Peer Effects and Local Congestion in Networks

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Abstract

We study linear quadratic games played on a network. Agents face peer effects with distance-one neighbors, and strategic substitution with distance-two neighbors (*local congestion*). For this class of games, we show that an interior equilibrium exists both in the high and in the low regions of the largest eigenvalue, but may not exist in the intermediate region. In the low region, equilibrium is proportional to a weighted version of Bonacich centrality, where weights are themselves centrality measures for the network. Local congestion has the effect of decreasing equilibrium behavior, potentially affecting the ranking of equilibrium actions. When strategic interaction extends beyond distance-two, equilibrium is characterized by a "nested" Bonacich centrality measure, and existence properties depend on the sign of strategic interaction at the furthest distance. We support the assumption of local congestion by presenting empirical evidence from a secondary school Dutch dataset.

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1 Introduction

Socio-economic decisions are typically taken in social networks defined by interpersonal, institutional and technological ties. Within these networks, neighbors jointly consume and produce goods, discuss political opinions, share information and beliefs. As a consequence, neighbors in the network tend to display correlation in behavior. Positive correlation (and in particular peer effects) has commanded substantial attention,[1] partly because of its pervasiveness in social interaction, and because it amplifies individual shocks acting as a "social multiplier" (Glaeser et al., 2003).

In this paper we study problems in which strategic interdependency extends beyond the social ties, to agents that are at distance-two or further away in the social network. Most of our analysis deals with the case in which agents face peer effects at distance-one, as well as strategic substitution at distance-two. We refer to this substitution effect as *local congestion*. The term "local" refers to the assumption that actions generate congestion through common neighbors, rather than at large in the social network.

Examples of local congestion abound in economics. In job-referral networks, agents get to know about vacancies *via* their social ties, and compete for the information that becomes available to common neighbors. As shown by Calvó-Armengol and Jackson (2004), in the short run the incentives of an unemployed agent to actively search for a job are negatively affected by how actively the other unemployed agents who share the same social contacts (and, thereby, have access to the same sources of information) search. A similar effect of local competition shapes the incentives structure in collaboration networks, where researchers compete for the limited time and attention of common co-authors.

Local congestion may also stem from the presence of negative local externalities. For instance, a smoker may decide to limit smoking in the presence of friends or relatives when these are already subject to large amounts of secondhand smoke. The incentive to limit smoking stems from the perception of a large marginal health damage suffered by the smoker's congested friends. Pecuniary externalities may sort similar effects. Suppose firms A and B use each other's output as factor of production. Therefore, an increase in firm A's output level increases the demand for B's product, and thus its price. This in turn raises the marginal cost of all other firms that use B's product as production factor. The increase in the marginal cost will reduce these firms' incentives to produce. Finally,

[1] Evans et al. (1992); Gaviria and Raphael (2001); Kirke (2004); Christakis and Fowler (2007); Clark and Loheac (2007); Christakis and Fowler (2008); Calvó-Armengol et al. (2009).

local congestion can also be created by free-riding incentives of agents sharing a common neighbor. In conflict networks, for instance, countries that share a common enemy tend to free ride on each other in the production of armaments.

The aim of this paper is to trace equilibrium behavior to the topology of the social network in problems with local congestion. This is a crucial issue for the design of network-based policies that affect behavior by incentivizing the creation and/or the destruction of social ties such as friendships, kinship, work relations and such. The key element of our analysis, and a common feature of all the above examples, is that the adjacency matrix of the social network *does not* coincide with the matrix of strategic interactions of the game. In fact, the presence of local congestion both affects the intensity of distance-one relationships (when two neighbors also share common neighbors), and creates channels of strategic interaction not accounted for by the adjacency matrix of the network (between agents who are not neighbors but share common neighbors). We show that the matrix of strategic interaction is a weighted sum of the social network's adjacency matrix and its second power.

A recent literature investigates the relation between equilibrium behavior and the pattern of social interaction in games with linear best replies. This relation has been studied in Ballester et al. (2006) and Bramoullé et al. (2014), and then applied to the analysis of various socio-economic problems (see for example Calvó-Armengol et al., 2009; Patacchini and Zenou, 2012, 2011; Ballester et al., 2010; Bloch and Quérou, 2013; Topa and Zenou, 2014). Ballester et al. (2006) is based on the observation that in this class of games, the matrix of strategic interactions can be decomposed into the sum of a local complementarity matrix and of a global substitution matrix, plus an idiosyncratic element. Ballester et al. (2006) shows that when an interior equilibrium exists, this is proportional to the vector of Bonacich centralities for the local complementarity matrix. Bramoullé et al. (2014) allows also for multiple non interior equilibria with active and inactive agents.

We follow this literature by focusing on games with linear best replies, and, as in Ballester et al. (2006), we look at interior equilibria. We note that, although the above decomposition can be applied to our problem with local congestion, the local complementarity matrix so obtained does not coincide with the adjacency matrix of the social network, and does not preserve its fundamental properties. The problem of tracing behavior to the topology of the social network cannot therefore be addressed by direct application of Ballester et al. (2006) results. As we discuss in detail below, this crucial difference implies that equilibrium behavior relates to the network *via* a variant of the

Bonacich centrality measure.

We frame our results in terms of the effect of local congestion on both the existence and the characterization of equilibrium. In particular, we take as benchmark the case of peer effects only, in which the adjacency matrix of the social network does coincide with the matrix of strategic interaction (and with the local complementarity matrix in Ballester et al., 2006). For this case, equilibrium is found only in the low range of the largest eigenvalue. We show that in the presence of local congestion, an interior equilibrium exists both in networks with high largest eigenvalue and in networks with low largest eigenvalue. The intuition for this result is immediate in the class of regular networks, whose eigenvalue equals the average degree: as the network becomes denser, distance-two channels of strategic substitution tend to grow faster than distance-one channels of peer effects, and this eventually bounds the magnitude of equilibrium feedbacks.

We then turn to the characterization of equilibrium. We show that in the low range of the largest eigenvalue, equilibrium is characterized by a weighted Bonacich centrality measure for the social network, where weights are themselves centrality measures for the same network. Using our characterization, we show that the introduction of local congestion always decreases equilibrium behavior, and that such a decrease is larger for agents who are more central in the social network. We provide an example where this produces a reversal in the ranking of agents' equilibrium actions with respect to the case of peer effects only. We then perform comparative statics with respect to the social network. Within the class of regular networks, we show that the relation between behavior and network density is non monotonic: behavior first increases and then decreases after a critical density level is attained. We also find that creating cliques unambiguously contracts aggregate behavior.

We extend the model with local congestion to encompass strategic effects beyond distance-two in the social network. This is relevant, for instance, in the adoption of safe behaviors in the presence of a transmittable disease, where someone's incentives to adopt the safe behavior depends on the probability that her neighbors are infected. This in turn depends on the adoption of safe behaviors by all other agents to whom these neighbors are directly or indirectly connected. We show that the existence of an interior equilibrium crucially depends on the sign of the strategic interaction taking place at the furthest distance. In particular, an interior equilibrium exists in the region of high largest eigenvalues when strategic interaction at the furthest distance is of the substitute type. We also show that in the region of low largest eigenvalues, equilibrium behavior is characterized in terms of a "nested" variant of Bonacich centrality. This generalizes the characterization result for the case of local congestion to this more complex case.

Finally, we provide empirical evidence of the existence of a negative correlation in behavior at distance-two, consistent with the presence of local congestion. We focus on doing homework, a behavior that generates interaction patterns of the type described for collaboration networks. The empirical analysis uses data from a novel dataset containing information about over 2,500 Dutch secondary school pupils.

We note that a notion similar to what we call local congestion is present in previous works on network economics, where it has been mainly formalized in terms of the effect of the "degree" of a node on the incentives of other nodes to link to that node. The "co-author model", first proposed by Jackson and Wolinsky (1996), contains the general idea that the benefits coming from a connection may be limited when this is incident to a very connected node. This idea is embedded in various models of network formation, and it is key in determining the agents' incentives to form and sever links. Examples of such models include Morrill (2011); Billand et al. (2012, 2013); Möhlmeier et al. (2016). Differently from these papers, we refer to local congestion as the impact of the "actions" taken by a neighbor of a given node on the incentives to act of the other neighbors of that same node. Finally, congestion in distance-two relations is behind Wahba and Zenou (2005) analysis of job market networks, where the probability of finding a job through social contacts (weak ties) decreases at high levels of network density. This is due to the relative speed of growth of distance-one and distance-two relations (and the ensuing competition effects), a mechanism at all similar to the one behind our non monotonicity result in the density-behaviour relation.

The paper is organized as follows. Section 2 sets up the formal model with local congestion. Section 3 studies equilibrium existence and characterization. Section 4 extends the basic model to encompass interaction at arbitrary distance in the network. Section 5 brings the model to the data. Section 6 concludes the paper.

2 A Model with Peer Effects and Local Congestion

We consider a set N of n agents, organized in a network $\mathbf{g}$, defined by a $n \times n$ matrix $\mathbf{G}$ whose generic entry $g_{ij} \in \{0, 1\}$ measures the presence of a social tie between agents i and j. We assume that the network is undirected, $g_{ij} = g_{ji}$ for all $i, j \in N$, and we let $g_{ii} = 0$ for all i. When $g_{ij} = 1$ we say that agents i and j are neighbors in $\mathbf{g}$. The number of neighbors of agent i in $\mathbf{g}$ is called the degree of agent i and is denoted by d_i. A *walk* of length m between agents i and j in $\mathbf{g}$ is defined as a finite sequence of agents $(k_n)_{n=1,...,m}$ such that $k_1 = i$, $k_m = j$ and $g_{k_n k_{n-1}} = 1$ for all $n = 2, ..., m$. The generic term $g_{ij}^{[2]}$ of the power matrix $\mathbf{G}^2$ counts the number of walks of length-two from node i to node j in $\mathbf{g}$ (notice that $g_{ii}^{[2]} = d_i$).

The payoff function in (1) defines the payoff of agent i given a vector $\mathbf{x} \in \mathbb{R}_+^n$ of actions and the network $\mathbf{g}$:

$$U_i(\mathbf{x}) = \alpha x_i - \frac{\sigma}{2} x_i^2 + \phi \sum_{j \in N} g_{ij} x_i x_j - \gamma \sum_{k \in N} g_{ik}^{[2]} x_i x_k \tag{1}$$

The first two terms of (1) capture the private benefits from one's own action. These benefits are the sum of a linear increasing part and a quadratic decreasing part, with intensity measured respectively by parameters α and σ. The third term captures the peer effect: the marginal incentive to act increases in the sum of the actions taken by neighbors. The intensity of such complementarity is measured by the parameter $\phi > 0$. In the fourth term, each entry $g_{ik}^{[2]}$ counts the number of lenght-two walks from i to k. This term describes an indirect strategic interdependence: if $\gamma > 0$, the marginal incentives to act decrease in the aggregate level of actions taken at distance-two in the network. We call *local congestion* this strategic substitution effect between agents at distance-two in $\mathbf{g}$. In Appendix A we sketch three micro-founded economic problems characterized by local congestion, yielding the utility function (1) as a reduced form.

