consider only sequences of qualified majority rules $\{\beta^n\}_{n\geq 1}$ such that $\lim_{n\to+\infty} \beta^n = \beta$ for some $\beta \in (0,1)$. We thus exclude from our analysis the unanimity rules. We also assume that the tie-breaking rule α is constant.

Electors vote independently, but the probability each elector votes for an alternative can depend on her identity. For every $i \geq 1$, let p_i, be the probability elector i votes correctly (which is, for alternative A). Probability p_i is called agent i *competence*. Let $\bar{p}_n = \frac{1}{n}\sum_{i=1}^n p_i$ be the average competence of a committee of size n and set $\sigma_n = \left(\sum_{i=1}^n p_i (1-p_i)\right)^{\frac{1}{2}}$. Let Φ be the standard normal distribution: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$.

Differently than most literature about the Condorcet Jury theorem, we do not assume that electors are more likely to be right than wrong, but simply that their competence is uniformly bounded below by a small positive number. Thus, we allow also for systematic mistakes and outright irrational behavior. Formally, we assume that there exist a real number $0 < \varepsilon < \frac{1}{2}$ and an integer m such that $p_i \geq \varepsilon$ for all for all $i \geq m$. Let $P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right)$ be the probability that the elections select A when a β^n-rule is used, the tie-breaking rule is α, when there are n electors and every agent i votes for

[6] The case $\alpha = 1$ is equivalent to an increase in β of $\frac{1}{n}$.

A with probability p_i.
If the classical statement of the asymptotical form of the Condorcet Jury Theorem holds, which is if $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = 1$, we say that the Condorcet Jury Theorem holds in its *strong form*. If elections are more likely to select the correct decision than the wrong one, which is if $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) > \frac{1}{2}$, we say that the Condorcet Jury Theorem holds in a *weak form*.

3 General Results

We open this section with a motivating example. Berend and Paroush (1998) prove that, under our hypothesis, the the Condorcet Jury Theorem holds in its strong form in majoritarian elections if and only if $\lim_{n\to+\infty} \sqrt{n}\left(\bar{p}_n - \frac{1}{2}\right) = +\infty$. However we do not know what happens when this condition does not hold.

Example 1 Consider majoritarian elections and the following committees.
(i) Let $p_i = \frac{1}{2} + \min\left\{\frac{0.6}{\sqrt{i}}, \frac{1}{4}\right\}$ for every $i \geq 1$. Summing up the inequalities $\frac{1}{\sqrt{i+1}} \leq \int_i^{i+1} \frac{1}{\sqrt{t}} dt \leq \frac{1}{\sqrt{i}}$ from $i = 1$ to $i = n$, we obtain $\sum_{i=1}^n \frac{1}{\sqrt{i}} \approx 2\sqrt{n}$ for large n. This fact implies that $\bar{p}_n - \frac{1}{2} \approx \frac{1.2}{\sqrt{n}}$ for large n. Thus, $\lim_{n\to\infty} \sqrt{n}\left(\bar{p}_n - \frac{1}{2}\right) = 1.2$.
(ii) Let $q_i = \frac{1}{2} + \min\left\{\frac{1}{i}, \frac{1}{4}\right\}$, for every $i \geq 1$. Summing up the inequalities $\frac{1}{i+1} \leq \int_i^{i+1} \frac{1}{t} dt \leq \frac{1}{i+1}$ from $i = 1$ to $i = n$, we obtain $\sum_{i=1}^n \frac{1}{i} \approx \log n$. We have $\bar{q}_n - \frac{1}{2} \approx \frac{1}{n}$. We have $\lim_{n\to\infty} \sqrt{n}\left(\bar{q}_n - \frac{1}{2}\right) = 0$.

