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Optimal control of the mean field game equilibrium for a pedestrian tourists' flow model

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Abstract Art heritage cities are popular tourist destinations but for many of them overcrowding is becoming an issue. In this paper, we address the problem of modeling and analytically studying the flow of tourists along the narrow alleys of the historic center of a heritage city. We initially present a mean field game model, where both continuous and switching decisional variables are introduced to respectively describe the position of a tourist and the point of interest that it may visit. We prove the existence of a mean field game equilibrium. A mean field game equilibrium is Nash-type equilibrium in the case of infinitely many players. Then, we study an optimization problem for an external controller who aims to induce a suitable mean field game equilibrium.

Keywords Tourist flow optimal control · mean field games · switching variables · dynamics on networks

Mathematics Subject Classification (2010) 91A13 · 49L20 · 90B20 · 91A80

1 Introduction

In the recent years, art heritage cities have experienced a continuous growth of tourists to the point that overcrowding is becoming an issue and local authorities start implementing countermeasures. In this paper, we address the

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problem of modeling and analytically studying the flow of tourists (more precisely pedestrian daily excursionists) along the narrow alleys of the historic center of a heritage city. Starting from the results in Bagagiolo-Pesenti [4], we recast this into a mean field game for controlled dynamics on a network, representing the possible paths in the city together with the possible attractive sites.

We suppose that the tourists have only two main attractions to visit. This is for simplicity only: situations with more attractions can be equally treated. We represent their possible paths inside the city by a circular network where three points are identified: S the train station where tourists arrive in the morning and to which they have to return in the evening; P_1 the first attraction; P_2 the second attraction (see Figure 1.a). We are given an external arrival flow at the station, represented by a continuous function $g : [0, T] \to [0, +\infty[$ (roughly speaking, $g(t)$ stays for the density of arriving tourists per unit time). Here $T > 0$ is the final horizon, the time before which everyone has to be returned to the station, after the tour of the city. Each single tourist (agent) controls its own dynamics represented by the equation

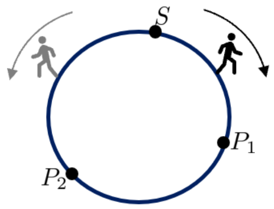
$$\theta' = u \tag{1}$$

where $\theta \in \mathbb{R}$ is a space-coordinate in the network and $t \mapsto u(t) \in \mathbb{R}$ is a measurable control. We denote by θ_S, θ_1 and θ_2, respectively, the position of the station, of the attraction P_1 and of the attraction P_2. To each tourist, we associate a time-varying label $(w_1, w_2) \in \{0, 1\} \times \{0, 1\}$. For $i \in \{1, 2\}$, $w_i(t) = 1$ means that, at the time t, the attraction P_i is not visited yet, and $w_i(t) = 0$ that the attraction was already visited. The state of an agent is then represented by the triple (θ, w_1, w_2), where θ is a time-continuous variable and w_1, w_2 are switching variables. In the following, we denote by $B = [0, 2\pi] \times \{0, 1\} \times \{0, 1\}$ the state space of variables (θ, w_1, w_2) and in particular, we call *(circle-)branch* any subset $B_{\hat{w}_1, \hat{w}_2}$ of B which includes the states $(\theta, \hat{w}_1, \hat{w}_2)$, with $\theta \in [0, 2\pi]$. Such branches also correspond to the edge of the switching networks in Figure 2, which is another way to represent our network model with its switching dynamics.

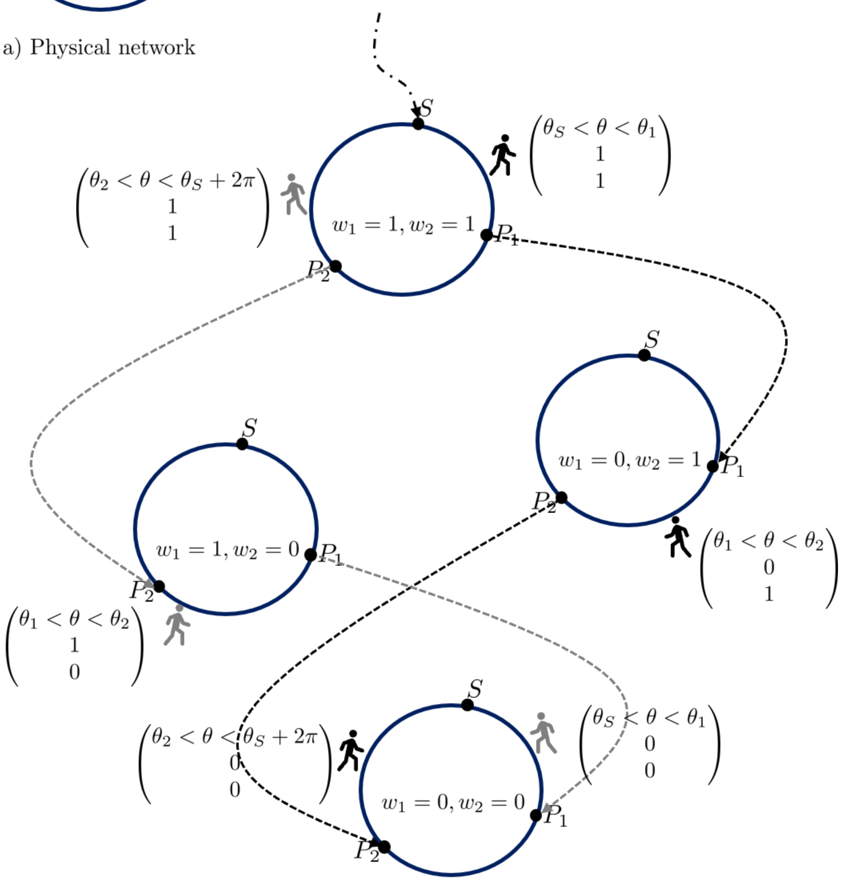
Indeed, while the evolution of the value of θ is governed by (1), the evolution of the values switching variables can be only from 1 to 0 and is described by the following condition

$$w_i = \begin{cases} 1 & \text{for } t \in [t, \tau_i] \\ 0 & \text{for } t \in]\tau_i, T] \end{cases}$$

being $\tau_i \in [t, T]$ is the first time instant at which the agent reaches and visits attraction P_i, $i \in \{1, 2\}$. This dynamics is represented by the arrows and the labels in Figure 1.b (see also Figure 2). In Figure 1.b, consider a tourist that arrives in the station and visits P_1 first, at time τ. It has state $(\theta_S, 1, 1) \in B_{1,1}$ when it arrives at the station. Then, at time $\tau \in [t, T]$, its state w_1 switches, it was equal to 1 in $[t, \tau_1]$, it assumes value $w_1 = 0$ in a left-open interval $]\tau_1, T]$.



a) Physical network



b) State network

Fig. 1 a) the physical network of the paths inside the city (with the three identified points: train station, S, and two attractions, $P_1, P2$) and two tourists visiting the city following opposite directions. b) the state network and the states (θ, w_1, w_2) of the tourists during their visits. Each singular circle-branch represents the network of the city paths as seen by tourist with given values of the switching variables w_1, w_2. The dashed arrows between the four circles-branches represent the switching of the four labels: at the beginning the label is $(1,1)$; when the attraction P_1 is reached (and hence visited), the label switches to $(0,1)$, and similarly when P_2 is reached; the last is $(0,0)$ which holds when both attractions are reached. The point-dashed arrow represents the external arrival flow in the station.

