Reversibility and (non)linearity in time series

Reversibilità e (non)linearità nelle serie storiche

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Abstract The recent literature has proposed a (limited) number of approaches to test for time reversibility, that is one of the main hypotheses in time series. A very interesting proposal is a Gini-based framework that, among other things, includes a test for time reversibility focussing on possible differences between backward and forward autocorrelations. This feature is indeed useful to identify models with underlying heavy tailed and non-normal innovations. In this paper we intend to shed some more light on this and investigate, via Monte Carlo simulations, on the possibility that this test can effectively have power in detecting some form of nonlinearity.

Key words: Reversibility, long memory, bootstrap

1 Introduction

Time reversibility is one of the main features of strictly stationary Gaussian linear stochastic processes. From an intuitive point of view, a stochastic process is said to be time-reversible if its probabilistic structure is invariant with respect to the reversal of the time indices. In an applications perspective, a check for time reversibility is a useful addition to existing diagnostics for stationary data since the absence of this feature (so the process is irreversible) signals the exclusion of serially independent or gaussian processes as candidate models.

Starting from the late Nineties the literature has discussed the issue of testing for time reversibility (Ramsey and Rothman, 1996; Hinich and Rothman, 1998; Chen *et al.*, 2000; Chen, 2003). Among the most recent contributes, Racine and Maasoumi (2007) proposed an approach based on an entropy measure of symmetry. Psaradakis

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(2008) introduces a sample index of the deviation from zero of the median of the one-dimensional law of differenced data. Shelef and Schechtman (2016) develop a framework for time series anaylsis based on a set of Gini-based equivalents that, among other things, introduces a test for time reversibility focussing on possible differences between backward and forward (in time) autocorrelation. This feature is indeed useful to identify models with underlying heavy tailed and non-normal innovations. One of the nice advantages of this method is that is based on only first-moment assumption.

While there are several finite sample experiment documenting the capability of this class of test for time reversibility to recognize situations that depart from gaussianity, in particular heavy tailed innovations, there is nothing to our knowledge about the possibility that this test can effectively have power in dectecting some form of nonlinearity. Indeed, although nonlinearity does not necessarily imply time-irreversability, (see, e.g., Lewis *et al.*, 1989) time reversible nonlinear processes appear to be the execption rather than the rule (Tong, 1990). With this in mind, in this paper we propose a Monte Carlo experiment, where time reversibility is tested for a variety of the most common non-linear models. For some type of models, in particular TAR and Markov Switching, the results are very promising and require further investigation that is in the future research lines.

The paper is organizes as follows. In section 2 we recall the Gini-based time reversibility test. In section 3 we propose our Monte Carlo experiment and some selected results. Section 4 concludes.

2 Gini-based reversibility test

2.1 Overview

From a formal point of view, a strictly stationary discrete-parameter stochastic process Y_t is said to be time reversible if $(Y_{t1},...,Y_{th})$ and $(Y_{-t1},...,Y_{-th})$ have the same joint distributions for every $h \in \mathscr{N}$ and any h-tuple $(t_1,...,t_h)$ such that $-\infty < t_1 < ... < t_h < \infty$. Put it differently, Y_t is time reversible if looking forward and backward at the time series result in similar probabilistic structure. Weiss (1975) showed that time reversibility for finite order ARMA processes is a typically Gaussian property.

Shelef and Schechtman (2016) shows that time reversibility can be associated to some form of symmetry. This, in turn, can be investigated by observing the behaviour of the two Gini autocovariances at lag s.

$$\gamma^{G_1}_{(t,t_{t-s})} = COV(Y_t, F(Y_{t-s})) \qquad \gamma^{G_2}_{(t,t_{t-s})} = COV(Y_{t-s}, F(Y_t)) \tag{1}$$

The above expressions can be viewed as Gini autocovariances looking backward and forward. Under strictly stationarity conditions, the following equality hold for all t and s

$$\gamma_{(s)}^{G_1} = COV(Y_t, F(Y_{t-s})) = COV(Y_{t-j}, F(Y_{t-j-s})) \tag{2}$$

and

$$\gamma_{(s)}^{G_2} = COV(Y_{(t-s)}, F(Y_t)) = COV(Y_{(t-j-s)}, F(Y_{t-j})) \tag{3}$$

where $\gamma_{(s)}^{G_1}$ and $\gamma_{(s)}^{G_2}$ are time independent. Note that $\gamma_{(s)}^{G_1}$ and $\gamma_{(s)}^{G_2}$ are equal in case of time reversibility.