3 Equilibrium

3.1 The Matrix of Strategic Interaction

We start by laying out the matrix of strategic interaction associated with the payoff function (1) and with a given adjacency matrix $\mathbf{G}$. This matrix, that we call $\tilde{\mathbf{G}}$, keeps track of both peer effects at distance-one and substitution effects at distance-two in the network $\mathbf{g}$. An interior equilibrium $\mathbf{x}$ is characterized by the following FOCs:

$$\alpha \cdot \mathbf{1} = \left[\sigma \mathbf{I} - \phi \tilde{\mathbf{G}}\right] \mathbf{x}, \tag{2}$$

where $\tilde{\mathbf{G}}$ is defined as:

$$\tilde{\mathbf{G}} \equiv \mathbf{G} - \frac{\gamma}{\phi}\mathbf{G}^2. \tag{3}$$

Strategic interaction in $\tilde{\mathbf{G}}$ is defined with respect to both the network $\mathbf{G}$ and its power matrix $\mathbf{G}^2$. Note that $\tilde{\mathbf{G}}$ is symmetric, being the sum of symmetric matrices, and therefore has real valued eigenvalues. The generic entry of $\tilde{\mathbf{G}}$ is given by:

$$\tilde{g}_{ij} = \begin{cases} 0 & \text{if } g_{ij} = 0 \text{ and } g_{ij}^{[2]} = 0 \\ 1 & \text{if } g_{ij} = 1 \text{ and } g_{ij}^{[2]} = 0 \\ -\frac{\gamma}{\phi} g_{ij}^{[2]} & \text{if } g_{ij} = 0 \text{ and } g_{ij}^{[2]} > 0 \\ 1 - \frac{\gamma}{\phi} g_{ij}^{[2]} & \text{if } g_{ij} = 1 \text{ and } g_{ij}^{[2]} > 0 \end{cases} \tag{4}$$

Let us consider each of the four possibilities of (4) in detail. In the first line, since i and j are neither directly nor indirectly connected in $\mathbf{g}$, they experience neither peer effects nor local congestion, and $\tilde{g}_{ij} = 0$. In the second line, i and j are neighbors in $\mathbf{g}$ but do not share any common neighbor; as a consequence, their interaction consists in the peer effect only. In the third line, i and j share a common neighbor but are not neighbors; there are no peer effects at work, but there is local congestion. Note that this entry grows in magnitude with the number of common neighbors. Note also that $\tilde{g}_{ii} = -\frac{\gamma}{\phi} d_i$, so that $\tilde{\mathbf{G}}$ always contains negative entries. Finally, in the fourth line i and j are both direct and indirect neighbors, and the sign of their interaction depends on the relative magnitude of peer effects and local congestion. Both the issues of existence and characterization of an interior equilibrium can be addressed by referring to the notion of Bonacich centrality.

Definition 1 (Bonacich Centrality) *Let $\mathbf{A}$ be an adjacency matrix, and let $a \in \mathbb{R}_+$ be a discount parameter. i) The Bonacich centrality matrix is given by $\mathbf{M}(\mathbf{A}, a) \equiv [\mathbf{I} - a\mathbf{A}]^{-1}$; ii) The vector of Bonacich centralities is given by $\mathbf{b}(\mathbf{A}, a) \equiv \mathbf{M}(\mathbf{A}, a) \cdot \mathbf{1}$; iii) The vector of weighted Bonacich centralities with weights vector $\mathbf{w}$ is given by $\mathbf{b}_{\mathbf{w}}(\mathbf{A}, a) = \mathbf{M}(\mathbf{A}, a) \cdot \mathbf{w}$.*

Our analysis of existence of a unique interior equilibrium makes use of Ballester et al. (2006) construction of the normalized $n \times n$ matrix $\mathbf{C}$ (that they call 'local interaction matrix"), whose generic entry $c_{ij} = \frac{\tilde{g}_{ij} + \theta}{\lambda} \in [0, 1]$ is defined in terms of our model by the following parameters: the absolute value θ of the maximal substitutability in $\tilde{\mathbf{G}}$; the maximal complementarity δ in $\tilde{\mathbf{G}}$; and their range $\lambda \equiv \delta + \theta$. Let us also denote by $\mu_1(\mathbf{C})$ the largest eigenvalue of $\mathbf{C}$. Ballester et al. (2006) have shown that if $\mu_1(\mathbf{C}) < \frac{\sigma}{\phi\lambda}$, there exists a unique interior equilibrium which is proportional to Bonacich centralities in $\mathbf{C}$. In the next subsection we use the above construction and result to obtain explicit conditions on the eigenvalues of the network $\mathbf{G}$ for the existence of an interior equilibrium.

3.2 Existence

In preparation of Proposition 1, containing our main existence results, we discuss the relationship between the spectral properties of the matrix $\mathbf{C}$ described above and the adjacency matrix $\mathbf{G}$ of the social network. For a generic square matrix $\mathbf{A}$, let $\boldsymbol{\mu}(\mathbf{A})$ the vector of its eigenvalues. From the definition of $\tilde{\mathbf{G}}$ in (3), for all $i = 1, ..., n$ we can associate the i^{th} eigenvalue of $\mathbf{G}$ with a j^{th} eigenvalue of $\tilde{\mathbf{G}}$. Formally, we define by $I(\mathbf{G})$ and $I(\tilde{\mathbf{G}})$ the set of indexes of the eigenvalues of the two matrices, and by $\rho : I(\mathbf{G}) \to I(\tilde{\mathbf{G}})$ the bijection between the elements of the two sets. With a little abuse of notation we define as $j(i)$ the index $j \in I(\tilde{\mathbf{G}})$ such that $j = \rho(i)$. Then we can write the map ρ as follows:[2]

$$\mu_{j(i)}(\tilde{\mathbf{G}}) = \mu_i(\mathbf{G}) - \frac{\gamma}{\phi}\mu_i^2(\mathbf{G}). \tag{5}$$

Two remarks are in order here. First, the mapping in (5) is non monotonic, and therefore it does not preserve the order of the eigenvalues of $\mathbf{G}$. It follows that, in general, $\mu_1(\mathbf{G})$ is not mapped into $\mu_1(\tilde{\mathbf{G}})$, that is $\rho(1) \neq 1$. Second, the map defined in (5) is strictly concave. Define $i^* \in I(\mathbf{G})$ as the index such that $\rho(i^*) = 1$. That is, $\mu_{i^*}(\mathbf{G})$ is mapped into $\mu_1(\tilde{\mathbf{G}})$. Let us then decompose $\mathbf{C}$ as follows:

$$\mathbf{C} = \frac{1}{\lambda}\tilde{\mathbf{G}} + \frac{\theta}{\lambda}\mathbf{U} \tag{6}$$

where $\mathbf{U}$ is a matrix of ones. It follows that for a vector of shifts $\mathbf{y}$ we can write:

$$\boldsymbol{\mu}(\mathbf{C}) = \frac{1}{\lambda}\boldsymbol{\mu}(\tilde{\mathbf{G}}) + \mathbf{y} \tag{7}$$

Since (7) defines a monotone map (see the proof of Proposition 1), it follows that the eigenvalue $\mu_{i^*}(\mathbf{G})$, is mapped into $\mu_1(\mathbf{C})$ *via* the relations (6) and (7).

In Proposition 1 we use the analysis above to provide a novel result relating existence of an interior equilibrium to conditions on the spectral properties of the network $\mathbf{g}$. These conditions bound the magnitude of the largest eigenvalue $\mu_1(\mathbf{G})$ and reflect the twofold

[2]The result come from the fact that, for a generic square matrix $\mathbf{A}$ and an associated polynomial $q(\mathbf{A})$, the eigenvectors $\boldsymbol{\mu}(q(\mathbf{A})) = q(\boldsymbol{\mu}(\mathbf{A}))$. Note also that, in the analysis to follow, the assumption of symmetry of $\mathbf{G}$ is key since it guarantees that $\mathbf{G}$ is Hermitian (and this property is used in the proof of Proposition 1).

effect of each link in $\mathbf{G}$: to create both one additional channel of peer effects *and* new channels of indirect substitution. Let $\bar{d} \equiv max\{d_i | i \in N\}$ and $\hat{d} \equiv \frac{\sigma}{n\phi\gamma} - \frac{\phi}{4n\phi\gamma^2}$.

Proposition 1 *A unique interior equilibrium exists if one of the following conditions hold:*

i. $\bar{d} < \hat{d}$;

ii. $\mu_1(\mathbf{G}) < \frac{\phi - \sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_1 + \phi^2}}{2\gamma}$;

iii. $\mu_1(\mathbf{G}) > \frac{\phi}{2\gamma}$, *and either* $\mu_{i^*}(\mathbf{G}) < \frac{\phi - \sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_{i^*} + \phi^2}}{2\gamma}$ *or* $\mu_{i^*}(\mathbf{G}) > \frac{\phi + \sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_{i^*} + \phi^2}}{2\gamma}$.

Proposition 1 points to a remarkable role for local congestion: an interior (positive) equilibrium exists when $\mu_1(\mathbf{G})$ is either large or small, while existence may fail for intermediate values of $\mu_1(\mathbf{G})$. This result is in stark contrast with the case of peer effects only (i.e., of $\gamma = 0$), where an interior equilibrium fails to exist for large values of $\mu_1(\mathbf{G})$. To get an intuition for this result, it is useful to consider the relation between the largest eigenvalue and the average degree, a rough measure of network density. Given that $\mathbf{G}$ is symmetric, the Min-Max theorem for Hermitian matrices directly implies that the average degree of $\mathbf{G}$ is a lower bound for the largest eigenvalue $\mu_1(\mathbf{G})$ (see Teschl, 2014, p.117). Point *iii.* in Proposition 1 can be therefore interpreted as an existence result for very dense networks. The role of local congestion is easy to grasp: as density increases, the additional distance-two interaction channels (of the substitute type) have the effect of mitigating the positive impact on behavior of the additional direct complementarity channels. Point *iii.* states that this effect bounds equilibrium actions when density is large enough. The intuition behind Proposition 1 is best illustrated for the class of regular networks in the following example.

Example 1 (Regular Networks) *In regular networks, the average degree d in* $\mathbf{g}$ *coincides with the largest eigenvalue $\mu_1(\mathbf{G})$ and comparative statics directly on d can be performed. The unique (symmetric) interior equilibrium is given by:*[3]

$$x_i^* = \frac{\alpha}{\sigma - \phi d + \gamma d^2} \tag{8}$$

[3]We focus on symmetric equilibria since, when they exist, they are the unique solution of the FOCs. An analysis of asymmetric equilibria could imply the study of the cases where equilibria are not interior, which is beyond the scope of this paper (see Bramoullé et al., 2014).

