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Abstract
This paper computes the non central moments of the Truncated Normal variable, that is, a Normal $u \sim N(\mu, \sigma^2)$ constrained to assume values in the interval $K = (k_1, k_2)$, with $-\infty \le k_1 < k_2 \le \infty$ and $k_1 < \mu < k_2$.
We define two recursive expressions where one can be expressed in closed form. Another closed form is defined using the Lower Incomplete Gamma Function.
Moreover, an upper bound for the absolute value of the noncentral moments is determined.
The numerical results of the expressions are compared and the different behavior for high value of the order of the moments is shown.

Keywords: truncated normal variable, non central moments, lower incomplete gamma function

JEL Codes: G11, G14, G24

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Non Central Moments of the Truncated Normal Variable

Fausto Corradin[(*)], Domenico Sartore[(**)]

ABSTRACT: This paper computes the non central moments of the Truncated Normal variable, that is, a Normal $u \sim N(\mu, \sigma^2)$ constrained to assume values in the interval $K = (k_1, k_2)$, with $-\infty \leq k_1 < k_2 \leq \infty$ and $k_1 < \mu < k_2$.

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1. Introduction

This paper shows a set of expressions for the Non central Moments (NcM) of a Truncated Normal variable. The choice of a particular expression depends on the different contexts for which those moments can be used.
In general, a truncated random variable refers to a random variable with reduced range from $(-\infty, \infty)$ to the interval (k_1, k_2), with $\infty \leq k_1 < k_2 \leq \infty$.
In this line, we choose a Normal $u \sim N(\mu, \sigma^2)$, constrained to assume only values in the interval $K = (k_1, k_2)$, with $-\infty \leq k_1 < k_2 \leq \infty$ and $k_1 < \mu < k_2$.
An upper bound for the absolute value of the NcM is determined.
We found 4 expressions for NcM, absolutely equivalent from a mathematical point of view; but, due to the fact that some of them have recursive expressions, they can diverge for high orders of the NcM for the limited precision of computer numbers.

The paper is organized as follows.
In Section 2 we find:
The upper bound (2.5) for the absolute value of the NcM.
A First Recursive Expression, (2.6), which is a second order difference equation, non-autonomous and non-homogenous, which is a generalization of Dhrymes 2005.
A Second Recursive Expression, (2.8), which is also a second order difference equation, non-autonomous and non-homogenous, which is a generalization of Burkardt 2014.

The closed form (2.9), which is a solution of the (2.8) equation.
The non recursive closed form (2.10), which is an expression based on the Lower Incomplete Gamma Function.

Section 3 reports the comparison between the expressions and sets out some examples where the recursive expression can diverge instead of providing finite results.

2. Computation of Non central Moments

Consider u as a Normal $u \sim N(\mu, \sigma^2)$ constrained to assume values in the interval $K = (k_1, k_2)$, with $-\infty \leq k_1 < k_2 \leq \infty$ and $k_1 < \mu < k_2$.
With the notations:

$$\phi(\xi) = \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}, \qquad \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\tau^2}{2}} d\tau$$

(2.1)

$$h_2 = \frac{k_2 - \mu}{\sigma}, \qquad h_1 = \frac{k_1 - \mu}{\sigma}, \qquad \Delta\Phi_K = \Phi(h_2) - \Phi(h_1)$$

where $\Delta\Phi_K$ is the probability that $u \subset K$. We define the Probability Density Function for the Truncated Normal *(TN)* distribution, $f_{TN}(u)$:

(2.2) $$f_{TN}(u) = \begin{cases} \dfrac{\phi\left(\frac{u-\mu}{\sigma}\right)}{\sigma\Delta\Phi_K} = \dfrac{e^{-(u-\mu)^2/2\sigma^2}}{\int_{k_1}^{k_2} e^{-(u-\mu)^2/2\sigma^2}\,du} & u \in K \\ 0 & u \notin K \end{cases}$$

The non central moments have the expression:

(2.3) $$E[\,u^n|\; k_1 < u < k_2\,] \equiv \mu_{TN,n} = \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2} \xi^n \;\phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi$$

Obviously, we have:

$\mu_{TN,0} = 1$

and it is possible to compute:

$$\mu_{TN,1} = \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2}\xi\;\phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi = \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2}\xi\,\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}}\;d\xi$$
$$= \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2}(\xi-\mu+\mu)\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}}\;d\xi = \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2}\frac{(\xi-\mu)}{\sigma}\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}}\;d\xi + \mu$$

Putting $\tau = \frac{(\xi-\mu)}{\sigma}$, using (2.1) and considering that $\frac{d\phi(\tau)}{d\tau} = \;-\tau\phi(\tau)$ we obtain:

$$\mu_{TN,1} = \frac{\sigma}{\Delta\Phi_K}\int_{h_1}^{h_2}\tau\frac{e^{-\tau^2/2}}{\sqrt{2\pi}}\;d\tau + \mu = \frac{\sigma}{\Delta\Phi_K}\int_{h_1}^{h_2}\tau\frac{e^{-\tau^2/2}}{\sqrt{2\pi}}\;d\tau + \mu$$
$$= -\frac{\sigma}{\Delta\Phi_K}\int_{h_1}^{h_2}\frac{d\phi(\tau)}{d\tau}\;d\tau + \mu$$

that is:

(2.4) $$\mu_{TN,1} = \mu + \frac{\sigma\phi(h_1)}{\Delta\Phi_K} - \frac{\sigma\phi(h_2)}{\Delta\Phi_K}$$

Proposition 2.1: *The non central moments are bounded in absolute value by the expression:*

(2.5) $$\left|\mu_{TN,n}\right| \le \frac{\left\{\dfrac{{k_2}^{2n+1} + (-k_1)^{2n+1}}{2n+1}\right\}^{1/2}\left\{\Delta\Phi_{K\sqrt{2}}\right\}^{1/2}}{\sqrt{2}\sigma\sqrt[4]{\pi}\Delta\Phi_K}$$

where:

$$\Delta\Phi_K = \;\Phi(h_2) - \;\Phi(h_1), \qquad \Delta\Phi_{K\sqrt{2}} = \;\Phi\big(\sqrt{2}h_2\big) - \Phi\big(\sqrt{2}h_1\big)$$

In the particular case $-1 < k_1 < 0 < k_2 < 1$, the bounds of the non central moments' absolute values are decreasing if n increases.

Proof: From the (2.3):

$$\mu_n \leq |\mu_n| \leq \frac{1}{\sigma\Delta\Phi_H}\left|\int_{k_1}^{k_2} \xi^n \ \phi\left(\frac{\xi-\mu}{\sigma}\right) d\xi\right|$$

and using the Schwarz's inequality we have:

$$\frac{1}{\sigma\Delta\Phi_H}\left|\int_{k_1}^{k_2} \xi^n \ \phi\left(\frac{\xi-\mu}{\sigma}\right) d\xi\right| \leq \frac{1}{\Delta\Phi_H}\left\{\int_{k_1}^{k_2} |\xi^n|^2 \, d\xi\right\}^{1/2} \left\{\int_{k_1}^{k_2} \left|\frac{\phi\left(\frac{\xi-\mu}{\sigma}\right)}{\sigma}\right|^2 \ \phi d\xi\right\}^{1/2}$$

The calculation of the integrals on the right-hand side of the inequality gives:

$$\left\{\int_{k_1}^{k_2} |\xi^n|^2 \, d\xi\right\}^{1/2} = \left\{\int_0^{k_2} \xi^{2n} \, d\xi + \int_0^{-k_1} \xi^{2n} \, d\xi\right\}^{1/2} = \left\{\frac{{k_2}^{2n+1} + (-k_1)^{2n+1}}{2n+1}\right\}^{1/2}$$

and

$$\left\{\int_{k_1}^{k_2} \left|\frac{\phi\left(\frac{\xi-\mu}{\sigma}\right)}{\sigma}\right|^2 \ d\xi\right\}^{1/2} = \left\{\int_{k_1}^{k_2} \frac{e^{-2(\xi-\mu)^2/2\sigma^2}}{2\pi\sigma^2} \ d\xi\right\}^{1/2}$$

