Incentives and implementation in marriage markets with externalities*

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Abstract

We study the implementability of stable correspondences in marriage markets with externalities. We prove that, contrary to what happens in markets without externalities, no stable revelation mechanism makes a dominant strategy for the agents on one side of the market to reveal their preferences. However, the stable correspondence is implementable in Nash equilibrium.

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1 Introduction

In this paper we study the implementation of stable correspondences in one-to-one matching markets, or marriage markets, with externalities. In those markets agents care not only about their partner but also about the partners of the other agents. Relevant examples include labor markets in which workers care about their colleagues, school choice problems in which families care about their children's classmates and partner dance competitions in which each couple care about how the other couples are formed.

A focal concept in matching theory is stability. A stable matching is defined by two requirements. The first one is individual rationality: no agent prefers to stay unmatched rather than accepting her/his assigned partner. The second condition is that the matching must not be blocked by a pair. That is, no pair of agents would both prefer to be matched together rather than to accept their allocation. Stability plays a central role in the success of centralized mechanisms (see, Abdulkadiroğlu and Sönmez, 2013 and Roth and Sotomayor, 1990). Defining stability in markets with externalities is not straightforward. Indeed, it has to take into account the expectations of a potential deviating agent or pair about the behavior of the other agents (see Bando and Muto, 2016). We consider a concept of stability based on prudent expectations. We assume that a pair blocks matching μ only if both agents strictly prefer any matching in which they are together to matching μ. This concept of stability, introduced by Sasaki and Toda (1996), guarantees the existence of stable matchings in marriage markets with externalities.

We start studying direct mechanisms and uncover a difference with respect to markets without externalities. In those markets the woman-optimal (resp. man-optimal) stable mechanism makes a dominant strategy for all women (resp. men) reveal their preferences (see Roth and Sotomayor, 1990). Instead, when there are externalities, there exists no stable revelation mechanism which makes truth-telling a dominant strategy for all women (resp.

men). In particular, no stable revelation mechanism makes truth-telling a dominant strategy for all agents.

We then consider Nash equilibria (NE, from now on) of preference revelation games and prove that, under a mild restriction, any stable revelation mechanism implements the set of individually rational matchings in NE, extending the findings by Alcalde (1996) (see also Shin and Suh, 1996).

Finally, we consider general mechanisms and investigate the implementation of stable correspondences in NE (see Maskin, 1999). Kara and Sönmez (1996) prove that, in a model without externalities, the stable correspondence is implementable in NE. We follow their same strategy and employ the characterization of Nash implementable allocations by Yamato (1992) to prove that the stable correspondence is implementable in NE if there are externalities.

The structure of the paper is as follows. Section 2 introduces the model. Section 3 considers implementation through stable revelation mechanism the results. Section 4 considers implementation in NE through general mechanisms. Section 5 concludes.

2 The model

There are two disjoint sets of women and men, W and M, respectively. Set $N = W \cup M$. A matching is a function $\mu : W \cup M \to W \cup M$ such that (i) $\mu(w) \in M \cup \{w\}$ for all $w \in W$, $\mu(m) \in W \cup \{m\}$ for all $m \in M$ and (ii) $\mu^2(i) = i$ for all $i \in N$.[1] We denote a matching μ by a set of pairs and single agents. By $\mu = \{(w_1, m_1), (w_2, m_2), ..., (w_r, w_r), i_1, i_2, .., i_k\}$, we denote matching μ in which $\mu(w_l) = m_l$ and $\mu(i_l) = i_l$ for all l. Let $\mathcal{A}$ be the set of matchings. For each $(w, m) \in W \times M$, let $(w, m) = \{\mu \in \mathcal{A} : \mu(w) = m\}$. Similarly, for each $i \in N$ let $A(i) = \{\mu \in \mathcal{A} : \mu(i) = i\}$. Agent i has strict, complete and transitive preferences over $\mathcal{A}$, denoted by

[1] Function μ^2 is defined by $\mu^2(i) = \mu(\mu(i))$.

P_i. Let R_i be the weak preference relation associated to P_i, which is $\mu R_i \mu'$ if either $\mu P_i \mu'$ or $\mu = \mu'$ for every $\mu, \mu' \in \mathcal{A}$. By $\mathcal{P}$ we denote the set of strict preferences over $\mathcal{A}$. A marriage market with externalities is a triple (W, M, P), where $P \in \mathcal{P}^{|N|}$.[2] We represent the preferences of agent $i \in N$ by a list of matchings. For example, $P_i : \mu_1, \mu_2, ..., \mu_k, ...$ means that $\mu_j P_i \mu_k$ for each $j < k$.