The above expression is well defined and positive if and only if either $d \leq \frac{\phi - \sqrt{\phi^2 - 4\gamma\sigma}}{2\gamma}$ *or* $d \geq \frac{\phi + \sqrt{\phi^2 - 4\gamma\sigma}}{2\gamma}$*. Note also that the small (large) root is decreasing (increasing) in* ϕ*, capturing the fact that strengthening local congestion allows for equilibrium in a smaller set of networks. Indeed, when peer effects become more intense, either sparser networks (with fewer channels of complementarity) or denser networks (where distance-two channels of interaction grow fast enough compared to the number of direct channels) are needed to recover existence of a positive equilibrium. Also, the small (large) root is increasing (decreasing) in* γ*, capturing the fact that strengthening local congestion allows for equilibrium in a larger set of networks.*

3.3 Characterization

In this section we explore the relation between equilibrium behavior and centrality in the network $\mathbf{g}$. We then build on this relation to discuss the effect of (small degrees of) local congestion on both the levels and the ranking of individual actions at equilibrium. We finally study the effect of changes in the network $\mathbf{g}$ on individual and aggregate behavior.

3.3.1 Equilibrium and Centrality

From system (2), an interior equilibrium solves the following equality:

$$\mathbf{x} = \alpha \left[\sigma \mathbf{I} - \phi \tilde{\mathbf{G}}\right]^{-1} \mathbf{1} \tag{9}$$

Under the sufficient conditions on the eigenvalues of $\mathbf{G}$ highlighted in Proposition 1, system (9) provides the full characterization of the interior equilibrium in terms of the matrix $\mathbf{G}$ and its power matrix $\mathbf{G^2}$.

In the next proposition, we provide a characterization of equilibrium which applies to networks with a small largest eigenvalue. We show that equilibrium is characterized by a weighted version of the Bonacich centrality vector of $\mathbf{G}$, where both the weighting vector and the discount factors take the strength of local congestion into account. To state the proposition, we first need to define the following two scalars[4]

$$a_1 = \frac{\phi + \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}, \; a_2 = \frac{\phi - \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}. \tag{10}$$

[4]We can also invert the order of the two scalars, i.e., $a_1 = \frac{\phi - \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}$ and $a_2 = \frac{\phi + \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}$

Proposition 2 *Let* $\phi > 2\sqrt{\gamma\sigma}$ *and* $\mu_1(\mathbf{G}) < \frac{2\sigma}{\phi+\sqrt{\phi^2-4\gamma\sigma}}$. *Then the unique interior Nash equilibrium of the game is given by:*

$$\mathbf{x} = \frac{\alpha}{\sigma}\mathbf{b}_{\mathbf{b}(\mathbf{G},a_1)}(\mathbf{G}, a_2) < 0 \tag{11}$$

The weighted centrality that in Proposition 2 characterizes equilibrium behavior is "nested", meaning that weights are themselves Bonacich centralities of the network **g**. Two remarks are in order. First, when $\gamma = 0$, the upper bound for $\mu_1(\mathbf{G})$ is $\frac{\sigma}{\phi}$. This means that Proposition 2 applies to all networks where, in the absence of congestion, a unique interior equilibrium exists by results from Ballester et al. (2006). Second, the upper bound is increasing in γ, which implies that an increase in γ enlarges the set of admissible networks. These two observations allow us to use Proposition 2 to measure the effect of introducing small levels of local congestion on equilibrium behavior.

Proposition 3 *The decrease in equilibrium behavior due to the introduction of small levels of local congestion is given by:*

$$\left.\frac{d\mathbf{x}}{d\gamma}\right|_{\gamma=0} = \frac{1}{\phi} \cdot \mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma}) \cdot \left(\mathbf{d} - \mathbf{G} \cdot \mathbf{b}(\mathbf{G}, \frac{\phi}{\sigma})\right) < 0. \tag{12}$$

The introduction of local congestion decreases equilibrium behavior for the following reason. The i^{th} entry of the vector $\mathbf{G} \cdot \mathbf{b}(\mathbf{G}, \frac{\phi}{\sigma})$ measures the sum of Bonacich centralities of all neighbors of i in $\mathbf{G}$. Thus, the i^{th} entry of the vector $\left(\mathbf{d} - \mathbf{G} \cdot \mathbf{b}(\mathbf{G}, \frac{\phi}{\sigma})\right)$ sums up, across all neighbors of i, the difference between 1 and each neighbor's Bonacich centrality. Since Bonacich centralities are strictly larger than 1, this difference is strictly negative for all agents. Proposition 3 also shows how the reduction in behavior is distributed across agents: the reduction is larger for those agents who are 'better connected" to agents whose neighbors are very central in **g** (i.e., for agents for which the matrix $\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma})$ associates a large entry to agents whose neighbors are very central). The intuition is clear: very central agents are characterized by large actions, and those who are closely connected to neighbors of very central agents are exposed to high levels of indirect substitution, and therefore suffer more than others from the introduction of local congestion.

One important issue is whether the implied modification of equilibrium behavior can result in a change in the *ranking* of individual actions. A general answer for arbitrary values of γ is complex due to the strong non linearity of centrality measures. However, we can address this issue in the context of simple network architectures, where central and peripheral agents in $\mathbf{g}$ are clearly identified. In Figure 1 and Table 1 we consider three different networks: the star, the butterfly and the connected star. Consistently with expression (12), the impact of γ is not uniform across agents and across networks (see the last two columns of Table 1). In particular, in the star and the butterfly, agent 1 has the largest behavior (as in the case of $\gamma = 0$), while this is not the case in the connected star, where the ranking of the agents' actions is inverted. Consistently with the intuition behind Proposition 3, the sharp decrease in agent 1's behavior in the connected star is due to his many links towards agents with sufficiently large degree, suffering therefore from consistent congestion levels.

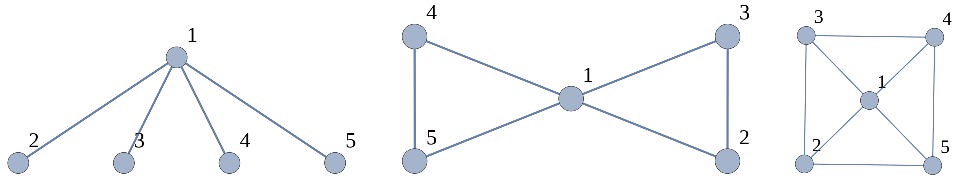

Figure 1: Star, Butterfly and Connected Star Networks

Table 1: Effect of γ on equilibrium actions

Network	**agents**		$\gamma = 0$	$\gamma = 0.51$	$\lvert x_0 - x_{0.51}\rvert$	$\lvert x_0 - x_{0.51}\rvert / x_0$
Star	1	Center	0.2916	0.2275	0.0641	0.2198
	2-5	Periphery	0.2291	0.1850	0.0441	0.1925
Butterfly	1	Center	0.3023	0.1956	0.1067	0.3530
	2-5	Periphery	0.2558	0.1814	0.0744	0.2909
Connected Star	1	Center	0.3157	0.1650	0.1507	0.4774
	2-5	Periphery	0.2894	0.1652	0.1242	0.4292

Parametrization: $\alpha = 2$, $\phi = 1$, $\sigma = 10$, in equation (1). Networks are shown in Figure 1.

3.3.2 Changing the Network

In this section we study the effect of changes in the topology of the network $\mathbf{g}$ on equilibrium behavior. We first focus on the impact of changes in the density of the network. While in the absence of local congestion, adding links unambiguously increases individual and aggregate behavior (see Ballester et al., 2006), the effect is ambiguous in the presence of local congestion, due to the additional substitution channels at distance-two. We address this issue by considering changes in the average degree of regular networks.

In a regular network, the symmetric equilibrium is characterized by equation (8). We can study the impact of an increase in d on equilibrium behavior by considering the sign of the following derivative:

$$\frac{\partial x^*}{\partial d} = \frac{\alpha(\phi - 2\gamma d)}{[\sigma - \phi d + \gamma d^2]^2}. \tag{13}$$

determined by the following regions:

$$\begin{cases} d < \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} > 0 \\ d = \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} = 0 \\ d > \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} < 0 \end{cases}$$

We see that equilibrium behavior and network density are related according to a non-monotonic pattern, with maximal behavior at $d = \frac{\phi}{2\gamma}$. The forces driving the non-monotonic pattern are the following. Distance-two connections (responsible for strategic substitution) grow at the square of the speed of direct connections (channeling the peer effects), and eventually take over, causing a decrease in overall behavior. Outside the class of regular networks, we look at which changes in the topology of a given network would unambiguously decrease (increase) aggregate behavior.

Proposition 4 *Consider the network* $\mathbf{g}'$ *obtained from* $\mathbf{g}$ *by fully connecting an independent set* Z *of nodes of cardinality* $|Z|$ *in* $\mathbf{g}$. *Let* $\mathbf{x}'$ *and* $\mathbf{x}$ *denote the associated equilibrium vectors. If* $|Z| \geq \frac{\phi}{\gamma} + 2$, *then* $\mathbf{x}' \leq \mathbf{x}$.

Proposition 4 shows that a sufficient condition to reduce behavior is the presence of a large enough set of agents who are not connected in $\mathbf{g}$; the number of such agents is inversely related to the intensity of local congestion γ. Behavior is reduced by creating very dense relations among these sparse agents, so that new direct ties come with enough new indirect interaction channels. If the number $|Z|$ of these agents is not high enough with respect to the complementarity/substitutability ratio $\frac{\phi}{\gamma}$, the new connections will create complementarity channels that are not counteracted by a large enough number of indirect substitution channels.

4 Strategic Interaction at Arbitrary Distance

In this section, we extend our framework to encompass strategic interaction at arbitrary distance in the network. Consider a situation where agent i is exposed to the transmission of a disease from his neighbors. Each interaction determines the transmission with some given probability. Agent i can take a (costly) action reducing the probability of transmission. The incentives of agent i to take action depend on the likelihood that his neighbors are infected, which in turns depends on the actions taken at distance-two and the implied risks of contagion. Differently from the set up in Section 2, in this example i's incentives are affected also by actions taken at distances larger than two, through the effect that these actions have on the probability of i's neighbors to be reached by the disease. Within the context of this example, it can be expected that all indirect interaction is of the strategic substitute type, since the larger the action at any distance, the smaller the probability of i's neighbors to be infected.

Yet, one could also envisage problems where the sign of strategic interaction alternates with distance. Consider for instance the following variant of the collaboration networks model sketched in Appendix A. Agents sharing a common collaborator compete for his time and effort; a busier collaborator (i.e, one with very active collaborators) is less attractive and provides weaker peer effects. Differently from the model in Appendix A, assume now that if i and j collaborate, the decrease in the peer effects enjoyed by i is milder when j's collaborators are themselves very busy. In this set-up, interaction extends beyond distance-two; in particular, we expect strategic substitution at distance-two, complementarity at distance-three, substitution again at distance-four, and so on.

To keep the general analysis tractable, we assume that the type of strategic interaction (substitution vs. complementarity) between two agents only depends on their distance in the network, and that all agents at the same distance experience the same kind of strategic interaction. The utility function is written by augmenting (1) with interaction at distance up to $R \geq 2$:

$$U_i = \alpha_i x_i - \frac{\sigma}{2} x_i^2 + \sum_{r=1}^{R} \sum_{j \in N} \phi_r g_{ij}^{[r]} x_i x_j. \tag{14}$$

In (14), $r = 1, \ldots, R$ denotes the distance, $g_{ij}^{[r]}$ is the generic entry of the power matrix $\mathbf{G}^r$ and ϕ_r is the associated coefficient. If $\phi_r > 0$ any two agents at distance r experience strategic complementarity, while if $\phi_r < 0$ they experience strategic substitution, and if

$\phi_r = 0$ there is no strategic interaction.