When stationarity holds, a Gini version of the autocorrelation function (ACF) between Y_t and Y_{t-s} can also be defined

$$\rho_{(s)}^{G_1} = \frac{\gamma_{(s)}^{G_1}}{\gamma_{(s=0)}^{G_1}} \qquad \rho_{(s)}^{G_2} = \frac{\gamma_{(s)}^{G_2}}{\gamma_{(s=0)}^{G_2}} \tag{4}$$

It is interesting to observe that for an $AR(1)$ process, $Y_t = \phi_0 + \phi_1 Y_{t-1} + \varepsilon_t$, we have that $\gamma_{(s)}^{G_1} = \phi^s \gamma_{(s=0)}^{G_1}$, then $\rho_{(s)}^{G_1} = \phi_1^s = \rho(s)$ and this indicates that the first Gini-ACF is equal to the traditional ACF, denoted by ρ_s.

Consistent Gini-ACFs estimates are the following (Shelef and Schechtman, 2016)

$$\hat{\rho}_{(s)}^{G_1} = \frac{\sum_{t=1}^{T-s}(Y_{t+s} - \bar{Y})(R(Y_t) - \bar{R}(Y_{1:(T-s)}))}{\sum_{t=1}^{T}(Y_t - \bar{Y})(R(Y_t) - \bar{R}(Y_{1:T}))} \tag{5}$$

and

$$\hat{\rho}_{(s)}^{G_2} = \frac{\sum_{t=1}^{T-s}(Y_t - \bar{Y})(R(Y_{t+s}) - \bar{R}(Y_{(s+1):T}))}{\sum_{t=1}^{T}(Y_t - \bar{Y})(R(Y_t) - \bar{R}(Y_{1:T}))} \tag{6}$$

where $R(Y_t)$ is the rank of Y_t and $\bar{R}(Y_{i:j}) = \sum_{t=i}^{j} R(Y_t)/(j-i+1)$.

The Gini-based framework by Shelef and Schechtman (2016) also includes Gini PACF, defined as the last coefficient of a partial Gini autoregression equation of order s

$$Y_t = \phi_{s1}^{G_1} Y_{t-1} + \phi_{s2}^{G_1} Y_{t-2} + \ldots + \phi_{ss}^{G_1} Y_{t-s} + \varepsilon_t$$

hence

$$\rho_{(j)}^{G_1} = \phi_{s1}^{G_1} \rho_{(j-1)}^{G_1} + \ldots + \phi_{ss}^{G_1} \rho_{(j-s)}^{G_1}$$

Plugging the second Gini ACF ($\rho_{(s)}^{G_2}$) in place of the first, leads to a second version of the Gini PACF, that can be called second Gini-PACF. As for the estimation, the two Gini-PACFs can be estimated solving for $\phi_{ss}^{G_1}$ and $\phi_{ss}^{G_2}$, $s = 1, 2, \ldots$ the implied two systems of equations.

2.2 Testing for time reversibility

Implied by the definition of time reversibility itself, a crucial feature of the Gini autocorrelations is that if the series is time reversible, the Gini-ACFs at the remarkable lags are equal. Hence, generally, if the Gini-ACFs differ, this should indicate that Y_t

and Y_{t-s} are not exchangeable and this in turn implies time irreversibility. In order to capture this feature in case of moving average, MA(q), processes whose ACFs cuts off after q lags, Gini-PACFs should also be taken into consideration. This leads (Shelef and Schechtman, 2016) to the following system of hypotheses at each lag s for the null hypothesis of time reversibility

$$H_{01}: \rho_{(s)}^{G_1} = \rho_{(s)}^{G_2} \quad and \quad H_{01}: \phi_{ss}^{G_1} = \phi_{ss}^{G_2} \tag{7}$$

The alternative hypothesis is that at least one of the two equation is violated (for further detail, see the original article by Shelef and Schechtman, 2016).

The logic followed by the authors aims at identifying large absolute differences between the sample Gini ACFs and PACFs, denoted by, respectively, $\hat{\theta}_{Gini-ACF(s)} = \hat{\rho}_{(s)}^{G_1} - \hat{\rho}_{(s)}^{G_2}$ and $\hat{\theta}_{Gini-PACF(s)} = \hat{\phi}_{ss}^{G_1} - \hat{\phi}_{ss}^{G_2}$. The test statistic then is

$$\sqrt{T}\left|\hat{\theta}_{Gini-ACF(s)} - \theta_{Gini-ACF(s)}, H_0\right| \quad and \quad \sqrt{T}\left|\hat{\theta}_{Gini-PACF(s)} - \theta_{Gini-PACF(s)}, H_0\right|$$

where under the null hypothesis the differences are equal to zero. In practice, large value of the test statistics support rejection of the null hypothesis.