We employ the concept of stability introduced by Sasaki and Toda (1996). A matching μ is individually rational if there is no $i \in N$ such that $\mu' P_i \mu$ for all $\mu' \in A(i)$. A pair $(w, m) \in W \times M$ blocks μ if $\mu' P_w \mu$ and $\mu' P_m \mu$ for all $\mu' \in A(w, m)$. The matching μ is stable if it is individually rational and no pair blocks it. Let $\mathcal{S}(P)$ be the set of stable matchings under P. We have $\mathcal{S}(P) \neq \emptyset$ for all $P \in \mathcal{P}^{|N|}$.[3]

Let $\mathcal{D} \subseteq \mathcal{P}^{|N|}$. A social choice correspondence $\Gamma : \mathcal{D} \rightrightarrows \mathcal{A}$ maps profiles of preferences into subsets of matchings. The stable correspondence is defined by $\Gamma(P) = \mathcal{S}(P)$ for all $P \in \mathcal{P}^{|N|}$. A mechanism is a pair (S, g) where $S = \prod_{i \in N} S_i$, S_i is the strategy space of agent $i \in N$ and $g : S \to \mathcal{A}$ is the outcome function. In a revelation mechanism, $S \subseteq \mathcal{P}^{|N|}$. In a stable revelation mechanisms $g(P) \subseteq \mathcal{S}(P)$ for all $P \in \mathcal{D}$. Each mechanism (S, g) induces a strategic form game, (W, M, P, S, g). Let $NE(P, S, g)$ denote the set of pure strategy Nash equilibria of game (W, M, P, S, g). Mechanism (S, g) implements Γ if $g(NE(P, S, g)) = \Gamma(P)$ for all $P \in \mathcal{D}$.

3 Revelation mechanisms

We start by considering dominant strategies. Without externalities, the woman-optimal stable mechanism makes truth-telling a dominant strategy for each woman. On the contrary, in markets with externalities, no stable revelation mechanism makes truth-telling a dominant strategy for the agents

[2] For every set X, $|X|$ denotes the cardinality of set X.
[3] See Theorem 4.1 in Sasaki and Toda (1996).

on one side of the market.

Proposition 1 *Let $\mathcal{D} = \mathcal{P}^{|N|}$. There is no stable revelation mechanism in which reporting the true preferences is a dominant strategy for women.*

Proof. The proof is by mean of an example. Let $W = \{w_1, w_2\}$ and let $M = \{m_1, m_2\}$. There are seven matchings: $\mu_1 = \{(w_1, m_1), (w_2, m_2)\}$, $\mu_2 = \{(w_1, m_1), w_2, m_2\}$, $\mu_3 = \{(w_2, m_1), (w_1, m_2)\}$, $\mu_4 = \{(w_2, m_1), w_1, m_2\}$, $\mu_5 = \{(w_1, m_2), w_2, m_1\}$, $\mu_6 = \{(w_2, m_2), w_1, m_1\}$ and $\mu_7 = \{w_1, w_2, m_1, m_2\}$. Consider their following preferences:

$$P_{w_1} : \mu_1, \mu_4, \mu_6, \mu_5, \mu_2, \mu_3, \mu_7; \qquad P_{w_2} : \mu_5, \mu_2, \mu_1, \mu_3, \mu_4, \mu_6, \mu_7;$$

$$P_{m_1} : \mu_1, \mu_5, \mu_6, \mu_4, \mu_7, \mu_2, \mu_3; \qquad P_{m_2} : \mu_4, \mu_1, \mu_2, \mu_3, \mu_5, \mu_6, \mu_7.$$

We have $S(P) = \{\mu_1, \mu_4, \mu_5, \mu_6\}$.[4] Let φ be a stable revelation mechanism, then $\varphi(P) \in S(P)$. Consider the following cases.

(i) $\varphi(P) = \mu_5$. Let $P'_{w_1} : \mu_1, \mu_4, \mu_6, \mu_7, ...$ Let $P' = (P'_{w_1}, P_{-w_1})$, we have $S(P') = \{\mu_1, \mu_4, \mu_6\}$.[5] Since $\mu' \, P_{w_1} \, \mu_5$ for all $\mu' \in S(P')$, woman w_1 has incentives to misrepresent her preferences.

(ii) Let $P''_{w_2} = \mu_2, \mu_5, \mu_7, ...$ Let $P'' = (P''_{w_2}, P_{-w_2})$, we have $S(P'') = \{\mu_5\}$.[6] Since $\mu_5 \, P_{w_2} \, \mu'$ for all $\mu' \in \{\mu_1, \mu_4, \mu_6\}$, woman w_2 has incentives to misrepresent her preferences.

The argument is easily generalized to any W and M.[7] Then, there is no stable revelation mechanism in which truth-telling is a dominant strategy for women. ■

[4] Matchings μ_2 and μ_3 are not individually rational for m_1, μ_7 is blocked by $\{w_1, m_2\}$ and $\{w_2, m_2\}$.