4.1 Existence

The FOCs for an interior equilibrium are:

$$\alpha \cdot \mathbf{1} = \left[\sigma \mathbf{I} - \phi_1 \tilde{\mathbf{G}}\right] \mathbf{x}, \tag{15}$$

where the adjacency matrix $\tilde{\mathbf{G}}$ of strategic interaction is defined as follows:

$$\tilde{\mathbf{G}} \equiv \mathbf{G} + \sum_{r=2}^{R} \frac{\phi_r}{\phi_1} \mathbf{G}^r. \tag{16}$$

As in (7), we relate the eigenvalues of the matrices $\mathbf{C}$ and $\tilde{\mathbf{G}}$ as follows:

$$\boldsymbol{\mu}(\mathbf{C}) = \frac{1}{\lambda} \boldsymbol{\mu}(\tilde{\mathbf{G}}) + \mathbf{y} \tag{17}$$

and, using the definition of $\tilde{\mathbf{G}}$ and recalling the mapping $\rho : I(\mathbf{G}) \to I(\tilde{\mathbf{G}})$ introduced in Section 3.2, the i^{th} eigenvalue of $\mathbf{C}$ and j^{th} eigenvalue of $\mathbf{G}$ as:

$$\mu_{j(i)}(\mathbf{C}) = \frac{1}{\lambda}[\mu_i(\mathbf{G}) + \sum_{r=2}^{R} \frac{\phi_r}{\phi_1} \mu_i^r(\mathbf{G})] + y_i \tag{18}$$

As in section 3.2, we define $\mu_{i^*}(\mathbf{G})$ as the i^{*th} eigenvalue of $\mathbf{G}$ that is mapped into $\mu_1(\mathbf{C})$. Following the steps of Proposition 1 we can state the following:

Proposition 5 *Consider the problem in (14). A sufficient condition for the existence of a unique interior equilibrium is that,* $\frac{1}{\lambda}[\mu_{i^*}(\mathbf{G}) + \sum_{r=2}^{R} \frac{\phi_r}{\phi_1} \mu_{i^*}^r(\mathbf{G})] + y_{i^*} < \frac{\sigma}{\phi_1 \lambda}$.

In order to derive specific bounds for the eigenvalues one would need to know the polynomial in (17). Yet, we can use Proposition 5 to infer the qualitative properties of these bounds. In particular, we argue that networks with high largest eigenvalues are consistent with an interior equilibrium provided interaction at furthest distance is of the substitute type. To the extent that the largest eigenvalue can be interpreted as an indicator of the density of the network, this result points to the idea that in dense networks the large number of indirect interaction channels of the substitute type can bound the equilibrium feedbacks and allow for an interior solution. Assume, for simplicity, that

$\mu_1(\mathbf{G})$ is mapped into $\mu_1(\tilde{\mathbf{G}})$, that is $\rho(1) = 1$ and $i^* = 1$. Then, as $\mu_1(\mathbf{G})$ grows, the term:

$$\mu_1(\mathbf{C}) = \frac{1}{\lambda}[\mu_1(\mathbf{G}) + \sum_{r=2}^{R} \frac{\phi_r}{\phi_1}\mu_1^r(\mathbf{G})] + y_1 \tag{19}$$

remains bounded above by the term $\frac{\sigma}{\phi\lambda}$ (as required by Proposition 5) if and only if $\phi_R < 0$, that is, if and only if interaction at the largest distance in the network is of the substitute type. This is in line with the results obtained for local congestion. As previously, we illustrate this result in the class of regular networks, where the largest eigenvalue coincides with the average degree.

Example 2 (Regular Networks) *We parametrize regular networks by their common degree d. A symmetric interior equilibrium takes the following form:*

$$x = \frac{\alpha}{1 - \sum_{r=1}^{R} \phi_r d^r}. \tag{20}$$

As the average degree d grows, expression (20) remains positive if and only if $\phi_R < 0$.

4.2 Characterization

We now turn to the characterization of an interior equilibrium. Proposition 6 extends Proposition 2 to the case of interaction up to an arbitrary distance R. Our characterization relies on the novel notion of "Nested Weighted Centrality", introduced below.

Definition 2 (Nested Weighted Bonacich Centrality) *The Nested Weighted Centrality of order s is the vector $\mathbf{b}^{[s]}(\mathbf{G}, a_s)$ defined recursively as follows:*

$$\mathbf{b}^{[1]}(\mathbf{G}, a_1) = \mathbf{b}(\mathbf{G}, a_1) \tag{21}$$

$$\mathbf{b}^{[2]}(\mathbf{G}, a_2) = \mathbf{b}_{\mathbf{b}^{[1]}(\mathbf{G}, a_1)}(\mathbf{G}, a_2) \tag{22}$$

and

$$\mathbf{b}^{[s]}(\mathbf{G}, a_s) = \mathbf{b}_{\mathbf{b}^{[s-1]}(\mathbf{G}, a_{s-1})}(\mathbf{G}, a_s), \ \ for \ s \in \mathbb{N} \tag{23}$$

In preparation of the next Proposition, we introduce the terms S_r for $r = 1, 2, ..., R$. For a given R and a given vector of parameters $\mathbf{a} \equiv (a_i)_{i=1,...,R}$, we denote by $T_r(\mathbf{a})$ the set of all unordered tuples of distinct r elements from the set $A \equiv \{a_i\}_{i=1,\ldots,R}$. A generic element of $T_r(\mathbf{a})$ is called $t_r(\mathbf{a})$. We then define:

$$S_r = (-1)^{r-1} \sum_{t_r(\mathbf{a}) \in T_r(\mathbf{a})} \prod_{a_i \in t_r(\mathbf{a})} a_i$$

So, S_1 denotes the sum of all the parameters of A, S_2 the (negative of the) sum of all products of (unordered) pairs of parameters of A, S_3 the sum of all products of (unordered) triples of parameters of A and so on.

Proposition 6 *Assume there exists a sequence of non-negative scalars $(a_i)_{i=1,\ldots,R}$ such that $S_r = \frac{\phi_k}{\sigma}$, for $k = 1, \ldots, R$. If $\mu_1(\mathbf{G}) \cdot \max\{a_i\}_{i=1,\ldots,R} < 1$, then the unique interior equilibrium satisfying the FOCs (15) can be written as follows:*

$$\mathbf{x} = \frac{\alpha}{\sigma} \mathbf{b}^{[R]}(\mathbf{G}, a_R). \tag{24}$$

The following example ($R = 3$), provides and explicit account of the terms S_k, of the associated constraints and of the nested structure underlying the above characterization.

Example 3 *Let R=3. The first order conditions in (15) take the following form:*

$$\alpha \cdot \mathbf{1} = \sigma[\mathbf{I} - \sum_{i=1}^{3} \frac{\phi_r}{\sigma} \mathbf{G}^r]\mathbf{x}, \tag{25}$$

where

$$[\mathbf{I} - \sum_{i=1}^{3} \frac{\phi_r}{\sigma} \mathbf{G}^r] = [\mathbf{I} - \frac{\phi_1}{\sigma}\mathbf{G} - \frac{\phi_2}{\sigma}\mathbf{G}^2 - \frac{\phi_3}{\sigma}\mathbf{G}^3]. \tag{26}$$

If there exists a vector $(a_1, a_2, a_3) \in \mathbb{R}^3_+$ satisfying the following constraints:

$$\begin{aligned} \tfrac{\phi_1}{\sigma} &= a_1 + a_2 + a_3 = S_1 \\ \tfrac{\phi_2}{\sigma} &= -(a_1 a_2 + a_1 a_3 + a_2 a_3) = S_2 \\ \tfrac{\phi_3}{\sigma} &= a_1 a_2 a_3 = S_3 \end{aligned}$$

then we can write:

$$[\mathbf{I} - \frac{\phi_1}{\sigma}\mathbf{G} - \frac{\phi_2}{\sigma}\mathbf{G}^2 - \frac{\phi_3}{\sigma}\mathbf{G}^3] = [\mathbf{I} - a_1\mathbf{G}][\mathbf{I} - a_2\mathbf{G}][\mathbf{I} - a_3\mathbf{G}]. \tag{27}$$

Using (23) and (27), equilibrium is characterized by:

$$\mathbf{x} = \frac{\alpha}{\sigma}\mathbf{b}^{[3]}(\mathbf{G}, a_3) \tag{28}$$

5 Local Congestion: Empirical Evidence

The aim of this section is to investigate the empirical relevance of local congestion. As an individual action we consider putting effort in doing homework, claiming that it originates both peer effects between friends and substitution effects at distance-two in the friendship network. Substitution at distance-two occurs if students face low incentives to work when their friends are busy doing homework with their own respective friends. The mechanism is the following. A student whose friends are very active in doing homework is expected to experience substantial opportunities of collaboration (peer effects). At the same time, a student may find it difficult to collaborate with 'busy' friends, that is friends surrounded by very active friends. As a consequence, the more active those students that share friends with i, the more likely i is isolated, and the lower i's incentives to do homework. Notice that incentives in this problems are similar to those in problems of scientific collaboration.

To the extent that the above mechanism is in place, we expect to find a positive association at distance-one in the network and a negative one at distance-two. We show that this is indeed the case, and adding interaction at further distance does not increase the explanatory power of the model. The empirical analysis is purely descriptive and aims at showing the existence of a negative correlation at distance-two. Given that the error term is spatially autocorrelated, we follow Bramoullé et al. (2009) who instrument peer effects with the matrix of distance-two neighbors' demographics (and its higher powers). Differently from Bramoullé et al. (2009), we need to instrument both the behavior of friends and that of friends of friends. As a consequence, the only available instruments are demographic characteristics at distance three and further. These instruments are valid because they are correlated both to the friends and distance-two neighbors's behaviors, and they do not directly affect the outcome of interest. However, we stress that while Bramoullé et al. (2009) focus on the identification of peer effect in models without local congestion, a structural estimation of the parameters of our model is beyond the scope of this paper.

We use data from Networks and Actor Attributes in Early Adolescence', a longitudinal survey collected in the Netherlands between 2003 and 2004 (see Knecht, 2004; Corten and

Knecht, 2013). The survey contains information about pupils enrolled in the first year of secondary school. Such pupils have been interviewed four times every three months between the first month after enrolment and the end of the first year. The sample consists of about 120 classes in 14 different schools. We use the last wave of the survey because we believe that the knowledge of distance-two friends (and the local congestion created by them) is crucial in our model and the relationships between the pupils are not well known in the first months of school.

Data contain also an indication of the network pupils are embedded in. We consider the network of best friends (up to 12 nominations), because links are strong, known and often bilateral.[5] To construct an undirected network, with a corresponding symmetric adjacency matrix, we give value 1 to the link between i and j if either i nominated j, or j nominated i (or both). Some descriptive network statistics are included in Table 2, while Figures 2 and 3 represent the network and its properties. The dataset contains a variable that measures whether the pupils always do homework. The variable is categorical and takes value from 1 to 5, where 1 indicates 'very true' and 5 'not true at all'. We reverse the order to have a variable increasing in effort. Descriptive statistics on the sample and the variable of interest are included in Table 3 and Figure 4.

