Ca' Foscari University of Venice

Department of Economics

Working Paper

Fausto Corradin
and Domenico Sartore

Risk Aversion: Differential Conditions for the Concavity in Transformed Two-Parameter Distributions

ISSN: 1827-3580
No. 30/WP/2016

Working Papers
Department of Economics
Ca' Foscari University of Venice
No. 30/WP/2016
ISSN 1827-3580

Risk Aversion: Differential Conditions for the Concavity in Transformed Two-Parameter Distributions

Fausto Corradin
GRETA Associati, Venice

Domenico Sartore
Ca' Foscari University of Venice, Department of Economics

Abstract
The condition of Risk Aversion implies that the Utility Function must be concave. We take into account the dependence of the Utility Function on the return that has any type of two-parameter distribution; it is possible to define Risk and Target, that usually is the Expected value of the return, as a generic function of these two parameters. This paper determines the Differential Conditions for the definitions of Risk and Target that maintain the Concavity of the Expected Utility Function downward in the 3D space of the Risk, Target and Expected Utility Function. As a particular case, in the paper we discuss these conditions in the case of the CRRA Utility Function and the Truncated Normal distribution. Furthermore, different measures of Risk are chosen, as Value at Risk (VaR) and Expected Shortfall (ES), to verify if these measures maintain the downward concavity property for the Expected Utility Function.

Keywords: utility function, expected utility function, risk aversion, transformation, parametric functions, differential condition, Jacobian, truncated normal distribution

JEL Codes: G11, G14, G23, G24

Address for correspondence:

Domenico Sartore
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
Fax: (++39) 041 2349176
E-mail: sartore@unive.it

The Working Paper Series
is available only on line
(http://www.unive.it/nqcontent.cfm?a_id=86302)
For editorial correspondence, please contact:
wp.dse@unive.it

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

Risk Aversion: Differential Conditions for the Concavity in Transformed Two-Parameter Distributions

Fausto Corradin(*), Domenico Sartore(**)

ABSTRACT: The condition of Risk Aversion implies that the Utility Function must be concave. We take into account the dependence of the Utility Function on the return that has any type of two-parameter distribution; it is possible to define Risk and Target, that usually is the Expected value of the return, as a generic function of these two parameters.
This paper determines the Differential Conditions for the definitions of Risk and Target that maintain the Concavity of the Expected Utility Function downward in the 3D space of the Risk, Target and Expected Utility Function. As a particular case, in the paper we discuss these conditions in the case of the CRRA Utility Function and the Truncated Normal distribution.
Furthermore, different measures of Risk are chosen, as Value at Risk (VaR) and Expected Shortfall (ES), to verify if these measures maintain the downward concavity property for the Expected Utility Function.

JEL Classification Numbers: G11, G14, G23, G24
Key Words: Concavity, CRRA, Differential Condition, Expected Shortfall, Expected Utility Function, Quadratic Utility Function, Risk Aversion, Standard Deviation, Transformation of Parametric Functions, Truncated Normal distribution, Value at Risk.

The paper is part of the research project MISURA (Multivariate Statistical Methods for Risk Assessment) of the Italian Ministry of Education (PRIN – cod. 2010RHAHPL), whose financial support is gratefully acknowledged.

(*) GRETA Associati, S. Polo 2605, 30125 Venice, Italy
(**) Department of Economics, Ca' Foscari University of Venice, Cannaregio 873, 30121 Venice, Italy (Corresponding author. Email: sartore@unive.it)

1. Introduction

Risk aversion is referred to as the amount an agent is willing to pay in order to avoid risk.

In the expected utility theory, the risk aversion measure is generally given by the Arrow-Pratt index, which requires the von Neumann-Morgenstern utility function.

There is no doubt that risk aversion is linked to the concavity of the utility function. For example, the Arrow–Pratt measure of absolute risk-aversion (ARA) relates the degree of concavity of a utility function measured by the curvature index known as the coefficient of absolute risk aversion. As underlined by Machina (1987), since someone with a concave utility function will always prefer receiving the expected value of a gamble to the gamble itself, concave utility functions are termed risk averse.

Machina affirms that in the case of non-expected utility function we can use calculus to extend the results obtained from the expected utility function. In particular he takes into account the concavity in the consequences of the partial derivatives with respect to probabilities of the preference function.

Other authors criticize the results obtained by this extension. For example, Montesano (1991) argues that, unlike what happens in the expected utility function, in non-expected utility function we can find examples of agents that prefer the lottery to its expected value (denoting risk attraction) while they prefer a smaller risk and vice versa. In this case, the concavity of the derivatives of the utility function cannot be considered an index of risk aversion for smaller risks.

Li Calzi and Sorato (2004), starting from the consideration that the existing parameterizations of prospect theory are not satisfactory, suggest a parameterization for utility and value functions that works across both the expected utility and prospect theory. With this parameterization the consequent family of functions are twice differentiable and are restricted to have only possible shapes: convex, concave, S-shaped and reverse S-shaped.

The drawback of the suggested parameterization is that the family includes utility (or value) functions which have no representation in closed-form, even thought their first derivatives always admit an explicit representation.

We have mentioned some articles that discuss the concavity and the risk aversion by considering properties of the functions in two-dimensional space.

In three-dimensional space, we can quote Lajeri and Nielsen (1998) whose aim is to determine whether one decision maker is more risk averse than another. For this purpose, Lajeri and Nielsen limit themselves to the two-parameter family of random variables and the risk aversion is measured considering the expected utility as a function of mean and standard deviation. In their analysis the concavity of the utility function plays an important role in determining the decision maker's attitude, measured by the marginal rate of substitution between mean and standard deviation, that is, by the slope of an indifference curve. The authors also establish the equivalence of the concept of decreasing absolute prudence (DAP), introduced by Kimball (1990), and the decreasing of the slope of the indifference curves of the utility function. Eichener and Wagener (2001) show that this latter result cannot be generalized for distributions other than the normal distribution.

The purpose of our paper is to determine the differential conditions for the downward concavity of the *Expected Utility Function* $E[U(W)]$ in the 3D space $[Risk, Target, E[U(W)]]$, when the *Utility Function U(W)* is *risk-averse*, Wealth is defined as $W = W_0(1 + r)$, r is the *return* with a generic distribution which depends on two parameters and *Risk* and *Target* are defined as a functions of these two parameters; *Target*, usually, is the Expected value of the return.

The *risk-averse* conditions are related to the first and the second derivatives of the *U(W)* and the degree of risk aversion can be measured by the curvature of the U(W).

These conditions are defined in two dimension and, taking the expectation of the *U(W), E[U(W)]*, these conditions do not necessarily imply that in three dimensional space $[Risk, Target, E[U(W)]]$ the *E[U(W)]* has a downward concavity. The downward concavity in 3D means that *E[U(W)]* depends decreasingly on $Risk$ and increasingly on $Target$.

As a particular case, the paper describes the *Constant Relative Risk Aversion Utility Function* (*CRRA*) applied to a return that has a Truncated Normal distribution.

The paper is organized as follows. Section 2 introduces the properties for the *Utility Function* when wealth depends on the return *r* that is a Normal variable. These properties are extended when the return *r* has a generic distribution which depends on two parameters and the definitions of *Risk* and *Target* are transformations of these two parameters.

Section 3 takes into consideration the *CRRA Utility Function* and the transformation of a Normal variable, e.g. a Truncated Normal variable, and illustrates that Standard Deviation and Mean of the starting Normal variable cannot be a correct definition for *Risk* and *Target*, due to the fact that the *E[U(W)]* has not the downward concavity in the space $[Risk, Target, E[U(W)]]$. This Section introduces the example to analyze the conditions for the concavity in 3D in a more general way.

Section 4 defines the Differential Conditions that must be respected when we consider a parametric representation of the surface concerning the *Risk*, *Target* and $E[U((W)]$ and we desire that the concavity of the $E[U((W)]$ remains downward, i.e. *E[U(W)]* depends decreasingly on $Risk$ and increasingly on $Target$. The Conditions pertain to any two-parameter distribution.

This is obtained without restrictions for the *U(W)* or definitions of *Risk* and *Target*.

As a particular case, taking in account the Truncated Normal variable for the return and using its Expected value for *Target*, Standard Deviation, VaR and Expected Shortfall of the return with *CRRA Utility Function* are analyzed. Only the Standard Deviation respects the Differential Conditions and maintains the concavity of $E[U(W)]$ downward.

Section 5 contains the conclusions.

2. Utility Function in the case of Normal distribution.

Let us consider the *Utility Function* $U(W)$, where W is wealth (or a quantity of the uncertain payment), given by:

(2.1) $$W = W_0(1 + r),$$

with the initial value W_0 and the return *r*.
If $U(W)$ represents a *risk-averse* person with insatiable appetite:

(2.2) $$U'(W) > 0\,; \qquad U''(W) < 0$$

(2.3) $$ARA = Absolute\ Risk\ Aversion = -\frac{U''(W)}{U'(W)} > 0$$

***Theorem 2.1**: Let ≽ be an expected utility preference relation on all normal distributions* $N(\mu, \sigma^2)$ *for the return r. Then there exists a mean-variance Expected Utility Function* $\psi(\sigma, \mu)$ *which describes ≽.*

In the case of risk aversion, $\psi(\sigma, \mu)$ *has the following partial derivatives and the first derivative of the implicit function* $\mu_\psi(\sigma)$*:*

(2.4) $$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} > 0, \quad \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0, \qquad \Rightarrow \frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\dfrac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\dfrac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

***Proof:** Appendix A.* ∎

The Theorem 2.1 describes a reasonable and intuitive behavior for the *risk-averse* investor translated in 3 dimensional space $[\sigma, \mu, \psi(\sigma, \mu)]$ when $r \sim N(\mu, \sigma^2)$.

More generally we consider the return $r \sim G(\sigma, \mu)$, where G is any two-parameter distribution and $g(r, \sigma, \mu)$ is the probability density function defined for $r \subseteq [\delta_1, \delta_2]$.
It is possible to compute the following *Expected Utility Function,* $\psi(\sigma, \mu)$.

(2.5) $$\psi(\sigma, \mu) \equiv E[U(W)] = E[U(1 + r)] = \int_{\delta_1}^{\delta_2} U\,(1 + r) g(r, \sigma, \mu) dr$$

The *Target* can be defined, as usual, as the *Expected value* of *r* :

$$Target = T(\sigma, \mu) = \int_{\delta_1}^{\delta_2} r\, g(r, \sigma, \mu) dr$$

and *Risk*, e.g., as the *Standard Deviation* of *r* :

$$Risk = R(\sigma, \mu) = \int_{\delta_1}^{\delta_2} [r - T(\sigma, \mu)]^2\, g(r, \sigma, \mu) dr$$

We can choose any other definition for *Risk* as a generic functions of (σ, μ), e.g. VaR or Expected Shortfall (ES). In the same line it is also possible to introduce a generic transformation to define the *Target :*

$$Risk = R(\sigma, \mu)$$

$$Target = T(\sigma, \mu)$$

where $R(\sigma, \mu)$ *and* $T(\sigma, \mu)$ are generic functions of (σ, μ) and we assume that they are at least once differentiable with continuous first derivatives.

For sake of simplicity we named the generic parameters as (σ, μ); later we will introduce the specific case of the Normal variable, and this choice allows us not to rename the parameters.

The question is: if we consider a *risk-averse Utility Function* and define *Risk* and *Target* as a generic functions of (σ, μ), which conditions must be satisfied by the three functions $R(\sigma, \mu), T(\sigma, \mu)$, $\psi(\sigma, \mu)$ so that in the parametric space $[R(\sigma, \mu), T(\sigma, \mu), \psi(\sigma, \mu)]$ the *Expected Utility Function* $\psi(\sigma, \mu)$ maintain the concavity downward? Is it sufficient that *U(W)* is *risk-averse* or is it necessary to introduce other conditions for the three functions mentioned above?

It is useful to recall that concavity downward means iso-utility curves with positive slope or, alternatively, a positive first derivative of the Implicit Function that is defined by the intercept of $\psi(\sigma, \mu)$ with a generic horizontal plane.

The conditions that will be determined later on also assure that the following inequalities are true :

$$\begin{cases} \frac{\partial \psi}{\partial R} < 0 \\ \frac{\partial \psi}{\partial T} > 0 \end{cases} \tag{2.6}$$

The following section describes which counterintuitive behavior may be encountered if the definition of *Risk* and *Target* are not correctly done and do not precisely respect some differential conditions.

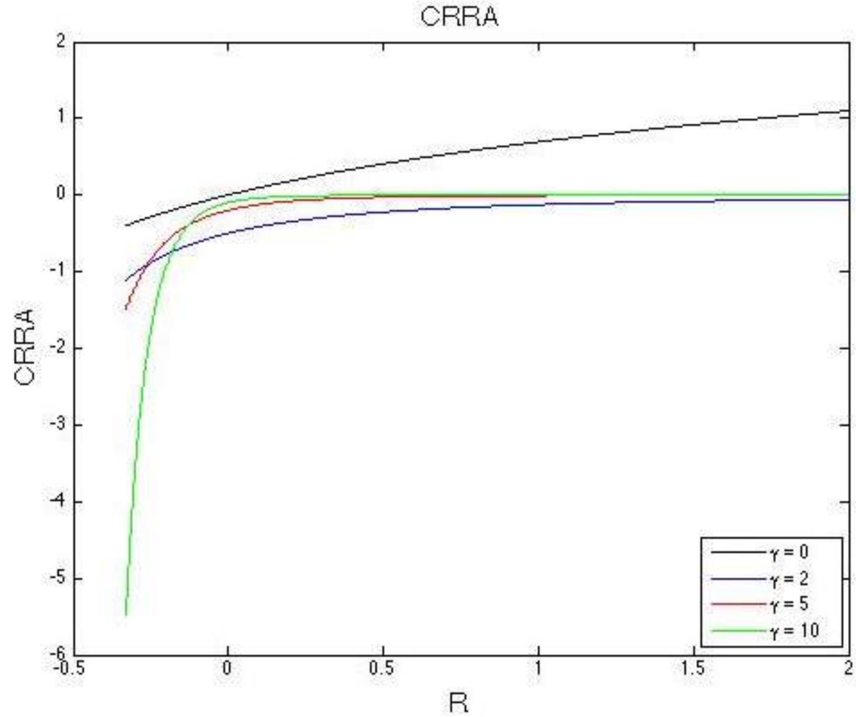
3. CRRA Utility Function and the Truncated Normal case

Consider a generic *CRRA* Utility Function:

(3.1) $$CRRA(\gamma) = \begin{cases} -W^{-\gamma}/\gamma, & \gamma > -1, \ \gamma \neq 0 \\ logW & \gamma = 0 \end{cases}$$

where W is defined by (2.1) and γ is a parameter that expresses an investor's sensitivity to risk.
The following Figure 3.1 shows the behavior of the CRRA with respect to different values of the γ parameter.

Figure 3.1: Constant Relative Risk Aversion Utility Functions



$\gamma < -1$: the investor is a risk lover rather than *risk-averse*.
$\gamma = -1$: means that the degree of risk aversion is zero, and the investor is indifferent between a risk-free choice and a risky choice so long as the arithmetic average expected return is the same.
$\gamma = 0$: the investor is indifferent between a risk-free choice and a risky choice so long as the geometric average expected return is the same.
$\gamma > 0$: the investor is *risk-averse* and calls a premium against his choice of a risky asset, the larger the value of γ the greater the risk penalty.

In this paper, we consider $\gamma = 2$.

Without any loss of generality we state $W_0 = 1$ in (2.1), therefore the *ARA (Absolute Risk Aversion)* and *RRA (Relative Risk Aversion)* for the CRRA have the following expressions:

$$ARA[CRRA(\gamma)] = \frac{\gamma + 1}{1 + r}\ , \qquad\qquad RRA[CRRA(\gamma)] = \gamma + 1$$

The value $r = -1$ represents a singular point for the (3.1), when $\gamma > 0$; this means that $r > -1$ is a condition that we have to pose. Furthermore, for $r < -1$ the *CRRA Utility Function* is not *risk-averse*.