[5] Matchings μ_2, μ_3, μ_5 are not individually rational for w_1, μ_7 is blocked by $\{w_2, m_2\}$.

[6] Matchings μ_1, μ_3, μ_4, μ_6 are not individually rational for w_2, μ_2 is not individually rational form_1, μ_7 is blocked by $\{w_1, m_2\}$.

[7] The proof is available upon request.

Consider the domain of preferences where each agent ranks consecutively all matchings where she/he is unmatched. Formally, let

$$\widetilde{\mathcal{P}} = \left\{ P \in \mathcal{P}^{|N|} : \forall\ i \in N,\ \nexists\ \mu', \mu'' \in A(i),\ \mu \notin A(i),\ \mu' P_i \mu P_i \mu'' \right\}.$$

We next characterize the incentive properties of stable revelation mechanisms.

Proposition 2 *Let φ be a stable revelation mechanism in the marriage market with externalities. If $\mathcal{D} \subseteq \mathcal{P}^{|N|}$, all NE outcomes are individually rational. In addition, if $\mathcal{D} = \widetilde{\mathcal{P}}$, φ implements the individually rational correspondence in NE.*

Proof. We first prove that any NE outcome is individually rational for all agents. By the definition of a stable matching, $\varphi(P)$ is individually rational for all $P \in \mathcal{P}^{|N|}$. In particular, if $\varphi(P')$ is not individually rational for agent i in market (W, M, P), then agent i has a profitable deviation: to state her/his true preference P_i, then P' is not a NE.
Assume $\mathcal{D} = \widetilde{\mathcal{P}}$. Let μ be an individually rational matching in market (W, M, P). We will prove that μ is a NE outcome. For all $i \in N$, consider preference P'_i such that (i) $\mu' P'_i \mu''$, $\forall \mu' \in A(i)$, $\forall \mu'' \in A(i, j)$ with $j \notin \{i, \mu(i)\}$; (ii) if $\mu(i) \neq i$ then $\mu'' P'_i \mu'$ $\forall \mu' \in A(i)$, $\forall \mu'' \in A(i, \mu(i))$. Let $P' = (P'_i)_{i \in N}$. Then $\{\mu\} = \mathcal{S}(P')$ and P' is a NE. ■

4 General mechanisms

Now we study the implementation of the stable correspondence $\mathcal{S}$ in NE. We first introduce additional notation. Let $L(\mu, R_i) = \{\mu' \in \mathcal{A} : \mu R_i \mu'\}$ be the lower contour set of $\mu \in \mathcal{A}$ at R_i. A preference profile P' is a monotonic transformation of P at $\mu \in \mathcal{A}$ if $L(\mu, R_i) \subseteq L(\mu, R'_i)$ for all $i \in N$. A social choice correspondence Γ is monotonic if, for all $P, P' \in \mathcal{D}$ and all $\mu \in \mathcal{A}$

such that $\mu \in \Gamma(P)$ and P' is a monotonic transformation of P at μ, then $\mu \in \Gamma(P')$. Let $i \in N$ and $X \subseteq \mathcal{A}$. A matching $\mu \in X$ is essential for agent $i \in N$ in the set X for Γ if $\mu \in \Gamma(P)$ for some preference profile P such that $L(\mu, R_i) \subseteq X$. The set of essential matchings is denoted by $Ess(\Gamma, i, X)$. Correspondence Γ is essentially monotonic if for all $P, P' \in \mathcal{D}$, for all $\mu \in \Gamma(P)$, if $Ess(\Gamma, i, L(\mu, R_i)) \subseteq L(\mu, R'_i)$ for all $i \in N$, then $\mu \in \Gamma(P')$.

The domain $\mathcal{P}^{|N|}$ satisfies Condition D in Yamato (1992).[8] Then, $\Gamma : \mathcal{P}^{|N|} \rightrightarrows \mathcal{A}$ is implementable in NE if and only if Γ is essentially monotonic, from Yamato (1992, Corollary, p. 490).[9] Then, we prove that $\mathcal{S}$ is implementable in NE, by proving that $\mathcal{S}$ is essentially monotonic (see Kara and Sönmez, 1996). We start proving that the stable correspondence is monotonic.

Lemma 1 *The stable correspondence $\mathcal{S}$ is monotonic.*

Proof. Let $P \in \mathcal{P}^{|N|}$ and assume $\mu \in \mathcal{S}(P)$. Let P' be a monotonic transformation of P at μ. We prove by contradiction that $\mu \in \mathcal{S}(P')$. Assume μ is not stable under P'. There exists an agent or a couple which blocks the matching μ under P'. Since P' is a monotonic transformation of P at μ, for each $i \in N$, $\mu' P'_i \mu$ implies $\mu' P_i \mu$. Then, μ is blocked in market (W, M, P), which yields a contradiction. ■

Correspondence $\mathcal{S}$ does not satisfy the "no veto-power" condition (see Maskin, 1999) so Lemma 1 does not imply the Nash implementability of $\mathcal{S}$, but it is an important tool in the proof of our main result. Before concluding, we prove an additional result.

