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Giorgio Fabbri, Silvia Faggian, and Giuseppe Freni

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Non-Existence of Optimal Programs in Continuous Time

Giorgio Fabbri
Aix-Marseille School of Economics, CNRS & EHES, France

Silvia Faggian
Ca' Foscari University of Venice, Department of Economics

Giuseppe Freni
University of Naples Parthenope, Department of Business and Economics

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Abstract: We report an example of a two-dimensional undiscounted convex optimal growth model in continuous time in which, although there is a unique "golden rule", no overtaking optimal solutions exists in a full neighbourhood of the steady state. The example confirms a conjecture advanced in 1976 that the minimum dimension for non-existence of overtaking optimal programs in continuous time is 2.

Keywords: Optimal growth, Overtaking, Continuous time models

JEL Codes: C61, D90

Address for correspondence:
Silvia Faggian
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
Phone: (++39) 041 2346913
e-mail: faggian@unive.it

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wp.dse@unive.it

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

1 Introduction

For the class of undiscounted convex models of optimal growth, it has been known since Gale (1967) that existence of optimal (in the sense of overtaking) solutions cannot be proved in general if the "golden rule" capital stock is not unique. Soon, however, it turned out that uniqueness is not sufficient for the existence of an optimal solution. Brock (1970), indeed, proved existence under this condition, but used the weaker optimality criterion know as maximality (or weak overtaking optimality) and presented an example of a maximal steady state that is not optimal. Peleg (1973) then pointed out that the same example can be used to prove non-existence of optimal paths, implying that, without additional assumptions, it is not possible to strength Brock's existence theorem.

To the best of our knowledge there are only three published examples of non-existence: the Brock-Peleg one, the one reported in Khan & Piazza (2010) and finally the one provided in a paper by Fabbri *et al.* (2015). Of these examples, the first two relate to different two-sector one capital good discrete models, whereas only the last one is in continuous time and, in addition, with an infinite-dimensional state space. So while it has been already established that in discrete time non-existence is possible even with a one-dimensional state space, it is not clear which is the minimum dimension for non-existence in continuous time. We here report a new example showing that the minimum dimension is 2. In other words, our example confirms the conjecture advanced in Brock & Haurie (1976) p. 345:

> We have not yet constructed an example where the steady state $\bar{x}$ is unique but no overtaking optimal program exists from some x^0 while a weakly overtaking optimal program exists from our x^0. Such an example will take some work to construct because it seems that the state space will have to be two dimensional whereas in discrete time as shown in Brock (1970) we can get by with a one-dimensional output space.

2 The Model

We consider the $(n+1)$-sector single-technique case of the discrete capital model introduced in Bruno (1967). In the system, there are $n+1$ commodities: n pure capital goods and a pure consumption good. The services of a primary factor of production, labor, are combined with the services of the stocks of capital to produce the $n+1$ commodities. Technology is of the discrete type, and only $n+1$ processes, one for each good, are available.[1]

A unit of the j-th capital good needs to be produced a_{ij} units of the i-th capital good and ℓ_j units of labour, whereas one unit of the consumption good needs α_i units of the i-th capital good and ℓ_c units of labour, so that the technology is described by a vector and a matrix of capital coefficients

$$A = [a_{ij}]_{i,j=1,..n}\,, \quad \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & ... & \alpha_n \end{bmatrix}^T.$$

and a vector ℓ and a scalar ℓ_c of labor input coefficients

[1] In Bruno (1967) the rate of discount is assumed to be strictly positive. Moreover, although choice of technique is allowed, a complete analysis is provided only for the homogeneous capital case.

$$\ell = \begin{bmatrix} \ell_1 & \ell_2 & \dots & \ell_n \end{bmatrix}^T, \quad \ell_c.$$

Let

$$k(t) = \begin{bmatrix} k_1(t) & k_2(t) & \dots & k_n(t) \end{bmatrix}^T$$

represent the stock of capital goods at a given time $t \geq 0$, and

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^T, \quad x_c(t)$$

be the intensities of activation of the production processes at that time (chosen by the social planner). Assuming that the flow of new capitals is accumulated and that capitals decay at a constant depreciation rate $\delta > 0$ (the same for all capital goods), and that the initial state of the system is $k_0 \geq 0$, then the state equation is given by the n-dimensional system

$$\begin{cases} \dot{k}(t) = -\delta k(t) + x(t), & t \geq 0 \\ k(0) = k_0. \end{cases} \tag{1}$$

Assume that the labour flow available at every t is constant and normalized to 1, and assume that every unit of capital good instantaneously provides one unit of production services. Then the production is subject to the following set of constraints, holding for all $t \geq 0$:

$$Ax(t) + x_c(t)\alpha \leq k(t), \tag{2}$$

$$\langle \ell, x(t) \rangle + x_c(t)\ell_c \leq 1, \tag{3}$$

$$x(t) \geq 0, x_c(t) \geq 0. \tag{4}$$

Assuming a linear utility and a positive discount factor $\rho \geq 0$, the problem is that of maximizing

$$J(x, x_c, k_0) = \int_0^{+\infty} e^{-\rho t} x_c(t)\, \mathrm{d}t \tag{5}$$

over the set of admissible controls

$$\mathcal{X}(k_0) = \{(x, x_c) \in L^1_{loc}(0, +\infty; \mathbb{R}^{n+1}_+) \ : \ (1) - (4) \text{ hold at all } t \geq 0.\} \tag{6}$$

We stress the fact that, due to the constraints (2)–(4), the admissible controls depend on the initial capital k_0.