Table 2: Network statistics

	Degree	**Distance-two Degree**	**Component Size**
Mean	5.3049	31.6596	16.1989
Max	15	106	29
Min	0	0	1
Median	5	29	19
Std	2.4286	19.5363	9.4742

'Networks and Actor Attributes in Early Adolescence' (Wave 4). Authors' calculations.

In our regressions we control for a set of demographic characteristics. Individual controls are: a) gender; b) a dummy variable indicating whether or not the language spoken at home is Dutch (to have a proxy of their ethnicity); c) age;[6] d) money pupils

[5]Networks are defined at the classroom level. Other network definitions would be a) classmates respondent receives practical support from; b) classmates respondent receives emotional support from; c) classmates respondent has been friends with at primary school; d) classmates respondent talks about personal things; e) classmates respondent would like to be friends with; f) classmates respondent meets outside school; g) classmates who likes same music as respondent; h) classmates respondent would lend 25 Euro; i) classmates whose opinion is important for respondent

[6]Age is pretty homogeneous given pupils are all enrolled in the first year of secondary school, but some some older pupils are also attending these classes.

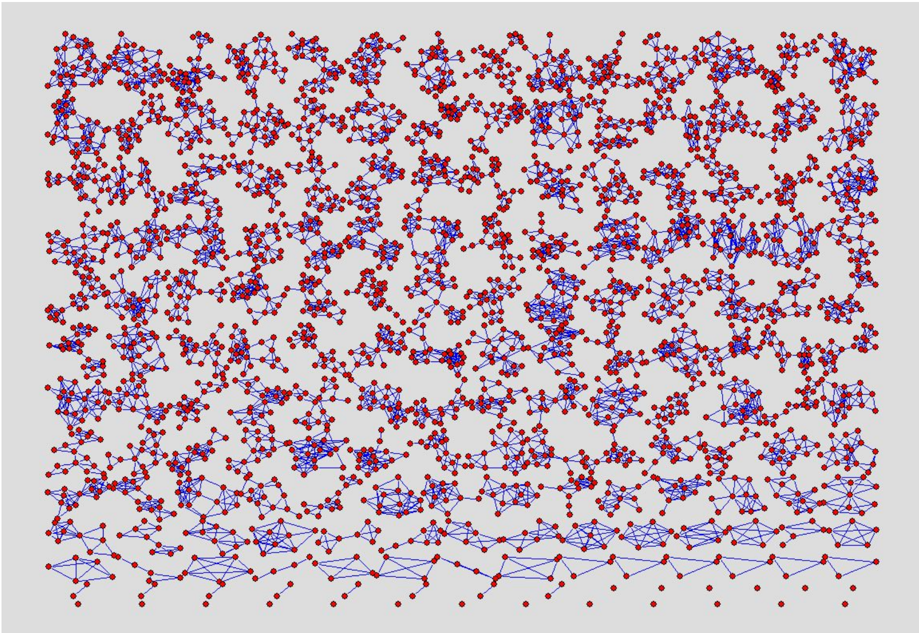

Figure 2: Network Representation. 'Networks and Actor Attributes in Early Adolescence' (Wave 4)

receive from their parents, as a proxy of family income. To control for the influence of network characteristics different from behavior, we include the average of the individual controls at the friends' level. Finally, to capture unobservables at the school and at the class level, we include school dummies (school fixed effects) and the kind of educational track the pupil is enrolled in.[7]

Table 4 displays correlations in behavior. $\mathbf{G}y$, $\mathbf{G}^2y$, and $\mathbf{G}^3y$ are respectively the sum of the action chosen by friends, distance-two friends and distance-three friends, with zeros on the principal diagonal of matrices $\mathbf{G}$, $\mathbf{G}^2$ and $\mathbf{G}^3$. Correlation between the actions is always positive due to the cascade generated by the peer effect. However, correlation between y and $\mathbf{G}y$ (0.083) is much stronger than that between y and both $\mathbf{G}^2y$ (0.040),

[7]The variable indicating different school tracks takes 9 values such as: 1= LWOO; 2=LWOO/ VMBO-Basis and Kaderberoepsgerichte; 3=VMBO-Basis and Kaderberoepsgerichte; 4=VMBO-Basis and Kaderberoepsgerichte / VMBO-theoretisch; 5=VMBO-theoretisch; 6=VMBO-theoretisch / HAVO; 7=HAVO; 8=HAVO / VWO; 9=VWO, where VMBO is 'voorbereidend middelbaar beroepsonderwijs', the middle-level applied education, divided in basic, middle-management (kaderberoepsgerichte) and theoretical (theoretisch). HAVO is 'Hoger algemeen voortgezet onderwijs' (higher general education), and VWO 'voorbereidend wetenschappelijk onderwijs' (preparatory scholarly education). Finally, LWOO is 'Leerwegondersteunend onderwijs' (learning path supporting education) for pupils with special needs.

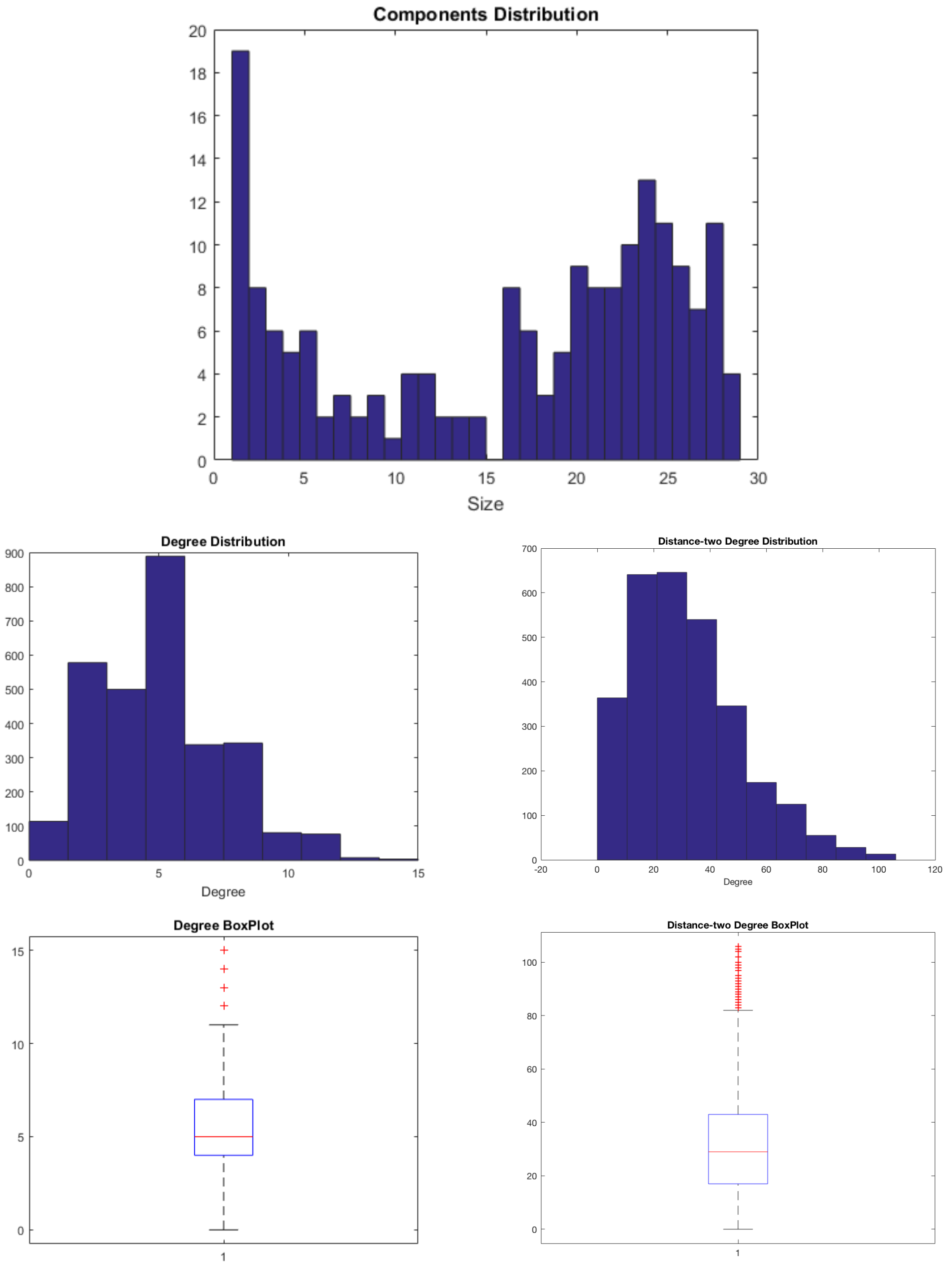

Figure 3: Network Features. 'Networks and Actor Attributes in Early Adolescence' (Wave 4).

Table 3: Descriptive statistics

Variable		Mean/Share	Std. deviation	Min	Max
Doing homework	*Very True*	0.1432		0	1
	True	0.3353		0	1
	Sometimes	0.4317		0	1
	not true	0.0797		0	1
	not true at all	0.0201		0	1
Female		0.4777		0	1
Dutch spoken at home		0.9111		0	1
Type of secondary school	*Track 1*	0.0342		0	1
	Track 2	0.0267		0	1
	Track 3	0.0302		0	1
	Track 4	0.0074		0	1
	Track 5	0.0801		0	1
	Track 6	0.3402		0	1
	Track 7	0.0320		0	1
	Track 8	0.3616		0	1
	Track 9	0.0876		0	1
At least one smoking parent		0.5152		0	1
Euros received from the parents		28.8520	44.7497	0	1000
Age		12.1025	0.4846	10	15
N. of observations	2284				
N. of schools	14				

'Networks and Actor Attributes in Early Adolescence' (Wave 4). Authors' calculations on the estimation sample.

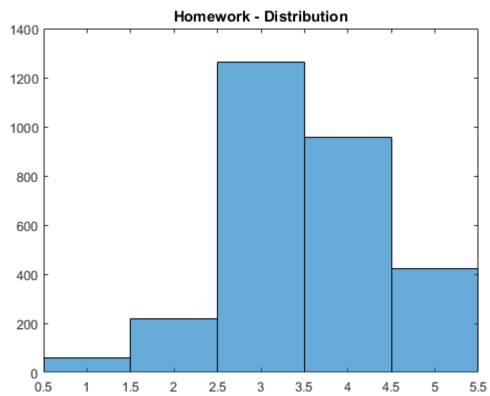

Figure 4: Doing homework: distribution, 'Networks and Actor Attributes in Early Adolescence' (Wave 4)

and $\mathbf{G}^3y$ (0.033). Notice also that $\mathbf{G}^2y$ and $\mathbf{G}^3y$ are almost perfectly correlated (0.984).