The Condorcet Jury Theorem does not hold in its strong form in case (i) and in case (ii). However the “centered competence” of the electorate $\sqrt{n}\left(\bar{p}_n - \frac{1}{2}\right)$ in case (i) is infinitely larger than the “centered competence” of the electorate $\sqrt{n}\left(\bar{q}_n - \frac{1}{2}\right)$ in case (ii) since $\lim_{n\to+\infty} \frac{\bar{p}_n - \frac{1}{2}}{\bar{q}_n - \frac{1}{2}} = +\infty$.

The first result result determines the the probability large elections yield the correct decision, depending on the asymptotic behavior of the ratio $\frac{n}{\sigma_n}\left(\bar{p}_n - \beta\right)$, for any sequence of qualified majority rules converging to β. Such a probability is a natural measure of the amount of information aggregated by large elections.

Theorem 1 *Let $\{\beta^n\}_{n\geq 1}$ be a sequence of qualified majority rules such that $\lim_{n\to+\infty} \beta^n = \beta$ for some $\beta \in (0,1)$ and let α be a tie-breaking rule. Let $t_n = \frac{n}{\sigma_n}\left(\bar{p}_n - \beta\right)$. Assume that $\lim_{n\to+\infty} t_n = t$ exists. Then $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = \Phi(t)$.*

Proof Let $\sigma = \lim_{n\to+\infty} \sigma_n$, there are two cases.
(i) First, assume $\sigma = +\infty$. Let $\hat{t}_n = \frac{\bar{p}_n - \frac{\lceil \beta^n n \rceil}{n}}{\sigma_n}$. Notice $\lim_{n\to+\infty} \hat{t}_n = \lim_{n\to+\infty} t_n = t$. For every voter i, define the following random variable:
$X_i = 1 - p_i$, if voter i votes for A;
$X_i = -p_i$, if voter i votes for B.
We have: $E[X_i] = p_i$, $E\left[X_i^2\right] = p_i(1-p_i)$, $\left|E\left|X_i^3\right|\right| = p_i(1-p_i)\left(1 + 2p_i^2 - 2p_i\right)$.
Set $S_n = \frac{\sum_{i=1}^n X_i}{\sigma_n}$.
Alternative A is selected with certainty if $S_n + \frac{\sum_{i=1}^n p_i}{\sigma_n} \geq \frac{\lceil \beta^n n \rceil}{\sigma_n}$, which is if and only

$S_n \geq n\frac{\frac{\lceil \beta^n n \rceil}{n} - \bar{p_n}}{\sigma_n} = -\hat{t}_n$. Alternative A is selected with probability α if $S_n = -\hat{t}_n - \frac{1}{n\sigma_n}$. Thus, the probability that alternative A is selected is

$$P(S_n \geq -\hat{t}_n) + \alpha \left[P\left(S_n = -\hat{t}_n - \frac{1}{n\sigma_n} \right) \right].$$

From the Berry-Esseen theorem (see Chow and Teicher, 1997, p. 322) it follows that there exists C, such that

$$|P(S_n \leq x) - \Phi(x)| \leq C \frac{\sum_{i=1}^n p_i (1-p_i) \left(1 + 2p_i^2 - 2p_i\right)}{\sigma_n^3}$$