Remark 2.1 Since from (1) one derives

$$k(t) = e^{-\delta t} k_0 + \int_0^t e^{-\delta(t-s)} x(s) ds$$

the solution k is in the space $W^{1,1}_{loc}(0, +\infty; \mathbb{R}^n)$. Note that in our assumption, trajectories k are always nonnegative, as they are bounded from below from $k_0 e^{-\delta t} \geq 0$. Moreover, if vector ℓ is strictly positive, then trajectories are uniformly bounded by a constant depending only on k_0. Indeed, if we define $c := \left(\sum_{i=1}^n \ell_i^{-2}\right)^{1/2}$ we may check that (3) implies $\|x\| \leq c$ so that

$$\|k(t)\| \leq \|k_0\| + c/\delta, \ \forall t \geq 0.$$

$\square$

Due to (3) and (4), when $\rho > 0$ the utility is finite for all admissible controls. On the contrary, when $\rho = 0$ the utility may be infinite valued. We take into consideration the following criteria of optimality.

Definition 2.2 *A control (x^*, x_c^*) in $\mathcal{X}(k_0)$ is said optimal (or overtaking optimal) at k_0 if, for every other control (x, x_c) in $\mathcal{X}(k_0)$*

$$\liminf_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\,\mathrm{d}t \geq 0.$$

If k^* is the trajectory starting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is said an optimal couple.

Definition 2.3 *A control (x^*, x_c^*) in $\mathcal{X}(k_0)$ is said maximal (or weakly overtaking) at k_0 if, for every other control (x, x_c) in $\mathcal{X}(k_0)$*

$$\limsup_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\,\mathrm{d}t \geq 0.$$

If k^* is the trajectory straiting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is said a maximal couple.

Every optimal control is maximal but the viceversa is false in general.

We here list the assumptions that will be used throughout the paper.

Hypothesis 2.4

1. *The matrix A is semipositive, that is, $a_{ij} \geq 0$ for all i and j and there is at least a positive element;*
2. *The vector α is semipositive, that is, $\alpha \geq 0$ and $\alpha_i > 0$ for at least one i.*
3. *The vector ℓ is positve, that is, $\ell_i > 0$ for all i; also $\ell_c > 0$.*
4. *A is indecomposable;*[2]

3 Golden Rules

The aim of this section is to define *golden rules*, that is, stationary solutions supported by stationary prices. Some properties of Hamiltonian functions will prove useful for the arguments developed afterwards. We define the *current value Hamiltonian* associated to the problem as $h : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$h(k, \lambda, x, x_c) = x_c + \langle \lambda, x - \delta k \rangle$$

and the *maximal value Hamiltonian* as

$$H(k, \lambda) = \sup\{h(k, \lambda, x, x_c) \ : \ (x, x_c) \geq 0, Ax + x_c\alpha \leq k, \langle \ell, x \rangle + x_c\ell_c \leq 1\}. \tag{7}$$

[2] In economic terms, this assumption means that each capital good enters directly or indirectly into the production of all capital goods. Since the vector α is semipositive, this also implies that each capital good enters directly or indirectly into the production of all goods. Indecomposable semipositive square matrices have some useful properties: their Perron Frobenious eigenvalue μ is positive, right and left eigenvectors associated to this root are unique (up to multiplication by a scalar), and $(I - \lambda A)^{-1} > 0$, whenever λ is a scalar such that $\mu\lambda < 1$ (see e. g., Kurz & Salvadori, 1995, Theorem A.3.5).

The maximization process through which the maximal value Hamiltonian is computed, corresponds to solving the following linear programming problem

$$\max[\langle \lambda, x\rangle + x_c] \tag{8}$$

subject to

$$\begin{array}{l} Ax + x_c\alpha \le k \\ \langle \ell, x\rangle + x_c\ell_c \le 1 \\ (x, x_c) \ge 0. \end{array} \tag{9}$$

which has feasible region

$$U(k) = \{(x, x_c) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \ : \ (9) \text{ holds}\}.$$

The corresponding dual problem is

$$\min[\langle q, k\rangle + w] \tag{10}$$

subject to

$$\begin{array}{l} \lambda \le A^T q + w\,\ell \\ 1 \le \langle \alpha, q\rangle + w\,\ell_c \\ q \ge 0, w \ge 0, \end{array} \tag{11}$$

where $(q,w) \in \mathbb{R}^n \times \mathbb{R}$ are dual control variables having the meaning, respectively, of rental rates of capital goods and wage rate (i.e., the multiplier associated to the constraint of availability of labour). We denote the feasible region of the dual problem by

$$V(\lambda) = \{(q, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \ : \ (11) \text{ holds}\}.$$

.