Table 4: Correlations in behaviour at distance 1, 2, 3

	y	$\mathbf{G}y$	$\mathbf{G}^2y$	$\mathbf{G}^3y$
y	1.000	0.083	0.040	0.033
$\mathbf{G}y$	0.083	1.000	0.894	0.845
$\mathbf{G}^2y$	0.040	0.894	1.000	0.984
$\mathbf{G}^3y$	0.033	0.845	0.984	1.000
N				2284

'Networks and Actor Attributes in Early Adolescence' (Wave 4). Authors' calculations on the estimation sample.

Table 5 reports the results of a OLS regression.[8] We run 6 specifications: in (1) we regress individual behavior on $\mathbf{G}y$. In (2) and (3) we add, respectively $\mathbf{G}^2y$ and $\mathbf{G}^3y$. In specifications (4)-(6) $\mathbf{G}^3y$ is excluded and controls are included. Consider specifications (1)-(3). In specification (1) the coefficient associated to $\mathbf{G}y$ is positive and highly significant, consistently with the presence of peer effects. In specification (2) $\mathbf{G}y$ and $\mathbf{G}^2y$ are significant at the 1% level. However, while the coefficient of the former is positive, the coefficient of the latter is negative. The introduction of $\mathbf{G}^2y$ makes the coefficient associated to $\mathbf{G}y$ increase, suggesting that in (1) it was downward biased due an omitted variable bias. The sign of the bias is perfectly in line with our theoretical model.[9]

Adding $\mathbf{G}^3y$ does not improve our model (see specification (3)). Indeed $\mathbf{G}^3y$ does not have any explanatory power, since its coefficient is almost negligible and not significant. Notice that in specification (3) $\mathbf{G}^2y$ is negative, but not significant. The result is probably due to the strong multicollinearity between $\mathbf{G}^2y$ and $\mathbf{G}^3y$ (see Table 4) inflating the standard errors, and making the t-test drop. Notice also that the introduction of $\mathbf{G}^3y$ does not make the R^2 increase, again suggesting that the variable does not have any explanatory power. For this reason, our preferred specification does not include $\mathbf{G}^3y$. In specifications (4)-(6) we enriched (2) by adding individual, friends', class and school controls, and the results are pretty stable and similar to those in specification (2).[10]

[8] As a robustness check we also ran all the specifications excluding singletons and using ordered probit. Results are robust and available upon request.

[9] Recall that the omitted variable bias is given by the product of the effect of $\mathbf{G}^2y$ on y (negative because of substitution at distance-two), and the coefficient of a regression on $\mathbf{G}y$ on $\mathbf{G}^2y$ (positive because of complementarity at distance-one)

[10] Regressions in Table 5 do not allow to unravel the direct effect of local congestion from the equilibrium cascade effect, nor to separate the effect of the peer effect/local congestion from the effects of similarities due to network formation. Such tests have been carried out and shown that network formation alone does not explain correlations in behavior. Results are available upon request.

Table 5: Behaviour at distance 1, 2, 3 (OLS)

	(1)		(2)		(3)		(4)		(5)		(6)	
	Coeff.		**Coeff.**		**Coeff.**		**Coeff.**		**Coeff.**		**Coeff.**	
Gy	0.00844	***	0.02395	***	0.02507	***	0.02346	***	0.02317	***	0.02206	***
	(0.04425)		(0.00472)		(0.00525)		(0.00473)		(0.00476)		(0.00477)	
$\mathbf{G}^2y$			-0.00234	***	-0.00333		-0.00225	***	-0.00231	***	-0.00225	***
			(0.00064)		(0.00214)		(0.00064)		(0.00065)		(0.00066)	
$\mathbf{G}^3y$					0.00009							
					(0.00020)							
Const.	3.33264	***	3.2809	***	3.29003	***	3.33857	***	3.70288	***	3.51034	***
	(0.00212)		(0.04631)		(0.04982)		(0.48889)		(0.53131)		(0.56114)	
$\mathbf{R^2}$	0.00687		0.01270		0.01281		0.01674		0.02375		0.04339	
Individual Controls	No		No		No		Yes		Yes		Yes	
Friends' and Class Controls	No		No		No		No		Yes		Yes	
School FE	No		No		No		No		No		Yes	
N	2284		2284		2284		2284		2284		2284	

'Networks and Actor Attributes in Early Adolescence' (Wave 4). Individual controls: female, Dutch spoken at home, quantity of money received from parents, at least one smoking parents. Friend's controls: average of the individual controls within the friends. Class controls: dummy variables indication the kind of secondary school. Standard errors in parentheses, $* = p < 0.1, ** = p < 0.05, *** = p < 0.01$.

Given OLS may not be consistent due to spatial correlation, we also performed an instrumental variable estimation (using two stages least squares, 2SLS) and instrumenting $\mathbf{G}^2 y$ and $\mathbf{G}^3 y$ with the the average demographics of the friends of friends, as well as those of the distance-three and distance-four friends. Our results are in line with those obtained via OLS.[11] Tests on the first stages suggest that instruments may be weak. To overcome this problem, we report the Anderson-Rubin (AR) test statistics for which identification of the coefficients is not assumed.[12] Figure 5 shows the area where the model is not misspecified, i.e. when the peer effect is positive and the effect of distance-two friends is negative, confirming the result of the 2SLS.

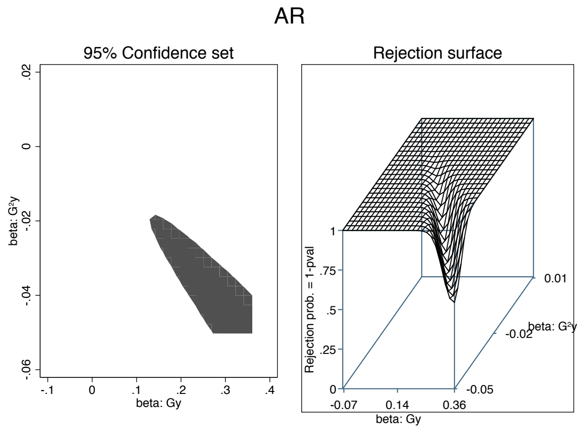

Figure 5: Anderson-Rubin (AR) test statistics for the overidentified IV model.

6 Conclusions

We have studied games where the pattern of agents' interaction is determined by a network of social relations. Along with peer effects between neighbors, the network induces strategic substitution between agents who share one or more neighbors. We have referred to this indirect effect as local congestion. We have looked at the predictions of this class of models, and have compared how individual and aggregate behaviors depart from a model

[11]Results are available upon request.

[12]The value of the under-identification, over-identification, and the F-test on excluded instruments are available under request. The AR test is a joint test on the parameters and on the exogeneity of the instruments and rejects the null when one or both conditions do not hold.

with peer effects only. In particular, we have focused on how equilibrium actions relate to network centrality, and to network density. We have also extended some of our results on local congestion to more general interaction patterns on the network.

We believe our results provide valuable insights on the relation between the topology of social networks and behavior. These insights should be taken into account in designing policies that target these relations. Moreover, our analysis can be used to interpret empirical evidence on the distribution of certain types of behaviors in social networks. For instance, our result concerning the reversal of the ranking of Bonacich centralities (see Proposition 3 and the ensuing example) provides a novel explanation of the prevalence of smoking at the periphery of the network recorded in Christakis and Fowler (2008). This explanation views the gradual marginalization of smokers as an equilibrium phenomenon due to the congestion of central players, rather than a result of changes in the structure of the network in terms of a progressive severance of relational links with heavy smokers.

We see at least two interesting extension of the model we propose. First, equilibria with strong substitution effects and inactive agents may be relevant in many economic problems, and an analysis of how local congestion would affect this class of equilibria would be certainly of interest. Second, an estimation of the structural parameters of the model would be needed to design effective policies in problems when local congestion plays a substantial role.

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References

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in the networks. *Econometrica*, 74(5):1403–1417.

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2010). Delinquent networks. *Journal of the European Economic Association*, 8(1):34–61.

Billand, P., Bravard, C., and Sarangi, S. (2012). Directed networks with spillovers. *Journal of Public Economic Theory*, 14(6):849–878.

Billand, P., Bravard, C., and Sarangi, S. (2013). A note on local spillovers, convexity, and the strategic substitutes property in networks. *Theory and Decision*, 75(2):293–304.

Bloch, F. and Quérou, N. (2013). Pricing in social networks. *Games and Economic Behavior*, 80:243–261.

Bramoullé, Y., Djebbari, H., and Fortin, B. (2009). Identification of peer effects through social networks. *Journal of Econometrics*, vol. 150(1):41–55.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic interaction and networks. *American Economic Review*, 104(3):898–930.

Calvó-Armengol, A. and Jackson, M. O. (2004). The effects of social networks on employment and inequality. *American Economic Review*, 94(3):426–454.

Calvó-Armengol, A., Patacchini, E., and Zenou, Y. (2009). Peer effects and social networks in education. *Review of Economic Studies*, 76(4):1239–1267.

Christakis, N. and Fowler, J. (2007). The spread of obesity in a large social network over 32 years. *The New England Journal of Medicine*, 357:370–9.

Christakis, N. and Fowler, J. (2008). The collective dynamics of smoking in a large social network. *The New England Journal of Medicine*, 358:2249–58.

Clark, E. and Loheac, Y. (2007). "it wasn't me, it was them!" social influence in risky behavior by adolescents. *Journal of Health Economics*, 26:763–784.

Corten, R. and Knecht, A. (2013). Alcohol use among adolescents as a coordination problem in a dynamic network. *Rationality and Society*, 25(2):146–177.

Evans, W., Oates, W., and Schab, R. (1992). Measuring peer group effect: a study of teenage behaviour. *Journal of Political Economy*, 100, No 5:966–991.

Gaviria, A. and Raphael, S. (2001). School-based peer effects and juvenile bahavior. *The Review of Economics and Statistics*, 83(2):257–268.

Glaeser, E., Sacerdote, B., and Scheinkman, J. (2003). The social multiplier. *Journal of the European Economic Association*, 1(2-3):345–353.

Horn, R. A. and Johnson, C. R. (2012). *Matrix analysis*. Cambridge university press.

Jackson, M. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Kirke, D. (2004). Chain reactions in adolescent's cigarette, alcohol and drug use: similarity through peer influence or the patterning of ties in peer networks? *Social Networks*, 26:3–28.

Knecht, A. (2004). Network and actor attributes in early adolescence. DANS. https://doi.org/10.17026/dans-z9b-h2bp.

McCann, B. and Folta, T. (2008). Location matters: Where we have been and where we might go in agglomeration research. *Journal of Management*, vol. 34 no. 3:532–565.

McCann, B. T. and Folta, T. B. (2009). Demand-and supply-side agglomerations: Distinguishing between fundamentally different manifestations of geographic concentration. *Journal of Management Studies*, 46(3):362–392.

Möhlmeier, P., Rusinowska, A., and Tanimura, E. (2016). Competition for the access to and use of information in networks. Documents de travail du Centre d'Economie de la Sorbonne 16033, Universit Panthon-Sorbonne (Paris 1), Centre d'Economie de la Sorbonne.

Morrill, T. (2011). Network formation under negative degree-based externalities. *International Journal of Game Theory*, 40(2):367–385.