for every x and for every n. Since $\lim_{n\to+\infty} \sigma_n = +\infty$ and $\sum_{i=1}^n p_i (1-p_i) \left(1+2p_i^2-2p_i\right) < \sigma_n^2$, we have $\lim_{n\to+\infty} |P(S_n \leq -\hat{t}_n) - \Phi(-\hat{t}_n)| = 0$ and $\lim_{n\to+\infty} P(S_n = -\hat{t}_n) = 0$.
(ii)Now, assume that $\sigma^2 < +\infty$. Since $p_i \geq \varepsilon$ for all i large enough, then $\lim_{i\to+\infty} p_i = 1$. It follows that $\lim_{n\to+\infty} t_n = +\infty$. We prove that $\lim_{n\to+\infty} P_{\beta^n \alpha}\left(\{p_i\}_{i\leq n}\right) = 1$.
Consider the following ancillary model where $\hat{p}_i = 1-\varepsilon$ if $p_i \geq 1-\varepsilon$ and $\hat{p}_i = p_i$ otherwise. Since $\hat{p}_i \geq 1-\varepsilon$ for large enough i, we have $\sum_{i=1}^n \hat{p}_i \approx n(1-\varepsilon)$ and $\sum_{i=1}^n \hat{p}_i (1-\hat{p}_i) \approx n\varepsilon(1-\varepsilon)$ as $n \to +\infty$. Thus, $\lim_{n\to+\infty} \frac{\sum_{i=1}^n \hat{p}_i(1-\hat{p}_i)|2\hat{p}_i - 1|}{\left[\sum_{i=1}^n \hat{p}_i(1-\hat{p}_i)\right]^{\frac{3}{2}}} = 0$ and $\lim_{n\to+\infty} \frac{\left(\sum_{i=1}^n \hat{p}_i\right) - \lceil \beta n \rceil}{\left[\sum_{i=1}^n \hat{p}_i(1-\hat{p}_i)\right]^{\frac{1}{2}}} = +\infty$. The Berry-Esseen theorem implies that the probability that at least $\lceil \beta n \rceil$ agents vote for A approaches one in this ancillary model, when $n \to +\infty$.
For every i, set $P^i_{\lceil \beta n \rceil - 1} = \sum_{k \geq \lceil \beta^n n \rceil - 1, \{i_1,\ldots,i_k\} \cup \{i_{k+1},\ldots,i_n\} = N\setminus\{i\}} \prod_{r=1}^k p_{i_r} \prod_{r=k+1}^n (1-p_{i_s})$ and set $P^i_{\lceil \beta^n n \rceil} = \sum_{k \geq \lceil \beta n \rceil, \{i_1,\ldots,i_k\} \cup \{i_{k+1},\ldots,i_n\} = N\setminus\{i\}} \prod_{r=1}^k p_{i_r} \prod_{r=k+1}^n (1-p_{i_s})$. Then the probability that at least $\lceil \beta n \rceil$ agents vote for A can be written as $p_i\left(P^i_{\lceil \beta n \rceil - 1} - P^i_{\lceil \beta n \rceil}\right) + P^i_{\lceil \beta n \rceil}$. It follows that the probability that A wins is increasing in p_i, $P^i_{\lceil \beta n \rceil - 1} \geq P^i_{\lceil \beta n \rceil}$ for every i,. Since $\hat{p}_i \leq p_i$ for all $i \leq n$, the probability probability that at least $\lceil \beta n \rceil$ agents vote for A approaches one when $n \to +\infty$ in the original model as well, which completes the proof of the claim.

In particular, Theorem **??** can help us in assessing the failures of the Condorcet Jury Theorem. For instance, it makes it possible to clearly distinguish between case (i) and case (ii) in Example **??**.

Example 2 Consider the committees of Example **??**.
(i) Notice that $\sigma_n \approx \frac{\sqrt{n}}{2}$. It follows that $\lim_{n\to+\infty} t_n = 2.4$. Thus Theorem **??** implies that the committee selects the best decision with probability above 0.99, when there are many voters.
(ii) In this case $\lim_{n\to+\infty} t_n = 0$. Thus, Theorem **??** implies that the committee selects the best decision only with probability arbitrarily close to $\frac{1}{2}$ when there are many voters.

Next, we consider the assumption that a sufficient number of voters are neither too dumb nor too smart (see Berend and Paroush 1998).

Assumption 1 *Assume that there exist $C > 0$, $\delta > 0$ such that $|\{i \leq n : \delta \leq p_i \leq 1-\delta\}| > Cn$ for large enough n.*

Under this assumption we can extend the finding of Theorem 1 in Berend and Paroush (1998) to the case of supermajority rules.