Immediately after τ_1, the tourist's state switches to the branch $B_{0,1}$. See also Figure 2.

The cost to be minimized by every agent, when it starts its tour from the station at time t, is given by a combination of terms representing: i) the hassle of running during the tour; ii) the pain of being entrapped in highly congested paths; iii) the frustration of not being able to visit some attractions; iv) the disappointment of not being able to reach the station by the final horizon T. Such a cost can be analytically represented by

$$\begin{aligned} &J(t,u) = \int_t^T \left(\frac{u(s)^2}{2} + \mathcal{F}^{w_1(s),w_2(s)}(\mathcal{M}(s)) \right) ds + \\ &c_1 w_1(T) + c_2 w_2(T) + c_S \xi_{\theta=\theta_S}(T) \end{aligned} \quad (2)$$

Here, $c_1, c_2, c_S > 0$ are fixed, and $\xi_{\theta=\theta_S}(t) \in \{0,1\}$ and it is equal to 0 if and only if $\theta(t) = \theta_S$. In (2), the quadratic term inside the integral stays for the cost i); the second term inside the integral stays for the congestion cost ii); the first two addenda outside the integral stay for the cost iii); the third addendum outside the integral stays for the cost iv). In particular, the congestion cost $\mathcal{F}^{w_1(s),w_2(s)}(\mathcal{M}(s))$ is instantaneously paid by an agent whose switching label at time s is $(w_1(s), w_2(s))$ being the actual distribution of the agents $\mathcal{M}(s)$. For any $(w_1, w_2) \in \{0,1\} \times \{0,1\}$, $\mathcal{F}^{w_1,w_2}$ is a positive function defined on the set of all admissible distribution of agents.

In Bagagiolo-Pesenti [4] the problem of minimizing the cost (2) by a large number of tourists is seen as a mean field game and, under some suitable assumptions, the existence of a mean field game equilibrium is proven. A mean field game equilibrium is a time-varying distribution of agents on the network, $t \mapsto \mathcal{M}^*(t)$ for $t \in [0,T]$, such that, when inserted it in (2), the returned optimal control $u^*(\cdot;t) : [t,T] \to \mathbb{R}$, $s \mapsto u(s;t)$, which is implemented by all agents who starts moving from the station at time $t \in [0,T]$, gives rise to optimal trajectories of the agents which exactly generate the time-varying distribution $\mathcal{M}^*$. A mean field game equilibrium can be seen as a fixed point, over a suitable set of time-varying distributions, of a map of the form

$$\mathcal{M} \longrightarrow u_{\mathcal{M}} \longrightarrow \mathcal{M}_{u_{\mathcal{M}}} \quad (3)$$

where $u_{\mathcal{M}}$ is the optimal control when $\mathcal{M}$ is inserted in (2), and $\mathcal{M}_{u_{\mathcal{M}}}$ is the corresponding evolution of the agents' distribution when all of them are moving implementing $u_{\mathcal{M}}$ as control. Of course, the problem must be coupled with an initial condition for the distribution $\mathcal{M}$, whereas the boundary condition is represented by the incoming external arrival flow g.

We remark that the concept of mean field equilibrium is a Nash-type equilibrium concept. Indeed, in the case of a large number of agents, even infinitely many, as the case of mean field games is, every single agent is irrelevant, the single agent has measure zero, it is lost in the crowd. Hence, in the case of equilibrium, for a single agent is not convenient to unilaterally change behavior, because such a single choice will not change the mean field $\mathcal{M}$, and the agent will not optimize.

The goal of the present paper is twofold. First we amend some stringent assumptions that were made in [4] in order to prove the existence of a mean field game equilibrium. Then we study a possible optimization problem for an external controller who aims to induce a suitable mean field game equilibrium. We suppose that such an external controller (the city administration, for example) may act on the congestion functions $\mathcal{F}^{w_1,w_2}$, choosing them among a suitable set of admissible functions.

Consider as an example the historical center of the city of Venice, Italy. Tourists typically enter the city at the train station and from there they have two main alternative routes, a shorter and a longer one, to the main monuments. In the recent years, the shorter route have been particularly congested on peak days. For this reason, local authorities have introduced both some gates to slow down the access to the shorter route and some street signs to divert at least part of the tourist flow long the longer route.

The mean field games theory goes back to the seminal work by Lasry-Lions [22] (see also Huang-Caines-Malhamé [19]), where the new standard terminology of Mean Field Games was introduced. This theory includes methods and techniques to study differential games with a large population of rational players and it is based on the assumption that the population influences the individuals' strategies through mean field parameters. Several application domains such as economic, physics, biology and network engineering accommodate mean field game theoretical models (see Achdou-Camilli-Capuzzo Dolcetta, [1], Lachapelle-Salomon-Turinici[21]). In particular, models to the study of dynamics on networks and/or pedestrian movement can be found for example in Camilli-Carlini-Marchi [8], in Cristiani-Priuli-Tosin [15], Camilli-De Maio-Tosin [9] and Bagagiolo-Bauso-Maggistro-Zoppello [5]. Our problem is also a sort of routing problem. Indeed to control the roadway congestion, different strategies was proposed as variable speed limits (Hegyi-Schutter-Hellendoorn [18]), ramp metering (Gomes-Horowitz [17]) or signal control (Brian Park-Yun-Ahn[7]). However, such mechanisms do not consider neither the agents' perspective nor affect the total amount of vehicles/people. A significant research effort was done to understand the agents' answer to external communications from intelligent traveler information devices (see Srinivasan-Mahmassani [25], Khattak-Polydoropoulou-Ben Akiva[20]) and, in particular, to study the effect of such technologies on the agents' route choice behaviour and on the dynamical properties of the transportation network (see Como-Savla-Acemoglu-Dahleh-Frazzoli [14]). Moreover, it is known that if individual agents make their own routing decisions to minimize their own experienced delays, overall network congestion can be considerably higher than if a central planner had the ability to explicitly direct traffic. From that, the idea to include in our problem an external controller which induces a suitable mean field game equilibrium. While, in the specific case of vehicular congestion, tolls payment is considered to influencing drivers to make routing choices that result in globally optimal routing, namely to force the Wardrop equilibrium to align with the system optimum network flow (see Smith [24], Morrison [23], Dial [16] and Cole-Dodis-Roughgarden [13]). The Wardop equilibrium [26] is a configuration in which the perceived cost associated to any source-destination path chosen by a nonzero fraction of drivers does not exceed the perceived cost associated to any other path. This is a stationary concept, indeed recently was developed its continuous counterpart (see Carlier-Jimenez-Santambrogio [11], Carlier-Santambrogio [12]) which fits the situation of pedestrian congestion (that in this paper we indeed treat using a mean field model) and also it is useful to look at large scale traffic problems, when one only want to identify average value of the traffic congestion in different zones of a large area. Actually, also the models using that continuous framework are essentially stationary because they only accounts for sort a cyclical movement, where every path is constantly occupied by the same density of vehicles, since those who arrive at destination are immediately replaced by others. This is the essentially difference with respect to the mean field models. Indeed in the latter, due to the explicit presence of time, the optimal evolution is given by a system coupling a transport equation and an Hamilton-Jacobi equation.

2 Preliminary results

In this section, we report some of the results introduced in Bagagiolo-Pesenti [4], and we present some new ways to approach the problem.