This last integral can be solved by substitution with the change of variable $t = \sqrt{2}(\xi-\mu)/\sigma$. We have:

$$\left\{\int_{k_1}^{k_2} \frac{e^{-2(\xi-\mu)^2/2\sigma^2}}{2\pi\sigma^2} \ d\xi\right\}^{1/2} = \left\{\int_{\sqrt{2}(k_1-\mu)/\sigma}^{\sqrt{2}(k_2-\mu)/\sigma} \frac{e^{-t^2/2}}{2\sqrt{2}\pi\sigma} \ dt\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{\Phi\left(\frac{\sqrt{2}(k_2-\mu)}{\sigma}\right) - \Phi\left(\frac{\sqrt{2}(k_1-\mu)}{\sigma}\right)}{2\sigma\sqrt{\pi}}\right\}^{1/2}$$

$$= \left\{\frac{\Delta\Phi_{K\sqrt{2}}}{2\sigma\sqrt{\pi}}\right\}^{1/2}$$

with $\Delta\Phi_{K\sqrt{2}} = \Phi\left(\sqrt{2}(k_2-\mu)/\sigma\right) - \Phi\left(\sqrt{2}(k_1-\mu)/\sigma\right) = \Phi\left(\sqrt{2}h_2\right) - \Phi\left(\sqrt{2}h_1\right)$ according with the notation in (2.1).
In conclusion:

$$\mu_{TN,n} \le |\mu_{TN,n}| \le \frac{\left\{\frac{{k_2}^{2n+1} + (-k_1)^{2n+1}}{2n+1}\right\}^{1/2} \left\{\Delta\Phi_{K\sqrt{2}}\right\}^{1/2}}{\sqrt{2\sigma}\sqrt[4]{\pi}\Delta\Phi_K}$$

In the particular case $-1 < k_1 < 0 < k_2 < 1$:,

$${k_2}^{2n+1} > 0 \textit{ and decreasing when n increases}$$

$$(-k_1)^{2n+1} > 0 \textit{ and decreasing when n increases}$$

$$2n+1 \textit{ increases when n increases}$$

$$\left\{\Delta\Phi_{K\sqrt{2}}\right\}^{1/2}, \sqrt{2\sigma}\sqrt[4]{\pi}\Delta\Phi_K \textit{ are not dependent by n}$$

Therefore, the bounds of the non central moments' absolute values are decreasing when n increases. □

Proposition 2.2: *A first recursive expression for the non central moments is:*

(2.6) $$\mu_{TN,n} = \Delta\rho_{b(n-1)} + \mu\mu_{TN,n-1} + (n-1)\sigma^2\mu_{TN,n-2}, \qquad n \ge 2$$

where

$$\rho_1 = \frac{\phi(h_1)}{\Delta\Phi_K};\ \rho_2 = \frac{\phi(h_2)}{\Delta\Phi_K};\ \Delta\rho_{b(n)} = \sigma(\rho_1 k_1^n - \rho_2 k_2^n).$$

Proof: From the expression (2.3):

$$\mu_{TN,n} = \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2} \xi^n\ \phi\left(\frac{\xi-\mu}{\sigma}\right) d\xi = \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^n\ \frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} d\xi$$

$$= \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-1}\xi\ \frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} d\xi$$

we derive the exponential term, obtaining:

$$\frac{d}{d\xi}\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} = \frac{-2(\xi-\mu)}{2\sigma^2}\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} = -\frac{\xi}{\sigma^2}\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} + \frac{\mu}{\sigma^2}\frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

$$\rightarrow \xi \frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \mu \frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} - \sigma^2 \frac{d}{d\xi}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

If we substitute this result in the expression of $\mu_{TN,n}$, we have:

$$\mu_{TN,n} = \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-1}\xi \, \frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\xi = \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-1}\left[\mu \frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} - \sigma^2 \frac{d}{d\xi}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma}\right] d\xi$$

With the integration by parts:

$$\frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-1}\left[-\sigma^2 \frac{d}{d\xi}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma}\right] d\xi$$
$$= \frac{-\sigma^2}{\Delta\Phi_K}\left[\xi^{n-1}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma}\right]_{k_1}^{k_2} + \frac{(n-1)\sigma^2}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-2}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\xi$$

we have:

$$\mu_{TN,n} = \frac{\sigma^2}{\Delta\Phi_K} k_1^{n-1}\frac{e^{-(k_1-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} - \frac{\sigma^2}{\Delta\Phi_K} k_2^{n-1}\frac{e^{-(k_2-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} + \frac{\mu}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-1}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\xi$$
$$+ \frac{(n-1)\sigma^2}{\Delta\Phi_K}\int_{k_1}^{k_2} \xi^{n-2}\frac{e^{-(\xi-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\xi$$

$$\mu_{TN,n} = \frac{\sigma^2}{\Delta\Phi_K} k_1^{n-1}\frac{e^{-(k_1-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} - \frac{\sigma^2}{\Delta\Phi_K} k_2^{n-1}\frac{e^{-(k_2-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} + \mu\mu_{n-1} + (n-1)\sigma^2\mu_{n-2}$$

Using the following definitions:

$$\rho_1 = \frac{\sigma}{\Delta\Phi_K}\frac{e^{-(k_1-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{\phi(h_1)}{\Delta\Phi_K}; \qquad \rho_2 = \frac{\sigma}{\Delta\Phi_K}\frac{e^{-(k_2-\mu)^2/_{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{\phi(h_2)}{\Delta\Phi_K};$$

(2.7)

$$\Delta\rho_{b(n)} = \sigma(\rho_1 k_1^n - \rho_2 k_2^n)$$

we can write:

$$\mu_{TN,n} = \Delta\rho_{b(n-1)} + \mu\mu_{TN,n-1} + (n-1)\sigma^2\mu_{TN,n-2} \qquad n \geq 2$$

which is a second order difference equation, non-autonomous and non-homogenous.

Due to the order of the equation, we need to know $\mu_{TN,0}$ and $\mu_{TN,1}$. Since for $n = 1$ the last term disappears, the relation can be considered true also for $n \geq 1$.
To make this recursive representation fully useful, we need to define the initial condition.

We know that: $\mu_{TN,0} = 1$

Using the definitions (2.7), for n = 1 the (2.6) leads again to the expression (2.4):

$$\mu_{TN,1} = \mu + \Delta\rho_{b(0)} = \mu + \sigma(\rho_1 - \rho_2)$$

The first iteration provides:

$$\mu_{TN,2} = \Delta\rho_{b(1)} + \mu\Delta\rho_{b(0)} + \mu^2 + (2-1)\sigma^2 = \Delta\rho_{b(1)} + \mu(\mu + \sigma\rho_1 - \sigma\rho_2) + \sigma^2$$

$$\sigma_{TN}^2 = \mu_{TN,2} - \left(\mu_{TN,1}\right)^2 = \Delta\rho_{b(1)} + \mu\Delta\rho_{b(0)} + \mu^2 + \sigma^2 - \left(\mu + \Delta\rho_{b(0)}\right)^2$$

$$= \Delta\rho_{b(1)} + \mu\Delta\rho_{b(0)} + \mu^2 + \sigma^2 - \mu^2 - 2\mu\Delta\rho_{b(0)} - \left(\Delta\rho_{b(0)}\right)^2$$

$$= \sigma^2 + \Delta\rho_{b(1)} - \mu\Delta\rho_{b(0)} - \left(\Delta\rho_{b(0)}\right)^2$$

Proposition 2.3: The *second recursive expression for the non central moments is:*

$$(2.8) \qquad \mu_{TN,n} = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k I_k, \quad n \geq 0$$

where:

$$\rho_1 = \frac{\phi(h_1)}{\Delta\Phi_K}; \qquad \rho_2 = \frac{\phi(h_2)}{\Delta\Phi_K}; \qquad \Delta\rho_k = \rho_1 h_1^k - \rho_2 h_2^k;$$

$$I_0 = 1; \qquad I_1 = \Delta\rho_0; \qquad I_k = \Delta\rho_{k-1} + (k-1)I_{k-2}$$

The closed form for the non central moments is:

$$(2.9) \quad \mu_{TN,n} = \mu^n + \mu^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\sigma}{\mu}\right)^k (k-1)!! \left\{ \sum_{r=0}^{\underline{(k-1)/2}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{[1+(-1)^k]}{2} \right\}, \quad n \geq 1$$

where $\underline{(k-1)/2}$ is the whole number preceding the value $(k-1)/2$.