Proposition 1 *Let $\{\beta^n\}_{n\geq 1}$ be a sequence of qualified majority rules such that $\lim_{n\to+\infty}\beta^n = \beta$ for some $\beta \in (0,1)$ and let α be a tie-breaking rule. Then, under assumption **??**, the Condorcet Jury Theorem holds in its strong form if and only if $\lim_{n\to+\infty}\sqrt{n}(\bar{p}_n - \beta) = +\infty$.*

Proof Notice that $\frac{n}{\sigma_n}(\bar{p}_n - \beta) = \frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\frac{\sigma_n}{n}}}$. Observe that $\sigma_n^2 \geq \{i \leq n : \delta \leq p_i \leq 1-\delta\}\,\delta > \delta Cn$. Furthermore $\sigma_n^2 \leq n$ because $p_i(1-p_i) \leq 1$ for all i. It follows that $\frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\delta C}} \leq \frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\frac{\sigma_n^2}{n}}} \leq \sqrt{n}(\bar{p}_n - \beta)$ which implies the claim.

Proposition **??** implies that when the mean competence is uniformly bounded above the β, the Condorcet Jury Theorem holds in its strong, which yields a classical statement of the Theorem for supermajority rules (see Fey 2003).

Corollary 1 *Let $\{\beta^n\}_{n\geq 1}$ be a sequence of qualified majority rules such that $\lim_{n\to+\infty}\beta^n = \beta$ for some $\beta \in (0,1)$ and let α be a tie-breaking rule. Assume there exists $\delta > 0$, such that $\bar{p}_n > \beta + \delta$ for large enough n, then the Condorcet Jury theorem holds.*

4 Information aggregation by a poorly informed electorate

The quality information voters have (which is their competence) can influence the performance of a voting body. The rational ignorance hypothesis suggests that we should expect voters to have small amounts of information. The reason is that electors are very unlikely to be pivotal so they acquire very little information (see Martinelli 2006 for non-cooperative foundations of this claim). Theorem **??** allows us to gauge the effect of this hypothesis on the performance of large elections. Example **??**, (i), shows that even a poorly informed electorate can take accurate decision with high probability. Indeed, the sufficient condition determined in Corollary **??** are not at all necessary for meaningful information aggregation.

Proposition 2 *Let $\{\beta^n\}_{n\geq 1}$ be a sequence of qualified majority rules such that $\lim_{n\to+\infty}\beta^n = \beta$ for some $\beta \in (0,1)$ and let α be a tie-breaking rule. Then, under Assumption **??**, the Condorcet Jury Theorem holds in a weak form if and only if $\lim_{n\to+\infty}\sqrt{n}(\bar{p}_n - \beta) > 0$.*

Proof Notice that $\frac{n}{\sqrt{\sigma_n}}(\bar{p}_n - \beta) = \frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\frac{\sigma_n}{n}}}$. Observe that $\sigma_n \geq \{i \leq n : \delta \leq p_i \leq 1-\delta\}\,\delta > \delta Cn$. Furthermore $\sigma_n \leq n$ because $p_i(1-p_i) \leq 1$ for all i. It follows that $\frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\delta C}} \leq \frac{\sqrt{n}(\bar{p}_n-\beta)}{\sqrt{\frac{\sigma_n}{n}}}$ which implies the claim.

Ben-Yashar and Zahavi (2011) present a model of committee with rational uninformed voters and study whether adding an informed voters improve the performance of the committee. However, both theoretical and empirical literature suggest that we should also consider situations where at least some agents base their votes on mistaken information (see Mandler 2012; Hayes et al. 2015; Lee et al. 2016). Thus, we generalize Ben-Yashar and Zahavi (2011) model in order to consider this case and study its asymptotical implications. More precisely, we consider an electorate of size n where there are K_n informed voters voting for the right alternative with probability $p > \frac{1}{2}$, H_n misinformed voters voting for the right alternative with probability $q < \frac{1}{2}$ and $n - K_n - H_n$ uninformed voters voting for the right alternative with probability $\frac{1}{2}$. In order to provide a clearer comparison of our results with the ones obtained in Ben-Yashar and Zahavi (2011), we only consider majority rule. However, the result can be easily generalized to any qualified majority rule.[7]