Suppose that a distributional evolution $t \mapsto \mathcal{M}(t)$ (the density of the agents) is given. We denote by $V(\theta, t, w_1, w_2)$ the value function (i.e. the infimum over the measurable controls of the cost functional (2)), faced by an agent that at time t is in the position (θ, w_1, w_2) and starts its evolution. We drop the dependence of V on $\mathcal{M}$. Assuming that the functions $t \mapsto F^{w_1,w_2}(\mathcal{M}(t))$ are all continuous and bounded, then $V : B \times [0, T] \to \mathbb{R}$ can be uniquely identified by the viscosity solutions of system of Hamilton-Jacobi equations (one per each branch) with boundary conditions mutually exchanged between the branches. On the switching points of the branch, the boundary conditions are given by the value function itself evaluated on the point where we switch on, in the new branch. This fact comes from the dynamic programming principle and for a suitable interpretation of the optimal control problem on a single branch as a suitable exit-time optimal control problem (see Bagagiolo-Danieli [3], and the references therein for similar problems.).

Indeed, when the distribution $\mathcal{M}$ is given (i.e. when the fixed point procedure described above is performed), on every branch (as in Figure 2) the optimal control problem faced by any agent is a rather standard optimal control given by some suitable combinations of reaching target (the station) problem and exit time problems. In particular, on every branch B_{w_1,w_2} we have the (θ, t)-depending continuous value function $V(\cdot, \cdot, w_1, w_2)$. The exit cost from $B_{1,1}$ at time τ is given by $V(\theta_1, \tau, 0, 1)$ in θ_1 and by $V(\theta_2, \tau, 1, 0)$ in θ_2. The exit costs from $B_{0,1}$ and from $B_{1,0}$ are, respectively, $V(\theta_2, \tau, 0, 0)$ and $V(\theta_1, \tau, 0, 0)$ (on both exit points of the branch). On the final branch $B_{0,0}$ we only have the problem of reaching the station by the final time T paying as less as possible. Note that, when $\mathcal{M}$ is given, the optimal control problem on $B_{0,0}$ can

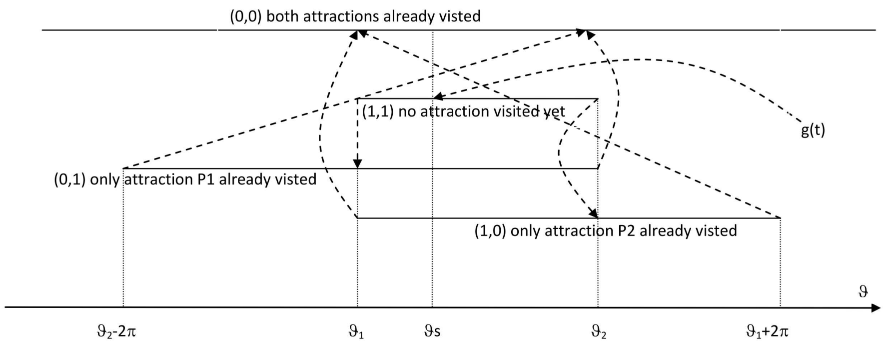


Fig. 2 Switching cross branches representation

be solved (and in particular $V(\cdot,\cdot,0,0)$ can be calculated) independently from the behavior on the other branches. Solved the problem on $B_{0,0}$ we can then solve the problem on $B_{0,1}$ and on $B_{1,0}$ and eventually on $B_{1,1}$.

Once the control problem is solved on every branch, then we tackle the optimal transport problem which, when the initial distribution is given, returns the evolution of the distribution of the agents in the interval $[0,T]$. We can then perform the fixed point analysis.

More formally, for every branch B_{w_1,w_2}, the optimal feedback control u^* is, at least in principle,

$$u^*(\theta,t,w_1,w_2) = -V_\theta(\theta,t,w_1,w_2), \quad \forall t, \forall \theta. \tag{4}$$

For every $(w_1,w_2) \in \{0,1\} \times \{0,1\}$ and for every $t \in [0,T]$, we indicate by $m^{w_1,w_2}(\cdot,t) : [0,2\pi] \to [0,+\infty[$ the time dependent measure that represents the agents' distribution on the branch B_{w_1,w_2} at time t. We assume that the city network is initially empty of tourist, that is $m^{w_1,w_2}(\cdot,0) = 0$ for $(w_1,w_2) \in \{0,1\} \times \{0,1\}$. By conservation of mass principle, $\mathcal{M}$ in point satisfies

$$\begin{aligned}\int_B d\mathcal{M}(t) = \int_{B_{1,1}} dm^{1,1}(t) + \int_{B_{1,0}} dm^{1,0}(t) + \\ + \int_{B_{0,1}} dm^{0,1}(t) + \int_{B_{0,0}} dm^{0,0}(t) = \int_0^t g(s)ds \quad t \in [0,T].\end{aligned}$$

There a transport equations for the density m^{w_1,w_2} must hold, as an example in case of $B_{1,0}$ we have:

$$\begin{aligned} &m_t^{1,0}(\theta,t) - [V_\theta(\theta,t,1,0)m^{1,0}(\theta,t)]_\theta = 0 \text{ in } B_{1,0} \times [0,T] \\ &m^{1,0}(\theta_2,t) = m^{1,1}(\theta_2,t),. \end{aligned} \tag{5}$$

The switching nature of the problem is reflected in the boundary conditions like (5), which tie the distributions m^{w_1,w_2} of the different branches B_{w_1,w_2}.

As in [4], there are some suitable hypotheses: the initial distribution is null (no-one is around the city at $t = 0$); the congestion functions $\mathcal{F}^{w_1,w_2}$ do not explicitly depend on θ (all agents in the same branch at the same instant equally suffer the same congestion pain). These hypotheses lead to fact that the control choice each agent makes in one of the point $(\theta_S; 1, 1)$, $(\theta_1, 0, 1)$, $(\theta_2, 1, 0)$, $(\theta_1, 0, 0)$ and $(\theta_2, 0, 0)$ do not change until the agent remains in the same branch, and it is constant in time. Hence, only those points are to be considered in the optimization procedure. In the sequel we will call them: significant points.

Now here, instead of treating coupled Hamilton-Jacobi equations and transport equation as it is common in mean field game theory), we write equivalent conditions but in a different and in some sense more operative ways.

An agent in $(\theta_1, 0, 0)$, at time $t \in [0, T]$, has two possibilities of behavior (i.e. the optimal behavior is certainly one of the following two): either to stay at θ_1 without moving or to move and reach θ_S by time T (it is certainly not convenient to reach θ_S before the time T and wait there; see the considerations made in [4]). In the first case the optimal control is $u \equiv 0$, in the second case it is $u \equiv \pm\frac{\theta_S - \theta_1}{T - t}$. Here $\theta_s - \theta_1$ stays for the length of the minimal path between θ_S and θ_1 in the circular network, and the sign is coherently chosen with (1), in order to run that path. Similarly when the agent is in $(\theta_2, 0, 0)$.

Hence we get, given the cost functional (2),

$$
\begin{aligned}
V(\theta_1, t, 0, 0) &= \min\left\{c_S, \frac{1}{2}\frac{(\theta_S - \theta_1)^2}{T - t}\right\} + \int_t^T \mathcal{F}^{0,0}\, ds \\
V(\theta_2, t, 0, 0) &= \min\left\{c_S, \frac{1}{2}\frac{(\theta_S - \theta_2)^2}{T - t}\right\} + \int_t^T \mathcal{F}^{0,0}\, ds
\end{aligned}
\tag{6}
$$

where we do not display the argument of $\mathcal{F}^{0,0}$, being it fixed (the distribution $\mathcal{M}$).