Therefore, as particular case of $r \sim G(\sigma, \mu)$, where G is any two-parameter distribution, consider the return r as a Truncated Normal variable, that is r is constrained to assume values only in the interval $K = (k_1, k_2)$, with $-1 < k_1 < 0 < k_2 \le \infty$ and $k_1 < \mu < k_2$; we call r_{TN} this constrained variable, where the suffix "*TN*" means Truncated Normal. In this paper the computations are done for $k_1 = -0.99$, $k_2 = \infty$. To define the density of the random variable r_{TN}, we use the following notations:

$$\phi(\xi) = \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}, \qquad \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\tau^2}{2}} d\tau$$

$$h_2 = \frac{k_2 - \mu}{\sigma}\ , \quad h_1 = \frac{k_1 - \mu}{\sigma}, \qquad \Delta\Phi_K = \Phi(h_2) - \Phi(h_1)$$

Then, the density of the random variable r_{TN} is given by:

$$f(r_{TN}) = \begin{cases} \dfrac{\phi\left(\frac{r_{TN} - \mu}{\sigma}\right)}{\sigma \Delta\Phi_K} = \dfrac{e^{-(r_{TN}-\mu)^2 / 2\sigma^2}}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2} dx} & r_{TN} \in K \\ 0 & r_{TN} \notin K \end{cases}$$

and the *Expected Utility Function*, defined as $\psi(\sigma, \mu)$ is:

$$\psi(\sigma, \mu) \equiv E[CRRA(\gamma)] = -\frac{1}{\gamma} E\left[\frac{1}{(1 + r_{TN})^\gamma}\right] = -\frac{1}{\gamma\sigma\sqrt{2\pi}\Delta\Phi_K} \int_{k_1}^{k_2} \frac{e^{-(x-\mu)^2 / 2\sigma^2}}{(1 + x)^\gamma} dx$$

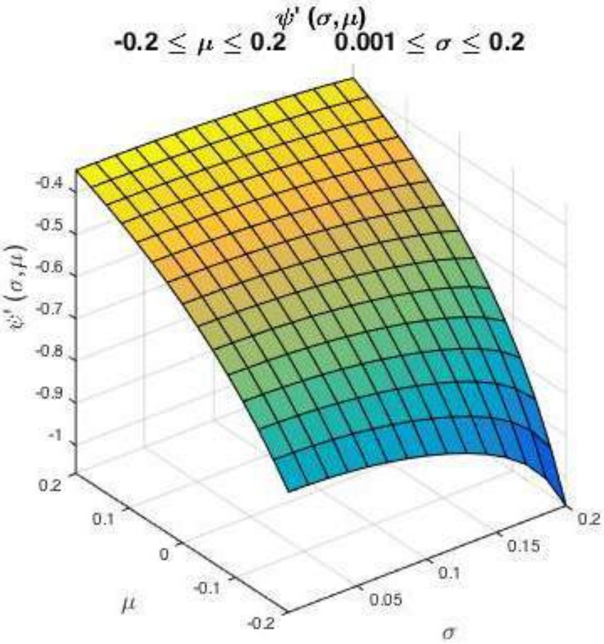
With the substitution $\tau = (x - \mu)/\sigma$ the function $\psi(\sigma, \mu)$ becomes:

$$(3.2) \qquad \psi(\sigma, \mu) = -\frac{1}{\gamma\sqrt{2\pi}\Delta\Phi_K} \int_{h_1}^{h_2} \frac{e^{-\tau^2/2}}{(1 + \mu + \sigma\tau)^\gamma} d\tau = -\frac{1}{\gamma} \frac{\int_{h_1}^{h_2} \frac{e^{-\tau^2/2}}{(1 + \mu + \sigma\tau)^\gamma} d\tau}{\int_{h_1}^{h_2} e^{-\tau^2/2} d\tau}$$

In this example, *Risk* and *Target* are given by the transformation $R(\sigma, \mu) = \sigma$, $T(\sigma, \mu) = \mu$.

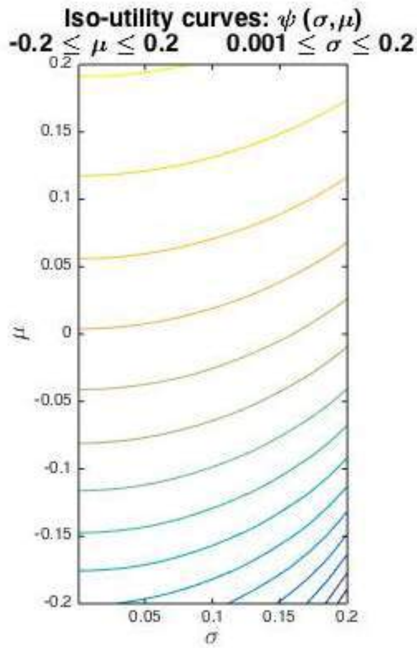
Now we give the three-dimensional representation with $(\sigma, \mu) \subseteq [(0.001 \le \sigma \le 0{,}2) \times (-0.2 \le \mu \le 0.2)]$, and we see that in this domain $\psi(\sigma, \mu)$ has the concavity downward in the space $[R(\sigma, \mu), T(\sigma, \mu), \psi(\sigma, \mu)] = [\sigma, \mu, \psi(\sigma, \mu)]$.

Figure 3.2: 3D $\psi(\sigma, \mu)$



The same representation in two dimension of iso-utility curves:

Figure 3.3: 2D Iso-utility curves of $\psi(\sigma, \mu)$



Looking at these results, a question can arise about the persistence of the concavity for $\psi(\sigma, \mu)$. If we increase the range space of (σ, μ) to $[(0.001 \leq \sigma \leq 1,2) \times (-0.9 \leq \mu \leq 0.3)]$ we have a counterintuitive behavior of the iso-utility curves, their slope becomes negative; for lower values of μ and greater values for σ we can observe that the change of the slope of the iso-utility curves is relevant. We graph this case in Figure 3.4, but again the anomalous behavior is more evident in Figure 3.5.

Figure 3.4: 3D $\psi(\sigma,\mu)$

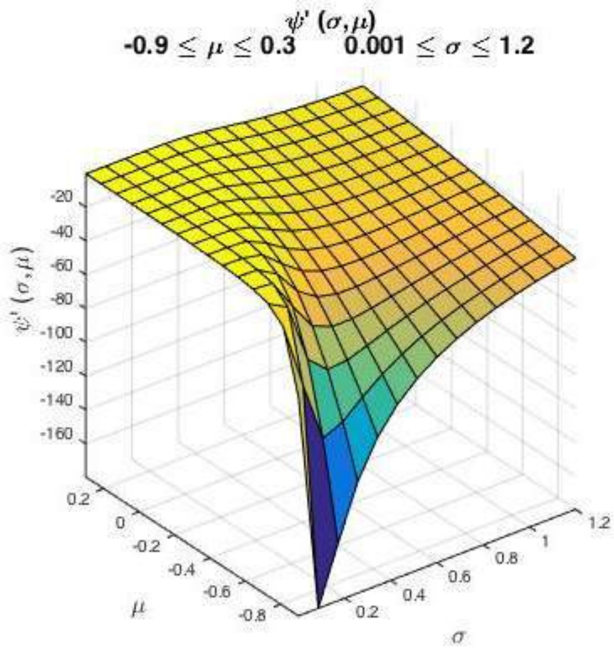


Figure 3.5: 2D Iso-utility curves of $\psi(\sigma,\mu)$

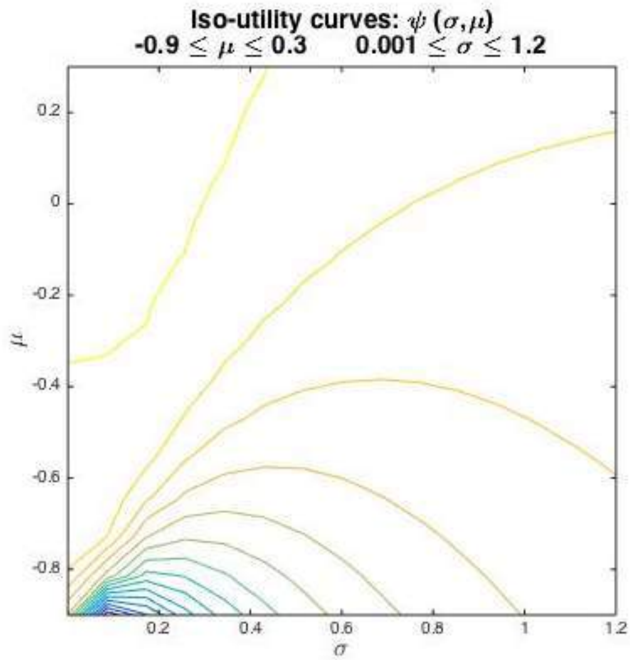


Figure 3.5 tells us that the iso-utility curves have negative slope in some region, that is the first derivatives of Implicit Function $\mu_{\psi}(\sigma)$ defined by the intercept of $\psi(\sigma,\mu)$ with a generic horizontal plane is negative. This counterintuitive behavior of the iso-utility curves was evident in studying the Morningstar's utility function used for Fund ranking (see Corradin and Sartore 2014).

From expression (3.2), it is possible to obtain the first derivative of the Implicit Function $\mu_\psi(\sigma)$:

$$\frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}}$$

and the result is reported in Appendix B. The related graph is given by Figure 3.6:

$Figure$ 3.6: $3D \;\; d\mu_\psi(\sigma)/d\sigma$

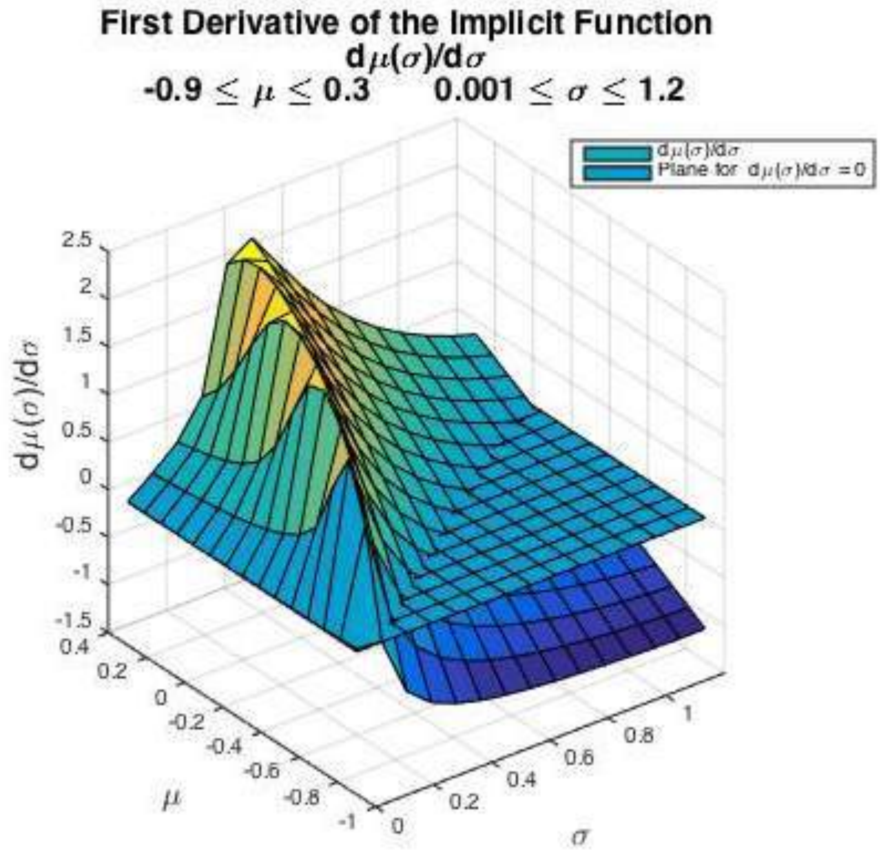


Figure 3.6 confirms that $d\mu_\psi(\sigma)/d\sigma$ can be negative.

This is due to the fact that the Cartesian system defined on (σ,μ), derived from the definitions of $R(\sigma,\mu)=\sigma$, $T(\sigma,\mu)=\mu$, is not the proper space in which to consider the *Expected Utility Function* $\psi(\sigma,\mu)$ of the CRRA, when the distribution of return is the Truncated Normal. We need to consider the proper alternative definitions for $R(\sigma,\mu), T(\sigma,\mu)$.

This is a particular case and in the following section we answer the more general question of Section 2.

4. Differential Conditions for the Concavity of the Expected Utility Functions. The specific case of the Truncated Normal.

As already introduced in Section 2, we consider $r \sim G(\sigma, \mu)$ and define *Risk* and *Target* as functions of (σ, μ):

(4.1) $$Risk = R(\sigma, \mu), \quad Target = T(\sigma, \mu)$$

The *Expected Utility Function* $\psi(\sigma, \mu)$ is defined in (2.5).
First of all we have to impose the condition that the transformation $[\sigma, \mu] \rightarrow [R(\sigma, \mu), T(\sigma, \mu)]$ defined by (4.1) is bijective.
This condition implies that the determinant of the Jacobian matrix J must be different from zero:

(4.2) $$detJ = det\begin{pmatrix} \dfrac{\partial R(\sigma, \mu)}{\partial \sigma} & \dfrac{\partial R(\sigma, \mu)}{\partial \mu} \\ \dfrac{\partial T(\sigma, \mu)}{\partial \sigma} & \dfrac{\partial T(\sigma, \mu)}{\partial \mu} \end{pmatrix} \neq 0$$

Now we want to find the conditions for *Risk* and *Target* so that the function $\psi(\sigma, \mu)$ maintains its concavity downward in the space $[R(\sigma, \mu), T(\sigma, \mu), \psi(\sigma, \mu)]$.

Consider a parametric representation of a surface:

x axis = *Risk* = $R(\sigma, \mu)$.
y axis = *Target* = $T(\sigma, \mu)$.
z axis = *Expected Utility Function* = $\psi(\sigma, \mu)$.

This surface is described in the space $[R(\sigma, \mu), T(\sigma, \mu), \psi(\sigma, \mu)]$ by the three functions $R(\sigma, \mu)$, $T(\sigma, \mu)$, $\psi(\sigma, \mu)$ that depends on (σ, μ) defined in $[(\sigma_{Min}, \sigma_{Max}) \times (\mu_{Min}, \mu_{Max})]$ in the cartesian space (σ, μ).

Using the vector notation, the surface is defined by vector $\boldsymbol{s}(\sigma, \mu)$ in the space $[R(\sigma, \mu), T(\sigma, \mu), \psi(\sigma, \mu)]$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the relative unit vectors:

(4.3) $$\boldsymbol{s}(\sigma, \mu) = R(\sigma, \mu)\boldsymbol{i} + T(\sigma, \mu)\boldsymbol{j} + \psi(\sigma, \mu)\boldsymbol{k}$$

For regularity of the surface, the Jacobian Matrix J_1:

(4.4) $$J_1 = \begin{pmatrix} \dfrac{\partial R(\sigma, \mu)}{\partial \sigma} & \dfrac{\partial R(\sigma, \mu)}{\partial \mu} \\ \dfrac{\partial T(\sigma, \mu)}{\partial \sigma} & \dfrac{\partial T(\sigma, \mu)}{\partial \mu} \\ \dfrac{\partial \psi(\sigma, \mu)}{\partial \sigma} & \dfrac{\partial \psi(\sigma, \mu)}{\partial \mu} \end{pmatrix}$$

must have rank two; e.g. this condition is satisfied if (4.2) is true.