Remark 3.1 The set $U(k)$ is nonempty and compact as a consequence of Hypothesis (2.4.4), for every given and positive k, so that the maximum is attained at some (x^*, x_c^*) and, equivalently (see e. g., Franklin, 2002, Section 1.8), there exists an optimal solution (q^*, w^*) of the corresponding dual problem, namely

$$(x^*, x_c^*) \in \arg\max\{\langle \lambda, x\rangle + x_c : (x, x_c) \in U(k)\}$$

if and only if there exists (q^*, w^*) in $\mathbb{R}^{n+1}$ such that

$$(q^*, w^*) \in \arg\min\{\langle k, q\rangle + w : (q, w) \in V(\lambda)\}$$

with both sets above being nonempty, moreover

$$\begin{cases} \langle \lambda, x\rangle + x_c \le \langle k, q\rangle + w, & \forall x, x_c, q, w \\ \langle \lambda, x^*\rangle + x_c^* = \langle k, q^*\rangle + w^*. \end{cases} \tag{12}$$

Remark 3.2 As a consequence of (12), one has

$$\partial_\lambda H = -\delta k + x^*, \quad \partial_k H = q^* - \delta\lambda.$$

Heuristically speaking, the candidate conditions of optimality associated to the problem are the following:

$$\begin{cases} \dot{k}(t) = -\delta k(t) + x(t), & t \geq 0 \\ k(0) = k_0 \\ \dot{\lambda}(t) = (\rho + \delta)\lambda(t) - q(t), & t \geq 0 \\ (x(t), x_c(t)) \in \arg\max\{\langle \lambda(t), x \rangle + x_c : (x, x_c) \in U(k(t))\}, & t \geq 0 \\ (q(t), w(t)) \in \arg\min\{\langle k(t), q \rangle + w : (q, w) \in V(\lambda(t))\}, & t \geq 0 \end{cases} \tag{13}$$

As a consequence of the previous remarks, we define golden rules as follows.

Definition 3.3 *We say that* $(\bar{k}, \bar{x}, \bar{x}_c, \bar{\lambda}, \bar{w}, \bar{q})$ *is a* golden rule *if it is a stationary solution of (13).*

Remark 3.4 In the literature, golden rules are sometimes called *modified* golden rules when $\rho > 0$ (see e. g., Mas-Colell *et al.*, 1995, Definition 20.E.2).

4 Sufficient conditions of optimality

Now we develop sufficient conditions of optimality which we will use to test optimality of specific admissible couples.

Theorem 4.1 *Let Hypothesis 2.4 be satisfied. Assume also* $k_0 \in \mathbb{R}^n_+$, $(x^*, x^*_c) \in \mathcal{X}(k_0)$, *and that there exists* $\lambda^*, q^* : \mathbb{R}_+ \to \mathbb{R}^n_+$ *and* $w^* : \mathbb{R}_+ \to \mathbb{R}_+$, *with* λ^* *absolutely continuous,* q^* *and* w^* *measurable and locally bounded, so that* $(k^*, \lambda^*, x^*, x^*_c, q^*, w^*)$ *satisfies* (13) *for almost every* $t \geq 0$. *If in addition*

$$\lim_{t \to +\infty} e^{-\rho t} \langle k(t), \lambda(t) \rangle = 0 \tag{14}$$

then $(k^*; (x^*, x^*_c))$ *is an optimal couple.*

Proof. The proof is standard but we write it here for the reader's convenience. Given (x, x_c) in $\mathcal{X}(k_0)$, let $k(t) = k(t; k_0, x, x_c)$ and define

$$\Delta_T = \int_0^T e^{-\rho t}(x^*_c(t) - x_c(t))dt$$

Then it is enough to show that $\lim_{T \to +\infty} \Delta_T \geq 0$. Note that by concavity in k of the Hamiltonian and by Remark 3.2 one has

$$\begin{aligned} x^*_c(t) - x_c(t) = \\ &= h(x^*(t), x^*_c(t), k^*(t), \lambda^*) - h(x(t), x_c(t), k(t), \lambda^*(t)) - \langle \lambda^*(t), \dot{k}^*(t) - \dot{k}(t) \rangle \\ &\geq \langle k^*(t) - k(t), \partial_k H(k^*(t)) \rangle - \langle \lambda^*(t), \dot{k}^*(t) - \dot{k}(t) \rangle \\ &\geq \langle k^*(t) - k(t), q^*(t) - \delta\lambda^*(t) \rangle - \langle \lambda^*(t), \dot{k}^*(t) - \dot{k}(t) \rangle. \end{aligned} \tag{15}$$