Proof: From the expression (2.3):

$$\mu_{TN,n} = \frac{1}{\sigma\Delta\Phi_K} \int_{k_1}^{k_2} \xi^n \, \phi\left(\frac{\xi - \mu}{\sigma}\right) d\xi = \frac{1}{\Delta\Phi_K} \int_{k_1}^{k_2} \xi^n \, \frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} d\xi$$

we define:

$$\tau = \frac{\xi - \mu}{\sigma}; \qquad I_k = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^k \ \phi(\tau) d\tau$$

If we substitute in (2.3) the following term:

$$(\sigma\tau + \mu)^n = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k \tau^k$$

we obtain:

$$\mu_{TN,n} = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k I_k$$

We can use the following result:

$$\frac{d\phi(\tau)}{d\tau} = -\tau\phi(\tau)$$

for the integration by parts of I_k:

$$I_k = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^k \ \phi(\tau) d\tau = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^{k-1} \left\{\frac{d}{d\tau} [-\phi(\tau)]\right\} d\tau$$

$$= \frac{1}{\Delta\Phi_H}\left\{[-\tau^{k-1}\phi(\tau)]_{h_1}^{h_2} + (k-1)\int_{h_1}^{h_2} \tau^{k-2} \ \phi(\tau) d\tau\right\}$$

$$= -h_2^{k-1}\frac{\phi(h_2)}{\Delta\Phi_H} + h_1^{k-1}\frac{\phi(h_1)}{\Delta\Phi_H} + (k-1) I_{k-2}$$

With the notation:

$$\rho_1 = \frac{\phi(h_1)}{\Delta\Phi_H}; \qquad \rho_2 = \frac{\phi(h_2)}{\Delta\Phi_H}; \qquad \Delta\rho_k = \rho_1 h_1^k - \rho_2 h_2^k$$

the previous expression can be simplified in:

$$I_k = \Delta\rho_{k-1} + (k-1) I_{k-2}$$

which is a non-autonomous non-homogeneous difference equation of second order.
The solution can be found recursively given the initial condition for I_0 and I_1. That is:

$$I_0 \; = \frac{1}{\Delta\Phi_K}\int_{h_1}^{h_2} \phi(\tau)\, d\tau \quad = 1$$

$$I_1 \; = \frac{1}{\Delta\Phi_K}\int_{h_1}^{h_2} \tau\; \phi(\tau) d\tau = \; -\frac{\phi(h_2)}{\Delta\Phi_K} + \frac{\phi(h_1)}{\Delta\Phi_K} = \; \Delta\rho_0$$

Alternatively, we can find the closed form solution for every I_k as function of the solely $\Delta\rho_k$. For this purpose, we consider the behavior of I_k for higher values of k , by following its recursive formula, that is:

$$I_2 \; = \Delta\rho_1 \; + (2-1)I_0 \quad = \; \Delta\rho_1 + 1$$

$$I_3 \; = \Delta\rho_2 \; + (3-1)I_1 \quad = \; \Delta\rho_2 + 2\Delta\rho_0$$

$$I_4 \; = \Delta\rho_3 \; + (4-1)I_2 \quad = \; \Delta\rho_3 + 3[\Delta\rho_1 + 1]$$

$$I_5 \; = \Delta\rho_4 \; + (5-1)I_3 \quad = \; \Delta\rho_4 + 4\Delta\rho_2 + 4\cdot 2\Delta\rho_0$$

$$I_6 \; = \Delta\rho_5 \; + (6-1)I_4 \quad = \; \Delta\rho_5 + 5\Delta\rho_3 + 5\cdot 3[\Delta\rho_1 + 1]$$

$$I_7 \; = \Delta\rho_6 \; + (7-1)I_5 \quad = \; \Delta\rho_6 + 6\Delta\rho_4 + 6\cdot 4\Delta\rho_2 + 6\cdot 4\cdot 2\Delta\rho_0$$

$$I_8 \; = \Delta\rho_7 \; + (8-1)I_6 \quad = \; \Delta\rho_7 + 7\Delta\rho_5 + 7\cdot 5\Delta\rho_5 + 7\cdot 5\cdot 3[\Delta\rho_1 + 1]$$

$$I_9 \; = \Delta\rho_8 \; + (9-1)I_7 \quad = \; \Delta\rho_8 + 8\Delta\rho_6 + 8\cdot 6\Delta\rho_4 + 8\cdot 6\cdot 4\Delta\rho_2 + 8\cdot 6\cdot 4\cdot 2\Delta\rho_0$$

$$I_{10} = \Delta\rho_9 \; + (10-1)I_8 \quad = \; \Delta\rho_9 + 9\Delta\rho_7 + 9\cdot 7\Delta\rho_5 + 9\cdot 7\cdot 5\Delta\rho_5 + 9\cdot 7\cdot 5\cdot 3[\Delta\rho_1 + 1]$$

and considering $0!! = 1$ and $1!! = 1$ we have:

$$I_0 \; = 1$$

$$I_1 \; = \; \Delta\rho_0$$

$$I_2 \; = \; \Delta\rho_1 + 1$$

$$I_3 \; = \; \Delta\rho_2 + \frac{2!!}{0!!}\Delta\rho_0 = \Delta\rho_2 + 2\Delta\rho_0$$

$$I_4 \; = \; \Delta\rho_3 + \frac{3!!}{1!!}\Delta\rho_1 + 3!! = \Delta\rho_3 + 3[\Delta\rho_1 + 1]$$

$$I_5 \; = \Delta\rho_4 \; + \frac{4!!}{2!!}\Delta\rho_2 + \frac{4!!}{0!!}\Delta\rho_0 = \; \Delta\rho_4 + 4\Delta\rho_2 + 4\cdot 2\Delta\rho_0$$

$$I_6 \; = \Delta\rho_5 \; + \frac{5!!}{3!!}\Delta\rho_3 + \frac{5!!}{1!!}\Delta\rho_1 + 5!! \; = \; \Delta\rho_5 + 5\Delta\rho_3 + 5\cdot 3[\Delta\rho_1 + 1]$$

$$I_7 = \Delta\rho_6 + \frac{6!!}{4!!}\Delta\rho_4 + \frac{6!!}{2!!}\Delta\rho_2 + \frac{6!!}{0!!}\Delta\rho_0 = \Delta\rho_6 + 6\Delta\rho_4 + 6\cdot 4\Delta\rho_2 + 6\cdot 4\cdot 2\Delta\rho_0$$

$$I_8 = \Delta\rho_7 + \frac{7!!}{5!!}\Delta\rho_5 + \frac{7!!}{3!!}\Delta\rho_3 + \frac{7!!}{1!!}\Delta\rho_1 + 7!! = \Delta\rho_7 + 7\Delta\rho_5 + 7\cdot 5\Delta\rho_3 + 7\cdot 5\cdot 3[\Delta\rho_1 + 1]$$

$$I_9 = \Delta\rho_8 + \frac{8!!}{6!!}\Delta\rho_6 + \frac{8!!}{4!!}\Delta\rho_4 + \frac{8!!}{2!!}\Delta\rho_2 + \frac{8!!}{0!!}\Delta\rho_0$$

$$= \Delta\rho_8 + 8\Delta\rho_6 + 8\cdot 6\Delta\rho_4 + 8\cdot 6\cdot 4\Delta\rho_2 + 8\cdot 6\cdot 4\cdot 2\Delta\rho_0$$

$$I_{10} = \Delta\rho_9 + \frac{9!!}{7!!}\Delta\rho_7 + \frac{9!!}{5!!}\Delta\rho_5 + \frac{9!!}{3!!}\Delta\rho_3 + \frac{9!!}{1!!}\Delta\rho_1 + 9!!$$

$$= \Delta\rho_9 + 9\Delta\rho_7 + 9\cdot 7\Delta\rho_5 + 9\cdot 7\cdot 5\Delta\rho_3 + 9\cdot 7\cdot 5\cdot 3[\Delta\rho_1 + 1]$$

therefore, for $k \geq 1$:

$$I_k = \sum_{r=0}^{\underline{(k-1)/2}} \Delta\rho_{k-1-2r}\frac{(k-1)!!}{(k-1-2r)!!} + \frac{[1+(-1)^k]}{2}(k-1)!!$$

$$= (k-1)!!\left\{\sum_{r=0}^{\underline{(k-1)/2}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{[1+(-1)^k]}{2}\right\}$$

where $\underline{(k-1)/2}$ is the whole number preceding the value $(k-1)/2$.