Proposition 3 *Let* $0 < q < \frac{1}{2} < p < 1$. *Let* $p_i \in \left\{\frac{1}{2}, p, q\right\}$, *for all* $i \geq 0$. *Let* $K_n = |\{i \leq n \mid p_i = p\}|$ *be the number of informed electors when the committee size is n. Let* $H_n = |\{i \leq n \mid p_i = q\}|$ *be the number of misinformed electors when the committee size is n.*[8] *Assume that the majority rule and any tie-breaking rule are used. Then*
(i) if $\lim_{n\to+\infty} \frac{K_n}{\sqrt{n}} = +\infty$ *and* $\lim_{n\to+\infty} K_n\left(p-\frac{1}{2}\right) + H_n\left(q-\frac{1}{2}\right) = +\infty$, *then the Condorcet Jury Theorem holds in its strong form;*
(ii) if $\lim_{n\to+\infty} \frac{K_n}{\sqrt{n}} = +\infty$ *and* $\lim_{n\to+\infty} K_n\left(p-\frac{1}{2}\right) + H_n\left(q-\frac{1}{2}\right) = C \in (-\infty+\infty)$, *then* $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = \frac{1}{2}$;
(iii) if $\lim_{n\to+\infty} \frac{K_n}{\sqrt{n}} = +\infty$ *and* $\lim_{n\to+\infty} K_n\left(p-\frac{1}{2}\right) + H_n\left(q-\frac{1}{2}\right) = -\infty$, *then* $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = 0$;
(iv) if $\lim_{n\to+\infty} \frac{K_n}{\sqrt{n}} = C_1 \in [0,+\infty)$ *and* $\lim_{n\to+\infty} \frac{H_n}{\sqrt{n}} = C_2 \in [0,+\infty)$, *then* $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = \Phi\left(\frac{C_1\left(p-\frac{1}{2}\right)+C_2\left(q-\frac{1}{2}\right)}{\sqrt{C_1 p(1-p)+C_2 q(1-q)+\frac{1}{4}}}\right) \in (0,1)$;
(v) if $\lim_{n\to+\infty} \frac{K_n}{\sqrt{n}} = C \in [0,+\infty)$ *and* $\lim_{n\to+\infty} \frac{H_n}{\sqrt{n}} = +\infty$, *then* $\lim_{n\to+\infty} P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right) = 0$.

Proof Observe that $\bar{p}_n = \frac{1}{2} + \frac{K_n}{n}\left(p-\frac{1}{2}\right) + \frac{H_n}{n}\left(q-\frac{1}{2}\right)$ and that $\sigma_n^2 = K_n p(1-p) + H_n q(1-q) + \frac{1}{4}(n-K_n-H_n)$.
It follows that $\frac{n}{\sigma_n}\left(\bar{p}_n - \frac{1}{2}\right) = \frac{K_n\left(p-\frac{1}{2}\right)+H_n\left(q-\frac{1}{2}\right)}{\sqrt{K_n p(1-p)+H_n q(1-q)+\frac{1}{4}(n-K_n-H_n)}}$. Thus, the claims follow from applying Theorem **??**

First, consider Ben-Yashar and Zahavi (2011) model, which is assume $H_n = 0$ for every n. Parts (a) and (c) in Proposition **??** imply that the asymptotical Condorcet Jury Theorem holds (in its strong form or in a weak form) if and only if the number of informed voters grows at least at the same rate as the square root of the number of voters. When we consider misinformed voters, this condition is necessary but not sufficient for meaningful information aggregation. Indeed, the number of misinformed

[7] The precise statementent and the proof is available upon request.
[8] The symbol $|X|$ denotes the cardinality of set X.

voters cannot grow too fast. More precisely, we need that $\lim_{n\to+\infty}\frac{K_n}{H_n}\geq\frac{\frac{1}{2}-q}{p-\frac{1}{2}}$. On the contrary, when this condition is not satisfied, due to the (relative) preponderance of misinformed electors, elections are more likely to reach the wrong decision than the right one.