Patacchini, E. and Zenou, Y. (2011). Social Networks and Parental Behavior in the Intergenerational Transmission of Religion. IZA Discussion Papers 5787, Institute for the Study of Labor (IZA).

Patacchini, E. and Zenou, Y. (2012). Juvenile delinquency and conformism. *Journal of Law, Economics, and Organization*, 28(1):1–31.

Teschl, G. (2014). *Mathematical methods in quantum mechanics*, volume 157. American Mathematical Soc.

Topa, G. and Zenou, Y. (2014). Neighborhood versus network effects. *Handbook of Regional and Urban Economics*, 4.

Wahba, J. and Zenou, Y. (2005). Density, social networks and job search methods: Theory and application to egypt. *Journal of Development Economics*, 78(2):443–473.

A Micro fundation of Utility

This appendix presents three examples of economic problems with different forms of local congestion producing utility function in (1).

Production networks. Consider a district where a set of monopolistic firms are linked by mutual supply relations. Firm i's product is both demanded by consumers in final market i and used as input by i's neighbors.[13] Each firm i produces according to a Leontief technology with constant returns to scale, transforming the set of employed inputs $Y_i \equiv \{y_j : g_{ij} = 1\}$ into the production level x_i:

$$f_i(Y_i) = \frac{1}{k} min\{y_j \in Y_i\} \tag{A.1}$$

Denoting by p_j the price for commodity j for $j = 1, 2, ..., n$, the marginal cost of each firm i is constant and equal to:

$$c_i = k \sum_{j \in N} g_{ij} p_j \tag{A.2}$$

Demand for commodity i is given by the following function:

$$x_i = A_i + D_i - p_i \tag{A.3}$$

[13] For simplicity, we are assuming that links are undirected, so that if a firm provides an input to another firm, also the latter provides an input to the former.

where A_i is the size of i's consumers' market, and D_i is the demand for input i coming from i's neighbors. From the Leontief technology specification, it follows that:

$$D_i = k \sum_{j \in N} g_{ij} x_j. \tag{A.4}$$

Each firm maximizes its profit as a monopolist:

$$\pi_i(\mathbf{x}) = (A_i + D_i - x_i - c_i)\, x_i = (A_i - x_i + k \sum_{j \in N} g_{ij} x_j - k \sum_{j \in N} g_{ij} p_j) x_i \tag{A.5}$$

Substituting the expression of each price p_j from the appropriate demand function $p_j = A_j - x_j + \sum_k g_{jk} x_k$, we obtain:

$$\pi_i(\mathbf{x}) = (A_i - k \sum_{j \in N} g_{ij} A_j) x_i - x_i^2 + 2k \sum_{j \in N} g_{ij} x_i x_j - k^2 \sum_{j \in N} \sum_{k \in N} g_{ij} g_{jk} x_i x_k \tag{A.6}$$

which can be written as (1) once we set $\alpha_i = (A_i - k \sum_j g_{ij} A_j)$, $\sigma = 2$, $\phi = 2k$ and $\gamma = k^2$. Note how firm i's production is increasing in i's neighbors' production (strategic complementarities) and linearly decreasing in the production of firms that share a common input provider with i (substitution at distance-two in the network), as in McCann and Folta (2008, 2009).

Scientific collaborations. The network $\mathbf{g}$ describes the pattern of collaborations between scientists. The action x_i measures the degree of research activity of scientist i. Collaborations are governed by complementarities, so that the larger the action of i's co-authors, the larger i's incentive to act. However, i's co-authors compete for the limited research effort of their co-authors, so the degree of complementarity decreases with the effort exerted at distance-two. We model the utility in the following linear quadratic form:

$$U_i(\mathbf{x}) = \alpha_i x_i - \frac{\sigma}{2} x_i^2 + \phi x_i \sum_{j \in N} g_{ij} [x_j - \gamma_1 \sum_{k \in N} g_{jk} x_k] \tag{A.7}$$

The parameter γ_1 measures the impact of competing projects in which a given co-author is involved on the benefits drawn from an ongoing project with that co-author. This expression is equivalent to (1) once we set $\gamma = \gamma_1 \phi$.

Local negative externalities. Consider a set of agents whose actions produce local negative externalities that accumulate in stocks (e.g, transfrontier pollution). The stock at an agent's location is given by the sum of her neighbors' actions. Each agent suffers a convex damage, which depends on her own stock and on a fraction of her neighbors' stocks. Possible examples of such situations are environmental games where the pollutant that accumulates from neighbors emissions leaks into neighboring locations. Alternative interpretations include social interaction problems where individual behavior has detrimental effects on friends and/or relatives (smoking, delinquency, skipping school), and where one's perceived damage is affected by the observation of friends' and relatives' conditions. For example, a smoker's awareness of his health risks may increase when a friend falls ill, which happens with higher probability the larger the amount of secondhand smoke this friend is exposed to. To model such situations, let for each i

$$Q_i \equiv \left(x_i + \sum_{k \in N} g_{ik} x_k \right) \tag{A.8}$$

denote the stock of pollutant that generates from local emissions in the neighborhood of i, which leaks into $i's$ neighborhood. Assuming quadratic damage, we get the following utility function:

$$U_i(\mathbf{x}) = \alpha_i x_i + \theta \sum_{j \in N} g_{ij} x_i x_j - \frac{(Q_i + \gamma_1 \sum_j g_{ij} Q_j)^2}{2} \tag{A.9}$$

The parameter γ_1 measures the amount of leakage between neighboring locations. Expanding the squared terms, we obtain the following expression:

$$\alpha_i x_i - \frac{x_i^2}{2} + (\theta - 1 - 3\gamma_1) \sum_{j \in N} g_{ij} x_i x_j - \gamma_1 x_i \sum_{j \in N} \sum_{k \in N} g_{ij} g_{jk} x_k + h(x_{-i}) \tag{A.10}$$

where the term $h(x_{-i})$ does not depend on x_i and thus does not affect optimal choices. Apart from the term $h(x_{-i})$, (A.10) can be rewritten as (1) by setting $\sigma = 1$, $\phi = (\theta - 1 - 3\gamma_1)$ and $\gamma = \gamma_1$.

B Proofs

Proof of Proposition 1. To prove the proposition we first state and prove the following lemma.

Lemma 1 *Let y_i^* the shift associated to $\mu_{i^*}(\mathbf{G})$ in (7). If either*

$$\mu_{i^*}(\mathbf{G}) < \frac{\phi - \sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_i^* + \phi^2}}{2\gamma}$$

or

$$\mu_{i^*}(\mathbf{G}) > \frac{\phi + \sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_i^* + \phi^2}}{2\gamma},$$

then a unique interior equilibrium exists.

Proof of Lemma 1.

Proof. Consider equation (5). Consider also $\boldsymbol{\mu}(\mathbf{C})$ and $\boldsymbol{\mu}(\tilde{\mathbf{G}})$. Recall that for all $i < n$ $\mu_i(\mathbf{C}) \geq \mu_{i+1}(\mathbf{C})$ and $\mu_i(\tilde{\mathbf{G}}) \geq \mu_{i+1}(\tilde{\mathbf{G}})$. We first prove that (7) defines a monotone map with the following property:

$$\mu_1(\mathbf{C}) \geq \mu_1(\tilde{\mathbf{G}}) \geq \mu_2(\mathbf{C}) \geq \mu_2(\tilde{\mathbf{G}}) \cdots \geq \mu_n(\mathbf{C}) \geq \mu_n(\tilde{\mathbf{G}}) \tag{B.1}$$

Given $\mathbf{C}$ and $\tilde{\mathbf{G}}$ are both Hermitian with n eigenvalues, by Weyl inequality[14] $\mu_i(\mathbf{C}) \leq \mu_{i-j}(\tilde{\mathbf{G}}) + \mu_{1+j}(\mathbf{U})$, for all $i = 1, \ldots, n$ and $j = 0, \ldots, n-1$. Recall also $\mathbf{U}$ is such that $\mu_1(\mathbf{U}) = n$ and $\mu_i(\mathbf{U}) = 0$ for all $i = 2, \ldots, n$, and consider the inequality above for $j \in \{0, 1\}$. If $j = 0$ then $\mu_i(\mathbf{C}) \leq \mu_i(\tilde{\mathbf{G}}) + \mu_1(\mathbf{U})$; if $j = 1$ then $\mu_i(\mathbf{C}) \leq \mu_{i-1}(\tilde{\mathbf{G}})$. It follows that $\mu_{i-j}(\tilde{\mathbf{G}}) \leq \mu_i(\mathbf{C}) \leq \mu_i(\tilde{\mathbf{G}}) + \mu_1(\mathbf{U})$. This implies that the map (7) transforming the eigenvalues of $\tilde{\mathbf{G}}$ into the eigenvalues of $\mathbf{C}$ preserves the ordering.

We are now able to derive the condition for existence. Recall that a sufficient condition for the existence of an interior equilibrium is $\mu_1(\mathbf{C}) < \frac{\sigma}{\phi\lambda}$. This condition, using (7), can be written as $\frac{1}{\lambda}\mu_1(\tilde{\mathbf{G}}) + y_1 < \frac{\sigma}{\phi\lambda}$. Consider the eigenvalue $\mu_{i^*}(\mathbf{G})$ which maps into $\mu_1(\tilde{\mathbf{G}})$ in (5), i.e., $1 = \rho(i^*)$. Using (5), we can rewrite the sufficient condition for the existence of an interior equilibrium as follows:

$$\frac{1}{\lambda}\mu_{i^*}(\mathbf{G}) - \frac{\gamma}{\lambda\phi}[\mu_{i^*}(\mathbf{G})]^2 + y_{i^*} < \frac{\sigma}{\phi\lambda} \tag{B.2}$$

[14] See, for example, Horn and Johnson (2012), p.241.

By noticing that $-\frac{\gamma}{\lambda\phi} < 0$ and solving for $\mu_{i^*}(\mathbf{G})$ the result immediately follows.

We can now prove Proposition 1.

Recall (6) and consider y_i, for all $i = 1, \ldots, n$. By the Wielandt-Hoffman theorem[15] $\mid \mu_i(\mathbf{C}) - \frac{1}{\lambda}\mu_i(\tilde{\mathbf{G}}) \mid = y_i \leq \frac{\theta n}{\lambda}$, being $n = \mu_1(\mathbf{U})$. Given $y_i \geq 0$ for each i, the following holds:

$$\frac{\phi\lambda}{\sigma}\mu_i(\mathbf{C}) = \frac{1}{\sigma}(\phi\mu_i(\mathbf{G}) - \gamma\mu_i^2(\mathbf{G}) + \phi\lambda y_i) \leq \frac{1}{\sigma}(\phi\mu_i(\mathbf{G}) - \gamma\mu_i^2(\mathbf{G}) + \phi\theta n) < 1$$

We recall that from the definition of $\tilde{\mathbf{G}}$ it follows that $\theta = \frac{\gamma}{\phi}\bar{d}$. Then the last inequality becomes

$$\frac{1}{\sigma}(\phi\mu_i(\mathbf{G}) - \gamma\mu_i^2(\mathbf{G}) + \gamma\bar{d}n) < 1. \tag{B.3}$$

If the roots of the polynomial on the left hand side are complex, the inequality is satisfied for all $\mu_i(\mathbf{G})$, which drives to the condition i) in the proposition.