The orthogonal unit vector of the surfaces is done by:

$$\frac{\dfrac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \sigma} \times \dfrac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \mu}}{\left\| \dfrac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \sigma} \times \dfrac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \mu} \right\|}$$

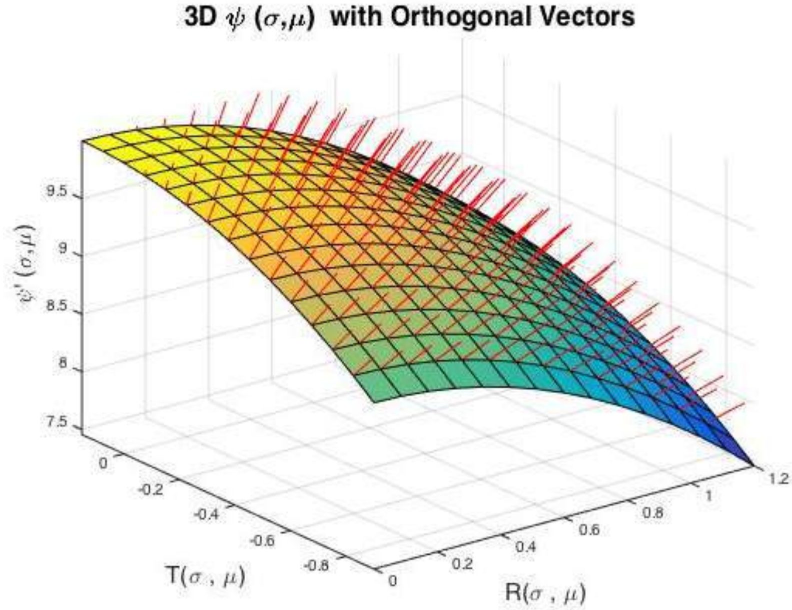
where:

$$(4.5)\quad \frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \sigma} \times \frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial \mu} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial R(\sigma,\mu)}{\partial \sigma} & \dfrac{\partial T(\sigma,\mu)}{\partial \sigma} & \dfrac{\partial \psi(\sigma,\mu)}{\partial \sigma} \\ \dfrac{\partial R(\sigma,\mu)}{\partial \mu} & \dfrac{\partial T(\sigma,\mu)}{\partial \mu} & \dfrac{\partial \psi(\sigma,\mu)}{\partial \mu} \end{vmatrix}$$

$$= \left[\frac{\partial T}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial T}{\partial \mu}\right]\boldsymbol{i} - \left[\frac{\partial R}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial R}{\partial \mu}\right]\boldsymbol{j} + \left[\frac{\partial R}{\partial \sigma}\frac{\partial T}{\partial \mu} - \frac{\partial T}{\partial \sigma}\frac{\partial R}{\partial \mu}\right]\boldsymbol{k}$$

where the dependence by (σ, μ) is omitted in the last formula. For a generic $\psi(\sigma, \mu)$:

Figure 4.1: 3D $\psi(\sigma, \mu)$ with Orthogonal Vectors



The surface is concave if the components of the orthogonal unit vectors are positive for $R - axis$ and $\psi - axis$ and negative for $T - axis$. We get the Differential Conditions:

$$(4.6)\quad \begin{cases} \dfrac{\partial T}{\partial \sigma}\dfrac{\partial \psi}{\partial \mu} - \dfrac{\partial \psi}{\partial \sigma}\dfrac{\partial T}{\partial \mu} > 0 & : Differential\ Condition\ 1 \equiv DC1 \\ \dfrac{\partial R}{\partial \sigma}\dfrac{\partial \psi}{\partial \mu} - \dfrac{\partial \psi}{\partial \sigma}\dfrac{\partial R}{\partial \mu} > 0 & : Differential\ Condition\ 2 \equiv DC2 \\ \dfrac{\partial R}{\partial \sigma}\dfrac{\partial T}{\partial \mu} - \dfrac{\partial T}{\partial \sigma}\dfrac{\partial R}{\partial \mu} > 0 & : Differential\ Condition\ 3 \equiv DC3 \end{cases}$$

Note the that *DC3* is the same as in the expression (4.2).

In the particular case of the Section 3, where $R(\sigma,\mu) = \sigma,\ T(\sigma,\mu) = \mu$:

$$\frac{\partial R}{\partial \sigma} = 1;\ \frac{\partial R}{\partial \mu} = 0;\ \frac{\partial T}{\partial \sigma} = 0;\ \frac{\partial T}{\partial \mu} = 1;$$

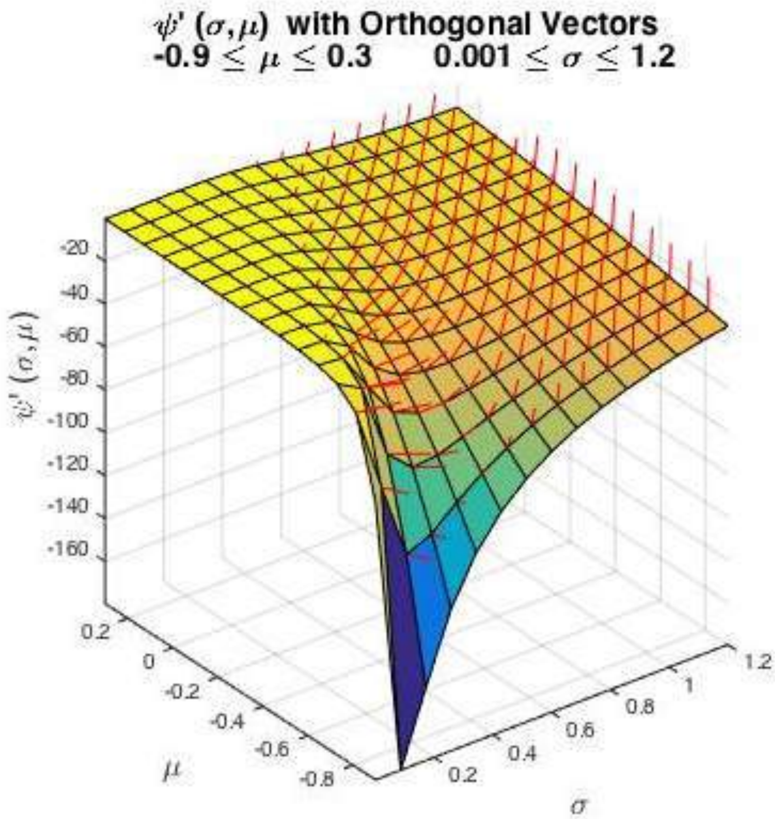
we have:

$$(4.7) \qquad \begin{cases} \frac{\partial T}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial T}{\partial \mu} > 0 \\ \frac{\partial R}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 \\ \frac{\partial R}{\partial \sigma}\frac{\partial T}{\partial \mu} - \frac{\partial T}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial \psi}{\partial \sigma} < 0 \\ \frac{\partial \psi}{\partial \mu} > 0 \\ 1 > 0 \end{cases} \Rightarrow \begin{cases} -\dfrac{\frac{\partial \psi(\sigma,\mu)}{\partial \sigma}}{\frac{\partial \psi(\sigma,\mu)}{\partial \mu}} = \dfrac{d\mu_\psi(\sigma)}{d\sigma} > 0 \\ 1 > 0 \end{cases}$$

where $\mu_\psi(\sigma)$ is the Implicit Function determined by $\psi(\sigma,\mu)$. The last condition $d\mu_\psi(\sigma)/d\sigma > 0$ is reasonable taking into consideration that $U(W)$ is *risk-averse*.

As we have seen in Figure 3.4, the case of the Truncated Normal does not maintain the concavity downward in the range space $[(0.001 \le \sigma \le 1{,}2) \times (-0.9 \le \mu \le 0.3)]$; this means that the conditions (4.7) are not satisfied and $d\mu_\psi(\sigma)/d\sigma$ can be negative as it is possible to see in Figure 3.5. Adding in Figure 3.4, the orthogonal unit vectors, we see that they have negative components along the $\sigma - axis$ in some regions of the surface.

Figure 4.2: 3D $\psi(\sigma,\mu)$ with Orthogonal Vectors



We give a possible economic interpretation for the Differential Conditions in (4.6).
The condition (4.2) that the transformation $[\sigma,\mu] \to [R,T]$ is bijective implies that the inverse transformation $\sigma(R,T)$, $\mu(R,T)$ exists locally:

$$\psi(\sigma,\mu) = \psi\big(\sigma(R,T)\,,\mu(R,T)\big) = \psi(R,T)$$

Computing the partial derivatives:

(4.8)
$$\frac{\partial\psi\big(\sigma(R,T)\,,\mu(R,T)\big)}{\partial R} = \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma}{\partial R} + \frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial R}$$
$$\frac{\partial\psi\big(\sigma(R,T)\,,\mu(R,T)\big)}{\partial T} = \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma}{\partial T} + \frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial T}$$

By the Theorem of the Inverse Function we have:

$$\begin{pmatrix} \dfrac{\partial\sigma(R,T)}{\partial R} & \dfrac{\partial\sigma(R,T)}{\partial T} \\ \dfrac{\partial\mu(R,T)}{\partial R} & \dfrac{\partial\mu(R,T)}{\partial T} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial R(\sigma,\mu)}{\partial\sigma} & \dfrac{\partial R(\sigma,\mu)}{\partial\mu} \\ \dfrac{\partial T(\sigma,\mu)}{\partial\sigma} & \dfrac{\partial T(\sigma,\mu)}{\partial\mu} \end{pmatrix}^{-1}$$

that has solution for the condition (4.2). We can write:

$$\begin{pmatrix} \dfrac{\partial\sigma(R,T)}{\partial R} & \dfrac{\partial\sigma(R,T)}{\partial T} \\ \dfrac{\partial\mu(R,T)}{\partial R} & \dfrac{\partial\mu(R,T)}{\partial T} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{detJ}\dfrac{\partial T}{\partial\mu} & -\dfrac{1}{detJ}\dfrac{\partial R}{\partial\mu} \\ -\dfrac{1}{detJ}\dfrac{\partial T}{\partial\sigma} & \dfrac{1}{detJ}\dfrac{\partial R}{\partial\sigma} \end{pmatrix}$$

and substituting in (4.8) we have:

$$\frac{\partial\psi}{\partial R} = \frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu} - \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial T}{\partial\sigma} \quad\Rightarrow\quad -detJ\frac{\partial\psi}{\partial R} = \frac{\partial T}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu}$$
$$\frac{\partial\psi}{\partial T} = -\frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu} + \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial R}{\partial\sigma} \quad\Rightarrow\quad detJ\frac{\partial\psi}{\partial T} = \frac{\partial R}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu}$$

Substituting in (4.6) we obtain:

(4.9)
$$\begin{cases} \dfrac{\partial T}{\partial\sigma}\dfrac{\partial\psi}{\partial\mu} - \dfrac{\partial\psi}{\partial\sigma}\dfrac{\partial T}{\partial\mu} > 0 \\ \dfrac{\partial R}{\partial\sigma}\dfrac{\partial\psi}{\partial\mu} - \dfrac{\partial\psi}{\partial\sigma}\dfrac{\partial R}{\partial\mu} > 0 \\ \dfrac{\partial R}{\partial\sigma}\dfrac{\partial T}{\partial\mu} - \dfrac{\partial T}{\partial\sigma}\dfrac{\partial R}{\partial\mu} > 0 \end{cases} \Rightarrow \begin{cases} -detJ\dfrac{\partial\psi}{\partial R} > 0 \\ detJ\dfrac{\partial\psi}{\partial T} > 0 \\ detJ > 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial\psi}{\partial R} < 0 \\ \dfrac{\partial\psi}{\partial T} > 0 \\ detJ > 0 \end{cases} \Rightarrow \begin{cases} \dfrac{dT_\psi(R)}{dR} = -\dfrac{\frac{\partial\psi}{\partial R}}{\frac{\partial\psi}{\partial T}} > 0 \\ detJ > 0 \end{cases}$$

The inequalities in (4.9) shed light the meaning of the Differential Conditions in (4.6): the *Expected Utility Function* $\psi(\sigma,\mu)$ depends decreasingly on R and increasingly on T.

The conditions $\partial\psi/\partial R < 0$ and $\partial\psi/\partial T > 0$ are not verifiable in closed form; they are a consequence of (4.6) and they imply that the first derivatives of the Implicit Function $T_\psi(R)$, defined by the intercept of $\psi(R,T)$ with a generic horizontal plane, is positive.
The inequalities (4.9) generalize the conditions given in *Theorem 2.1* for the Normal distribution because they apply to any two-parameter distribution and to any definition of *Risk* and *Target*.

The condition, $det\,J > 0$ implies that the transformation defined by $R(\sigma,\mu)$ and $T(\sigma,\mu)$ does not change direction. If we walk around the border of the range defined by $[(\sigma_{Min},\sigma_{Max})\times(\mu_{Min},\mu_{Max})]$ counterclockwise, then in the same direction we walk on the border of transformed range of the $[R(\sigma,\mu),T(\sigma,\mu)]$ space.

It is possible to rewrite (4.9) to determine a geometric explanation.

We use, e.g., the hypothesis:

(4.10) $$\frac{\partial R}{\partial\sigma} > 0;\quad \frac{\partial R}{\partial\mu} < 0;\quad \frac{\partial T}{\partial\sigma} < 0;\quad \frac{\partial T}{\partial\mu} > 0;\quad \frac{\partial\psi}{\partial\mu} > 0;\quad \frac{\partial\psi}{\partial\sigma} < 0$$

From the first Differential Condition we have:

$$DC1 = \frac{\partial T}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu} > 0 \Rightarrow -\frac{\partial\psi/\partial\sigma}{\partial\psi/\partial\mu} > -\frac{\partial T/\partial\sigma}{\partial T/\partial\mu} \Rightarrow \mu'_{\psi}(\sigma) > \mu'_{T}(\sigma)$$

This means that the first derivative of the Implicit Function $\mu_T(\sigma)$ determined by the definition of $Target = T(\sigma,\mu)$ is lower than the first derivative of the Implicit Function $\mu_\psi(\sigma)$ defined by the *Expected Utility Function* $= \psi(\sigma,\mu)$.

From the second Differential Condition we have:

$$DC2 = \frac{\partial R}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu} > 0 \Rightarrow -\frac{\partial\psi/\partial\sigma}{\partial\psi/\partial\mu} < -\frac{\partial R/\partial\sigma}{\partial R/\partial\mu} \Rightarrow \mu'_{R}(\sigma) > \mu'_{\psi}(\sigma)$$

and by the third Differential Condition:

$$DC3 = \frac{\partial R}{\partial\sigma}\frac{\partial T}{\partial\mu} - \frac{\partial T}{\partial\sigma}\frac{\partial R}{\partial\mu} > 0 \Rightarrow -\frac{\partial T/\partial\sigma}{\partial T/\partial\mu} < -\frac{\partial R/\partial\sigma}{\partial R/\partial\mu} \Rightarrow \mu'_{R}(\sigma) > \mu'_{T}(\sigma)$$

Summing up:

(4.11) $$\mu'_{T}(\sigma) < \mu'_{\psi}(\sigma) < \mu'_{R}(\sigma)$$

which is an inequality between first derivatives of the Implicit Functions, which come from $T(\sigma,\mu), \psi(\sigma,\mu), R(\sigma,\mu)$ respectively, and indicates the constraints that the curvature with respect to σ of these three Implicit Functions measured in a plane parallel to the plane (σ,μ) must satisfy.

Until now $R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)$ are supposed to be generic functions. It is interesting to discuss three cases of definition of *Risk* when the *return* is a Truncated Normal variable r_{TN} defined in Section 3 and we assume the *CRRA Expected Utility Function* (3.2) with $\gamma = 2$. *Target* is defined, as usual, as Expected value of r_{TN}, more briefly *Expected Return*.

