Simultaneous calibrated prediction intervals for time series

Intervalli di previsione simultanei calibrati per serie storiche

Giovanni Fonseca, Federica Giummolè and Paolo Vidoni

Abstract This paper deals with simultaneous prediction for time series models. In particular, it presents a simple procedure which gives well-calibrated simultaneous predictive intervals with coverage probability equal or close to the target nominal value. Although the exact computation of the proposed intervals is usually not feasible, an approximation can be easily obtained by means of a suitable bootstrap simulation procedure. This new predictive solution is much simpler to compute than those ones already proposed in the literature based on asymptotic calculations. An application of the bootstrap calibrated procedure to first order autoregressive models is presented.

Abstract *Questo lavoro riguarda la costruzione di intervalli di previsione simultanei per serie storiche. In particolare, presenta una semplice procedura per ottenere intervalli di previsione simultanei calibrati con probabilità di copertura uguale o molto vicina al valore nominale. Sebbene il calcolo esatto di questi intervalli non sia sempre possibile, essi si possono approssimare tramite un'opportuna procedura bootstrap. Le approssimazioni così ottenute hanno il vantaggio di essere molto più semplici da calcolare delle soluzioni asintotiche già note. Viene infine presentata un'applicazione della procedura di calibrazione bootstrap per la previsione in modelli autoregressivi del primo ordine.*

Key words: Calibration, coverage probability, prediction intervals, time series.

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1 Introduction

In the statistical analysis of time series, a key problem concerns prediction of future values. Although, in the literature, great attention has been received by pointwise predictive solutions, in this paper we deal with the notion of prediction intervals, which explicitly takes account of the uncertainty related to the forecasting procedure. In particular, we assume a parametric statistical model and we follow the frequentist viewpoint, with the aim of constructing prediction intervals having good coverage accuracy.

It is well-known that the estimative or plug-in solution, though simple to derive, is usually not adequate. In fact, it does not properly take account of the sampling variability of the estimated parameters, so that the (conditional) coverage probability of the estimative prediction intervals may substantially differ from the nominal value.

Improved prediction intervals based on complicated asymptotic corrections have been proposed in a general framework by Barndorff-Nielsen and Cox (1996) and Corcuera and Giummolè (2006) and, for the case of time series models, by Giummolè and Vidoni (2010) and Vidoni (2004). A calibrating approach has been suggested by Beran (1990) and applied, for example, by Hall *et al.* (1999), using a suitable bootstrap procedure. Fonseca *et al.* (2011, 2014) extended this proposal and presented applications to one-step ahead prediction intervals for stationary time series models. Simulation-based prediction intervals for autoregressive processes are also considered by Kabaila and Syuhada (2007). Finally, there is an extensive literature on non-parametric bootstrap prediction intervals for autoregressive time series (see, for example, Clements and Kim, 2007 and references therein).

In this paper we extend the bootstrap calibration procedure proposed in Fonseca *et al.* (2011, 2014) to the multidimensional case. In particular this work is dedicated to the construction of joint prediction regions which are expected to contain a future sequence of observations with the required coverage probability. Although the specification of a multivariate prediction region may be quite general, we restrict our attention to joint regions of rectangular form, which are usually considered for forecasting future paths of time series observations. Recently, Wolf and Wunderli (2015) introduce a similar system of simultaneous prediction limits. In the last section, an application to simultaneous prediction within AR(1) models is presented.

2 Simultaneous calibrated prediction intervals

Given a discrete-time stochastic process $\{Y_t\}_{t\geq 1}$, we assume that $Y = (Y_1, \ldots, Y_n)$, $n > 1$, is observable, while $Z = (Z_1, \ldots, Z_m) = (Y_{n+1}, \ldots, Y_{n+m})$, $m \geq 1$, is a future or not yet available random vector, corresponding to an m-dimensional sequence of future observations. The vector (Y, Z) is continuous with $g(z|y;\theta)$ and $G(z|y;\theta)$, $\theta \in \Theta$, the conditional multivariate density and distribution function of vector Z given

$Y = y$. In the presence of a transitive statistic U, y is substituted by the observed value u of U.

Given the observed sample $y = (y_1, \ldots, y_n)$, a system of simultaneous α-prediction limits for vector Z is a set of functions $c_\alpha^j(y)$, $j = 1, \ldots, m$, such that, exactly or approximately,

$$P_{Y,Z}\{Z_j \leq c_\alpha^j(Y), j = 1, \ldots, m; \theta\} = \alpha, \tag{1}$$

for every $\theta \in \Theta$ and for any fixed $\alpha \in (0,1)$. In the presence of a finite dimensional transitive statistic, we usually consider the conditional coverage probability

$$P_{Y,Z|U}\{Z_j \leq c_\alpha^j(Y), j = 1, \ldots, m | U = u; \theta\} = \alpha. \tag{2}$$

An α-level joint prediction region of rectangular form is readily obtained by specifying two suitable systems of lower and upper simultaneous prediction limits.