In conclusion, we have:

$$\mu_{TN,n} = \mu^n + \sum_{k=1}^{n}\binom{n}{k}\mu^{n-k}\sigma^k(k-1)!!\left\{\sum_{r=0}^{\underline{(k-1)/2}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{[1+(-1)^k]}{2}\right\}$$

$$\mu_{TN,n} = \mu^n + \mu^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k(k-1)!!\left\{\sum_{r=0}^{\underline{(k-1)/2}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{[1+(-1)^k]}{2}\right\}$$

Proposition 2.5: *The closed form with the Lower Incomplete Gamma Function of the non central moments is:*

(2.10) $$\mu_{TN,n} = \mu^n + \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_K}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2}\left[(-1)^k\gamma\left(v,\frac{(h_1)^2}{2}\right)+\gamma\left(v,\frac{(h_2)^2}{2}\right)\right], n \geq 1$$

where:

$$\gamma(v,x) = \int_0^x t^{v-1}\, e^{-t} dt \qquad \textit{is the Lower Incomplete Gamma Function}$$

Proof: From the expression (2.3):

$$\mu_{TN,n} = \frac{1}{\sigma\Delta\Phi_K}\int_{k_1}^{k_2}\xi^n\ \phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi = \frac{1}{\Delta\Phi_K}\int_{k_1}^{k_2}\xi^n\ \frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}d\xi$$

we define the following function:

$$I_k = \frac{1}{\Delta\Phi_K}\int_{h_1}^{h_2}\tau^k\ \phi(\tau)d\tau$$

so that:

$$\mu_{TN,n} = \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k\ I_k$$

and with the definition of

$$G_k = \int_{h_1}^{h_2}\tau^k e^{-\tau^2/2}\ d\tau$$

we have:

$$\mu_{TN,n} = \frac{1}{\sqrt{2\pi}\Delta\Phi_H}\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k\ G_k$$

From the constraints $-\infty \leq k_1 < k_2 \leq \infty$ and $k_1 < \mu < k_2$ follows that $h_1 < 0$, $h_2 > 0$.

If $k > 0$ is even, then:

$$\int_{h_1}^{h_2}\tau^k e^{-\tau^2/2}\ d\tau = \int_0^{-h_1}\tau^k e^{-\tau^2/2}\ d\tau + \int_0^{h_2}\tau^k e^{-\tau^2/2}\ d\tau$$

If $k > 0$ is odd, then:

$$\int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} \; d\tau = -\int_0^{-h_1} \tau^k e^{-\tau^2/2} \; d\tau + \int_0^{h_2} \tau^k e^{-\tau^2/2} \; d\tau$$

Therefore:

$$\int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} \; d\tau = (-1)^k \int_0^{-h_1} \tau^k e^{-\tau^2/2} \; d\tau + \int_0^{h_2} \tau^k e^{-\tau^2/2} \; d\tau$$

We can use the definition of the Lower Incomplete Gamma Function:

$$\gamma(\upsilon, x) = \int_0^x t^{\upsilon-1} \, e^{-t} dt$$

to obtain the following relation:

$$\int_0^x \tau^k e^{-\beta\tau^p} \; d\tau = \frac{\gamma(\upsilon, \beta x^p)}{p\beta^\upsilon}\,, \qquad \upsilon = \frac{k+1}{p}, \;\; \beta > 0\,, p > 0$$

Now we can write the expression for the moments as follows:

$$(2.11) \qquad \mu_{TN,n} = \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_K} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sigma}{\mu}\right)^k \; 2^{(k-1)/2} \left[(-1)^k \gamma\left(\upsilon, \frac{(h_1)^2}{2}\right) + \gamma\left(\upsilon, \frac{(h_2)^2}{2}\right)\right]$$

If $k = 0$, the Gamma functions inside the square brackets becomes:

$$= \left[\gamma\left(\frac{1}{2}, \frac{(h_1)^2}{2}\right) + \gamma\left(\frac{1}{2}, \frac{(h_2)^2}{2}\right)\right]$$

$$= \int_0^{\frac{(h_1)^2}{2}} t^{-1/2} \, e^{-t} dt + \int_0^{\frac{(h_2)^2}{2}} t^{-1/2} \, e^{-t} dt$$

with the substitution: $t = z^2/2$ we have:

$$= \int_0^{-h_1} \frac{\sqrt{2}}{z} e^{-z^2/2} z dz + \int_0^{h_2} \frac{\sqrt{2}}{z} e^{-z^2/2} z dz$$

$$= \int_{h_1}^{h_2} \sqrt{2} e^{-z^2/2} \; dz = \sqrt{2}\sqrt{2\pi}\Delta\Phi_H$$

and taking in consideration that $\binom{0}{0} = 1$ the (2.11) for $n = 0$ becomes:

$$\mu_{TN,0} = \frac{\mu^0}{\sqrt{2\pi}\Delta\Phi_K}\sum_{k=0}^{0}\binom{0}{0}\left(\frac{\sigma}{\mu}\right)^0 \ 2^{-1/2}\left[\sqrt{2}\sqrt{2\pi}\Delta\Phi_H\right] = 1$$

This means that (2.11) is true for every $n \geq 0$.

We can rewrite the expression as:

$$\mu_{TN,n} = \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H}\left\{\sqrt{2\pi}\Delta\Phi_H + \sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k \ 2^{(k-1)/2}\left[(-1)^k\gamma\left(\upsilon,\frac{(h_1)^2}{2}\right) + \gamma\left(\upsilon,\frac{(h_2)^2}{2}\right)\right]\right\}$$

$$= \mu^n + \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k \ 2^{(k-1)/2}\left[(-1)^k\gamma\left(\upsilon,\frac{(h_1)^2}{2}\right) + \gamma\left(\upsilon,\frac{(h_2)^2}{2}\right)\right], n \geq 1$$

3. Simulations and Conclusions

As a measure of the accuracy we define the following expressions:

$$\Delta\,(2.6) = \big((2.6) - (2.3)\big)$$
$$\Delta\,(2.8) = \big((2.8) - (2.3)\big)$$
$$\Delta\,(2.9) = \big((2.9) - (2.3)\big)$$
$$\Delta\,(2.10) = \big((2.10) - (2.3)\big)$$

where Δ (2.6) means that Delta, referred to the formula (2.6), is equal to the result of the formula (2.6) minus the result of the formula (2.3). Formula (2.3) is chosen as a benchmark because it is characterised by direct computation.
Formulas (2.3), (2.6), (2.8), (2.9), (2.10) give theoretically the same results.
The expressions (2.6), (2.8), (2.9) contain a factorial term; this means that they are sensible to the accuracy of floating-point of the computer. The imprecision created by the truncation may be so strong that its result multiplied by a factorial can diverge even if the formulas should not allow it.

When we compute the NcM for the case $-1 < k_1 < k_2 < 1$ we have seen in the *Proposition 2.1* that the NcM are decreasing.
For the simulations we chose the value $k_1 = -0.9$, $k_2 = 0.9$, $\mu = 0.3$, $\sigma = 0.9$. In what follows is possible to see graphs and tables of the NcM behaviors.

Figure 3.1: NcM until 15'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

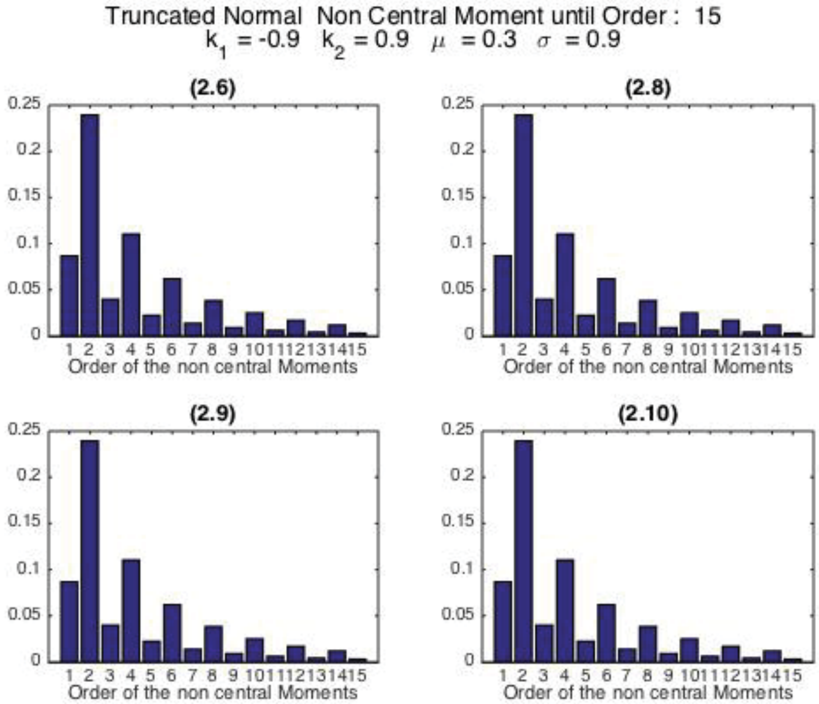

Table 3.1: NcM until 15'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(2.6)	0,0868	0,2394	0,0398	0,1103	0,0223	0,0620	0,0138	0,0383	0,0090	0,0251	0,0061	0,0170	0,0043	0,0119	0,0030
(2.8)	0,0868	0,2394	0,0398	0,1103	0,0223	0,0620	0,0138	0,0383	0,0090	0,0251	0,0061	0,0170	0,0043	0,0119	0,0030
(2.9)	0,0868	0,2394	0,0398	0,1103	0,0223	0,0620	0,0138	0,0383	0,0090	0,0251	0,0061	0,0170	0,0043	0,0119	0,0030
(2.10)	0,0868	0,2394	0,0398	0,1103	0,0223	0,0620	0,0138	0,0383	0,0090	0,0251	0,0061	0,0170	0,0043	0,0119	0,0030