When the agent starts from $(\theta_1, 0, 1)$ at the time t, it has three choices: to stay at θ_1 without moving; to reach θ_S at the time T; to reach θ_2 at some $\tau \in]t, T]$. Similarly as before, in the first case it uses the control $u \equiv 0$; in the second case $u \equiv \pm\frac{\theta_S - \theta_1}{T - t}$; and in the third case $u \equiv \pm\frac{\theta_2 - \theta_1}{\tau - t}$. Similarly for $(\theta_2, 1, 0)$. Hence:

$$
\begin{aligned}
V(\theta_1, t, 0, 1) = \min_{\tau \in [t,T]} \Bigg\{ & c_2 + c_S + \int_t^T \mathcal{F}^{0,1}\, ds, c_2 + \frac{1}{2}\frac{(\theta_S - \theta_1)^2}{T - t} + \int_t^T \mathcal{F}^{0,1}\, ds, \\
& \frac{1}{2}\frac{(\theta_2 - \theta_1)^2}{\tau - t} + \int_t^\tau \mathcal{F}^{0,1}\, ds + V(\theta_2, \tau, 0, 0) \Bigg\} \\
V(\theta_2, t, 1, 0) = \min_{\tau \in [t,T]} \Bigg\{ & c_1 + c_S + \int_t^T \mathcal{F}^{1,0}\, ds, c_1 + \frac{1}{2}\frac{(\theta_S - \theta_2)^2}{T - t} + \int_t^T \mathcal{F}^{1,0}\, ds, \\
& \frac{1}{2}\frac{(\theta_1 - \theta_2)^2}{\tau - t} + \int_t^\tau \mathcal{F}^{1,0}\, ds + V(\theta_1, \tau, 0, 0) \Bigg\}
\end{aligned}
\tag{7}
$$

Finally, in $(\theta_S, 1, 1)$ at the time t, the possibilities are: to stay there without moving, to reach θ_1 at a certain $\tau \in [t, T]$; and to reach θ_2 at a certain $\eta \in [t, T]$. Hence, similarly as before,

$$V(\theta_S, t, 1, 1) = \min_{\tau, \eta \in [t,T]} \left\{ c_1 + c_2 + \int_t^T \mathcal{F}^{1,1}\, ds, \frac{1}{2}\frac{(\theta_1 - \theta_S)^2}{\tau - t} + \int_t^\tau \mathcal{F}^{1,1}\, ds + \right.$$
$$\left. + V(\theta_1, \tau, 0, 1), \frac{1}{2}\frac{(\theta_2 - \theta_S)^2}{\eta - t} + \int_t^\eta \mathcal{F}^{1,1}\, ds + V(\theta_2, \eta, 1, 0) \right\} \quad (8)$$

Note that from (6)–(8), when $\mathcal{M}$ is fixed, performing the minimization procedures we get an optimal behavior: to stay or to move towards some of the other points. In doing that we also detect the possible arrival time (τ and η), and finally the corresponding optimal control u.

3 Existence of a mean field game equilibrium

In this section we give a proof of the existence of a mean field game equilibrium.

By the considerations of the previous section, the only significant points are $(\theta_S, 1, 1)$, $(\theta_1, 0, 1)$, $(\theta_2, 1, 0)$, $(\theta_1, 0, 0)$ and $(\theta_2, 0, 0)$. Together with them, we have also to consider the flows of arrivals on them (see Figure 2): g the external arrival flow at the station (it is a datum); g_{01} the arrival flow in $(\theta_1, 0, 1)$; g_{10} the arrival flow in $(\theta_2, 1, 0)$; g_{12} the arrival flow in $(\theta_2, 0, 0)$; g_{21} the arrival flow in $(\theta_1, 0, 0)$ (the last four being part of the solution). Such flows must satisfy the conservation conditions, for all $t \in [0, T]$:

$$\begin{aligned} &\int_0^t g(\tau) d\tau \ge \int_0^t g_{01}(\tau) d\tau + \int_0^t g_{10}(\tau) d\tau, \\ &\int_0^t g_{01}(\tau) d\tau \ge \int_0^t g_{12}(\tau) d\tau, \\ &\int_0^t g_{10}(\tau) d\tau \ge \int_0^t g_{21}(\tau) d\tau. \end{aligned} \quad (9)$$

Denoting by $\rho^{w_1,w_2}(t)$ the actual total mass of agents on the branch B_{w_1,w_2}, we then have

$$\begin{aligned} \rho^{1,1}(t) &= \int_0^t g(\tau) d\tau - \int_0^t g_{01}(\tau) d\tau - \int_0^t g_{10}(\tau) d\tau, \\ \rho^{0,1}(t) &= \int_0^t g_{01}(\tau) d\tau - \int_0^t g_{12}(\tau) d\tau, \\ \rho^{1,0}(t) &= \int_0^t g_{10}(\tau) d\tau - \int_0^t g_{21}(\tau) d\tau, \\ \rho^{0,0}(t) &= \int_0^t g_{12}(\tau) d\tau + \int_0^t g_{21}(\tau) d\tau. \end{aligned} \quad (10)$$

Actually, using the notation of the previous section, it is

$$\rho^{w_1,w_2}(t) = \int_{B_{w_1,w_2}} m^{w_1,w_2}(\theta,t)d\theta.$$

Form now on, we suppose that $\mathcal{F}^{w_1,w_2}$ only depends, in a continuous manner, from ρ^{w_1,w_2} and denote by ρ the 4-uple $(\rho^{1,1},\rho^{0,1},\rho^{1,0},\rho^{0,0})$. However, more general situations may be taken into account. Hence, for what concerns our problem, to give the distribution $\mathcal{M}$ is equivalent to give ρ. Accordingly, the idea is to perform a fixed-point procedure on ρ sent to a new ρ' given by the transport law given by the minimization (6)–(8) with ρ inserted on it. To this end, we need some convexity and compactness properties that we introduce in the following.

First of all note that the flow functions $g_{\cdot,\cdot}$ are bounded. Their bounds come from the bound of the external arrival flow g at θ_S (see also the Appendix for more details on such flows). Then functions ρ^{w_1,w_2} are Lipschitz continuous, and moreover they are also bounded by $K = \int_0^T g(s)ds$. Hereinafter, we denote by L the Lipschitz constant of any Lipschitz function we are going to consider. We then have, for the quadruplet ρ,

$$\rho \in \{f : [0,T] \to [0,+\infty[\text{ s.t. Lipschitz constant } \ L, \text{ bounded by } K\}^4 := X.$$

Space X is convex and compact for the uniform topology.

Next, obserserve that the function for which we need to exhibit the existence of a fixed point should be a continuous function $\psi : X \to X$ which acts in the following way: it takes $\rho \in X$, inserts ρ in (6)–(8), obtains the optimal controls u, constructs the flow-functions $g_{\cdot,\cdot}$ and gives the new corresponding $\rho' \in X$. Note that the flow-functions g_{01}, g_{10}, g_{12} and g_{21} as well as the external flow g are time densities which, in order to run along the branches B_{w_1,w_2} with the optimal control, at first must transform in spatial densities and then become again time densities on the switching points. Hence, the function ψ need to built the spatial and temporal components of the mentioned flow-functions in order to obtain the new $\rho' \in X$ (we report the computation of both spatial and temporal components of the above flow-functions in the Appendix.)