Since by assumption on λ^* one has

$$
\begin{aligned}
\frac{d}{dt}&\left[e^{-\rho t}\langle \lambda^*(t), k^*(t)-k(t)\rangle\right] = \\
&= e^{-\rho t}\left[\langle \lambda^*(t), \dot{k}^*(t)-\dot{k}(t)\rangle + \langle \dot{\lambda}^*(t)-\rho\lambda^*(t), k^*(t)-k(t)\rangle\right] \\
&= e^{-\rho t}\left[\langle \lambda^*(t), \dot{k}^*(t)-\dot{k}(t)\rangle - \langle q^*(t)-\delta\lambda^*(t), k^*(t)-k(t)\rangle\right] \quad (16)
\end{aligned}
$$

comparing the last identity with (15), and then recalling that $k(0)=k^*(0)=k_0$ and that k and λ^* are both nonnegative, one obtains

$$\Delta_T \geq -\left[e^{-\rho t}\langle \lambda^*(t), k^*(t)-k(t)\rangle\right]_{t=0}^{t=T} \geq -e^{-\rho T}\langle \lambda^*(T), k^*(T)\rangle,$$

so that, taking the limits as T tends to $+\infty$, the conclusion is drawn by means of the transversality condition (14). $\square$

Theorem 4.2 *Assume Hypothesis 2.4. Denote by μ the Perron-Frobenius eigenvalue of A. Suppose that $\delta<\mu^{-1}$ and $0\leq\rho<\mu^{-1}-\delta$. Then there exists a unique golden rule $(\bar{k},\bar{x},\bar{x}_c,\bar{\lambda},\bar{w},\bar{q})$, given by*

$$
\begin{aligned}
\bar{x}_c &= \left[\delta\langle \ell, (I-\delta A)^{-1}\alpha\rangle + \ell_c\right]^{-1} && (17)\\
\bar{x} &= \delta\bar{x}_c(I-\delta A)^{-1}\alpha && (18)\\
\bar{k} &= \bar{x}_c(I-\delta A)^{-1}\alpha && (19)\\
\bar{w} &= \left[(\delta+\rho)\langle\alpha, \left[I-(\delta+\rho)A^T\right]^{-1}\ell\rangle + \ell_c\right]^{-1} && (20)\\
\bar{\lambda} &= \bar{w}\left[I-(\delta+\rho)A^T\right]^{-1}\ell && (21)\\
\bar{q} &= (\rho+\delta)\bar{\lambda}. && (22)
\end{aligned}
$$

Moreover, for $\rho>0$, $(\bar{k},\bar{x},\bar{x}_c)$ is optimal.

Remark 4.3 Note that the assumption $\delta<\mu^{-1}$ says that the system is vital, meaning that the production can be strictly greater than mere reproduction of capital goods after decay. As a consequence, the matrix $(I-\delta A)$ is invertible, with positive inverse $(I-\delta A)^{-1}$, as A is indecomposable. Similarly $0\leq\rho<\mu^{-1}-\delta$ implies $(I-(\delta+\rho)A^T)$ is invertible with positive inverse $(I-(\delta+\rho)A^T)^{-1}$.

Remark 4.4 Note that $\bar{x}, \bar{x}_c$ and $\bar{k}$ in the previous theorem do not depend on the discount ρ.

Proof of Theorem 4.2. We show first that (13) is uniquely satisfied (among stationary solutions) by $(\bar{k},\bar{x},\bar{x}_c,\bar{\lambda},\bar{w},\bar{q})$. Note that the first and third equation in (13) imply

$$\bar{x}(t)\equiv\bar{x}=\delta\bar{k}, \quad \bar{q}(t)\equiv\bar{q}=(\rho+\delta)\bar{\lambda}.$$

Note also that the argmax/argmin conditions in (13) coincide with (9) (11). We then multiply the first inequality in (9) by $\bar{q}$, the second by $\bar{w}$ and sum them up to obtain

$$\langle A\bar{x},\bar{q}\rangle + \bar{x}_c\langle\alpha,\bar{q}\rangle + \langle\ell,\bar{x}\rangle\bar{w} + \bar{x}_c\ell_c\bar{w} \leq \langle\bar{k},\bar{q}\rangle + \bar{w}. \qquad (23)$$

Similarly, we multiply the first inequality in (11) by $\bar{x}$, the second by $\bar{x}_c$ and sum them up to obtain

$$\langle\bar{\lambda},\bar{x}\rangle + \bar{x}_c \le \langle\bar{x}, A^T\bar{q}\rangle + \bar{x}_c\langle\alpha,\bar{q}\rangle + \langle\ell,\bar{x}\rangle\bar{w} + \bar{x}_c\ell_c\bar{w}. \tag{24}$$