The four results seems equal but we can note in the next figures that $\Delta(2.10)$ is lower than all the others Δ. Indeed:

Figure 3.2: Δ until 15'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

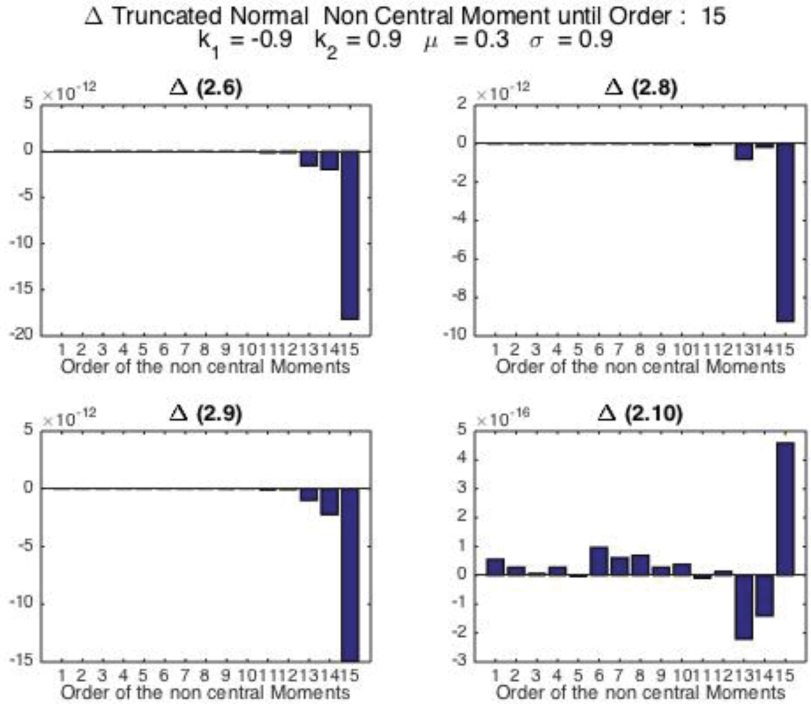

Table 3.2: Δ until 15'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Δ(2.6)	-8,E-17	3,E-17	-2,E-16	6,E-17	-6,E-16	7,E-18	-3,E-15	-7,E-16	-2,E-14	-1,E-14	-2,E-13	-1,E-13	-2,E-12	-2,E-12	-2,E-11
Δ(2.8)	0,E+00	0,E+00	-8,E-17	6,E-17	-4,E-16	2,E-16	-2,E-15	8,E-16	-1,E-14	3,E-15	-8,E-14	5,E-15	-8,E-13	-2,E-13	-9,E-12
Δ(2.9)	0,E+00	0,E+00	-8,E-17	6,E-17	-5,E-16	-1,E-16	-2,E-15	-1,E-15	-2,E-14	-5,E-15	-9,E-14	-5,E-14	-1,E-12	-2,E-12	-1,E-11
Δ(2.10)	6,E-17	3,E-17	7,E-18	3,E-17	-3,E-18	1,E-16	6,E-17	7,E-17	3,E-17	4,E-17	-1,E-17	1,E-17	-2,E-16	-1,E-16	5,E-16

We can see that Δ(2.10) is $\approx 10^{-16}$, whereas for the other Δ the values are $\approx 10^{-11}$.

The accuracy becomes worse, increasing the order of the NcM:

Figure 3.3: NcM until 30'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

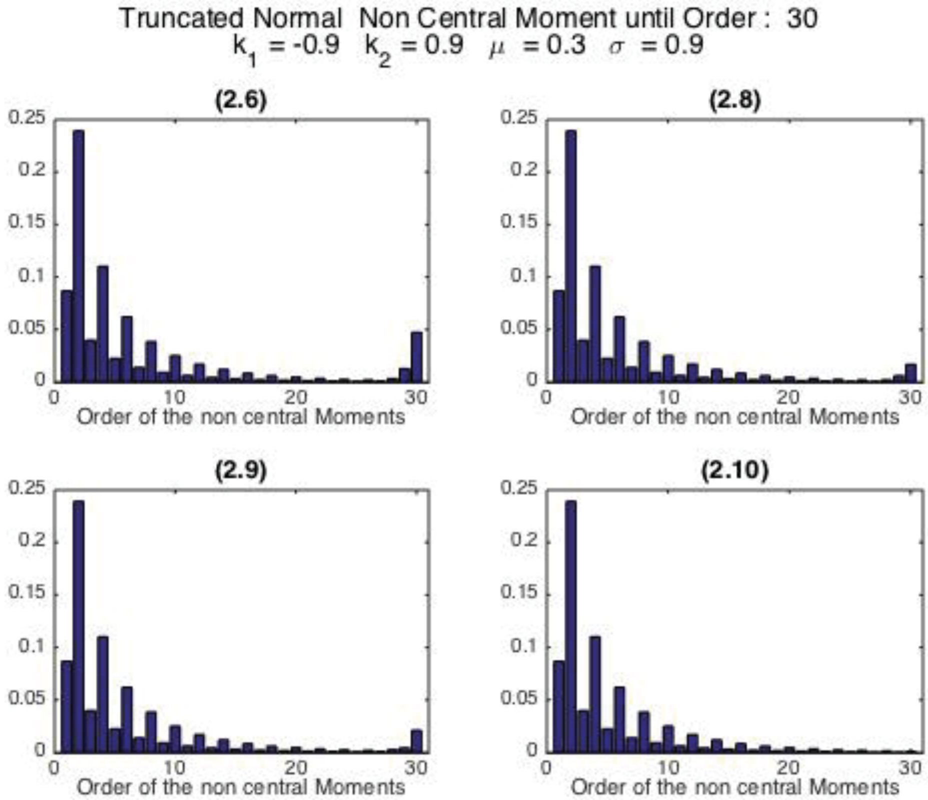

Table 3.3: NcM from 16'th to 30'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
(2.6)	0,0084	0,0022	0,0061	0,0016	0,0044	0,0012	0,0033	0,0009	0,0024	0,0007	0,0019	0,0010	0,0032	0,0127	0,0471
(2.8)	0,0084	0,0022	0,0061	0,0016	0,0044	0,0012	0,0033	0,0009	0,0024	0,0007	0,0018	0,0007	0,0020	0,0059	0,0166
(2.9)	0,0084	0,0022	0,0061	0,0016	0,0044	0,0012	0,0033	0,0009	0,0024	0,0007	0,0019	0,0010	0,0026	0,0044	0,0210
(2.10)	0,0084	0,0022	0,0061	0,0016	0,0044	0,0012	0,0033	0,0009	0,0024	0,0007	0,0018	0,0005	0,0014	0,0004	0,0010

We can observe that for the 26'th order we start to have a different values.
Δ reach higher values than the previous ones:

Figure 3.4: Δ until 30'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

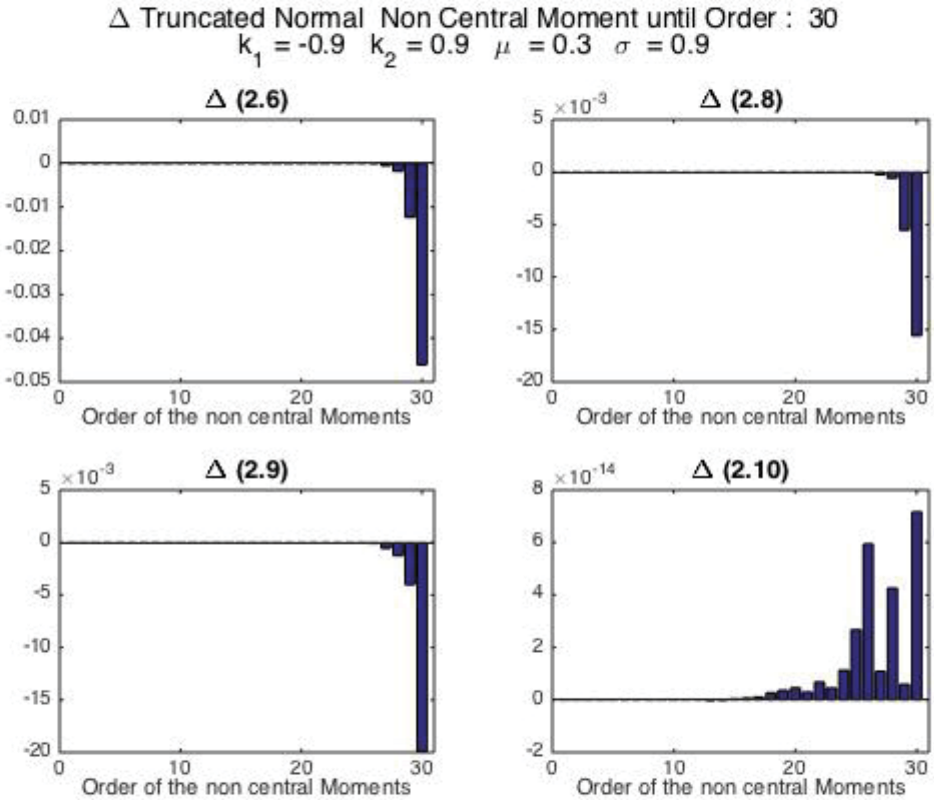

Table 3.4: Δ from 16'th to 30'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Δ(2.6)	-3,E-11	-2,E-10	-5,E-10	-4,E-09	-8,E-09	-6,E-08	-2,E-07	-1,E-06	-3,E-06	-2,E-05	-8,E-05	-5,E-04	-2,E-03	-1,E-02	-5,E-02
Δ(2.8)	-5,E-12	-1,E-10	-1,E-10	-2,E-09	-2,E-09	-3,E-08	-5,E-08	-5,E-07	-1,E-06	-1,E-05	-2,E-05	-2,E-04	-6,E-04	-6,E-03	-2,E-02
Δ(2.9)	-3,E-11	-2,E-10	-2,E-10	-1,E-09	-2,E-09	-4,E-08	-1,E-07	-8,E-07	-3,E-06	-2,E-05	-9,E-05	-6,E-04	-1,E-03	-4,E-03	-2,E-02
Δ(2.10)	7,E-16	1,E-15	3,E-15	4,E-15	5,E-15	3,E-15	7,E-15	4,E-15	1,E-14	3,E-14	6,E-14	1,E-14	4,E-14	6,E-15	7,E-14

We can notice that Δ(2.10) is $\approx 10^{-14}$, whereas for the other Δ the error increases until $\approx 10^{-2}$.

Finally, we consider:

Figure 3.5: NcM until 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

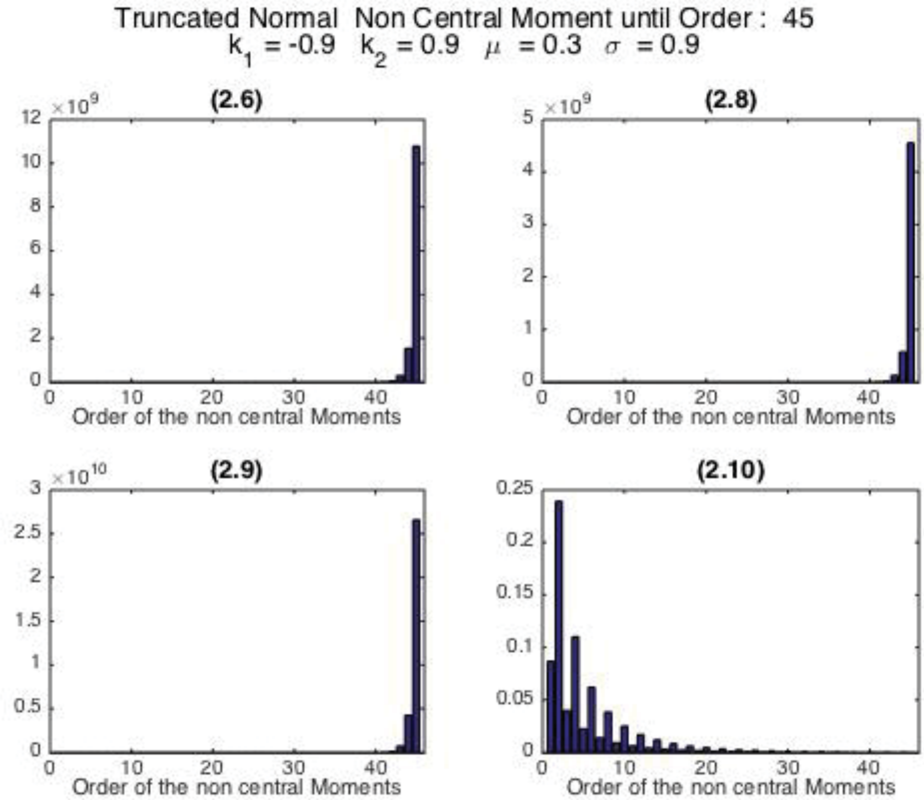

Table 3.5: NcM from 31'th to 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
(2.6)	3,E-01	1,E+00	9,E+00	4,E+01	2,E+02	1,E+03	7,E+03	4,E+04	2,E+05	1,E+06	8,E+06	4,E+07	3,E+08	2,E+09	1,E+10
(2.8)	1,E-01	4,E-01	4,E+00	1,E+01	1,E+02	4,E+02	3,E+03	1,E+04	1,E+05	4,E+05	3,E+06	2,E+07	1,E+08	6,E+08	5,E+09
(2.9)	2,E-01	1,E+00	7,E+00	4,E+01	4,E+02	2,E+03	2,E+04	9,E+04	6,E+05	3,E+06	2,E+07	1,E+08	7,E+08	4,E+09	3,E+10
(2.10)	3,E-04	8,E-04	2,E-04	6,E-04	2,E-04	5,E-04	1,E-04	4,E-04	1,E-04	3,E-04	7,E-05	2,E-04	6,E-05	2,E-04	4,E-05

We have divergent values for NcM computed with the (2.6), (2.8), (2.9) formulas that contain factorial term, only (2.10) remains coherent with the *Proposition 2.1*, which states the decreasing of NcM when $-1 < k_1 < k_2 < 1$.
Regarding Δ we have:

Figure 3.6: Δ until 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

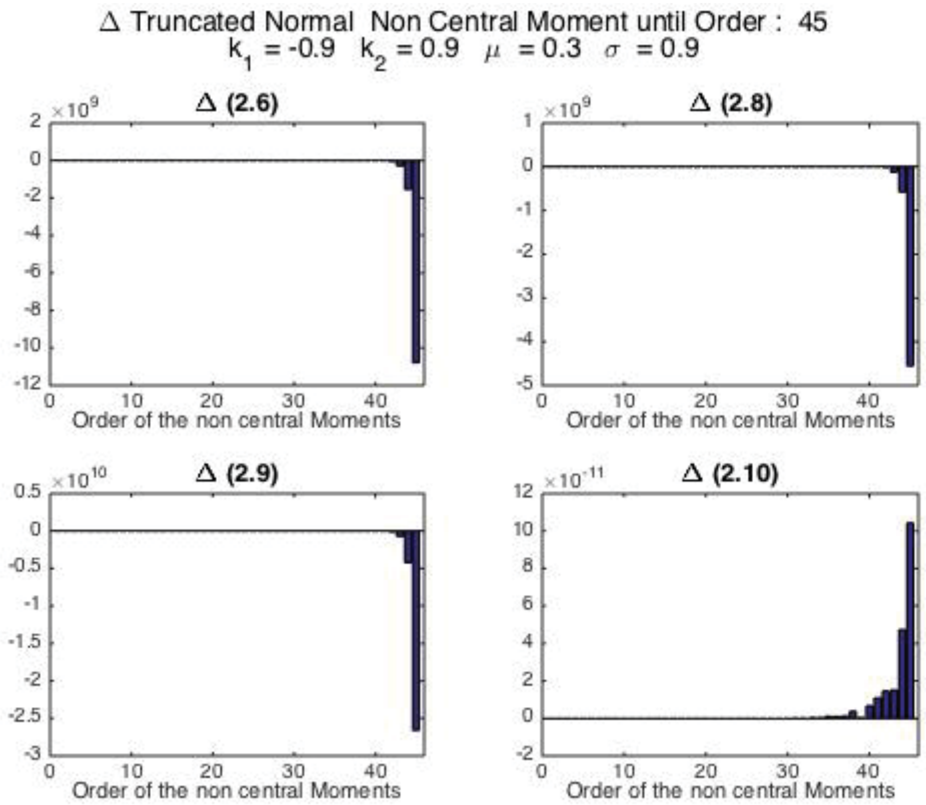

Table 3.6: Δ from 31'th to 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.9$