Lemma 2 *For all $P \in \mathcal{P}^{|N|}$, $\mu \in \mathcal{S}(P)$ and $i \in N$:*

$$Ess(\mathcal{S}, i, L(\mu, R_i)) = L(\mu, R_i).$$

[8]A domain $\mathcal{D} \subseteq \mathcal{P}^{|N|}$ satisfies condition D if, for all $\mu \in \mathcal{A}$, $P \in \mathcal{D}$, $i \in N$ and $\mu' \in L(\mu, R_i)$, there exists $P' \in \mathcal{D}$ such that $L(\mu, R_i) = L(\mu', R'_i)$ and for all $j \neq i$, $L(\mu', R'_j) = \mathcal{A}$.

[9]See also Danilov (1992).

Proof. Let $i \in N$, $P \in \mathcal{P}^{|N|}$, $\mu \in \mathcal{S}(P)$. The proof of is in two steps.

1. We prove $Ess(\mathcal{S}, i, L(\mu, R_i)) \subseteq L(\mu, R_i)$. Assume $\mu' \in Ess(\mathcal{S}, i, L(\mu, R_i))$. By definition of $Ess(\mathcal{S}, i, L(\mu, R_i))$, there exists a preference profile $P' \in \mathcal{P}^{|N|}$ such that $L(\mu', R'_i) \subseteq L(\mu, R_i)$. In particular, $\mu' \in L(\mu, R_i)$.

2. We prove $L(\mu, R_i) \subseteq Ess(\mathcal{S}, i, L(\mu, R_i))$. Let $\mu' \in L(\mu, R_i)$. Notice that there exists $\mu^* \in L(\mu, R_i)$ such that $\mu^*(i) = i$, otherwise μ would not be individually rational for i. Consider the strategy profile P' such that (i) $\mu' P'_j \mu''$ for all $\mu'' \neq \mathcal{A} \setminus \{\mu'\}$ and all $j \neq i$; (ii) $\mu'' P'_i \mu' P'_i \mu^*$ for all $\mu'' \in \mathcal{A} \setminus \{\mu', \mu^*\}$. By (ii) $\mu^* \in L(\mu', R'_i) \subseteq L(\mu, R_i)$. By (i) μ' is individually rational for all agents $j \neq i$ and no pair (w, m) blocks μ' under P'. By (ii), μ' is individually rational for i under P'. Then $\mu' \in \mathcal{S}(P')$. Then, $\mu' \in Ess(\mathcal{S}, i, L(\mu, R_i))$.

■

Applying Lemmas 1 and 2, we prove the main result of the paper.

Theorem 1 *The stable correspondence $\mathcal{S}$ is implementable in NE.*

Proof. The monotonicity of $\mathcal{S}$ (Lemma 1) implies that it is essentially monotonic by Lemma 2. Then, the result follows from Yamato (1992). ■

5 Concluding Remarks

We have studied incentive problems in marriage markets with externalities and proved the implementability of the stable correspondence in NE. Future research should establish the possibility of implementing stable matchings through simple mechanisms.

References

[1] Abdulkadiroğlu, A. Sönmez, T., 2013. Matching Markets: Theory and Practice. in D. Acemoğlu, M. Arellano and E. Dekel (eds), Advances in Economics and Econometrics, Vol. 1, 3–47, Cambridge, Cambridge University Press.

[2] Alcalde, J., 1996, Implementation of stable solutions to marriage problems. J. Econ. Theory 69, 240–254.

[3] Bando, K., Kawasaki, R., Muto, S., 2016. Two-sided matching with externalities: A survey. J. Operational Res. Society Japan 59, 35–71.

[4] Danilov, V., 1992. Implementation via Nash equilibria. Econometrica 60, 43–56.

[5] Kara, T., Sönmez, T., 1996. Nash implementation of matching rules. J. Econ. Theory 68, 425–439.

[6] Maskin, E., 1999. Nash equilibrium and welfare optimality. Rev. Econ. Stud. 66, 23–38.

[7] Roth, A. E., Sotomayor, M., 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge, Cambridge University Press.

[8] Sasaki, H., Toda, M., 1996. Two-sided matching problems with externalities. J. Econ. Theory 70, 93–108.

[9] Shin, S., Suh, S. C., 1996 A mechanism implementing the stable rule in marriage problems. Econ. Lett. 51, 185–189.

[10] Yamato, T., 1992. On Nash implementation of social choice correspondences. Games Econ. Behav. 4, 484–492.