Case 1: $R(\sigma,\mu) = Standard\ Deviation = SD_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

We have the transformation:

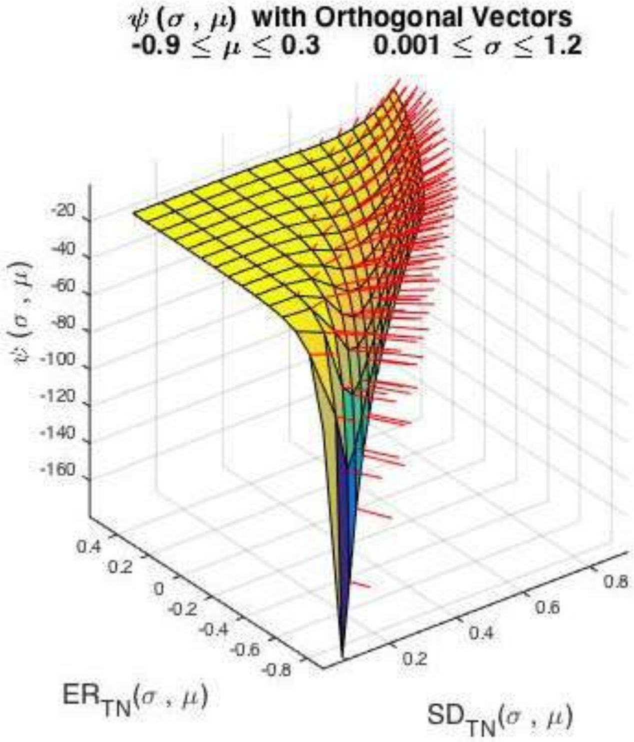
$$SD_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^2}$$

(4.12)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

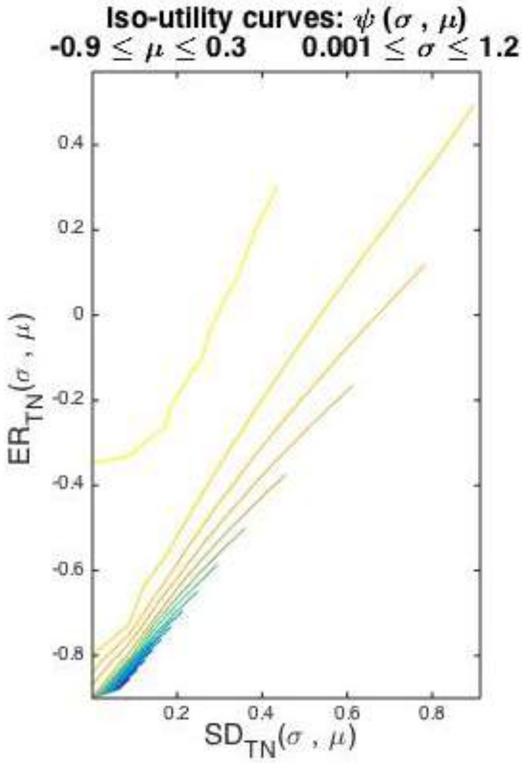
with the parametric representation for $\psi(\sigma,\mu)$ given by the following:

Figure 4.3: 3D $[SD_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$ with Orthogonal Vectors



The related Iso-utility curves are represented by:

Figure 4.4: Iso-utility curves of $\psi(\sigma,\mu)$ in 2D $[SD_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu)]$



Figures (4.3) and (4.4) show that (4.12) are coherent definitions of *Risk* and *Target*, according with the Differential Conditions (4.6).
The Differential Conditions are greater than zero in all the domain as is shown in Appendix C.

Case 2: $R(\sigma,\mu) = Value\ at\ Risk = VaR_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

$\alpha = Confidence\ Level = 0.95$

In Appendix D we compute the Value at Risk for a Truncated Normal, VaR_{TN}. We have the transformation:

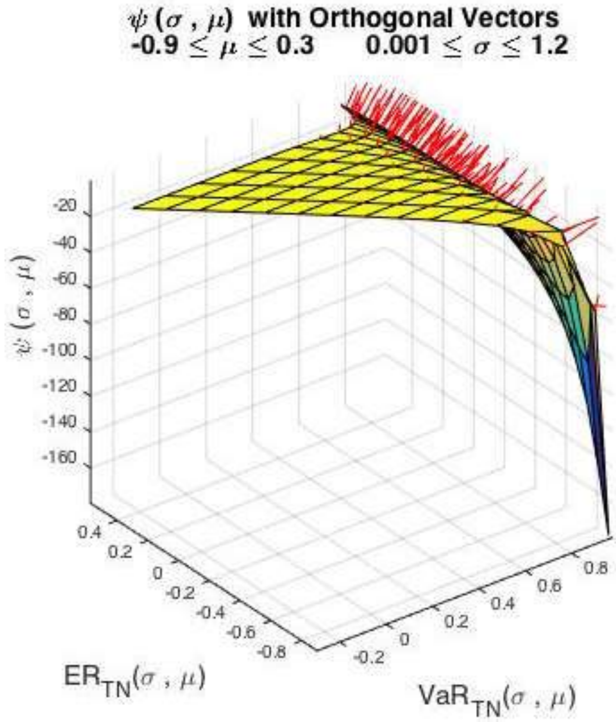
$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

(4.13)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

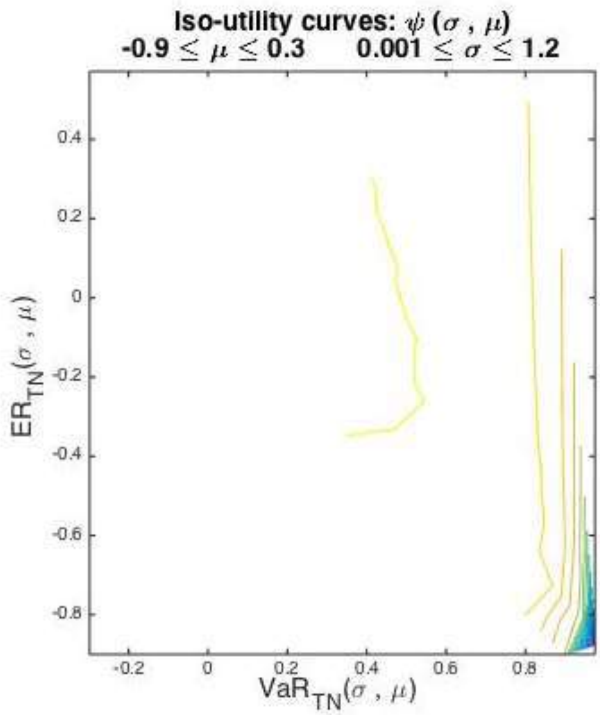
with the following parametric representation for $\psi(\sigma,\mu)$:

Figure 4.5: 3D $[VaR_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$ with Orthogonal Vectors



and the iso-utility curves represented by:

Figure 4.6: Iso-utility curves of $\psi(\sigma,\mu)$ in 2D $[VaR_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu)]$



This case demonstrates that VaR_{TN} is not a coherent Risk measure in the (4.6) sense, some iso-utility curves have a negative slope. Indeed, the Differential Conditions computed for VaR_{TN} are not respected in all the *3D* space $[VaR_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$.

To be more precise, Differential Condition 2, relative to the component of the axis of ER_{TN} of the Normal unit vector in (4.5), is negative (see Appendix D). The conclusion is that, in some region of the domain the behavior of the $\psi(\sigma,\mu)$ is not concave .
Writing ER_{TN} instead of T in (4.9) we have:

$$\frac{\partial\psi}{\partial ER_{TN}} < 0$$

that disagrees with (4.9) constraint.

Case 3: $R(\sigma,\mu) = Expected\ Shortfall = ES_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

$\alpha \quad = Confidence\ Level = 0.95$

In Appendix E we compute the Expected Shortfall for a Truncated Normal, ES_{TN}. We have the transformation:

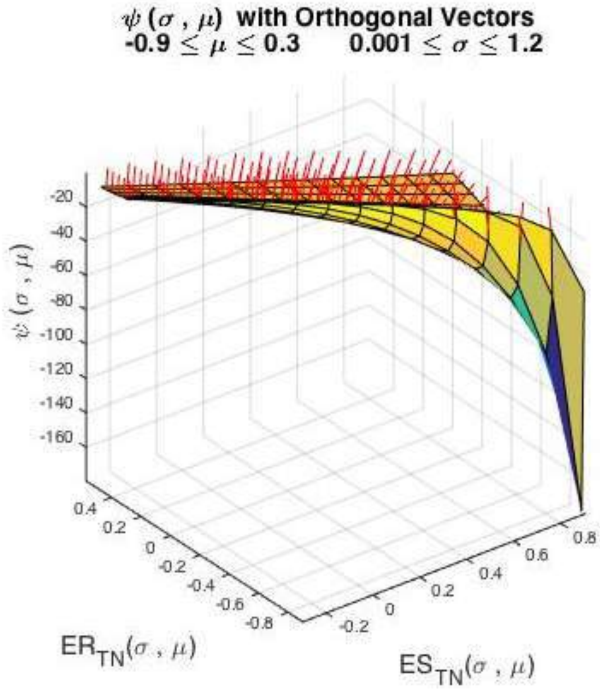
$$ES_{TN}(\sigma,\mu) = -\mu - \frac{\sigma[\phi(h_1) - \phi[\Phi_{inv}(b)]]}{(1-\alpha)\Delta\Phi_K}$$

(4.14)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

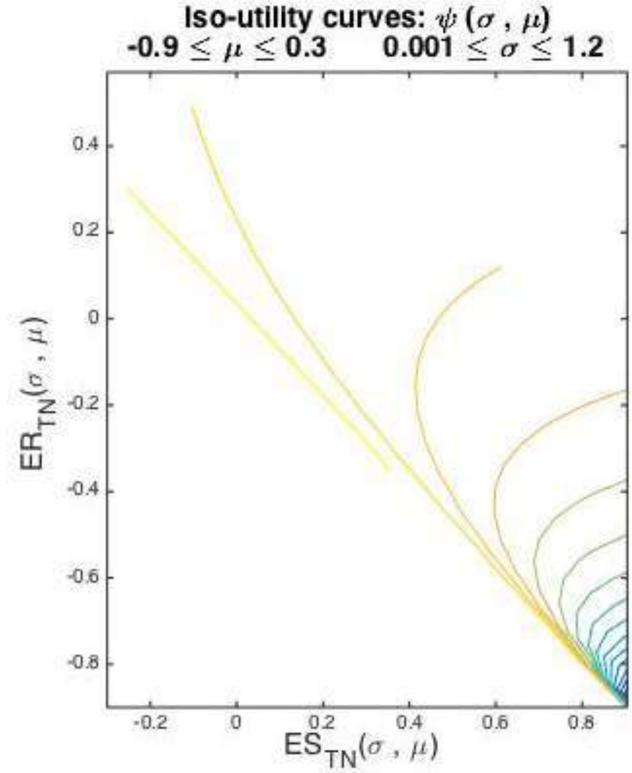
with the parametric representation for $\psi(\sigma,\mu)$:

Figure 4.7: 3D $[ES_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$ with Orthogonal Vectors



and with the Iso-utility curves represented by:

Figure 4.8: Iso-utility curves of $\psi(\sigma,\mu)$ *in 2D* $[ES_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu)]$



that also demonstrates ES_{TN} is not a coherent Risk measure in the (4.6) sense.
Indeed, the Differential Conditions computed for ES_{TN} are not respected in all the domain *3D* $[ES_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$. To be more precise, Differential Condition 2, relative to the component of the axis of ES_{TN} of the Normal unit vector in (4.5), is negative (see Appendix E).

The Quadratic Utility Function case is developed in Appendix F, G, H. This is an interesting case because it is possible to compute analytically the region in which the Differential Conditions are satisfied. We show how they represent the border between the *risk-averse* and *risk-lover* regions.

5. Conclusions

Starting with a *risk-averse Utility Function U(W)* with a wealth $W = W_0(1+r)$, where $r \sim G(\sigma,\mu)$ with G a generic distribution depending on two parameters, we consider the generic definitions of $Risk = R(\sigma,\mu), Target = T(\sigma,\mu)$. We find that the three functions $R(\sigma,\mu), T(\sigma,\mu)$ and *Expected Utility Function* $\psi(\sigma,\mu)$ must satisfy the Differential Conditions (4.6) so that $\psi(\sigma,\mu)$ has the concavity downward on the entire three dimensional space $[R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)]$.
These Conditions are verifiable because the analytic expressions of $R(\sigma,\mu)$, $T(\sigma,\mu)$ and $\psi(\sigma,\mu)$ are known.

The (4.6) imply the (4.9), that is $\psi\big(\sigma(R,T)\,,\mu(R,T)\big)$ has $\partial\psi/\partial R < 0$, $\partial\psi/\partial T > 0$. A third necessary condition is $det\ J > 0$, which is the determinant of the Jacobian matrix J of the transformation defined by $R(\sigma,\mu)$ and $T(\sigma,\mu)$, must be positive. In other words, the transformation does not make a change direction.

We present some cases in which the Differential Conditions show that not all the *Risk* and *Target* definitions are coherent with the chosen form of the *Utility Function*.

More precisely, if we consider the Truncated Normal case and define the *Target* as the Expected Return, ER_{TN}, then neither VaR nor Expected Shortfall (named VaR_{TN} and ES_{TN} in Case 2 and Case 3 respectively, discussed on the previous section) are a coherent definition of *Risk,* in the sense that some iso-utility curves have negative slope when we take into consideration the *CRRA Utility Function*.

Only the most elementary definition of *Risk*, the Standard Deviation, SD_{TN} in Case 1 of the previous section, respects the Differential Conditions.

References

Corradin, F. and Sartore, D. (2014), Fund Ratings: The method reconsidered, http://EconPapers.repec.org/RePEc:ven:wpaper:2014:17.

Corradin, F. and Sartore, D. (2016), Weak Dependence of CRRA on Standard Deviation in the Case of Truncated Normal Distribution of Returns, http://EconPapers.repec.org/RePEc:ven:wpaper:2016:18

Eichner, T. and Wagener, A. (2003), More on parametric characterization of risk aversion and prudence, *Economic Theory*, 21, 895-900.

Kimball, M. S. (1990), Precautionary Saving in the Small and in the Large, *Econometrica*, 58, 1, 53-73.

Lajeri, F. and Nielsen, L. T. (2000), Economic Theory 15, 469-476 (2000), Parametric characterizations of risk aversion and prudence, *Economic Theory*, 15, 469-476.

LiCalzi, M. and Sorato, A. (2006), The Pearson system of utility functions, *European Journal of Operational Research*, 172, 560-573.

Montesano, A. (1991), Measures of Risk Aversion with Expected and Nonexpected Utility, *Journal of Risk and Uncertainty*, 4, 271-283.

Appendix A. Proof of Theorem 2.1

Theorem 2.1*: Let $\succcurlyeq$ be an expected utility preference relation on all normal distributions $N(\mu, \sigma^2)$ for the return r. Then there exists a mean-variance Expected Utility Function $\psi(\sigma, \mu)$ which describes $\succcurlyeq$.*

In the case of risk-aversion, $\psi(\sigma, \mu)$ has the following partial derivatives and the first derivative of the implicit function $\mu_\psi(\sigma)$:

$$(2.4) \qquad \frac{\partial\psi(\sigma,\mu)}{\partial\mu} > 0, \quad \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0, \qquad \Rightarrow \frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

Proof:
Consider (2.1) here reported:

$$W = W_0(1 + r)$$

and without loss of generality pose $W_0 = 1$. We have:

$$r \backsim N(\mu, \sigma^2) \implies W \backsim N(1 + \mu, \sigma^2)$$

We prove at first the existence of $\psi(\sigma, \mu)$:

$$E[U(W)] = \int_{-\infty}^{\infty} \frac{U(W)e^{-(W-1-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dW$$

changing variable $z = (W - 1 - \mu)/\sigma$:

$$E[U(W)] = \int_{-\infty}^{\infty} \frac{U(1 + \mu + \sigma z)e^{-z^2/2}}{\sqrt{2\pi}} dz = \int_{-\infty}^{\infty} U(1 + \mu + \sigma z)\phi(z)\, dz = \psi(\sigma, \mu)$$

where $\phi(z)$ is the probability density function of the standard normal distribution.

Therefore, $E[U(W)]$ can be expressed as $\psi(\sigma, \mu)$, function of (σ, μ).
Now we can prove (2.4) when $U(W)$is *risk-averse*:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} = \int_{-\infty}^{\infty} U'(1 + \mu + \sigma z)\phi(z)\, dz > 0$$

from (2.2). And:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma} = \int_{-\infty}^{\infty} zU'(1 + \mu + \sigma z)\phi(z)\, dz$$

$$= \int_{-\infty}^{0} zU'(1 + \mu + \sigma z)\phi(z)\, dz + \int_{0}^{\infty} zU'(1 + \mu + \sigma z)\phi(z)\, dz$$

$$= \int_0^{\infty} z[U'(1+\mu+\sigma z) \; - U'(1+\mu-\sigma z)]\phi(z)\, dz$$

where the last line follows by the symmetry of $\phi(z)$.
By risk aversion $U''(W) < 0$ for all W, so that we have $U'(1+\mu+\sigma z) < U'(1+\mu-\sigma z)$ for $z > 0$, thus

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0$$

i.e., risk aversion imply that investor likes higher expected returns and dislikes higher standard deviation. Differentiating implicity:

$$\frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

Not surprisingly, indifference curves are upward in (σ,μ) cartesian plane.

Appendix B. First Derivative of the Implicit Function for the Truncated Normal case.