As regards the continuity of ψ the main difficulty is that the optimal feedback controls may be not unique: for any fixed t, the minimization procedure in (6)–(8) may return more than one minimizers, when more than one of the arguments of the considered $\min\{.,.\}$ functions reaches the minimum value. In particular, this situation may happen for several t, even accumulating or in a whole interval, in (7)–(8). Differently, in (6) it may occur at most in a single instant and in this case is not a problem.

When such a multiplicity situation occurs, then the agents naturally split in more communities using one of the optimal behaviors. In some sense they use mixed-strategies, that is they convexify. The function ψ described above, is defined assuming that the agents do not split: all agents, in the same position and at the same instant makes the same choice for their behavior (this is indeed a characteristic feature of the mean field games model: the agents are homogeneous and indistinguishable). But this is coherent only with the uniqueness of the optimal feedback. When, instead, we have multiplicity, this is not more true and, when such a multiplicity accumulates in time, the continuity of the function ψ is not evident. In particular there are more than one way to construct it: just taking different optimal behaviors.

In Bagagiolo-Pesenti [4], the above difficulty is overcome by a-priori assuming that there exists a finite number N that bounds the maximum number of times at which those multiplicities appear, independently on ρ. This assumption guarantees a sort of continuity of the function ψ by considering a multifunction whose image is the closed convex hull of all possible outputs of the function ψ. Specifically, the assumption on the uniform bound N, guarantees that such a closed convex hull is given by a set of a a-priori bounded number of extremal points, and hence getting compact image and closed graph.

Here we let such an assumption drop and we prove the continuity and convexity of ψ by defining a sequence of continuous convex functions approximating of ψ, based on the assumption that an agent can make sub-optimal decisions as described in the following.

Take $\varepsilon > 0$ and, whenever you make an optimal choice for all agents in one of the minimization problems (6)–(8), then maintain the same choice, i.e., consider the same argument within the $\min\{.,.\}$ function as minimizing one until, possibly, its difference with a new minimizing one is greater than ε. In this latter case, select the new minimizer as the new choice. Calling ψ_ε such a new correspondence we now describe it in a precise way.

In (8), let us call (8)-1, (8)-2, (8)-3 respectively the first, the second and the third term inside the parenthesis in the right-hand side. Note that, when (8)-1 is the minimizer, then it means that to stay at θ_S without moving till the final time T is an optimal strategy. When the minimum is realized by (8)-2 (for some τ), then to move and reach θ_1 at time τ is an optimal strategy. When the minimum is given by (8)-3 (for some η), then to move and to reach θ_2 at time η is an optimal strategy. In the sequel, when we say, for example, "at the time t the agents make the choice (8)-2", it means that they also choose the corresponding minimizing τ and move towards θ_1 with constant velocity $|\theta_1 - \theta_S|/(\tau - t)$. But it does not mean that it is necessary the best choice at all.

At $(\theta_S, 1, 1)$ at the time $t_0 = 0$, all the present agents make the same optimal choice, among the optimal ones, for example (8)-2. Then, all other agents, at $(\theta_S, 1, 1)$ at every subsequent times t, continue to make the same choice (8)-2, as long as, at t, (8)-2 realizes the minimum up to an error lower than ε (i.e. they continue to choose (8)-2 even if it is not more optimal, but anyway ε-optimal). Let $t_1 \in [0, T]$ be the possible first instant at which the error is equal to ε. Then the agents at $(\theta_S, 1, 1)$ at t_1 make a new optimal choice (certainly different from (8)-2) among the optimal ones, for example (8)-3. Then, all other agents, at $(\theta_S, 1, 1)$ at every subsequent times t, continue to make the same choice (8)-3, as long as, at t, (8)-3 realizes the minimum up to an error lower than ε. We proceed in this way, taking a possible new instant t_2, and so on.

We similarly argue for the other points $(\theta_2, 1, 0)$, $(\theta_1, 0, 1)$, $(\theta_2, 0, 0)$, $(\theta_1, 0, 0)$ and for the whole time interval $[0, T]$. Note that the arguing above is not influenced by the actual presence of the agents in the points.

Of course, starting from $t_0 = 0$, we may have more than one of such possible constructions. For example, at $(\theta_S, 1, 1)$ we may have more than one possible optimal choices at the time $t_0 = 0$. Different choices will generate different first instants t_1 for which other optimal choices will be considered, which will generate different second instants t_2 and so on. However, the number of all such possible instants is finite. Indeed, the functions involved in the minimization (6)–(8) can be considered as Lipschitz functions of their time-entries: t, τ, η. Actually, the terms of the form, for example, $(\theta_1 - \theta_2)^2/(\tau - t)$ are not Lipschitz in $t, \tau \in [0, T]$, but they play a role in the minimization certainly until they are not grater than $c_s + c_1 + c_2$, and until it happens, they are uniformly Lipschitz. Hence, at every significant points as above, the distance of two subsequent instants in anyone of the sequences constructed as before is not less than ε/L. This means that, in the time interval $[0, T]$, we have a uniform a-priori bound N_ε for the number of all those possible instants. Hereinafter N_ε will denote any possible bound of the number of some specific quantities.

The above arguments allow us to state that for each one of the significant points as above we can divide the interval $[0, T]$ in a finite number of sub-intervals, in the interior of which the number and the type of the ε-optimal choices does not change. Such choices are indeed ε-optimal because they realize the minimum in the corresponding formula (6)–(8) up to an ε-error, and those minima are exactly the optima.

Now we describe as ψ_ε acts. Take $\rho = (\rho^{1,1}, \rho^{0,1}, \rho^{1,0}, \rho^{0,0}) \in X$ and insert it in (6)–(8); for each significant point construct all possible sequences of ε-optimal behaviors choosing one of them inside anyone of the time sub-intervals (the number of all such sequences, as well as the number of their terms is bounded by N_ε). For each one of those sequences, starting by the given external arrival flow g, construct the flow functions $g_{\cdot,\cdot}$ (see the Appendix) and then the new mass concentration $\rho' = (\rho'^{1,1}, \rho'^{0,1}; \rho'^{0,1}, \rho'^{0,0}) \in X$, using (10). Define $\psi_\varepsilon(\rho) \subseteq X$ as the set of all possible mass concentrations ρ' constructed in this way. Note that $\psi_\varepsilon(\rho)$ is a finite set with no more than N_ε elements, independently on ρ.