To prove condition ii), consider now the case in which $\bar{d} > \widehat{d}$. Assume $\mu_1(\mathbf{G}) < \frac{\phi}{2\gamma}$, so the LHS of (B.3) is monotonically increasing in $\mu_i(\mathbf{G})$, and consequently $\mu_i(\mathbf{G}) < \frac{\phi}{2\gamma}$ for all $i = 2, \ldots, n$. Notice that $\frac{1}{\lambda}\mu_1(\mathbf{G}) - \frac{\gamma}{\lambda\phi}[\mu_1(\mathbf{G})]^2$ is strictly monotone and increasing in $\mu_1(\mathbf{G})$, once its domain is restricted to the interval $(-\infty, \frac{\phi}{2\gamma}]$. This monotonicity implies $i^* = 1$ so that $\mu_{i^*}(\mathbf{G}) = \mu_1(\mathbf{G})$. Then the existence conditions can be written just in terms of $\mu_1(\mathbf{G})$. Moreover, being $\bar{d} > \widehat{d} \equiv \frac{\sigma}{n\phi\gamma} - \frac{\phi}{4n\phi\gamma^2}$, then $\sqrt{-4\gamma\sigma + 4\gamma\lambda\phi y_1 + \phi^2}$ is real-valued, and $\frac{\phi - \sqrt{-4\gamma\sigma+4\gamma\lambda\phi y_1+\phi^2}}{2\gamma} < \frac{\phi}{2\gamma} < \frac{\phi + \sqrt{-4\gamma\sigma+4\gamma\lambda\phi y_1+\phi^2}}{2\gamma}$. Being $\mu_1(\mathbf{G}) < \frac{\phi}{2\gamma}$ only the first part of the inequality is binding and the result immediately follows by applying Proposition 1.

Consider now condition iii). This immediately follows from the fact that if $\mu_1(\mathbf{G}) > \frac{\phi}{2\gamma}$ then it is generally true that $\mu_1(\mathbf{G}) \neq \mu_{i^*}(\mathbf{G})$. Then we just need to apply Lemma 1 to the correct $\mu_{i^*}(\mathbf{G})$. ■

Proof of Proposition 2.

Proof. FOCs in (2) yield

$$\frac{\alpha}{\sigma} \cdot \mathbf{1} = \left[\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G} + \frac{\gamma}{\sigma}\mathbf{G^2}\right]\mathbf{x}. \tag{B.4}$$

Consider now the symmetric matrix

[15] See, for example, Horn and Johnson (2012), p.40.

$$\left[\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G} + \frac{\gamma}{\sigma}\mathbf{G^2}\right]. \tag{B.5}$$

If two scalars a_1 and a_2 exist that solve the following system:

$$a_1 + a_2 = \frac{\phi}{\sigma} \tag{B.6}$$

$$a_1 a_2 = \frac{\gamma}{\sigma} \tag{B.7}$$

then (B.5) can be written as:

$$\left[\mathbf{I} - (a_1 + a_2)\mathbf{G} + a_1 a_2 \mathbf{G}^2\right] = [\mathbf{I} - a_1\mathbf{G}] \cdot [\mathbf{I} - a_2\mathbf{G}] \tag{B.8}$$

Solving the constraints in (B.6) and (B.7) we get two pairs (a_1, a_2):

$$a_1 = \frac{\phi \pm \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma} \tag{B.9}$$

$$a_2 = \frac{\phi \mp \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma} \tag{B.10}$$

well defined if and only if $\phi > 2\sqrt{\gamma\sigma}$.

We can now rewrite (B.4) as follows:

$$\frac{\alpha}{\sigma} \cdot \mathbf{1} = [\mathbf{I} - a_1\mathbf{G}] \cdot [\mathbf{I} - a_2\mathbf{G}]\,\mathbf{x}. \tag{B.11}$$

If $\mu_1(\mathbf{G}) < \frac{1}{max\{a_1, a_2\}}$, then both inverses $[\mathbf{I} - a_1\mathbf{G}]^{-1}$ and $[\mathbf{I} - a_2\mathbf{G}]^{-1}$ are well defined, and (B.11) yields:

$$\mathbf{x} = \frac{\alpha}{\sigma}\,[\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot [\mathbf{I} - a_1\mathbf{G}]^{-1} \cdot \mathbf{1} \tag{B.12}$$

Since by definition of Bonacich centrality:

$$[\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot \mathbf{1} = \mathbf{b}(\mathbf{G}, a_2) \tag{B.13}$$

we can then apply the definition of weighted Bonacich centrality and write:

$$\mathbf{x} = \frac{\alpha}{\sigma}\mathbf{b}_{\mathbf{b}(\mathbf{G},a_1)}(\mathbf{G}, a_2) \tag{B.14}$$

■

Proof of Proposition 3. Proof. We study the derivative of (11) with respect to γ at the point $\gamma = 0$. Note first that when $\gamma = 0$ then $a_1 = \frac{\phi}{\gamma}$ and $a_2 = 0$. Immediate computations also give the following expressions when $\gamma = 0$:

$$\mathbf{b}(\mathbf{G}, a_2)\Big|_{\gamma=0} = \mathbf{1}; \quad \frac{\partial a_1}{\partial \gamma}\Big|_{\gamma=0} = -\frac{1}{\phi}; \quad \frac{\partial a_2}{\partial \gamma}\Big|_{\gamma=0} = \frac{1}{\phi}; \quad \frac{\partial \mathbf{b}(\mathbf{G}, a_2)}{\partial a_2}\Big|_{\gamma=0} = \mathbf{d}, \tag{B.15}$$

where in the last expression $\mathbf{d}$ denotes the vector of degrees in $\mathbf{G}$.

We can then write the total derivative of the equilibrium actions' vector with respect to γ as follows:

$$\frac{d\mathbf{x}}{d\gamma}\Big|_{\gamma=0} = \frac{\partial \mathbf{M}(\mathbf{G}, a_1)}{\partial \gamma}\Big|_{\gamma=0} \cdot \mathbf{b}(\mathbf{G}, a_2) + \mathbf{M}(\mathbf{G}, a_1) \cdot \frac{\partial \mathbf{b}(\mathbf{G}, a_2)}{\partial \gamma}\Big|_{\gamma=0} \tag{B.16}$$

Replacing terms from (B.15) we obtain:

$$\frac{d\mathbf{x}}{d\gamma}\Big|_{\gamma=0} = \frac{\partial \mathbf{M}(\mathbf{G}, a_1)}{\partial a_1}(-\frac{1}{\phi})\Big|_{\gamma=0} \cdot \mathbf{1} + \mathbf{M}(\mathbf{G}, a_1) \cdot \frac{\mathbf{d}}{\phi} \tag{B.17}$$

The term $\frac{\partial \mathbf{M}(\mathbf{G}, a_1)}{\partial a_1}$ is computed by using the expression of the matrix $\mathbf{M}(\mathbf{G}, a_1)$:

$$\frac{\partial \mathbf{M}(\mathbf{G}, a_1)}{\partial a_1} = \frac{\partial}{\partial a_1}[\mathbf{I} - a_1\mathbf{G}]^{-1} = -[\mathbf{I} - a\mathbf{G}]^{-1} \cdot \frac{\partial[\mathbf{I} - a\mathbf{G}]}{\partial a_1} \cdot [\mathbf{I} - a\mathbf{G}]^{-1} \tag{B.18}$$

or, equivalently,

$$-[\mathbf{I} - a_1\mathbf{G}]^{-1} \cdot -\mathbf{G} \cdot [\mathbf{I} - a_1\mathbf{G}]^{-1} \tag{B.19}$$

Substituting back in (B.17) and factorizing terms we obtain:

$$\frac{d\mathbf{x}}{d\gamma}\Big|_{\gamma=0} = \frac{\mathbf{1}}{\phi} \cdot \mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma}) \cdot \left(\mathbf{d} - \mathbf{G} \cdot \mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma}) \cdot \mathbf{1}\right) \tag{B.20}$$

or, using the definition of Bonacich centrality vector,

$$\frac{d\mathbf{x}}{d\gamma}\Big|_{\gamma=0} = \frac{\mathbf{1}}{\phi} \cdot \mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma}) \cdot \left(\mathbf{d} - \mathbf{G} \cdot \mathbf{b}(\mathbf{G}, \frac{\phi}{\sigma})\right) \tag{B.21}$$

■

Proof of Proposition 4.

Proof. Consider first a node $k \notin Z$ such that $g_{kz} = 0$ for all $z \in Z$. We have $\tilde{g}_{ki} = \tilde{g}'_{ki}$ for all $i \in N$. Consider then a node $k \notin Z$ such that $g_{ki} = 1$ for at least one $i \in Z$. We have that $\tilde{g}'_{ki} < \tilde{g}_{ki}$ and $\tilde{g}'_{kz} \leq \tilde{g}_{kz}$ for all $z \in Z$. Consider now any two nodes $i, j \in Z$, for which, by construction, $g'_{ij} - g_{ij} = 1$. We also have $g'^{[2]}_{ij} - g^{[2]}_{ij} = |Z| - 2$, since all nodes in Z are now linked with each other. Thus $\tilde{g}'_{ij} - \tilde{g}_{ij} = 1 - \frac{\rho\gamma}{\phi} \leq 0$ since we have assumed that $\phi \leq (|Z| - 2)\gamma$. Thus, $\tilde{g}'_{ij} \leq \tilde{g}_{ij}$ for all $i, j \in Z$ with at least one strict inequality.[16] ■

Proof of Proposition 6. Proof. Consider the FOCs we report below here

$$\alpha \cdot \mathbf{1} = \left[\sigma \mathbf{I} - \phi_1 \tilde{\mathbf{G}}\right] \mathbf{x}, \tag{B.22}$$

where the adjacency matrix $\tilde{\mathbf{G}}$ of strategic interaction is defined as follows:

$$\tilde{\mathbf{G}} \equiv \mathbf{G} + \sum_{r=2}^{R} \frac{\phi_r}{\phi_1} \mathbf{G}^{\mathbf{r}}. \tag{B.23}$$

Then we can write

$$\left[\sigma \mathbf{I} - \phi_1 \tilde{\mathbf{G}}\right] = \prod_{r=1}^{R} [\mathbf{I} - a_r \mathbf{G}^r] \tag{B.24}$$

provided there exists a set $A = \{a_r\}_{r=1,\dots,R}$ satisfying it. Moreover notice that (B.24) holds independently of the order of the product since matrices are symmetric. By induction it can be shown that the set A is such that $\frac{\phi_1}{\sigma}$ is the sum of all a_r, $\frac{\phi_2}{\sigma}$ is the sum of the double products of elements in A times -1, $\frac{\phi_3}{\sigma}$ is the sum of the triple products, $\frac{\phi_4}{\sigma}$ is the sum of the quadruple products times -1, and so on. Formally we get $\frac{\phi_r}{\sigma} = S_r$.
■

[16] Our result builds on Theorem 2 in Ballester et al. (2006), showing that increasing all entries in the network of social interactions unambiguously increases equilibrium behavior of all agents.