We give the following definitions that will be useful in the next expressions:

(B.1)

$$\tau = \frac{x-\mu}{\sigma}, \qquad h_2 = \frac{k_2-\mu}{\sigma}\,, \qquad h_1 = \frac{k_1-\mu}{\sigma},$$

$$I1 = \sigma\int_{h_1}^{h_2} e^{-\tau^2/2}\, d\tau, \qquad I2 = \int_{h_1}^{h_2} \tau e^{-\tau^2/2}\, d\tau \qquad I3 = \int_{h_1}^{h_2} \tau^2 e^{-\tau^2/2}\, d\tau$$

$$I4 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau) e^{-\tau^2/2}\, d\tau, \qquad I5 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau e^{-\tau^2/2}\, d\tau, \qquad I6 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau^2 e^{-\tau^2/2}\, d\tau,$$

$$I7 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau)^2 e^{-\tau^2/2}\, d\tau, \qquad I8 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau e^{-\tau^2/2}\, d\tau, \qquad I9 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau^2 e^{-\tau^2/2}\, d\tau,$$

$$I10 = \sigma\int_{h_1}^{h_2} \frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}\, d\tau, \qquad I11 = \int_{h_1}^{h_2} \tau\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}\, d\tau, \qquad I12 = \int_{h_1}^{h_2} \tau^2\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}\, d\tau$$

Consider function (3.2) here reported for brevity:

$$\psi(\sigma,\mu) = -\frac{1}{\gamma}\frac{\int_{k_1}^{k_2} \frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}\, dx}{\int_{k_1}^{k_2} e^{-(\xi-\mu)^2/2\sigma^2}\, d\tau}$$

As a first step, we compute the first derivatives of the integrals in the numerator and denominator with respect to σ and μ:

$$\frac{\partial}{\partial\sigma}\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=I12$$

$$\frac{\partial}{\partial\mu}\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\ =I11$$

$$\frac{\partial}{\partial\sigma}\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx=\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(\xi-\mu)^2/2\sigma^2}dx=I3$$

$$\frac{\partial}{\partial\mu}\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx=\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(\xi-\mu)^2/2\sigma^2}dx\ =I2$$

Therefore, the derivative of the function $\psi(\sigma,\mu)$ with respect to σ is:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}=$$

$$-\frac{1}{\gamma}\frac{\left[\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]-\left[\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]}{\left[\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]^2}$$

From (3.2):

$$\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=-\psi(\sigma,\mu)\gamma\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx$$

and we get:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}=$$

$$-\frac{1}{\gamma}\frac{\left[\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]\left\{\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]+\psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]\right\}}{\left[\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]^2}$$

$$=-\frac{1}{\gamma}\frac{\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]+\psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(\xi-\mu)^2/2\sigma^2}dx\right]}{\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx}$$

Finally, by using *(B.1)* we have:

$$(B.2) \qquad \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} = -\frac{1}{\gamma}\frac{I12 + \psi(\sigma,\mu)\gamma I3}{I1}$$

The derivative of the function $\psi(\sigma,\mu)$ with respect to μ is:

$$(B.3) \ \frac{\partial\psi(\sigma,\mu)}{\partial\mu} =$$

$$-\frac{1}{\gamma}\frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/_{2\sigma^2}}dx\right]\left\{\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/_{2\sigma^2}}}{(1+x)^\gamma}dx\right] + \psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(x-\mu)^2/_{2\sigma^2}}dx\right]\right\}}{\left[\int_{k_1}^{k_2} e^{-(\xi-\mu)^2/_{2\sigma^2}}dx\right]^2}$$

$$= -\frac{1}{\gamma}\frac{\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/_{2\sigma^2}}}{(1+x)^\gamma}dx\right] + \psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(x-\mu)^2/_{2\sigma^2}}dx\right]}{\int_{k_1}^{k_2} e^{-(\xi-\mu)^2/_{2\sigma^2}}dx}$$

$$= -\frac{1}{\gamma}\frac{I11 + \psi(\sigma,\mu)\gamma I2}{I1}$$

In the end, the first derivative of the Implicit Function $\mu_\psi(\sigma)$ is:

$$\frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}}$$

$$= -\frac{\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/_{2\sigma^2}}}{(1+x)^\gamma}dx\right] + \psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/_{2\sigma^2}}dx\right]}{\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/_{2\sigma^2}}}{(1+x)^\gamma}dx\right] + \psi(\sigma,\mu)\gamma\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(x-\mu)^2/_{2\sigma^2}}dx\right]}$$

$$= -\frac{I12 + \psi(\sigma,\mu)\gamma I3}{I11 + \psi(\sigma,\mu)\gamma I2}$$

Appendix C.

Case 1: $R(\sigma,\mu) = Standard\ Deviation = SD_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

To compute the Standard Deviation and the Expected Return of the Truncated Normal variable, it is preferable to start with the following definitions:

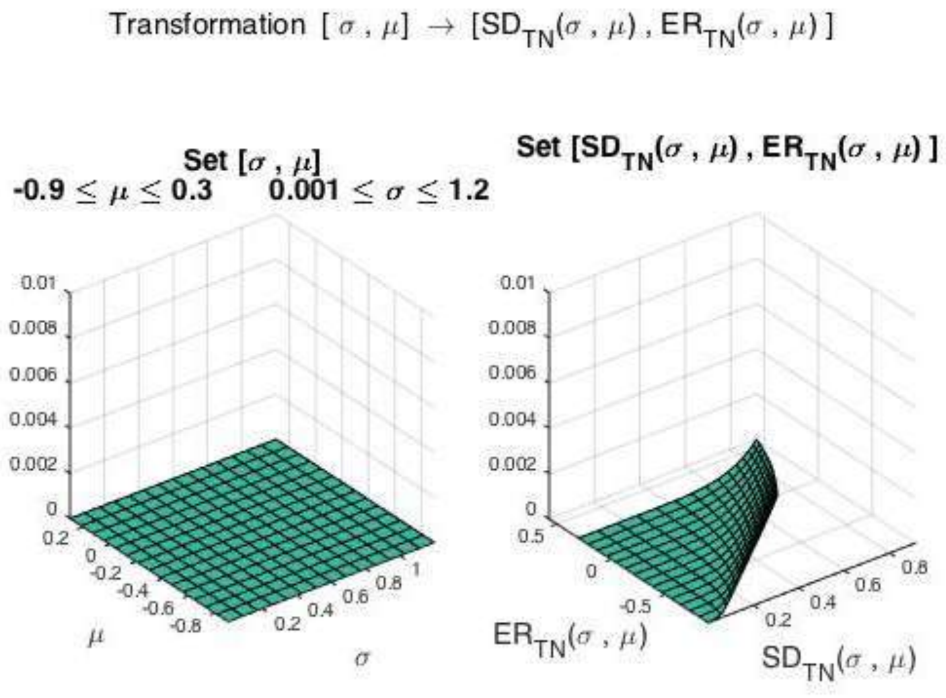
$$SD_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^2}$$

(C.1)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

The (C.1) formulas transform the set $[\sigma,\mu]$ into the set $[SD_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see from the following *Figure C.1*:

Figure C.1: Transformation $[\sigma,\mu] \rightarrow [SD_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$



The partial derivatives, using (B.1) are:

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\sigma}=\frac{\partial}{\partial\sigma}\left\{\frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right\}$$

$$=\frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/2\sigma^2}\,dx\right]-\left[\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\sigma}=\frac{I1*I6-I3*I4}{(I1)^2};$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\mu}=\frac{\partial}{\partial\mu}\left\{\frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right\}$$

$$=\frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x\left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/2\sigma^2}\,dx\right]-\left[\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2}\left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\mu}=\frac{I1*I5-I2*I4}{(I1)^2};$$

To compute the partial derivatives of SD_{TN} we consider:

$$\frac{\partial}{\partial\sigma}\frac{\int_{k_1}^{k_2} x^2e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}=$$

$$=\frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x^2\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/2\sigma^2}\,dx\right]-\left[\int_{k_1}^{k_2} x^2e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$=\frac{I1*I9-I3*I7}{(I1)^2}\Longrightarrow:$$

$$\frac{\partial SD_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{1}{2\,SD_{TN}(\sigma,\mu)}\left[\frac{I1 * I9 - I3 * I7}{(I1)^2} - 2\,ME_{TN}(\sigma,\mu)\,\frac{\partial ME_{TN}(\sigma,\mu)}{\partial\sigma}\right]$$

and:

$$\frac{\partial}{\partial\mu}\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/_{2\sigma^2}}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/_{2\sigma^2}}\,dx} =$$

$$= \frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/_{2\sigma^2}}\,dx\right]\left[\int_{k_1}^{k_2} x^2\left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/_{2\sigma^2}}\,dx\right] - \left[\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/_{2\sigma^2}}\,dx\right]\left[\int_{k_1}^{k_2}\left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/_{2\sigma^2}}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/_{2\sigma^2}}\,dx\right]^2}$$

$$= \frac{I1 * I8 - I2 * I7}{(I1)^2} \Longrightarrow$$

$$\frac{\partial SD_{TN}(\sigma,\mu)}{\partial\mu} = \frac{1}{2\,SD_{TN}(\sigma,\mu)}\left[\frac{I1I8 - I2I7}{I1^2} - 2\,ME_{TN}(\sigma,\mu)\,\frac{\partial ME_{TN}(\sigma,\mu)}{\partial\mu}\right]$$

It is possible now to compute and to graph the Differential Conditions (4.6):

Figure C.2: Differential Condition 1 for $[SD_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$

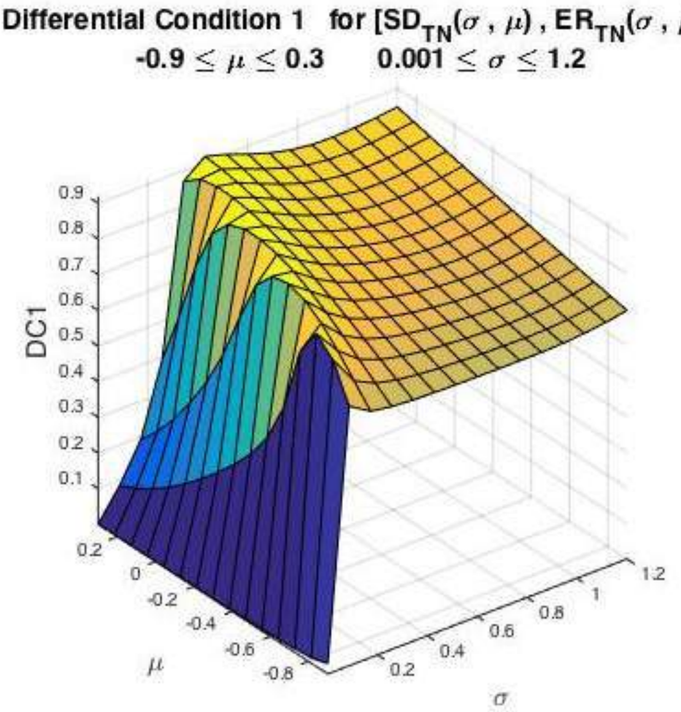


Figure C.3: Differential Condition 2 for $[SD_{TN}(\sigma, \mu),\ ER_{TN}(\sigma, \mu)]$

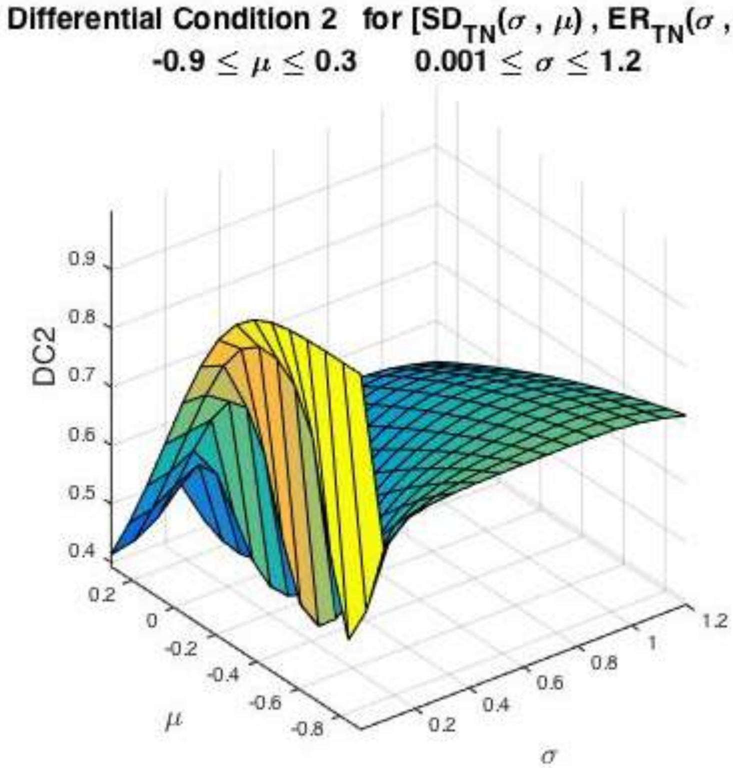
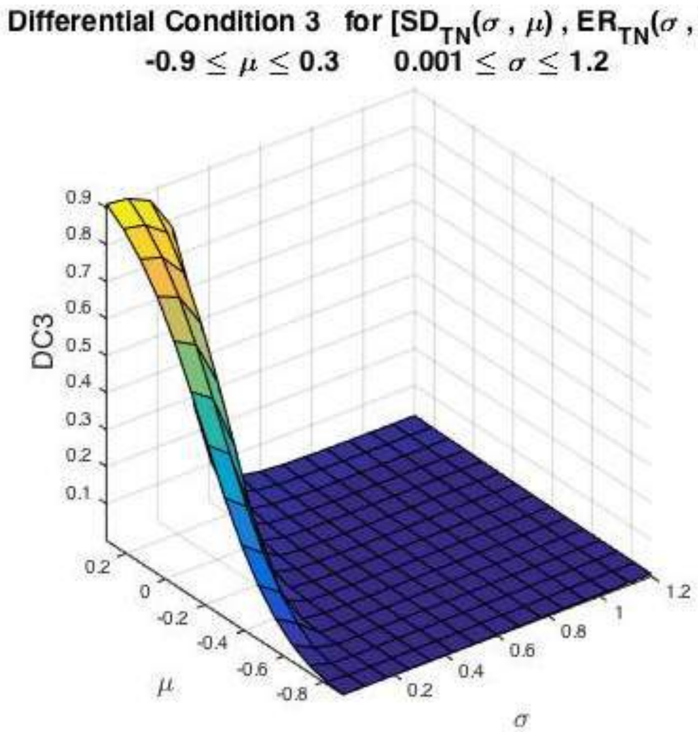


Figure C.4: Differential Condition 3 for $[SD_{TN}(\sigma, \mu),\ ER_{TN}(\sigma, \mu)]$



The three Figures above tell us that (4.6) are satisfied, all Differential Conditions are greater than zero.

Appendix D.