Since ψ_ε is a multifunction and we want to apply the fixed-point Kakutani Theorem, we need compact and convex images and closed graph. The set $\psi_\varepsilon(\rho)$ being finite is certainly not convex, hence we have to convexify. We define as $\tilde{\psi}_\varepsilon(\rho) \subseteq X$ the set of all quadruplets constructed in the following way. At $(\theta_S, 1, 1)$, for each one of the sub-intervals of $[0, T]$ suppose that the agents (arriving with the given flow g) split in some fractions each one of them following one of the possible ε-optimal choices in the time sub-intervals. This gives a mass concentration $\tilde{\rho}^{1,1}$ on the branch $B_{1,1}$, and, considering all the admissible splitting, we get the first components of all elements of $\tilde{\psi}_\varepsilon(\rho)$. For any of the possible splitting we also get the corresponding flow functions g_{01}, g_{10}, and then, as above, we consider all admissible splitting in the points

$(\theta_1, 0, 1)$ and $(\theta_2, 1, 0)$ getting the mass concentrations in the corresponding branches. This gives the second and the third components of the elements of $\tilde{\psi}_\varepsilon(\rho)$. In a similar way we construct the fourth components. The set $\tilde{\psi}_\varepsilon(\rho)$ is then a convex subset of X (any split is a convex combination of masses). It is a sort of convex hull of $\psi_\varepsilon(\rho)$, suitably performed branch by branch. The fact that $\psi_\varepsilon(\rho)$ is a finite set, also gives the closedness (and hence the compactness, being X compact) of $\tilde{\psi}_\varepsilon(\rho)$ in X (for the uniform topology). Similarly, the multifunction $\rho \mapsto \tilde{\psi}_\varepsilon(\rho)$ has closed graph. Here, we sketch such a proof.

We have to prove that whenever ρ_n converges to ρ in X, and $\xi^n \in \tilde{\psi}_\varepsilon(\rho^n)$ converges to ξ in X, then $\xi \in \tilde{\psi}_\varepsilon(\rho)$. By definition, the first component of the quadruplet ξ^n is given by a suitable split of the external arrival flow g in every sub-intervals of $[0, T]$ and hence by the transport with the corresponding choices (8)-1, (8)-2, (8)-3. Since, independently on ρ^n, the number of sub-intervals is bounded by N_ε, and the number of choices is always not more than three (in other words, the number of the element of $\psi_\varepsilon(\rho^n)$ is uniformly bounded with respect to ρ), we may suppose that, at least for a subsequence, the first component of all ξ^n is constructed by using the same number of sub-intervals with exactly the same type of choices for every sub-interval. Moreover, the sub-interval depending triple, $(\lambda_1^n, \lambda_2^n, \lambda_3^n)$ representing the sequences of split-fractions of the mass of agents that choose one of the admissible choices (if in the sub-interval a choice is not admissible, then the corresponding λ is zero) converge, sub-interval by sub-interval, to a limit fraction $(\lambda_1, \lambda_2, \lambda_3)$. Due to the convergence of ρ^n to ρ, using (8) we have that the first sub-interval of every step n converges to a limit sub-interval, which is coherent with (8) and ρ. Similarly, for the second sub-intervals and so on. Hence we have a candidate limit partition of $[0, T]$ in sub-intervals, with the corresponding choices and the corresponding split-fractions. Since ξ^n uniformly converges to ξ, it is easy to see that the first component of ξ is generated by those sub-intervals, choices and split-fractions. Moreover, sub-interval by sub-interval, the flow functions to the branches $B_{0,1}$ and $B_{1,0}$ converges to some some flow functions which, reasoning as above in the new branches gives the construction (sub-intervals, choices, split-fraction) of the second and third component of ξ. Similarly for the fourth component. Hence $\xi \in \tilde{\psi}_\varepsilon(\rho)$.

By the Kakutani fixed point theorem, for every $\varepsilon > 0$ there exists a fixed point $\rho_\varepsilon \in X$ for $\tilde{\psi}_\varepsilon$, that is $\rho_\varepsilon \in \tilde{\psi}_\varepsilon(\rho_\varepsilon)$. This means that, if all agents suppose that the realized congestion is given by ρ_ε than there is a suitable split choice of behaviors such that ρ_ε is actually realized and moreover, by construction, every agent, when moving by one of the admissible choices (split fractions not null), is subject to a cost (2) whose difference with the optimal one is infinitesimal as ε. In some sense ρ_ε is a ε- mean field games equilibrium.

Definition 1 A mean field game equilibrium for our problem is a pair (ρ, λ) where $\rho \in X$ and $\lambda \in L^\infty(0, T)^{13}$ is a string of 13 split-fractions functions, that is

$$\lambda = \Big(\lambda_1^{(\theta_S,1,1)}, \lambda_2^{(\theta_S,1,1)}, \lambda_3^{(\theta_S,1,1)}, \\ \lambda_1^{(\theta_1,0,1)}, \lambda_2^{(\theta_1,0,1)}, \lambda_3^{(\theta_1,0,1)}, \lambda_1^{(\theta_2,1,0)}, \lambda_2^{(\theta_2,1,0)}, \lambda_3^{(\theta_2,1,0)}, \\ \lambda_1^{(\theta_1,0,0)}, \lambda_2^{(\theta_1,0,0)}, \lambda_1^{(\theta_2,0,0)}, \lambda_2^{(\theta_2,0,0)}\Big) \tag{11}$$

which means, for example, that in $(\theta_S, 1, 1)$ we consider an arrival flow of density $\lambda_1^{(\theta_S,1,1)} g$ of agents which will make the choice (8)-1, an arrival flow of density $\lambda_2^{(\theta_S,1,1)} g$ of agents which will make the choice (8)-2, and so on. This will generate flows functions on the branches $B_{0,1}$ and $B_{1,0}$ which will be correspondingly split by the fractions in the second line of (11), and so on.

The pair (ρ, λ) is a mean field game equilibrium if the behavior explained above generates exactly the mass density ρ (by (10)) and if, putting ρ inside the cost (2), that behavior makes each agent optimize. This means that for almost every instants t, whenever one of the split fractions is not null, then the corresponding choice is an optimal one.

Remark 1 Similarly to the definition of λ for the mean field games equilibrium (11), we have the split fractions functions λ_ε for the ε-problem. In that kind of problem, such functions are, by construction, constant in the sub-intervals. Denoting by $\overline{\lambda}_\varepsilon$ the string of 13 of split functions we used in the construction of the element of $\psi_\varepsilon(\rho)$ as well as of $\tilde{\psi}_\varepsilon(\rho)$, the corresponding notion of split functions λ_ε as in (11) is given, in some sense, by the product of elements of $\overline{\lambda}_\varepsilon$. For example

$$\begin{array}{l} (\lambda_\varepsilon)_2^{(\theta_S,1,1)} = (\overline{\lambda}_\varepsilon)_2^{(\theta_S,1,1)} \\ (\lambda_\varepsilon)_2^{(\theta_1,0,1)} = (\overline{\lambda}_\varepsilon)_2^{(\theta_S,1,1)} (\overline{\lambda}_\varepsilon)_2^{(\theta_1,0,1)} \\ \text{(the product between the arriving fraction and the departing fraction)} \\ (\lambda_\varepsilon)_2^{(\theta_1,0,0)} = (\overline{\lambda}_\varepsilon)_2^{(\theta_S,1,1)} (\overline{\lambda}_\varepsilon)_2^{(\theta_1,0,1)} (\overline{\lambda}_\varepsilon)_2^{(\theta_1,0,0)} \\ \text{(here the arriving fraction is the product of the first two factors)} \end{array} \tag{12}$$

All the fractions functions must applied to the global flow g transported, branch by branch, by the chosen control. We refer to the Appendix for more details in the subject of transport on networks.