By (12), the right hand side in (23) coincides with the left hand side in (24), so that all inequalities hold as equalities. As a consequence, we have a golden rule for any solution $(\bar{k},\bar{x}_c,\bar{\lambda},\bar{w})$ of the following simplified system

$$\begin{cases} \delta A\bar{k} + \bar{x}_c\alpha \le \bar{k}, \quad \langle\delta A\bar{k} + \bar{x}_c\alpha - \bar{k},\bar{\lambda}\rangle = 0 \\ \delta\langle\ell,\bar{k}\rangle + \bar{x}_c\ell_c \le 1, \quad (\delta\langle\ell,\bar{k}\rangle + \bar{x}_c\ell_c - 1)\bar{w} = 0 \\ \bar{\lambda} \le (\delta+\rho)A^T\bar{\lambda} + \bar{w}\,\ell, \quad \langle\bar{\lambda} - (\delta+\rho)A^T\bar{\lambda} - \bar{w}\,\ell,\bar{k}\rangle = 0 \\ 1 \le (\delta+\rho)\langle\alpha,\bar{\lambda}\rangle + \bar{w}\,\ell_c, \quad (1-(\delta+\rho)\langle\alpha,\bar{\lambda}\rangle + \bar{w}\,\ell_c)\bar{x}_c = 0 \\ \bar{k}\ge 0, \quad \bar{x}_c \ge 0, \quad \bar{\lambda}\ge 0, \quad \bar{w}\ge 0. \end{cases} \tag{25}$$

We claim that $\bar{w} > 0$. Indeed, assume by contradiction that $\bar{w} = 0$ and that e_μ is the eigenvector associated with the Perron-Frobenius eigenvalue μ. Then, from the third line in (25) we derive

$$\langle e_\mu,\bar{\lambda}\rangle \le \mu(\rho+\delta)\langle e_\mu,\bar{\lambda}\rangle$$

which implies $1 \le \mu(\rho+\delta)$, in contradiction with the assumptions. From $\bar{w} > 0$ we deduce that the inequality in the second line of (25) is an equality. Moreover, since (12) implies $\bar{x}_c = \bar{w} + \rho\langle\bar{\lambda},\bar{k}\rangle$, also $\bar{x}_c > 0$, so that the inequality in the the fourth line inequality of (25) is an equality. Next we show that $\bar{k} > 0$. In fact, as $(I-\delta A)^{-1}$ is positive, the first inequality in (25) is equivalent to

$$\bar{k} \ge \bar{x}_c(I-\delta A)^{-1}\alpha > 0.$$

The fact that $\bar{k} > 0$ implies that the inequality in the third line of (25) is satisfied as equality. Then, from Remark 4.3,

$$\bar{\lambda} = \bar{w}(I-(\delta+\rho)A^T)^{-1}\ell,$$

so that also $\bar{\lambda} > 0$. As a consequence, the inequality in the first line of (25) is satisfied as equality, that is

$$\bar{k} = \bar{x}_c(I-\delta A)^{-1}\alpha.$$

Summing up, the unique solution of (25) is obtained by solving as equalities the inequalities of the system, that is

$$\begin{cases} \bar{k} = \bar{x}_c(I-\delta A)^{-1}\alpha \\ \delta\langle\ell,\bar{k}\rangle + \bar{x}_c\ell_c = 1 \\ \bar{\lambda} = \bar{w}(I-(\delta+\rho)A^T)^{-1}\ell \\ 1 = (\delta+\rho)\langle\alpha,\bar{\lambda}\rangle + \bar{w}\,\ell_c \end{cases} \tag{26}$$

which has $(\bar{k},\bar{x},\bar{x}_c,\bar{\lambda},\bar{w},\bar{q})$ as unique solution. When $\rho > 0$, (14) is trivially satisfied by $(\bar{k},\bar{\lambda})$, and the golden rule is optimal as a consequence of Theorem 4.1. $\square$

4.1 The undiscounted case $\rho = 0$

Throughout this subsection we assume $\rho = 0$. In this case, the application of the results by Brock & Haurie (1976) (see also Carlson *et al.*, 1991, chapter 4) will provide the existence of a maximal (or weakly overtaking) couple starting at those k_0 from which the steady state $\bar{k}$ can be reached in finite time along an admissible trajectory. Through the same techniques, we will see that a sufficient criterium for maximality for an admissible strategy is that of being a minimizer of the integral of a suitable *value-loss function*. Some preliminary work is needed.

Proposition 4.5 *Assume $\rho = 0$. The maximal Hamiltonian H defined by (7) has a saddle point at $(\bar{k}, \bar{\lambda})$.*

Proof. We first show that $H(\bar{\lambda}, k)$ has a maximum at $\bar{k}$. Note that

$$H(\bar{\lambda}, k) = -\delta \left\langle \bar{\lambda}, k \right\rangle + \max\{\langle \bar{\lambda}, x\rangle + x_c : (x, x_c) \in U(k)\} = \langle \bar{\lambda}, x^* - \delta k\rangle + x_c^*$$

where $(x^*, x_c^*) = (x^*(k), x_c^*(k))$ is the admissible strategy where the maximum is attained (see Remark 3.1). From (26) we derive

$$x_c^* = x_c^* \delta \ \langle \bar{\lambda}, \alpha\rangle + x_c^* \bar{w} l_c$$

while from (25) we get

$$\langle \bar{\lambda}, x^*\rangle = \delta \ \langle \bar{\lambda}, Ax^*\rangle + \bar{w}\langle \ell, x^*\rangle$$