Case 2: $R(\sigma,\mu) = Value\ at\ Risk = \ VaR_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma\ = 2.$

$\alpha \quad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $VaR_{TN} \equiv VaR_{TN}(\sigma,\mu)$. Starting from its definitions:

$$1-\alpha = \frac{1}{\sigma\sqrt{2\pi}\Delta\Phi_K}\int_{k_1}^{-VaR_{TN}} e^{-(\xi-\mu)^2/2\sigma^2}\,d\xi$$

we have:

$$1-\alpha = \frac{1}{\Delta\Phi_K}\left[\Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) - \Phi(h_1)\right] \Rightarrow \Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) = (1-\alpha)\Delta\Phi_K + \Phi(h_1);$$

$$\Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) = (1-\alpha)\Phi(h_2) - (1-\alpha)\Phi(h_1) + \Phi(h_1) = \alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)$$

$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

obtaining the transformation:

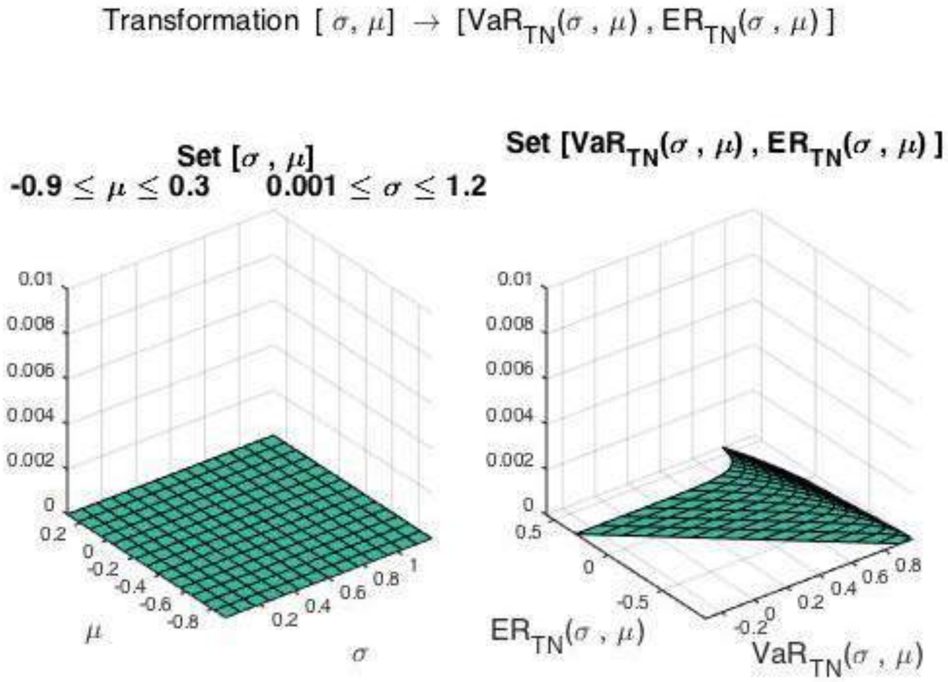
$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

(D.1)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

The (D.1) transforms the set $[\sigma,\mu]$ in the set $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see in the following *Figure D.1*:

Figure D.1: Transformation $[\sigma,\mu] \rightarrow [VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$



Defining:

$$(D.2) \qquad b = \alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\,, \qquad c = \Phi_{inv}(b)\,, \quad \Phi_{inv}(b) = -(VaR_{TN} + \mu)/\sigma$$

and computing:

$$(D.3) \qquad \frac{\partial \mathrm{b}}{\partial \mu} = -\frac{\alpha}{\sigma}\phi(h_1) - \frac{(1-\alpha)}{\sigma}\phi(h_2); \quad \frac{\partial \mathrm{b}}{\partial \sigma} = -\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)$$

we can use the Theorem of derivative of the inverse function:

$$\frac{d\Phi_{inv}(b)}{db} = \frac{1}{\frac{d\Phi(c)}{dc}} \quad iif \quad \frac{d\Phi(c)}{dc} \neq 0$$

to compute the partial derivatives of:

$$(D.4) \qquad \frac{\partial \Phi_{inv}(b)}{\partial \sigma} = \frac{d\Phi_{inv}(b)}{db} \cdot \frac{\partial \mathrm{b}}{\partial \sigma} = \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

By the definition of Φ:

$$\frac{d\Phi(c)}{dc} = \frac{d}{dc}\int_{-\infty}^{c} \phi(\tau)\,d\tau = \phi(c) = \phi\big(\Phi_{inv}(b)\big)$$

we have:

$$\frac{\partial \Phi_{inv}(b)}{\partial \sigma} = \frac{1}{\phi\big(\Phi_{inv}(b)\big)} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

and consequently:

$$\frac{\partial \Phi_{inv}(b)}{\partial \mu} = \frac{1}{\phi\big(\Phi_{inv}(b)\big)} \cdot \frac{\partial \mathrm{b}}{\partial \mu}$$

So we can compute the partial derivatives of VaR_{TN}:

$$\frac{\partial VaR_{TN}}{\partial \mu} = -1 - \sigma \frac{\left(-\frac{\alpha}{\sigma}\phi(h_1) - \frac{(1-\alpha)}{\sigma}\phi(h_2)\right)}{\phi\big(\Phi_{inv}(b)\big)}$$

$$= -1 + \frac{\alpha\phi(h_1) + (1-\alpha)\phi(h_2)}{\phi(-(VaR_{TN}+\mu)/\sigma)}$$

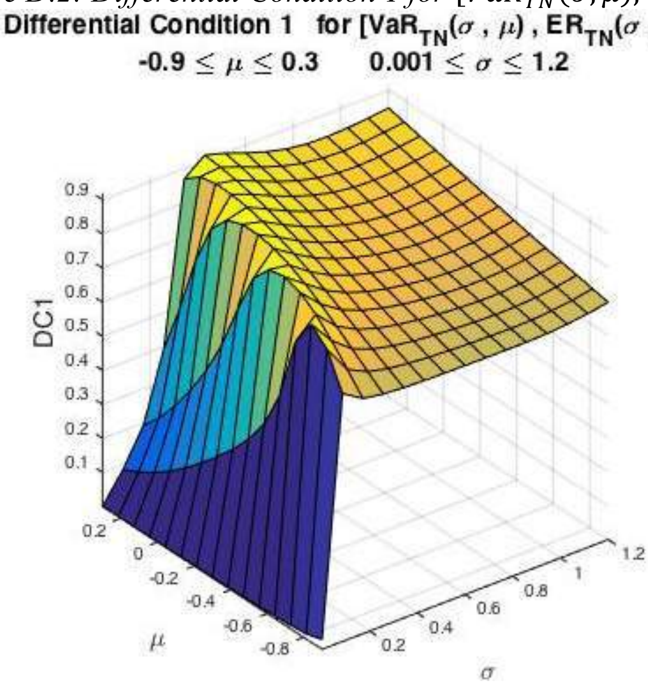
$$\frac{\partial VaR_{TN}}{\partial \sigma} = -\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big) - \sigma \frac{\left(-\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)\right)}{\phi\big(\Phi_{inv}(b)\big)}$$

$$= \frac{VaR_{TN}+\mu}{\sigma} - \sigma \frac{\left(-\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)\right)}{\phi\big(\Phi_{inv}(b)\big)}$$

$$= \frac{VaR_{TN}+\mu}{\sigma} + \frac{\big(\alpha h_1\phi(h_1) + (1-\alpha)h_2\phi(h_2)\big)}{\phi(-(VaR_{TN}+\mu)/\sigma)}$$

Now, it is possible to compute the Differential Conditions (4.6) and to graph them.

Figure D.2: Differential Condition 1 for $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$



DC1 is satisfied, it is > *0*.

Figure D.3: Differential Condition 2 for $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$

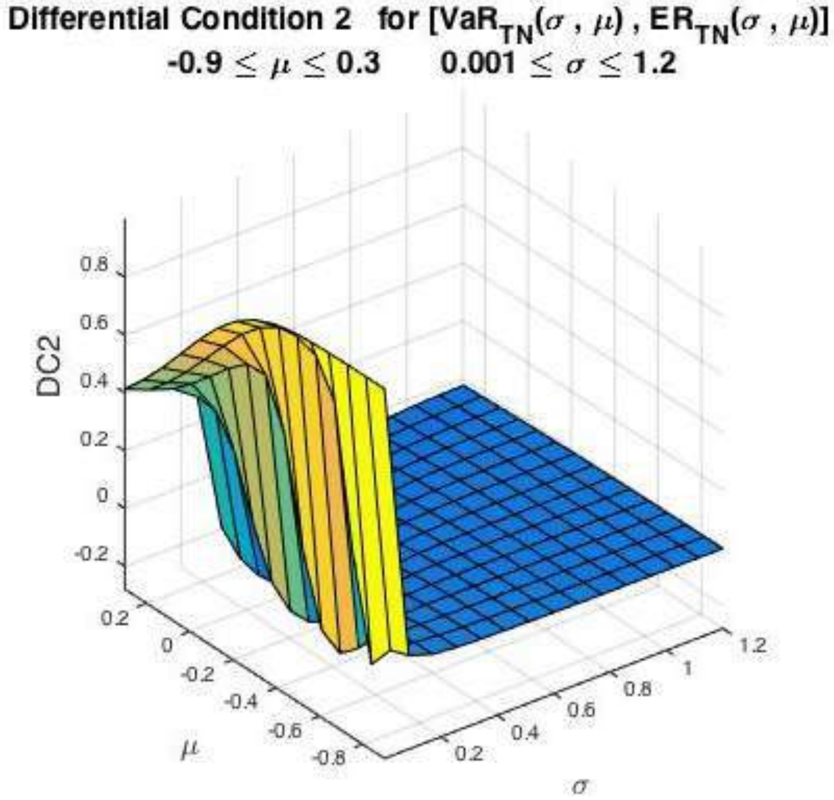


Table D.1: Values of the Differential Condition 2 for $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$

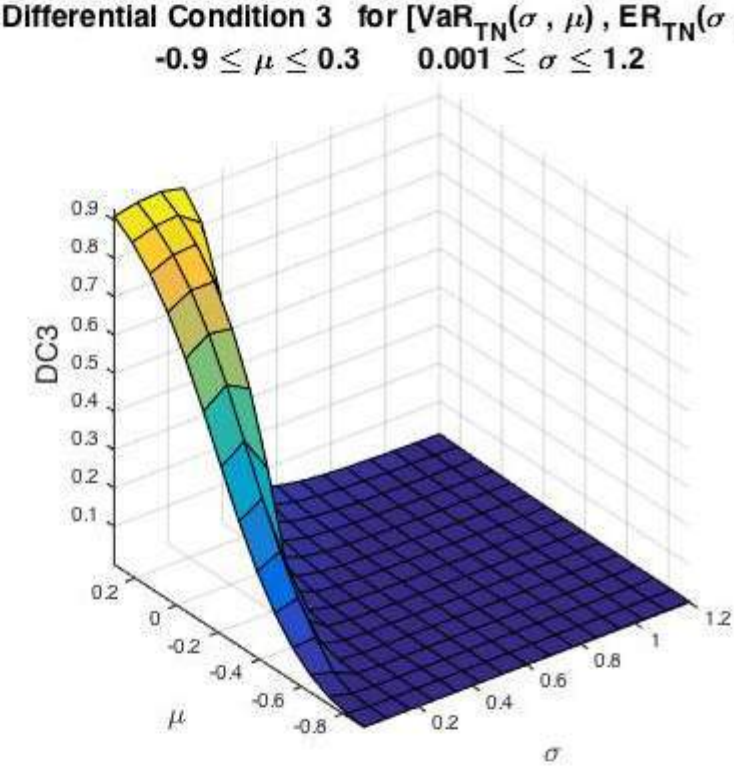
0,300	0,414	0,379	0,350	0,291	-0,192	-0,279	-0,205	-0,134	-0,088	-0,060	-0,044	-0,035	-0,029	-0,025	-0,023
0,214	0,487	0,446	0,410	0,283	-0,258	-0,252	-0,170	-0,106	-0,068	-0,047	-0,035	-0,028	-0,024	-0,021	-0,019
0,129	0,570	0,524	0,477	0,182	-0,276	-0,217	-0,134	-0,081	-0,052	-0,037	-0,028	-0,022	-0,019	-0,017	-0,016
0,043	0,661	0,610	0,547	-0,024	-0,263	-0,176	-0,101	-0,060	-0,039	-0,028	-0,022	-0,018	-0,016	-0,014	-0,014
-0,043	0,751	0,698	0,603	-0,182	-0,230	-0,134	-0,073	-0,044	-0,029	-0,021	-0,017	-0,014	-0,013	-0,012	-0,012
-0,129	0,833	0,780	0,607	-0,241	-0,184	-0,095	-0,051	-0,031	-0,021	-0,016	-0,013	-0,012	-0,011	-0,010	-0,010
-0,214	0,899	0,846	0,445	-0,237	-0,133	-0,063	-0,034	-0,021	-0,015	-0,012	-0,010	-0,009	-0,009	-0,008	-0,008
-0,300	0,946	0,887	0,080	-0,194	-0,086	-0,038	-0,021	-0,014	-0,011	-0,009	-0,008	-0,007	-0,007	-0,007	-0,007
-0,386	0,974	0,900	-0,151	-0,132	-0,048	-0,022	-0,013	-0,009	-0,007	-0,006	-0,006	-0,006	-0,006	-0,006	-0,006
-0,471	0,989	0,874	-0,189	-0,071	-0,023	-0,011	-0,007	-0,006	-0,005	-0,005	-0,004	-0,004	-0,004	-0,005	-0,005
-0,557	0,996	0,757	-0,129	-0,027	-0,008	-0,004	-0,003	-0,003	-0,003	-0,003	-0,003	-0,003	-0,004	-0,004	-0,004
-0,643	0,999	0,265	-0,045	-0,004	-0,000	-0,000	-0,001	-0,001	-0,002	-0,002	-0,002	-0,003	-0,003	-0,003	-0,003
-0,729	1,000	-0,070	0,004	0,006	0,004	0,002	0,001	0,000	-0,000	-0,001	-0,001	-0,002	-0,002	-0,002	-0,003
-0,814	1,000	0,022	0,017	0,009	0,006	0,004	0,002	0,001	0,000	-0,000	-0,001	-0,001	-0,002	-0,002	-0,002
-0,900	1,000	0,046	0,018	0,010	0,007	0,004	0,003	0,002	0,001	0,000	-0,000	-0,001	-0,001	-0,001	-0,002
μ↑σ→	**0,001**	**0,087**	**0,172**	**0,258**	**0,344**	**0,429**	**0,515**	**0,601**	**0,686**	**0,772**	**0,857**	**0,943**	**1,029**	**1,114**	**1,200**

Condition DC2 is not satisfied, as it is possible to see from Figure D.3 and *Table D.1*, where its values are reported. This means that this transformation, even if it is based on *Risk Averse Utility Function*, does not preserve the concavity property and there are regions of its domain where

$$\frac{\partial \psi}{\partial ER_{TN}} < 0$$

(see (4.9) and pose ER_{TN} instead of *T*) .

Figure D.4: Differential Condition 3 for $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$



DC3 is satisfied, it is > 0.

Appendix E.

Case 3: $R(\sigma,\mu) = Expected\ Shortfall = \ ES_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \ \gamma\ = 2.$

$\alpha \qquad = Confidence\ Level = 0.95$

Starting from the definitions of Expected Shortfall of a Truncated Normal:

$$-ES_{TN} = \frac{1}{(1-\alpha)\Delta\Phi_K}\int_{k_1}^{-VaR_{TN}} \frac{xe^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx$$

we have:

$$-ES_{TN} = \frac{1}{(1-\alpha)\Delta\Phi_K}\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} (\sigma\tau+\mu)\phi(\tau)\,d\tau$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} \tau\phi(\tau)\,d\tau + \mu\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} \phi(\tau)\,d\tau\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma[-\phi(\tau)]_{h_1}^{(-VaR_{TN}-\mu)/\sigma} + \mu[\Phi(-(VaR_{TN}+\mu)/\sigma) - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\{\sigma[\phi(h_1) - \phi(-(VaR_{TN}+\mu)/\sigma)] + \mu[\Phi(-(VaR_{TN}+\mu)/\sigma) - \Phi(h_1)]\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu[\Phi[\Phi_{inv}(b)] - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu[\mathrm{b} - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu[\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2) - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu(1-\alpha)[\Phi(h_2) - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu(1-\alpha)\Delta\Phi_K\right\}$$

and finally:

$$ES_{TN} = -\mu - \frac{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right]}{(1-\alpha)\Delta\Phi_K}$$

Therefore, the transformation becomes:

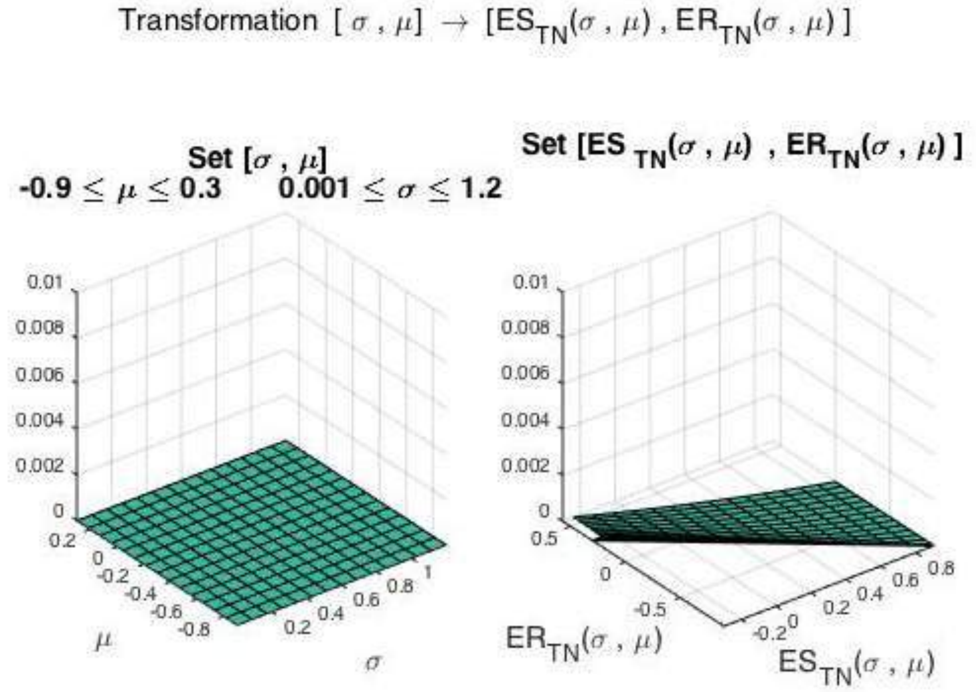
(E.1)

$$ES_{TN}(\sigma,\mu) = -\mu - \frac{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right]}{(1-\alpha)\Delta\Phi_K}$$

$$ME_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2 / 2\sigma^2}\, dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2}\, dx}$$

Formulas (E.1) transform the set $[\sigma,\mu]$ into the set $[ES_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see from the following representations:

Figure E.1: Transformation $[\sigma, \mu] \rightarrow [ES_{TN}(\sigma, \mu),\ ER_{TN}(\sigma, \mu)]$



Using the definitions (D.2) and the Theorem of derivative of the inverse functions (D.4):

$$\frac{d\Phi_{inv}(b)}{db} = \frac{1}{\frac{d\Phi(c)}{dc}} \quad iif \quad \frac{d\Phi(c)}{dc} \neq 0$$

we can compute the partial derivatives of:

$$\frac{\partial \phi\big(\Phi_{inv}(b)\big)(\sigma, \mu)}{\partial \sigma} = \frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial \sigma} = \frac{1}{\sqrt{2\pi}} \frac{\partial \exp\left(-\,{}^{\Phi_{inv}^2(b)}/_2\right)}{\partial \sigma}$$

We have:

$$\frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial \sigma} = -\frac{1}{\sqrt{2\pi}} exp\left(-\,{}^{\Phi_{inv}^2(b)}/_2\right) \cdot \Phi_{inv}(b) \cdot \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

$$= -\phi\big(\Phi_{inv}(b)\big) \cdot \Phi_{inv}(b) \cdot \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

By the definition of Φ:

$$\frac{d\Phi(c)}{dc} = \frac{d}{dc} \int_{-\infty}^{c} \phi(\tau)\, d\tau = \phi(c) = \phi\big(\Phi_{inv}(b)\big)$$

we have:

$$\frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = -\phi\big(\Phi_{inv}(b)\big)\cdot\Phi_{inv}(b)\cdot\frac{1}{\phi\big(\Phi_{inv}(b)\big)}\cdot\frac{\partial \mathrm{b}}{\partial\sigma} = -\Phi_{inv}(b)\cdot\frac{\partial \mathrm{b}}{\partial\sigma}$$

Using (D.3):

$$(E.2)\qquad \frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = \frac{\Phi_{inv}(b)}{\sigma}\left[\alpha h_1\phi(h_1) + (1-\alpha)h_2\phi(h_2)\right]$$

and with the same rationale:

$$(E.3)\qquad \frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\mu} = \frac{\Phi_{inv}(b)}{\sigma}\left[\alpha\phi(h_1) + (1-\alpha)\phi(h_2)\right]$$

We rewrite ES_{TN} as:

$$ES_{TN} = -\mu - \frac{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big]}{(1-\alpha)\Delta\Phi_K} = -\mu - \frac{\sigma\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)\int_{h_1}^{h_2} e^{-\tau^2/2}\,d\tau}$$

and using (B.1) we compute the partial derivatives of ES_{TN}:

$$\frac{\partial ES_{TN}}{\partial\mu} = -1 - \frac{\partial}{\partial\mu}\left\{\frac{\sigma^2\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)I1}\right\}$$

$$= -1 - \frac{\sigma^2\left\{I1\left[\frac{h_1}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\mu}\right] - \left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]\int_{h_1}^{h_2}\tau e^{-\tau^2/2}\,d\tau\right\}}{(1-\alpha)(I1)^2}$$

$$= -1 - \frac{\sigma^2\left\{I1\left[\frac{h_1}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\mu}\right] - \left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]I2\right\}}{(1-\alpha)(I1)^2}$$

Here, we can use (E.3) instead of $\partial\phi\big(\Phi_{inv}(b)\big)/\partial\mu$.

$$\frac{\partial ES_{TN}}{\partial\sigma} = -\frac{\partial}{\partial\sigma}\left\{\frac{\sigma^2\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)I1}\right\}$$

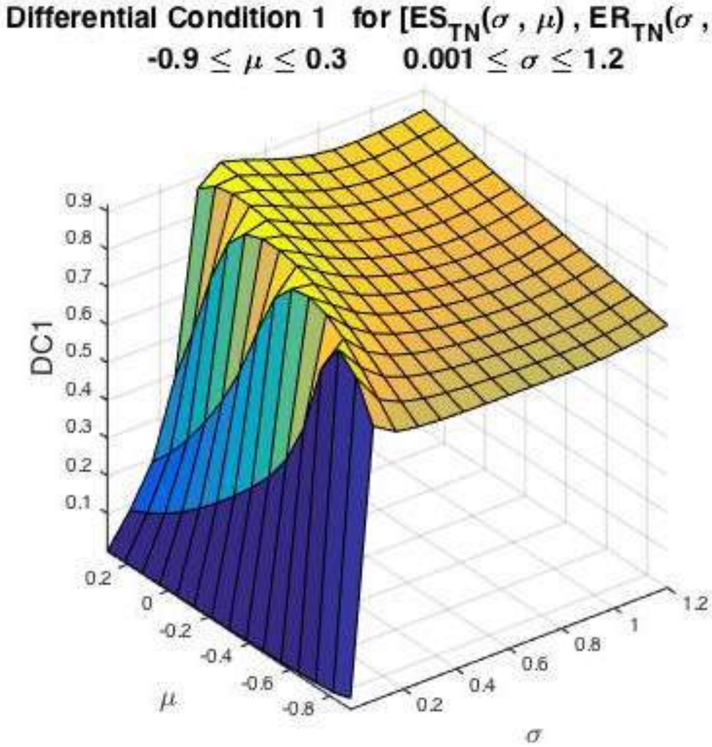
$$= -\frac{I1\left\{2\sigma\left[e^{-h_1^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right] + \sigma^2\left[\frac{h_1^2}{\sigma}e^{-h_1^2/2} - \sqrt{2\pi}\frac{\partial\phi\left(\Phi_{inv}(b)\right)}{\partial\sigma}\right]\right\} + \cdots}{(1-\alpha)(I1)^2}$$

$$\frac{\ldots - \sigma^2\left[e^{-h_1^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]\int_{h_1}^{h_2}\tau^2 e^{-\tau^2/2}\,d\tau}{(1-\alpha)(I1)^2}$$

$$= -\frac{\sigma\left\{\left[e^{-h_1^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right][2I1 - \sigma I3] + \sigma I1\left[\frac{h_1^2}{\sigma}e^{-h_1^2/2} - \sqrt{2\pi}\frac{\partial\phi\left(\Phi_{inv}(b)\right)}{\partial\sigma}\right]\right\}}{(1-\alpha)(I1)^2}$$

and then we have the following figures and tables:

Figure E.2: Differential Condition 1 for $[ES_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$



DC1 is satisfied, it is > *0*.

Figure E.3: Differential Condition 2 for $[ES_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$

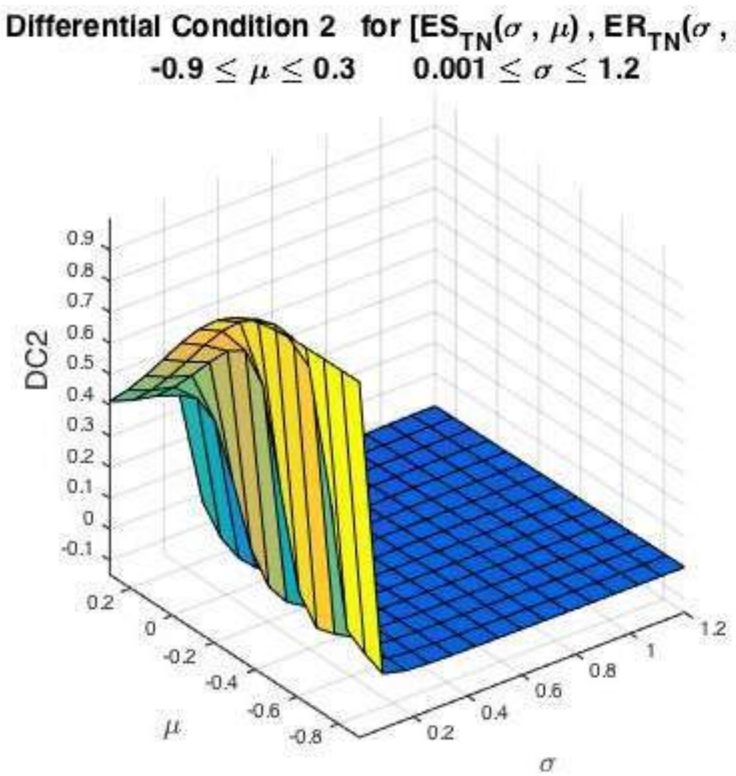
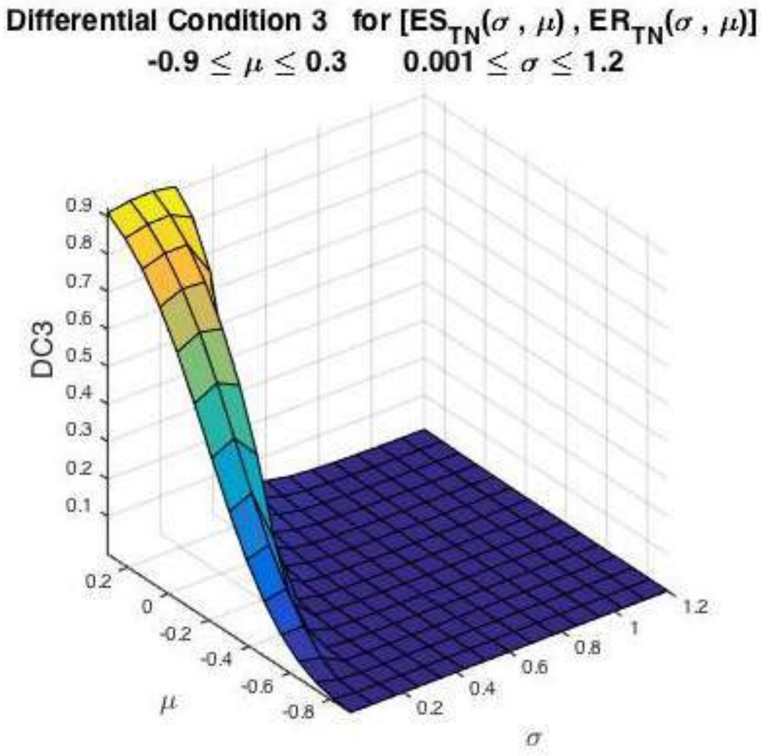


Table E.1: Values of the Differential Condition 2 for $[ES_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$

0,300	0,414	0,388	0,373	0,346	-0,027	-0,146	-0,093	-0,052	-0,030	-0,018	-0,013	-0,009	-0,007	-0,006	-0,006
0,214	0,487	0,457	0,439	0,374	-0,103	-0,125	-0,070	-0,037	-0,021	-0,013	-0,009	-0,007	-0,006	-0,005	-0,004
0,129	0,571	0,537	0,514	0,337	-0,131	-0,099	-0,050	-0,026	-0,015	-0,009	-0,007	-0,005	-0,004	-0,004	-0,004
0,043	0,661	0,625	0,595	0,180	-0,126	-0,072	-0,034	-0,017	-0,010	-0,007	-0,005	-0,004	-0,003	-0,003	-0,003
-0,043	0,751	0,715	0,668	0,003	-0,103	-0,048	-0,021	-0,011	-0,006	-0,004	-0,003	-0,003	-0,002	-0,002	-0,002
-0,129	0,834	0,799	0,707	-0,083	-0,073	-0,029	-0,012	-0,006	-0,004	-0,003	-0,002	-0,002	-0,002	-0,002	-0,002
-0,214	0,899	0,866	0,629	-0,095	-0,044	-0,015	-0,006	-0,003	-0,002	-0,002	-0,001	-0,001	-0,001	-0,001	-0,001
-0,300	0,946	0,911	0,321	-0,071	-0,022	-0,007	-0,003	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001
-0,386	0,974	0,931	0,040	-0,038	-0,008	-0,002	-0,000	0,000	0,000	0,000	-0,000	-0,000	-0,000	-0,000	-0,001
-0,471	0,989	0,924	-0,039	-0,011	0,000	0,001	0,001	0,001	0,001	0,001	0,000	0,000	-0,000	-0,000	-0,000
-0,557	0,996	0,861	-0,020	0,003	0,004	0,003	0,002	0,002	0,001	0,001	0,001	0,000	0,000	0,000	-0,000
-0,643	0,999	0,507	0,008	0,008	0,005	0,004	0,003	0,002	0,002	0,001	0,001	0,001	0,000	0,000	0,000
-0,729	1,000	0,096	0,017	0,009	0,005	0,004	0,003	0,002	0,002	0,001	0,001	0,001	0,001	0,000	0,000
-0,814	1,000	0,058	0,015	0,008	0,005	0,004	0,003	0,002	0,002	0,001	0,001	0,001	0,001	0,001	0,000
-0,900	1,000	0,031	0,012	0,007	0,005	0,004	0,003	0,002	0,002	0,002	0,001	0,001	0,001	0,001	0,000
μ↑σ→	**0,001**	**0,087**	**0,172**	**0,258**	**0,344**	**0,429**	**0,515**	**0,601**	**0,686**	**0,772**	**0,857**	**0,943**	**1,029**	**1,114**	**1,200**

Also in this case condition DC2 is not satisfied, see *Figure E.3* and *Table E.1*, and we can conclude with the same considerations done for *Differential Condition 2* of Appendix D.

Figure E.4: Differential Condition 3 for $[ES_{TN}(\sigma,\mu),\ ME_{TN}(\sigma,\mu)]$



DC3 is satisfied, it is > 0.

Appendix F
Quadratic Utility Function

Consider the following general Quadratic Utility Function (QUF):

$$(F.1) \qquad QUF(W) \equiv QUF = a + bW - cW^2 \qquad b,c > 0$$

where W is defined as in (2.1).

If the function (4.1) has positive first derivative and negative second derivative, it represents a risk-averse person with insatiable appetite, that is:

$$QUF' = b - 2cW > 0 \Rightarrow W < \frac{b}{2c} \equiv W_0(1+\mu_M)$$
$$QUF'' = -2c < 0 \qquad \Rightarrow c > 0$$

$$ARA[QUF] = -\frac{QUF''}{QUF'} = \frac{2c}{b-2cW} > 0 \quad , \qquad RRA[QUF] = \frac{2cW}{b-2cW}$$

In the Appendices F, G, H we take into consideration $r \backsim N(\mu, \sigma^2)$.

$W_0(1+\mu_M)$ is the maximum value allowed for W such that (F.1) maintains its characteristic of Risk aversion.

Proposition F.1: *With the definition* $b = 2cW_0(1+\mu_M)$, *the expected value of QUF in* (4.1), $E[Q(\mu_M)](\sigma,\mu)$, *is a function of Standard Deviation σ and Expected Return μ represented by a paraboloid in the space* $(\sigma, \mu, E[Q(\mu_M)](\sigma,\mu))$ *with downward concavity, whose vertex is given by the point* $\left(0, \mu_M, E[Q(\mu_M)](0,\mu_M)\right)$. *That is:*

$$E[Q(\mu_M)](\sigma,\mu) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu-\mu_M)^2]$$

where $QUF(W_0) = a + bW_0 - cW_0^2 = a + 2cW_0(1+\mu_M)W_0 - cW_0^2$.