Theorem 1 *There exists a mean field game equilibrium*

Proof. For every $\varepsilon > 0$ we have a fixed point ρ_ε for $\tilde{\psi}_\varepsilon$, and we have the corresponding split-fractions function $\lambda_\varepsilon \in L^\infty(0,T)^{13}$ which, for every ε is constant in each sub-intervals (note that the numbers of sub-intervals is not in general bounded uniformly in ε). Let $\lambda \in L^\infty(0,T)^{13}$ be the weak-star limit of λ_ε as $\varepsilon \to 0$ (i.e. $\int_0^t \lambda_\varepsilon f \to \int_0^T \lambda f$ for all integrable function f) and $\rho \in X$ be the uniform limit of ρ_ε (at least for a subsequence). By the construction of λ_ε, by the weak-star convergence of λ_ε to λ, and by the uniform convergence of ρ_ε to ρ, we get that λ is a split-fractions function generating the mass density ρ (it generates flow functions $g_{\cdot,\cdot}$ which by (10) generate ρ). Finally, at every step ε, if the agents exactly perform the behavior given by the split-fractions λ_ε, then they get an ε-optimal cost. We then conclude that the pair (ρ, λ) is a mean field game equilibrium. Indeed, for almost every instants t such that one of the components of λ is not null, by the weak-star convergence there is, at least for a sub-sequence, a sequence of instants t_ε converging to t such that the corresponding component of $\lambda_\varepsilon(t_\varepsilon)$ is not null. This means that the corresponding choice is ε-optimal with respect to ρ_ε and we conclude. □

Remark 2 The introduction of the ε-problem, is mostly due to an analytical purpose: the possibility of the construction of an ε-mean field games equilibrium $\rho_\varepsilon \in \tilde{\psi}_\varepsilon(\rho_\varepsilon)$ by the fact that the number of sub-intervals is finite, and then the passage to the limit.

However, from an applicative modeling point of view, the ε-problem is also interesting by itself. Indeed, it takes account of a typical human decision phenomenon: herd behavior (see, e.g., [6]), where an agent looks at and is influenced by the decisions made by other agents in taking its own decision, even if it is not the optimal choice for him. In these situations a second thought occurs when discrepancy from the optimal choice becomes too large (more than ε) and than it is evident that it is better to change choice.

4 The optimization problem

In this section we introduce a possible optimization problem faced by a local authority, hereinafter referred as to controller, that intends to manage the flow of tourists.

We restrict our analysis to congestion cost functions of the form

$$\mathcal{F}^{w_1,w_2}(\rho) = \alpha_{w_1,w_2}\rho^{w_1,w_2}(s) + \beta_{w_1,w_2} \tag{13}$$

with $\rho = (\rho^{1,1}, \rho^{0,1}, \rho^{1,0}, \rho^{0,0}) \in X$. In (13), the coefficients $(\alpha_{w_1,w_2}, \beta_{w_1,w_2})$ are at disposal of an external controller whose aim is to force the realized mean field game string ρ to be as close as possible (in the uniform topology) to a reference string $\overline{\rho} \in X$, i.e. to minimize:

$$\max_{w_1,w_2\in\{0,1\}} \left\{ \max_{t\in[0,T]} |\rho^{w_1,w_2}(t) - \overline{\rho}^{w_1,w_2}(t)| \right\} = \|\rho - \overline{\rho}\|_X \tag{14}$$

Formally, let us denote by $\chi_{\alpha,\beta}$ the set of all possible (mass component) mean field games equilibrium corresponding to the choice of the parameters $\alpha = (\alpha_{1,1}, \alpha_{0,1}, \alpha_{1,0}, \alpha_{0,0})$ and $\beta = (\beta_{1,1}, \beta_{0,1}, \beta_{1,0}, \beta_{0,0})$, and let us suppose that the parameters are restricted to belong to a compact set $K \subset \mathbb{R}^4 \times \mathbb{R}^4$. Then, the controller faces the optimization problem given by

$$\inf_{(\alpha,\beta)\in K} \mathcal{E}(\alpha,\beta) = \inf_{(\alpha,\beta)\in K} \left(\inf_{\rho\in\chi_{\alpha,\beta}} \|\rho - \overline{\rho}\|_X \right) \tag{15}$$

Theorem 2 *There exists an optimal pair* $(\alpha, \beta) \in K$.

Proof. Let $(\alpha^n, \beta^n) \in K$ be a minimizing sequence for $\mathcal{E}$, and for every n let $\rho^n \in \chi_{\alpha_n,\beta_n}$ realize the infimum in (15) up to $1/n$. By compactness we may suppose that (α^n, β^n) converges to $(\tilde{\alpha}, \tilde{\beta}) \in K$, and that ρ^n uniformly converges to $\tilde{\rho} \in X$. To prove the statement, we only need to prove that $\tilde{\rho} \in \chi_{\tilde{\alpha},\tilde{\beta}}$.

1) $\tilde{\rho} \in \chi_{\tilde{\alpha},\tilde{\beta}}$. This means to prove that $\tilde{\rho}$ is a mean field game equilibrium for the problem. Let λ^n be the split-fraction functions associated to the mean field game equilibrium ρ^n, and let $\tilde{\lambda}$ be a weak-star limit of it. Then, due to the considered convergences, arguing as in the proof before, we get that the pair $(\tilde{\rho}, \tilde{\lambda})$ is a mean field equilibrium for $(\tilde{\alpha}, \tilde{\beta})$

□

Remark 3 To consider in (15) an optimization problem of the form

$$\inf_{(\alpha,\beta)\in K} \left(\sup_{\rho \in \chi_{\alpha,\beta}} \|\rho - \overline{\rho}\|_X \right)$$

would be certainly more interesting and robust from the point of view of an external controller. However, the existence of an optimal pair, due to the possibilities of more than one mean field game equilibrium is not evident. Certainly there exists a pair (α, β) and $\rho \in \mathcal{X}_{\alpha,\beta}$ such that $\|\rho - \overline{\rho}\|_X$ is equal to that infimum, but this does not guarantee that the pair (α, β) is optimal.

If we would be able, under some suitable hypotheses, to guarantee the uniqueness of the mean field games equilibrium, then such a problem will be bypassed. Maybe, stronger hypotheses on the costs $\mathcal{F}$ could be useful. We leave such a question to future investigations.

So far we have assumed that the coefficients $(\alpha_{w_1,w_2}, \beta_{w_1,w_2})$ of congestion cost functions $\mathcal{F}^{w_1,w_2}$ are not time-depending. This hypothesis plays an essential role in the analysis of the problem, in particular on the nature of the optimal controls. Here we little bit amend such a hypothesis by assuming that $(\alpha_{w_1,w_2}, \beta_{w_1,w_2})$ may be piecewise constant functions and that the control is implemented at the significant points of each branch of the network, e.g., through gates.

We consider a finite sequence of fixed instant $t_0 = 0 < t_1 < t_2 < ... < t_N = T$, and for every $i = 0, ..N-1$, the coefficients $(\alpha^i, \beta^i) \in K$ at disposal of the external controller in the interval $[t_i, t_{i+1}[$. The congestion cost paid by an agent at time s becomes

$$\tilde{\mathcal{F}}^{w_1,w_2}(s) = \alpha_{w_1,w_2}(\tau(s))\rho^{w_1,w_2}(s) + \beta_{w_1,w_2}(\tau(s))$$

where $\alpha_{w_1,w_2}(\tau) = \alpha^i_{w_1,w_2}$, respectively $\beta_{w_1,w_2}(\tau) = \beta^i_{w_1,w_2}$, for $t_i \leq \tau < t_{i+1}$, $i = 0, ..., N-1$; and $\tau(s)$ is the last switching instant not greater than s along the agent trajectory, i.e., the instant at which the agent state entered B_{w_1,w_2}.