which summed up give, after subtracting $\delta\langle \hat{k}, \bar{\lambda}\rangle + \bar{w}$, the following equality

$$x_c^* + \langle \bar{\lambda}, x^* - \delta k\rangle - \bar{w} = \delta\langle \bar{\lambda}, Ax^* + x_c^* \alpha - k\rangle + \bar{w}(x_c^* \ell_c + \langle \ell, x^*\rangle - 1)$$

but since $(x^*, x_c^*) \in U(k)$, the right hand side is less or equal than 0, implying

$$H(\bar{\lambda}, k) \leq \bar{w}$$

On the other hand, applying (12) and the formula for q given by Theorem 4.2, one has

$$\bar{w} = \bar{x}_c + \langle \bar{\lambda}, \bar{x}\rangle - \langle \bar{q}, \bar{k}\rangle = \bar{x}_c + \langle \bar{\bar{\lambda}}, \bar{x} - \delta \bar{k}\rangle = \bar{x}_c$$

so that

$$H(\bar{\lambda}, k) \leq \bar{x}_c = H(\bar{\lambda}, \bar{k}) \tag{27}$$

We then show that $H(\lambda, \bar{k})$ has a minimum at $\bar{\lambda}$. Note that from the definition, and the fact that $(\bar{x}, \bar{x}_c)$ is admissible at $\bar{k}$, and $\bar{x} = \delta \bar{k}$, one gets

$$\begin{aligned} H(\lambda, \bar{k}) &= -\delta \left\langle \lambda, \bar{k} \right\rangle + \max\{\langle \lambda, x\rangle + x_c : (x, x_c) \in U(\bar{k})\} \\ &\geq -\delta \left\langle \lambda, \bar{k} \right\rangle + \delta \left\langle \lambda, \bar{k} \right\rangle + \bar{x}_c \\ &= \bar{x}_c = H(\bar{\lambda}, \bar{k}). \end{aligned}$$

□

Now, in analogy with Carlson *et al.* (1991), set

$$V(k,v)=\{(x,x_c):(x,x_c)\in U(k), x=\delta k+v\}$$

which is a compact and convex, possibly empty, set, and

$$\mathcal{L}(k,v)=\begin{cases}\max\{x_c:(x,x_c)\in V(k,v)\} & V(k,v)\neq\emptyset\\ -\infty & V(k,v)=\emptyset\end{cases}$$

and the value-loss function as

$$\theta(k,v)=\mathcal{L}(\bar{k},0)-\mathcal{L}(k,v)-\left\langle\bar{\lambda},v\right\rangle. \tag{28}$$

This function, which gives the value-loss of any admissible couple at the steady state competitive prices, is the analogous of the value-loss function commonly used for discrete time optimal growth problems.

Remark 4.6 Note that

$$\theta(k,v)\geq 0,\ \forall(k,v)\in\mathbb{R}^n\times\mathbb{R}^n. \tag{29}$$

Indeed, $\mathcal{L}(k,v)$ is concave in both variable (see Carlson *et al.*, 1991, Lemma 4.3), and $(\bar{k},\bar{\lambda})$ is a saddle point for the Hamiltoninan H, with $\bar{k}$ the steady state of the unique golden rule, as it is shown in proposition 4.5.[3] Then Assumption 4.5 in Carlson *et al.* (1991) holds, and (29) is a consequence of Rockafellar (1970), Theorem 37.5.

In Carlson *et al.* (1991), the authors pair the original problem with an associated Lagrange Problem (briefly, ALP), that of minimizing the integral of the value-loss function along $(k(t),\dot{k}(t))$, namely

$$\int_0^\infty \theta(k(t),\dot{k}(t))dt \tag{30}$$

A solution is defined as an absolutely continuous function $k^*:[0,\infty)\to\mathbb{R}^n$, such that $k(0)=k_0$, and

$$\liminf_{T\to\infty}\int_0^T\left[\theta(k(t),\dot{k}(t))-\theta(k^*(t),\dot{k}^*(t))\right]dt\geq 0$$

Theorem 4.7 *Assume $k_0\in\mathbb{R}^n_+$, and that $\bar{k}$ is reachable from k_0, along an admissible trajectory, in finite time. Then:*

- (i) *there exists a solution of the ALP;*
- (ii) *all solutions of ALP are maximal (that is, weakly overtaking) trajectories for the original problem. In particular the golden rule is a maximal solution.*

Proof. The proof of (i) follows from Theorem 4.7 in Carlson *et al.* (1991), as $\mathcal{L}(k,v)$ is concave, and $(\bar{k},\bar{\lambda})$ is a saddle point for the Hamiltoninan H. Moreover the set of velocities $\varphi(k)=\{x-\delta k:(x,x_c)\in U(k)\}$ is a compact convex set (indeed $\varphi(k)=-\delta k+\pi_1(U(k))$), that is, the translated by $-\delta k$ of the projection on the first coordinate of the compact convex set $U(k)$). The proof of (ii) can be deduced from the proof of Theorem 4.9 p.69, where the fact is shown under Assumption 4.5 (and not 4.4 as erroneously reported there) p.64. □

[3] Note that the golden rule stock is also the unique stationary solution of the maximization problem (4.34) defined in Carlson *et al.* (1991), moreover our H coincides with the Hamiltonian $\mathcal{H}$ there defined in (4.81), $\mathcal{H}(k,\lambda)=\sup_{v\in\mathbb{R}^n}\{\mathcal{L}(k,v)+\langle\lambda,v\rangle\}$.