As we can see, for example in Ravishanker *et al.* (1991) and Alpuim (1997), simultaneous prediction limits for vector Z in a time series context are usually defined as

$$z_{j,\alpha} = z_{j,\alpha}(Y; \theta) = P_j + h_\alpha(\theta)\, se_j(\theta), \quad j = 1, \ldots, m, \tag{3}$$

evaluated at $\theta = \hat{\theta}$, where $\hat{\theta} = \hat{\theta}(Y)$ is the maximum likelihood estimator for θ, or an asymptotically equivalent alternative. Here $P_j = P_j(Y; \theta)$ is a suitable unbiased point predictor for Z_j, such that $E_{Y,Z_j}(Z_j - P_j) = 0$, with prediction standard error $se_j(\theta) = \sqrt{V_{Y,Z_j}(Z_j - P_j)}$. Indeed, $h_\alpha(\theta)$ is a quantity satisfying

$$P_{Z|Y}\{\mathscr{E}_j \leq h_\alpha(\theta), j = 1, \ldots, m | Y = y; \theta\} = F\{h_\alpha(\theta), \ldots, h_\alpha(\theta) | y; \theta\} = \alpha,$$

with $\mathscr{E}_j = (Z_j - P_j)/se_j(\theta)$, $j = 1, \ldots, m$, the standardized forecast errors with joint distribution function $F(e_1, \ldots, e_m | y; \theta)$, conditional on $Y = y$.

In order to compute the prediction limits specified by relation (3), we need a vector of unbiased point predictors $P = (P_1, \ldots, P_m)$, the associated vector of prediction standard errors $se(\theta) = \{se_1(\theta), \ldots, se_m(\theta)\}$ and the quantity $h_\alpha(\theta) = \varphi^{-1}(\alpha | y; \theta)$, where $\varphi^{-1}(\cdot | y; \theta)$ is the inverse of function $\varphi(x | y; \theta) = F(x, \ldots, x | y; \theta)$, which corresponds to the conditional distribution function $F(e_1, \ldots, e_m | y; \theta)$ constrained to $\{(e_1, \ldots, e_m) \in \mathbf{R}^m | e_1 = \cdots = e_m = x\}$. For stationary linear models, we usually consider the optimal predictors $P_j = E(Z_j | Y)$, $j = 1, \ldots, m$. Indeed, with this choice for the point forecasts, provided that we have a linear or a Gaussian process, the vector of the (standardized) forecasts errors $(\mathscr{E}_1, \ldots, \mathscr{E}_m)$ is independent of Y.

The (unconditional) coverage probability of the estimative simultaneous prediction limits $\hat{z}_{j,\alpha} = z_{j,\alpha}(Y; \hat{\theta})$, $j = 1, \ldots, m$, corresponds to

$$\begin{aligned} P_{Y,Z}\{Z_j \leq \hat{z}_{j,\alpha}, j = 1, \ldots, m; \theta\} &= E_Y[P_{Z|Y}\{Z_j \leq \hat{z}_{j,\alpha}, j = 1, \ldots, m | Y; \theta\}; \theta] \\ &= E_Y[P_{Z|Y}\{\mathscr{E}_j \leq (\hat{z}_{j,\alpha} - P_j)/se_j(\theta), j = 1, \ldots, m | Y; \theta\}; \theta] \\ &= E_Y[F\{a_1 + h_\alpha(\hat{\theta}) b_1, \ldots, a_m + h_\alpha(\hat{\theta}) b_m | Y; \theta\}; \theta] = D(\alpha, \theta), \end{aligned}$$

where $a_j = a_j(Y,\theta) = (\hat{P}_j - P_j)/se_j(\theta)$ and $b_j = b_j(Y,\theta) = se_j(\hat{\theta})/se_j(\theta)$, $j = 1,\ldots,m$, with $\hat{P}_j = P_j(Y;\hat{\theta})$.

Following the calibrating procedure proposed in Fonseca *et al.* (2014) for univariate prediction limits, we may consider function

$$\varphi_c(x|y;\hat{\theta},\theta) = D\{F(x,\ldots,x|y;\hat{\theta}),\theta\} = D\{\varphi(x|y;\hat{\theta}),\theta\} \tag{4}$$

instead of $\varphi(x|y;\hat{\theta})$, in order to specify the quantity

$$h^c_\alpha(\hat{\theta},\theta) = \varphi_c^{-1}(\alpha|y;\hat{\theta},\theta) = \varphi^{-1}(D^{-1}(\alpha,\theta)|y;\hat{\theta}) = h_{D^{-1}(\alpha,\theta)}(\hat{\theta}),$$