In this situation the total cost payed by an agent is the usual

$$\begin{array}{l} J(t,u) = \int_t^T \left(\frac{u(s)^2}{2} + \tilde{\mathcal{F}}^{w_1(s),w_2(s)}(s) \right) ds + \\ c_1 w_1(T) + c_2 w_2(T) + c_S \xi_{\theta=\theta_S}(T) \end{array} \qquad (16)$$

but we have an explicit dependence on the time of the congestion cost through $\alpha_{w_1,w_2}(.)$ and $\beta_{w_1,w_2}(.)$. Nevertheless, such a dependence is compatible with the structure of the choices of the agents in our model. Indeed, by the cost (16), the agents that arrive at $(\theta_S, 1, 1)$ at the beginning feel the actual value of the parameters $\alpha^{1,1}$ and $\beta^{1,1}$, and act as they are constant until the possibly exit from the branch $B_{1,1}$. Suppose that they switch on the branch $B_{0,1}$. Then, at that moment, they feel the actual value of the parameters $\alpha^{0,1}$ and $\beta^{0,1}$ and, again, act as they are constant until the possibly exit from the branch, and so on. It is here worth recalling, that we assume that the controller applies its controls at the beginning of the network branches and that this latter model as the previous one implies that the agents make their decisions each time they enter a new branch, then a possible change in the parameters (α^i, β^i) is not perceived by the agents that are "on the road" along a branch. Consequently the considered time-dependence of the cost does not change the reasoning we did before about properties of the optimal controls. In particular, formulas (6)–(8) remain valid, and the existence of a mean field games equilibrium is guaranteed. Moreover a similar optimization problems as above can be successfully performed.

5 Conclusions

In this paper we have introduced a mean field game model as a possible way of representing the flows of tourists in the alleys of an heritage city. Then, we used this model as a basis of the optimization problem that aims at managing the tourist flows according to some targets posed by the local authorities.

Many further refinements of both the model and the optimization problem presented are possible from both a theoretical and an applicative perspective. As an example, the assumption of the existence of surveillance cameras, to count people entering and leaving areas of interest, may suggest the description of a new type of information in the model and justify the definition of a feedback control policy on the coefficients $(\alpha_{w_1,w_2}, \beta_{w_1,w_2})$ at disposal of an external controller. In addition different kinds of local authority management measures can be modeled and more than two places of interests may be considered. Note that in the latter case the number of alternative paths would increase exponentially. However the direct experience of the authors in Venice suggests that the vast majority of the tourists is interested in very few attractions in each city.

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A Appendix: on some flows and transports on networks

In this appendix, we discuss the characteristics of the flow functions g and $g_{\cdot,\cdot}$ both from a spatial and a temporal perspective.

Le us consider what happens in our model on the branch $B_{1,1}$. At the point θ_S there is an entering flow of agents, represented by the function of time g. This is a "density with respect to time", that is, from a dimensional point of view, something like "number of agents" divided by time. When instead we calculate the actual total mass $\rho^{1,1}(t)$ on the branch, we have to integrate on space (over the whole branch) a "density with respect to space ", that is something as "number of agents" divided by space. Moreover, in the two switching points θ_1, θ_2 of the branch, we need to recover again a "density with respect to time" which has to represent the entering flow in the other branches. Actually, in our model, being the congestion cost depending only on the total mass on the branch, we do not really need to pass through the determination of density with respect to space, but the computation of the exiting flows is enough, see (10). However, in a more general setting we need both computations. In the sequel, we call them the spatial and temporal component of the flow. Obviously, this argument is linked to what is called, for example, disintegration of a measure (see for instance Ambrosio-Gigli-Savaré [2] and Camilli-De Maio-Tosin [9]).

Still considering the branch $B_{1,1}$ as example, we recall that, in our model, the agents arriving in θ_S at time t choose their behavior at exactly that time and they do not change it until they, possibly, leave the branch.

Now we sketch how, in our situation, to split the flow in the two components. If an agent at time t is in the position θ and it is going towards θ_1, then there is a time $\tau < t$ at which it has arrived at θ_S, and there is a time $\overline{\tau} > t$ at which it will arrive at θ_1. And it is using a constant velocity $u(\tau)$ of absolute value $|\theta_1 - \theta_S|/(\overline{\tau} - \tau)$. In particular here $\overline{\tau}$ is calculated as the minimizing time for the second term in (8) with t replaced by τ. Let us indicate by $\Lambda(\theta, t)$ the function that gives the entering time τ. Note that, when the mass concentration is given (i.e. when performing the fixed point procedure) the function Λ is evaluable using (6)–(8) which give all possible constant velocities $u(\tau)$.

In the present heuristic arguing we suppose that Λ and u are differentiable (actually, in our setting they are Lipschitz and hence at least almost everywhere differentiable). Assuming for simplicity $\theta_S = 0$, $\theta > 0$, and denote $\lambda(\tau) = 1/u(\tau)$, we have the following relation

$$\Lambda(\theta, t) + \lambda(\Lambda(\theta, t))\theta = t, \tag{17}$$

Deriving with respect to t and θ we have the relations

$$\Lambda_t(\theta, t) = \frac{1}{1 + \lambda'(\Lambda(\theta, t))\theta} = -\frac{\Lambda_x(\theta, t)}{\lambda(\Lambda(\theta, t))}$$

Now, supposing that all agents arriving in θ_S with flow g move towards θ_1, then the density in time g at θ_S is spread on the spatial density component $-g(\Lambda(\theta, t))\Lambda_x(\theta, t)$ and on the temporal density component $g(\Lambda(\theta, t))\Lambda_t(\theta, t)$ (which act only if in the point θ at time t actually there are agents, some agents are already arrived, otherwise they are zero). Such relations may be verified by standard mass balance/conservation arguments. In particular, at the point θ_1 the arriving flow (density in time) which will be the entering flow in the new branch $B_{0,1}$ is $t \mapsto g(\Lambda(\theta_1, t))\Lambda_t(\theta_1, t)$. If moreover, at θ_S there is a split of the agents among different choices, and the corresponding split fraction is $\lambda_2^{(\theta_S,1,1)}$ (see (11)), then the entering flow in $B_{0,1}$ through θ_1 is $t \mapsto g(\Lambda(\theta_1, t))\lambda_2^{(\theta_S,1,1)}(t)\Lambda_t(\theta_1, t)$. And this is exactly the flow denoted by g_{01} in (10). Similar considerations (with different function Λ and λ) hold in the case of agents moving towards θ_2 in the branch $B_{1,1}$ and for all other cases in the other branches.

The above argument is applied in the fixed point procedure in Section 3 for $\tilde{\psi}_\varepsilon$ where, starting from $\rho \in X$ we obtain another $\rho' \in X$, passing through the flow functions. For example the flow entering in the branch $B_{0,0}$ through $(\theta_2, 0, 1)$, that is the flow g_{12} in (10), using the notation in (12), is (for the corresponding function Λ of the motion from θ_1 to θ_2 on the branch $B_{0,1}$) $t \mapsto g_{0,1}(\Lambda(\theta_2, t))(\overline{\lambda}_\varepsilon)_2^{(\theta_1,0,1)}(t)\Lambda_t(\theta_2, t)$, where $g_{0,1}$ is evaluated as above.