For the proof of Theorem 5.6 the following result contained in Carlson *et al.* (1991) will prove useful.

Lemma 4.8 *Consider a trajectory k of system* (1)(2)(3)(4), *such that*

$$\liminf_{T\to\infty}\int_0^T \left[\mathcal{L}(k(t),\dot k(t))-\mathcal{L}(\bar k,0)\right]dt>-\infty. \tag{31}$$

Then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T k(t)dt=\bar k. \tag{32}$$

Proof. It is enough to note that all trajectories of the system are bounded (see Remark 2.1) and apply Lemma 4.7 p.69 in Carlson *et al.* (1991). □

Definition 4.9 *A control (x,x_c) is good if the associated trajectory $k(\cdot,(x,x_c))$ satisfies (31).*

Lemma 4.10 *Assume that $k_0\in\mathbb{R}^n_+$ is such that there exists an admissible control stirring k_0 into $\bar k$ in finite time. Any maximal (respectively, optimal) control at k_0 is good.*

Proof. Assume (x,x_c) is an admissible control at k_0 which is not good, and let k be the associated trajectory. Note that by definition of $\mathcal{L}$

$$\int_0^T(\bar x_c-x_c(t))\,dt\ge\int_0^T\left[\mathcal{L}(\bar k,0)-\mathcal{L}(k(t),\dot k(t))\right]dt$$

Consider now $(k^y,(y,y_c))$ admissible at k_0, with (y,y_c) stirring k_0 into $\bar k$ in $[0,T_0]$ and then coinciding with $(\bar x,\bar x_c)$ in $(T_0,+\infty)$. Then for all $T\ge T_0$:

$$\begin{aligned}\int_0^T(y_c(t)-x_c(t))\,dt&=\int_0^{T_0}(y_c(t)-x_c(t))\,dt+\int_{T_0}^T(\bar x_c-x_c(t))\,dt\\&\ge\int_0^{T_0}(y_c(t)-x_c(t))\,dt-\int_0^{T_0}|\bar x_c-x_c(t)|\,dt+\int_0^T\left[\mathcal{L}(\bar k,0)-\mathcal{L}(k(t),\dot k(t))\right]dt\end{aligned}$$

where the first and second addenda are bounded for all $T>0$ whereas, when taking the limsup (or liminf) of both sides as T tends to $+\infty$, the third is unbounded from above by assumption. Hence we proved that for every nongood control one may build another control which overtakes the first. Hence a maximal (or optimal) control need be good. □

5 The example of nonexistence

We introduce the following example and study the behaviour of specific solutions both in the discounted and undiscounted case. We set

$$n=2,\ \delta=\frac{2}{3},\ \ell_c=1,\ A=\begin{bmatrix}1/2&3/4\\1/4&7/8\end{bmatrix},\quad\alpha=\begin{bmatrix}1/4\\3/4\end{bmatrix}\quad\ell=\begin{bmatrix}1\\1\end{bmatrix}. \tag{33}$$

Note that Hypothesis 2.4 is verified and A has eigenvalues $(\pm\sqrt{57}+11)/16$, with $\mu^{-1} \simeq 0.86254 > 2/3$. By means of the Theorem 4.2, the golden rule can be explicitely computed and it is given by

$$\bar{x}_c = \frac{2}{9}, \quad \bar{x} = \begin{bmatrix} \frac{23}{63} \\ \\ \frac{26}{63} \end{bmatrix}, \quad \bar{k} = \begin{bmatrix} \frac{23}{42} \\ \\ \frac{13}{21} \end{bmatrix} \tag{34}$$

The golden rule above is associated to a triple $(\bar{\lambda}, \bar{q}, \bar{w})$ that can be explicitely computed as a function of ρ:

$$\bar{w} = \frac{72\rho^2 - 300\rho + 56}{81\rho^2 + 252}, \quad \bar{\lambda} = \begin{bmatrix} \frac{56-60\rho}{27\rho^2+84} \\ \\ \frac{24\rho+112}{27\rho^2+84} \end{bmatrix}, \quad \bar{q} = \frac{2+\rho}{3} \begin{bmatrix} \frac{56-60\rho}{27\rho^2+84} \\ \\ \frac{24\rho+112}{27\rho^2+84} \end{bmatrix} \tag{35}$$