Proof: Consider the expected value of the Quadratic Utility Function (F.1):

$$\begin{aligned} E[Q(\mu_M)] &= E[a + bW - cW^2] \\ &= E[a + bW_0(1+r) - cW_0^2(1+r)^2] \\ &= a + bW_0(1+E[r]) - cW_0^2(1 + 2E[r] + E[r^2]) \\ &= a + bW_0 + bW_0\mu - cW_0^2 - 2cW_0^2\mu - cW_0^2(\sigma^2+\mu^2) \\ &= QUF(W_0) + W_0\mu(b - 2cW_0) - cW_0^2(\sigma^2+\mu^2) \end{aligned}$$

Substituting parameter b with its expression, we have:

$$\begin{aligned} E[Q(\mu_M)] &= QUF(W_0) + W_0\mu(2cW_0 + 2c\mu_M W_0 - 2cW_0) - cW_0^2(\sigma^2+\mu^2) \\ &= QUF(W_0) + 2cW_0^2\mu\mu_M - cW_0^2(\sigma^2+\mu^2) \end{aligned}$$

Adding and subtracting the same quantity $cW_0^2\mu_M^2$ and considering the $E[Q(\mu_M)]$ as a function of σ and μ we obtain:

$$(F.2) \qquad E[Q(\mu_M)](\sigma,\mu) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu-\mu_M)^2]$$

The expression (F.2) represents a paraboloid in the space $(\sigma, \mu, E[Q(\mu_M)](\sigma,\mu))$ with downward concavity, whose vertex is the point $\left(0, \mu_M, E[Q(\mu_M)](0,\mu_M)\right)$.

We assume for simplicity $W_0 = 1$:

$$E[Q(\mu_M)](\sigma,\mu) = \psi(\sigma,\mu) = \ QUF(W_0) + c\mu_M^2 - c[\sigma^2 + (\mu - \mu_M)^2]$$

And we have

$$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} = -2c(\mu - \mu_M) \quad , \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} = -2c\sigma$$

that will be used for to compute the (4.6) for the Quadratic Utility Function case.

Appendix G

Case QUF 1: $R(\sigma,\mu) = Value\ at\ Risk = \ VaR(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = \mu$

$\psi(\sigma,\mu) = Expected\ QUF\ \ with\ \ \mu_M = 0.3,\ a = 10,\ b = 3,\ c = 15.$

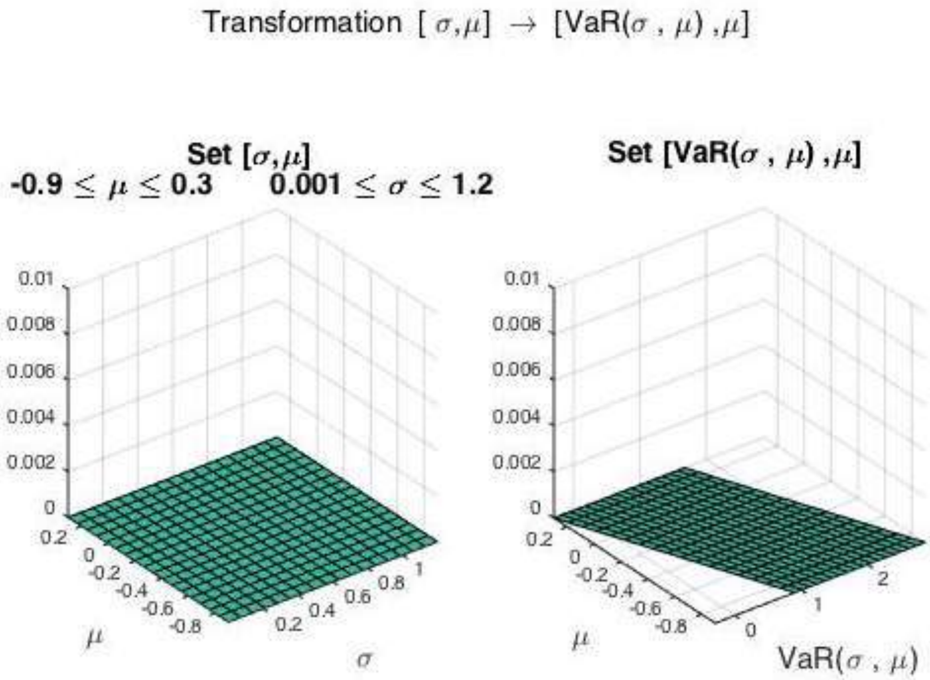
$\alpha \qquad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $VaR \equiv VaR(\sigma,\mu)$, starting from the transformation:

$$(G.1) \qquad R(\sigma,\mu) = VaR(\sigma,\mu) = -\mu + \sigma\Phi_{-1}(\alpha)\ , \quad T(\sigma,\mu) = \mu$$

The (G.1) transforms the set $[\sigma,\mu]$ in the set $[VaR(\sigma,\mu),\ \mu]$ as is possible to see:

Figure G.1: Transformation $[\sigma,\mu] \rightarrow [VaR(\sigma,\mu),\ \mu]$



The partial derivatives, using (G.1):

$$\frac{\partial \sigma_T}{\partial \sigma} = \Phi_{-1}(\alpha); \quad \frac{\partial \sigma_T}{\partial \mu} = -1; \quad \frac{\partial \mu_T}{\partial \mu} = 1; \quad \frac{\partial \mu_T}{\partial \sigma} = 0;$$

By (4.6), DC1: $2c\sigma > 0$ is true.

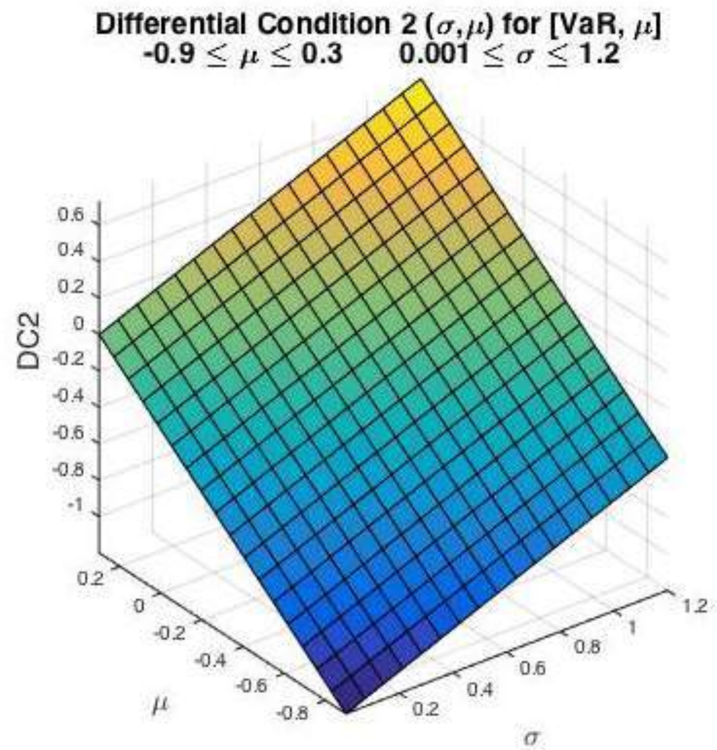
By (4.6), DC2:

$$-2c\Phi_{-1}(\alpha)(\mu - \mu_M) - (-2c\sigma)(-1) > 0$$

$$(G.2) \qquad \frac{\sigma}{\Phi_{-1}(\alpha)} < -(\mu - \mu_M) \rightarrow \frac{\sigma}{\Phi_{-1}(\alpha)} + \mu < \mu_M$$

we can represent the DC2 in closed form:

Figure G.2: Differential Condition2 for $[VaR(\sigma, \mu),\ \mu]$



DC2 is not satisfied, as is possible to see by *Figure G.1*. This means that this transformation, even if started by *Risk Averse Utility Function*, does not preserve the characteristic of the concavity and there are regions in the domain where:

$$\frac{\partial \psi}{\partial T} = \frac{\partial \psi}{\partial \mu} < 0$$

that is not typical of the *Risk Averse Utility Function, Theorem 2.1*.

By (G.2), taking in consideration that:

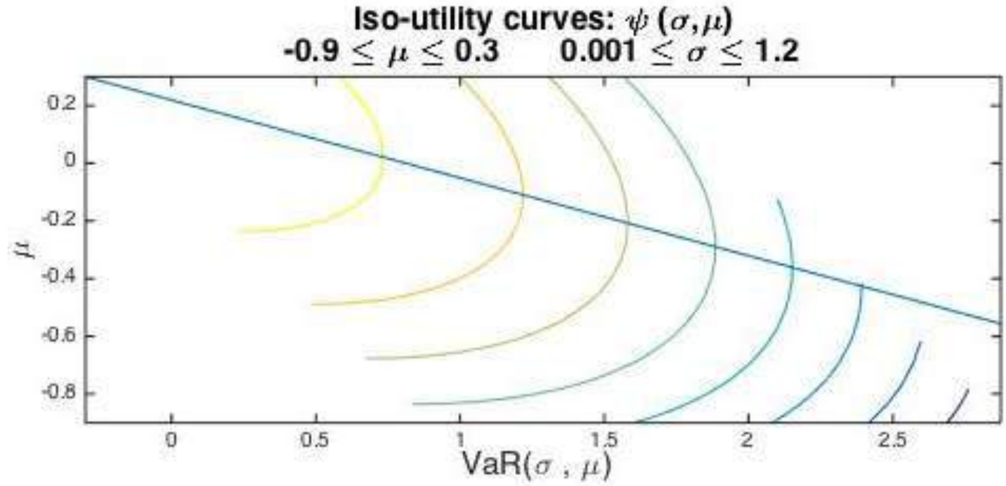
$$\frac{VaR + \mu}{\Phi_{-1}(\alpha)} = \sigma$$

we have:

$$(G.3) \qquad \frac{VaR + \mu}{[\Phi_{-1}(\alpha)]^2} + \mu < \mu_M \;\rightarrow\; \mu(1 + [\Phi_{-1}(\alpha)]^2) < -VaR + \mu_M[\Phi_{-1}(\alpha)]^2$$

The DC2 is respected only below the straight line (G.3), above the straight-line the iso-utility curves have negative slope.

Figure G.3: Iso-utility curves of $\psi(\sigma,\mu)$ in 2D $[VaR(\sigma,\mu),\mu]$



By (4.6), DC3: $\Phi_{-1}(\alpha) > 0$ is true.

Appendix H

Case QUF 2: $R(\sigma,\mu) = Expected\ Shortfall = ES(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = \mu$

$\psi(\sigma,\mu) = Expected\ QUF\ \ with\ \ \mu_M = 0.3,\ a = 10,\ b = 3,\ \mathrm{c} = 15.$

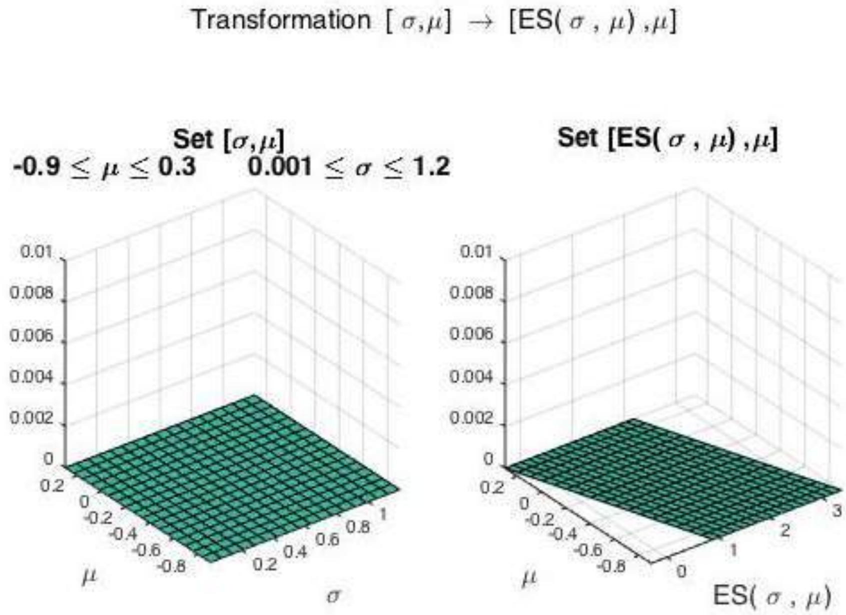
$\alpha \qquad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $ES \equiv ES(\sigma,\mu)$, starting from the transformation:

$$(H.1) \qquad R(\sigma,\mu) = ES(\sigma,\mu) = -\mu + \frac{\sigma}{1-\alpha}\phi[\Phi_{-1}(\alpha)], \quad T(\sigma,\mu) = \mu$$

The (H.1) transforms the set $[\sigma,\mu]$ in the set $[ES(\sigma,\mu),\ \mu]$ as is possible to see:

Figure H.1: Transformation $[\sigma, \mu] \rightarrow [ES(\sigma, \mu),\ \mu]$



By (4.6), DC1: $2c\sigma > 0$ is true.

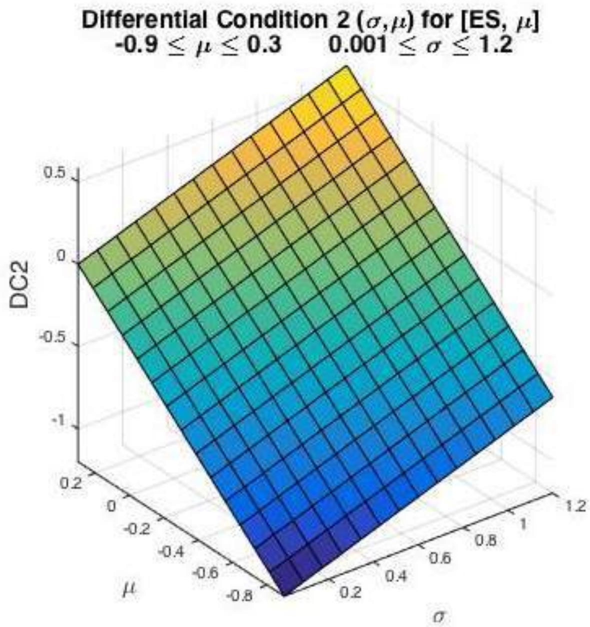
By (4.6), DC2:

$$-2c\frac{\phi[\Phi_{-1}(\alpha)]}{1-\alpha}(\mu - \mu_M) - (-2c\sigma)(-1) > 0$$

$$(H.2) \qquad \frac{\sigma(1-\alpha)}{\phi[\Phi_{-1}(\alpha)]} < -(\mu - \mu_M) \rightarrow \frac{\sigma(1-\alpha)}{\phi[\Phi_{-1}(\alpha)]} + \mu < \mu_M$$

We can represent the DC2 in closed form:

Figure H.2: Differential Condition2 for $[ES(\sigma, \mu),\ \mu]$



DC2 is not satisfied, as it is possible to see by *Figure H.2*. This means that this transformation, even if it starts with the *Risk Averse Utility Function*, does not preserve the characteristic of the concavity and there are regions in the domain where

$$\frac{\partial \psi}{\partial T} = \frac{\partial \psi}{\partial \mu} < 0$$

that is not typical of the *Risk Averse Utility Function (Theorem2.1).*

By (H.2), taking into consideration that:

$$\frac{(ES + \mu)(1 - \alpha)}{\phi[\Phi_{-1}(\alpha)]} = \sigma$$

we have:

$$(H.3) \quad \frac{(ES + \mu)(1 - \alpha)^2}{\phi[\Phi_{-1}(\alpha)]^2} + \mu < \mu_M \;\rightarrow\; \mu\{(1 - \alpha)^2 + \phi[\Phi_{-1}(\alpha)]^2\} < -ES + \mu_M \frac{\phi[\Phi_{-1}(\alpha)]^2}{(1 - \alpha)^2}$$

The DC2 is respected only below the straight line (H.3), above the straight-line the iso-utility curves have negative slope.

Figure H.3: Iso-utility curves of $\psi(\sigma, \mu)$ *in 2D* $[ES(\sigma, \mu), \mu]$