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Δ(2.6)	-0,314	-1,252	-8,507	-36,01	-245,1	-1095	-7475	-35045	-2E+05	-1E+06	-8E+06	-4E+07	-3E+08	-2E+09	-1E+10
Δ(2.8)	-0,139	-0,433	-3,743	-12,7	-106,9	-392,1	-3235	-12722	-1E+05	-4E+05	-3E+06	-2E+07	-1E+08	-6E+08	-5E+09
Δ(2.9)	-0,231	-1,027	-6,516	-41,64	-351,4	-2383	-16101	-94377	-6E+05	-3E+06	-2E+07	-1E+08	-7E+08	-4E+09	-3E+10
Δ(2.10)	1E-13	1E-13	5E-13	5E-13	1E-12	1E-12	1E-12	4E-12	4E-13	7E-12	1E-11	1E-11	2E-11	5E-11	1E-10

Here, Δ(2.10) is $\approx 10^{-10}$, and all the other Δ have values greater than $\approx 10^{9}$; this means that a truncation error becomes dangerous for the accuracy of the formulas.
Using $\sigma = 0.2$ we have:

Figure 3.7: NcM until 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

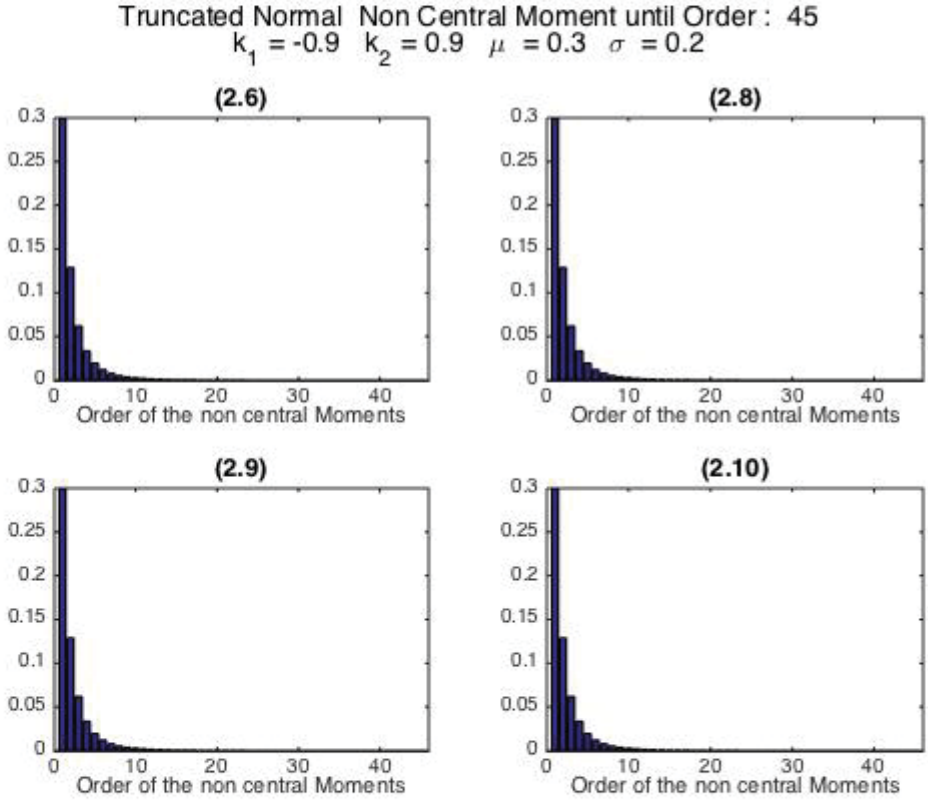

Table 3.7: NcM from 31'th to 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
(2.6)	4E-05	3E-05	3E-05	2E-05	2E-05	2E-05	2E-05	1E-05	1E-05	1E-05	9E-06	8E-06	7E-06	6E-06	5E-06
(2.8)	4E-05	3E-05	3E-05	2E-05	2E-05	2E-05	2E-05	1E-05	1E-05	1E-05	9E-06	8E-06	7E-06	6E-06	5E-06
(2.9)	4E-05	3E-05	3E-05	2E-05	2E-05	2E-05	2E-05	1E-05	1E-05	1E-05	9E-06	8E-06	7E-06	6E-06	5E-06
(2.10)	4E-05	3E-05	3E-05	2E-05	2E-05	2E-05	2E-05	1E-05	1E-05	1E-05	9E-06	8E-06	7E-06	6E-06	5E-06

The results seem stable until 45'th NcM. The Δ is:

Figure 3.8: Δ until 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

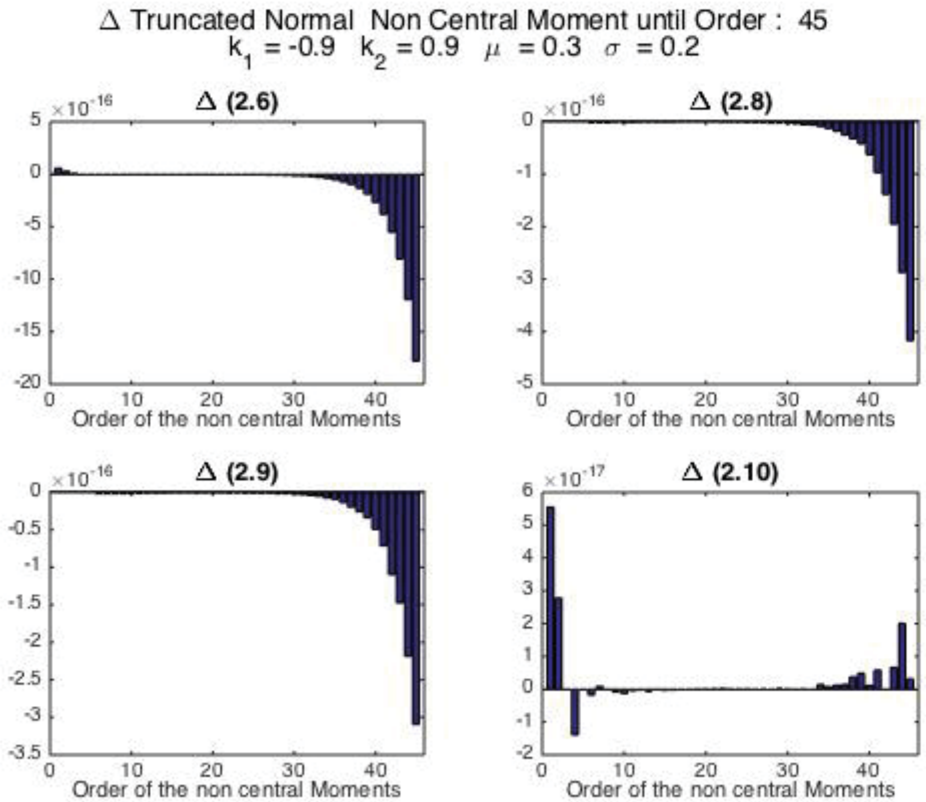

Table 3.8: Δ from 31'th to 45'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Δ(2.6)	-2E-17	-2E-17	-3E-17	-4E-17	-5E-17	-7E-17	-1E-16	-1E-16	-2E-16	-3E-16	-4E-16	-6E-16	-8E-16	-1E-15	-2E-15
Δ(2.8)	-4E-18	-6E-18	-6E-18	-9E-18	-1E-17	-2E-17	-3E-17	-3E-17	-4E-17	-6E-17	-1E-16	-1E-16	-2E-16	-3E-16	-4E-16
Δ(2.9)	-3E-18	-4E-18	-5E-18	-7E-18	-9E-18	-1E-17	-2E-17	-3E-17	-3E-17	-5E-17	-7E-17	-1E-16	-1E-16	-2E-16	-3E-16
Δ(2.10)	3E-20	7E-20	-1E-19	1E-18	8E-19	1E-18	1E-18	4E-18	5E-18	1E-18	6E-18	-3E-20	7E-18	2E-17	3E-18

All the Δ have small values, even if the Δ(2.10) remain lower than the others; if we increase the order of the NcM, also in this case the NcM for (2.6), (2.8), (2.9) become divergent.
In fact, for the 115'th order of the NcM:

Figure 3.9: NcM until 115'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

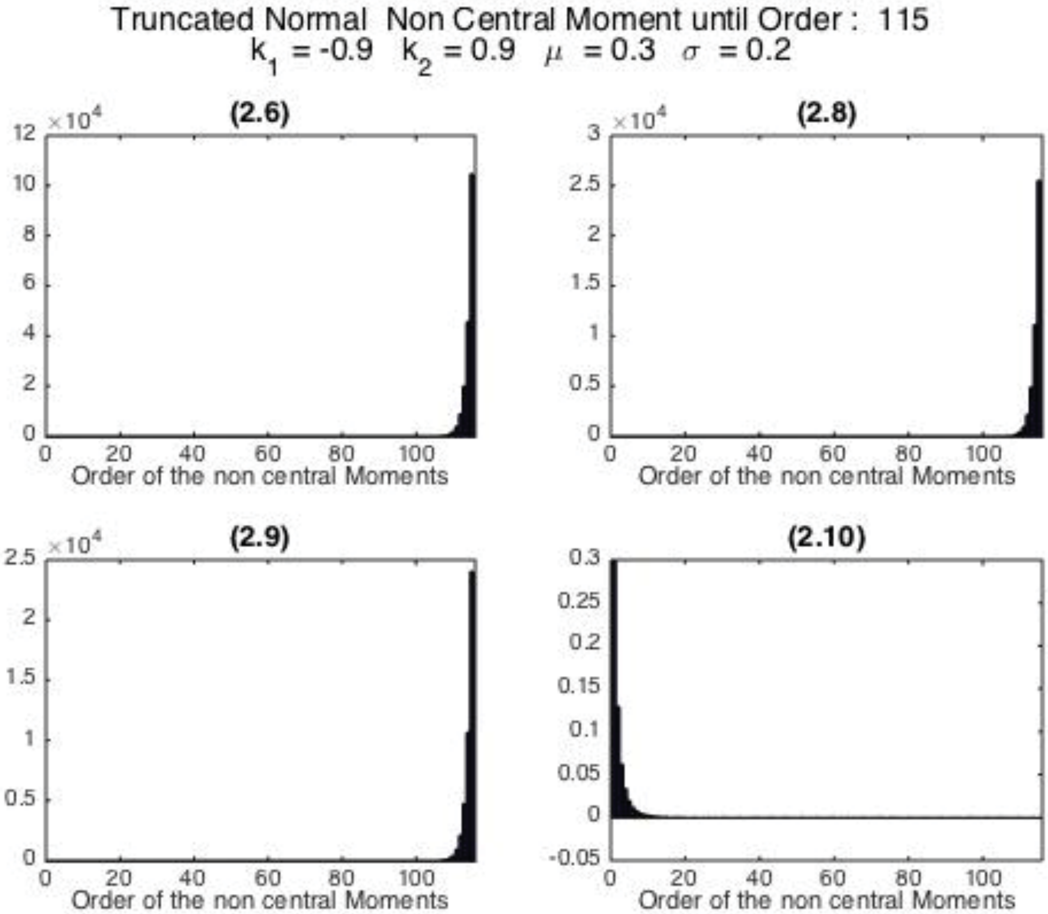

Table 3.9: NcM from 100'th to 105'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115
(2.6)	6,E-01	1,E+00	3,E+00	6,E+00	1,E+01	3,E+01	7,E+01	2,E+02	3,E+02	8,E+02	2,E+03	4,E+03	9,E+03	2,E+04	5,E+04	1,E+05
(2.8)	2,E-01	3,E-01	7,E-01	2,E+00	3,E+00	8,E+00	2,E+01	4,E+01	8,E+01	2,E+02	4,E+02	9,E+02	2,E+03	5,E+03	1,E+04	3,E+04
(2.9)	1,E-01	3,E-01	7,E-01	1,E+00	3,E+00	7,E+00	2,E+01	4,E+01	8,E+01	2,E+02	4,E+02	9,E+02	2,E+03	5,E+03	1,E+04	2,E+04
(2.10)	-4,E-09	8,E-09	8,E-09	1,E-08	9,E-08	9,E-08	4,E-07	4,E-07	7,E-07	9,E-07	-1,E-07	2,E-07	-2,E-07	-2,E-06	-7,E-06	-1,E-05

The values for (2.6), (2.8) and (2.9) diverge to $\approx 10^4$, only (2.10) remain bounded.
About Δ:

Figure 3.10: Δ until 115'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

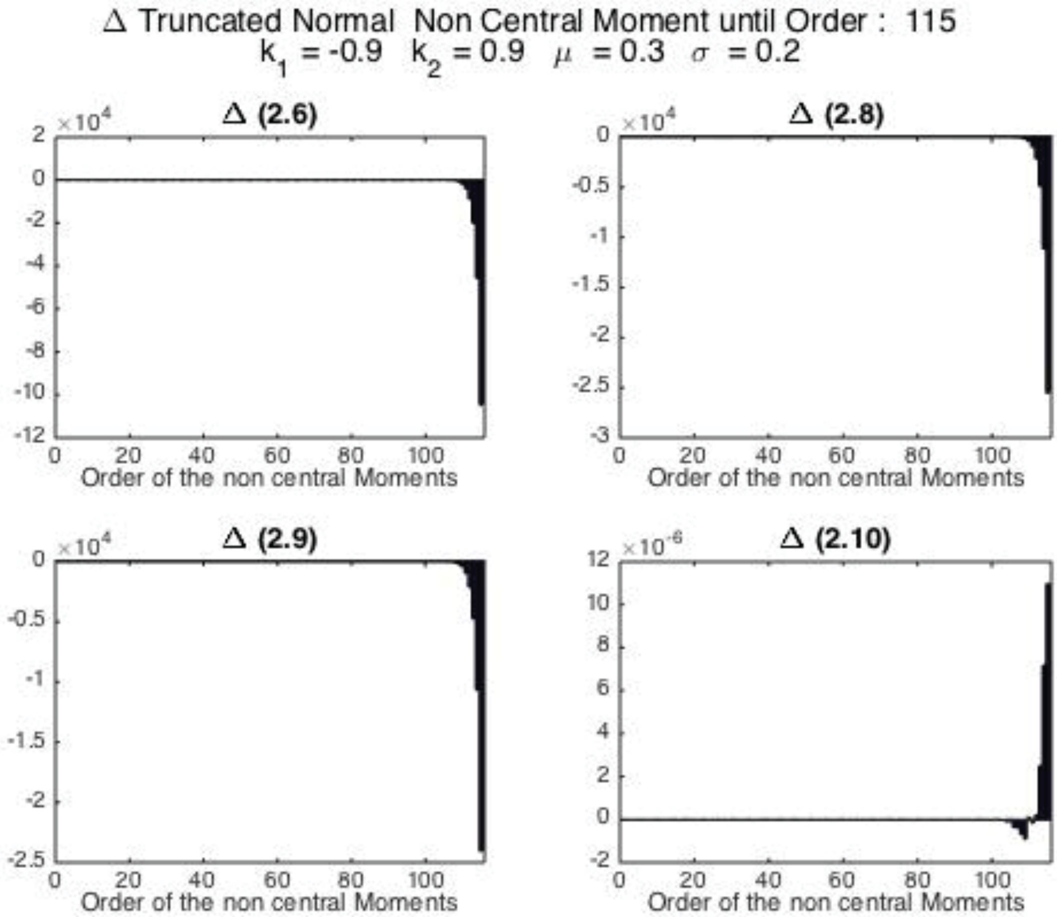

Table 3.10: Δ's values from 100'th to 115'th order $k_1 = -0.9,\ k_2 = 0.9,\ \mu = 0.3,\ \sigma = 0.2$

	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115
Δ(2.6)	-6,E-01	-1,E+00	-3,E+00	-6,E+00	-1,E+01	-3,E+01	-7,E+01	-2,E+02	-3,E+02	-8,E+02	-2,E+03	-4,E+03	-9,E+03	-2,E+04	-5,E+04	-1,E+05
Δ(2.8)	-2,E-01	-3,E-01	-7,E-01	-2,E+00	-3,E+00	-8,E+00	-2,E+01	-4,E+01	-8,E+01	-2,E+02	-4,E+02	-9,E+02	-2,E+03	-5,E+03	-1,E+04	-3,E+04
Δ(2.9)	-1,E-01	-3,E-01	-7,E-01	-1,E+00	-3,E+00	-7,E+00	-2,E+01	-4,E+01	-8,E+01	-2,E+02	-4,E+02	-9,E+02	-2,E+03	-5,E+03	-1,E+04	-2,E+04
Δ(2.10)	1,E-08	-3,E-09	-4,E-09	-8,E-09	-9,E-08	-9,E-08	-4,E-07	-4,E-07	-7,E-07	-9,E-07	1,E-07	-2,E-07	2,E-07	2,E-06	7,E-06	1,E-05

and again Δ(2.6), Δ(2.8) and Δ(2.9) diverge, whereas Δ(2.10) remains bounded.

A truncated normal distribution with $-1 < k_1 < 0 < k_2 \leq \infty$ is a useful distribution when we consider financial products that cannot lose more that 100% of their value: stocks, bonds, funds, etc. It should not utilized for the derivatives products, because they can lose more than 100% (e.g. selling a call option might be expose to "infinite" losses).

In addition, if we use a CRRA utility function in the Risk Averse range, the use of the normal distribution is unrealistic in that the distribution of returns can be skewed with possible extreme fat tails: we are obliged to consider returns whose range is greater than -1.

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