with $\varphi_c^{-1}(\cdot|y;\hat{\theta},\theta)$ and $D^{-1}(\cdot,\theta)$ the inverse functions of $\varphi_c(\cdot|y;\hat{\theta},\theta)$ and $D(\cdot,\theta)$, respectively. It is easy to show that the calibrated simultaneous prediction limits thus obtained, namely

$$z^c_{j,\alpha}(Y;\hat{\theta},\theta) = \hat{P}_j + h^c_\alpha(\hat{\theta},\theta)\, se_j(\hat{\theta}), \quad j = 1,\ldots,m, \tag{5}$$

present a coverage probability equal to the target nominal value α. Indeed, the specification of quantities $h^c_\alpha(\hat{\theta},\theta)$ from (4) determines simultaneous prediction limits satisfying relation (1) exactly for all $\alpha \in (0,1)$.

Whenever a closed form expression for $\varphi_c(x|y;\hat{\theta},\theta)$ is not available, we may consider a suitable parametric bootstrap estimator. Let y^b, $b = 1,\ldots,B$, be parametric bootstrap samples generated from the estimative distribution of the data and let $\hat{\theta}^b$, $b = 1,\ldots,B$, be the corresponding maximum likelihood estimates. Since $D(\alpha,\theta)$ is defined as an expectation, we define the following bootstrap estimator for (4)

$$\varphi^b_c(x|y;\hat{\theta}) = \frac{1}{B}\sum_{b=1}^{B} F\{\hat{a}^b_1 + h_\alpha(\hat{\theta}^b)\hat{b}^b_1,\ldots,\hat{a}^b_m + h_\alpha(\hat{\theta}^b)\hat{b}^b_m|Y;\hat{\theta}\}|_{\alpha=\varphi(x|y;\hat{\theta})},$$

where $\hat{a}^b_j = (\hat{P}^b_j - \hat{P}_j)/se_j(\hat{\theta})$ and $\hat{b}^b_j = se_j(\hat{\theta}^b)/se_j(\hat{\theta})$, $j = 1,\ldots,m$, with $\hat{P}^b_j = P_j(Y;\hat{\theta}^b)$. In this case, the associated α-level quantity permits the definition of a system of simultaneous prediction limits with coverage probability equal to α, apart from an error term depending on the efficiency of the bootstrap procedure.

3 Example: AR(1)

Let $\{Y_t\}_{t\geq 1}$ be a first-order Gaussian autoregressive process with

$$Y_t = \mu + \rho(Y_{t-1} - \mu) + \varepsilon_t, \qquad t \geq 1,$$

where μ and ρ are unknown parameters and $\{\varepsilon_t\}_{t\geq 1}$ is a sequence of independent Gaussian random variables with zero mean and unknown variance σ^2. We assume $|\rho| < 1$ so that the process is stationary. The observable random vector is

$Y = (Y_1, \ldots, Y_n)$ and the next m realizations of the process are $Z = (Y_{n+1}, \ldots, Y_{n+m})$. The conditional distribution of Z given $Y = y$ is Gaussian with mean $\mu_{Z|Y} = (\mu_{n+1}, \ldots, \mu_{n+m})$, where $\mu_{n+1} = \mu + \rho(y_n - \mu)$, $\mu_{n+k} = \mu + \rho(\mu_{n+k-1} - \mu)$, $k = 2, \ldots m$, and variance-covariance matrix Σ where $\Sigma_{ij} = \sigma^2 \rho^{|i-j|}$. Indeed, Y_n is a transitive statistic and we evaluate the performance of simultaneous prediction limits by means of their coverage probability conditioned on the observed value y_n of Y_n, as in (2).

A simulation study shows the performance of the proposed predictive solution (5). Conditional coverage probabilities for estimative and bootstrap calibrated prediction limits of level $\alpha = 0.9$ are calculated by means of the simulation technique presented in Kabaila (1999), keeping the last observed value fixed to $y_n = 1$. The results for $m = 2, 5$ future variables are collected in Table 1 and show that the bootstrap solution remarkably improve on the estimative one.

Table 1 AR(1) Gaussian model. Conditional coverage probabilities for simultaneous estimative and bootstrap calibrated prediction limits of level $\alpha = 0.9$, conditioned on $y_n = 1$; $m = 2, 5$ future observations are predicted; μ is considered to be known and equal to 0; $\rho = 0.5, 0.8$, $\sigma^2 = 1$ and $n = 20$. Estimation is based on 1,000 Monte Carlo replications. Bootstrap procedure is based on 500 bootstrap samples. Estimated standard errors are always smaller than 0.011.

m	ρ	Estimative	Calibrated
2	0.5	0.872	0.898
	0.8	0.859	0.895
5	0.5	0.859	0.893
	0.8	0.857	0.884

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