An alternative view of a rationally uninformed electorate is the one where every voter has a small amount of information, formally a situation where $\lim_{n\to+\infty}\bar{p}_n\leq\frac{1}{2}$.[9] In this situtation, electors are only slightly more likely to vote for the correct alternative than for the wrong one. Paroush (1998) and Example **??** show that under this condition the Condorcet Jury Theorem does not hold in its strong form, in general. However, Example **??** also shows that this failure includes very different situations, from the case where the committee takes the right decision 99% of the times to the case where the committee does not perform better than the toss of a fair coin. Actually, for any precision level x there exists a poorly informed electorate (which is an electorate such that $\lim_{n\to+\infty}\bar{p}_n\leq\frac{1}{2}$) that selects the correct decision with probability arbitrarily close to x. The proof of the result is constructive.

Proposition 4 *Let $\{\beta^n\}_{n\geq 1}$ be a sequence of qualified majority rules such that $\lim_{n\to+\infty}\beta^n=\beta$ for some $\beta\in(0,1)$ and let α be a tie-breaking rule. Let $t\in[0,1]$. Then there exists $\{p_i\}_{i\geq 1}$ such that $\lim_{n\to+\infty}\bar{p}_n\leq\beta$ and $\lim_{n\to+\infty}P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right)=t$.*

Proof First consider the case $t=0$ and let $0<p<\beta$. Set $p_i=p$ for all $i\geq 1$. Then, Theorem **??** implies that $\lim_{n\to\infty}P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right)=0$.
The case $t=\frac{1}{2}$ is covered by Example **??**, (ii) and Example **??**, (ii).
Now consider $t=1$ and let ρ be such that $\beta<\rho<1$. Set $p_i=\min\left\{\beta+\frac{1}{\sqrt[3]{i}},\rho\right\}$ for all $i\geq 1$. Summing up the inequalities $\frac{1}{\sqrt[3]{i+1}}\leq\int_i^{i+1}\frac{1}{\sqrt[3]{t}}dt\leq\frac{1}{\sqrt[3]{i}}$ from $i=1$ to $i=n$, we obtain $\bar{p}_n-\beta\approx\frac{2}{3\sqrt[3]{n}}$. Then, Theorem **??** implies that $\lim_{n\to\infty}P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right)=1$.
Now assume that $t\in(0,1)\setminus\left\{\frac{1}{2}\right\}$. Set $\sqrt{\beta(1-\beta)}\frac{\Phi^{-1}(t)}{2}$, let $0<\rho<\min\{\beta,1-\beta\}$ and let $p_i=\max\left\{\rho,\min\left\{\beta+\frac{C}{\sqrt{i}},1-\rho\right\}\right\}$. For large n, $\bar{p}_n-\beta\approx\frac{2C}{\sqrt{n}}$ (see Example**??**). Furthermore, for large n, $\sigma_n\approx\sqrt{n\beta(1-\beta)}$.
It follows that $\lim_{n\to\infty}\frac{n}{\sigma_n}(\bar{p}_n-\beta)=\frac{2C}{\sqrt{\beta(1-\beta)}}$. Then, Theorem **??** implies that

$\lim_{n\to\infty}P_{\beta^n,\alpha}\left(\{p_i\}_{i\geq 1}\right)=\Phi\left(\frac{2C}{\sqrt{\beta(1-\beta)}}\right)=t.$

An analogous result result can be obtained employing rationally uninformed electorates as introduced by Ben-Yashar and Zahavi (2011) (after adjusting for misinformed voters) as an almost straightforward application of Proposition **??**.[10]

5 Conclusions

In this paper we extend the Condorcet Jury Theorem by deriving the probability that an electorate reaches the correct decision, under any qualified majority rule,

[9] This model is consistent with the predictions of Martinelli (2006) for a strategic setup.
[10] The proof is available upon request.

which provides a measure of the amount of information aggregated by large elections. The results shed new light on information aggregation by rationally uninformed and poorly informed electorates.

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