Due to the complicated sampling distribution of the Gini-based estimators, that also involve additional restrictive assumption on the time series, critical values for this test tests are obtained via moving block bootstrap. All details about the algorithms are in Shelef and Schechtman (2016).

3 Reversibility and (non)linearity: preliminary Monte Carlo evidence

In their original paper, Shelef and Schechtman (2016) conduct a Monte Carlo experiment showing that at least at the first lags the proposed Gini-based time reversibility test reaches a reasonably high power when the innovations of the ARMA models are not Gaussian, but Pareto, lognormal and α–stable.

Here we intend to study the power of this test under a different setting, *i.e* in case the data generating process (DGP) is nonlinear. This is a very preliminary study and at this beginning stage we consider only some nonlinear DGPs. They are listed below, innovations are distributed as $N(0,1)$:

1. TAR(1,1), where

$$X_t = \begin{cases} -0.5X_{t-1} + a_t & X_{t-1} \leq 1 \\ 0.4X_{t-1} + a_t & X_{t-1} > 1 \end{cases}$$

$$X_t = \begin{cases} 2 + 0.5X_{t-1} + a_t & X_{t-1} \leq 1 \\ 0.5 - 0.4X_{t-1} + a_t & X_{t-1} > 1 \end{cases}$$

$$X_t = \begin{cases} 1 - 0.5X_{t-1} + a_t & X_{t-1} \leq 1 \\ 1 + a_t & X_{t-1} > 1 \end{cases}$$

2. MS(1), where

$$X_t = \begin{cases} -0.5X_{t-1} + a_t & s_t = 1 \\ 0.4X_{t-1} + a_t & s_t = 2 \end{cases}$$

with $p_{11} = p_{22} = 0.5,\ 0.9$.
3. $BL(0,0,1,1)$, where $X_t = a_t + 0.5X_{t-1}a_{t-1}$
4. $BL(0,0,2,1)$, where $X_t = a_t + 0.8X_{t-1}a_{t-1} + 0.5X_{t-2}a_{t-1}$

The number of Monte Carlo simulations is 2000, the number of bootstrap replications for the moving block bootstrap is 500 and the block size is 30 (following the findings by Shelef and Schechtman, 2016). The considered sample sizes are $T = 200, 500, 1000$.

Table 1 Percentages of rejection ($m = 1, 2$ number of lags, nominal level 0.05)

	T=200		*T=500*		*T=1000*	
	$m=1$	$m=2$	$m=1$	$m=2$	$m=1$	$m=2$
DGP1: TAR(1,1)	37.7	20.0	72.8	35.3	96.8	50.3
DGP2: TAR(1,1)	42.8	18.8	88.8	41.2	100	61.4
DGP3: TAR(1,1)	45.8	24.8	79.5	49.6	91.6	50.4
DGP4: BIL(0,0,1,1)	22.4	12.4	45.6	17.4	62.8	35.5
DGP5: BIL(0,0,2,1)	27.6	17.4	57.6	28.7	73.6	43.6
DGP6: MS(1)	100	62.4	100	82.6	100	92.3
DGP7: MS(1)	100	90.8	100	93.8	100	95.7

Our empirical power results are shown in table 1. They clearly reveal a very interesting capability of the test to detect nonlinearity with the increase of T. As expected, the percentage of rejections is very high for the largest sample size ($T = 1000$), especially for TAR and MS model, but also for smaller values of T the behaviour is fairly good. In particular, for the MS DGPs the test detects successfully the nonlinear feature even at $T = 200$. Moreover, results are in line with the performance of the majority of nonlinearity test in literature (see Bisaglia and Gerolimetto (2014) for a recent survey on nonlinearity tests and a comparative Monte Carlo experiment). It should be remarked that, as emphasized by Shelef and Schechtman (2016), the performance tends to deteriorate with the increase of m.

As said at the beginning of this section, this is only a preliminary Monte Carlo experiment. Yet, we find these results promising, in particular the test appears to perform well for Markov Switching models (followed by TAR models). We reckon that this could be effectively an alternative route to check for nonlinearity. Some other investigations are in order both in terms of Monte Carlo simulations (e.g. comparison with analogous test provided in the literature) and possible improvement of the performance of the test, for instance by considering other resampling methods to obtain the critical values.

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