Now consider system (1) and choose the admissible controls that satisfy (2) (3) as equalities. By inverting those relations, one obtains

$$\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} A & \alpha \\ \ell^T & \ell_c \end{bmatrix}^{-1} \begin{bmatrix} k(t) \\ 1 \end{bmatrix} \tag{36}$$

giving x, x_c as functions of k, which substituted into (1) imply

$$\begin{cases} \dot{k}_1(t) = -\frac{2}{9}k_1(t) - \frac{16}{9}k_2(t) + \frac{11}{9} \\ \dot{k}_2(t) = \frac{16}{9}k_1(t) + \frac{2}{9}k_2(t) - \frac{10}{9}. \end{cases} \tag{37}$$

The system above can be explicitely solved. The matrix of the system has purely imaginary eigenvalues so that one obtains periodic solutions

$$\begin{aligned} \hat{k}_1(t) &= c_1 \cos\frac{2\sqrt{7}t}{3} - c_2 \sin\frac{2\sqrt{7}t}{3} + \frac{23}{42} \\ \hat{k}_2(t) &= \frac{1}{8}\left(c_2 + 3\sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{1}{8}\left(3\sqrt{7}c_2 - c_1\right)\cos\frac{2\sqrt{7}t}{3} + \frac{13}{21} \end{aligned} \tag{38}$$

where the constants c_1 and c_2 depend on $k_0 = (k_1^0, k_2^0)$:

$$k_1^0 - \bar{k}_1 = c_1, \qquad k_2^0 - \bar{k}_2 = \frac{3\sqrt{7}}{8}c_2 - \frac{1}{8}c_1. \tag{39}$$

The associated controls $(\hat{x}, \hat{x}_c)$ can also be computed by means of (36),

$$\begin{aligned} \hat{x}_1(t) &= \frac{1}{3}(c_1 - 2\sqrt{7}c_2)\cos\frac{2\sqrt{7}t}{3} - \frac{1}{3}\left(2c_2 + 2\sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{23}{63} \\ \hat{x}_2(t) &= \frac{1}{3}\left(5c_1 + \sqrt{7}c_2\right)\cos\frac{2\sqrt{7}t}{3} + \frac{1}{3}\left(\sqrt{7}c_1 - 5c_2\right)\sin\frac{2\sqrt{7}t}{3} + \frac{26}{63} \\ \hat{x}_c(t) &= \frac{1}{3}(\sqrt{7}c_2 - 7c_1)\cos\frac{2\sqrt{7}t}{3} + \frac{1}{3}\left(7c_2 + \sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{2}{9}. \end{aligned} \tag{40}$$

Note that (39) imply that c_1 and c_2 are small for small differences of k_0 from $\bar{k}$. As a consequence, for c_1 and c_2 small enough:

- the whole trajectory is contained in a ball, centered at $\bar{k}$ and of arbitrarily small radius;
- $(\hat{x}, \hat{x}_c)$ is also cycling at an arbitrarily small neighborhood of $(\bar{x}, \bar{x}_c)$;
- since for $k_0 = \bar{k}$ the trajectory $k(t) \equiv \bar{k}$ satisfies strictly the positivity constraints (4), that remains true also for k_0 close enough to $\bar{k}$; hence, the constraints on the associated dual variables (the argmin condition in (13)) hold unchanged, and support prices associated to $\hat{k}, \hat{x}, \hat{x}_c$ coincide with $\bar{\lambda}, \bar{q}, \bar{w}$.

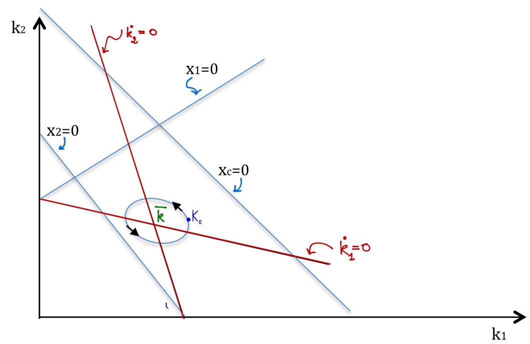

Figure 1: The cycles (38).

Assume now $\rho > 0$. As a consequence of the previous arguments, for k_0 close enough to $\bar{k}$, $(\hat{k}, (\hat{x}, \hat{x}_c), \bar{\lambda}, \bar{q}, \bar{w})$ satisfy the assumptions of Theorem 4.1 and is hence optimal.

We have then proved the following result.

Proposition 5.1 *Let $k_0 > 0$. The system* (1)(2)(3)(4)*, with data* (33)*, has a periodic solution $(\hat{k}, (\hat{x}, \hat{x}_c))$ given by* (38)(40)*. For $\rho > 0$ and for k_0 sufficiently close to $\bar{k}$, the admissible couple $(\hat{k}, (\hat{x}, \hat{x}_c))$ is optimal at k_0, and it is supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$.*

Now we prove that, in the specific case here described, any initial condition $k_0 > 0$ can be driven to the steady state $\bar{k}$ in finite time by means of an admissible control. The property holds in general for any $\rho \geq 0$ although we will make use of it only in the next subsection, where we analyse the case of a null ρ.

Lemma 5.2 *Let $k_0 \in \mathbb{R}^2$, $k_0 > 0$, be fixed. Then there exists $T(k_0) \geq 0$ and a control $(\tilde{x}, \tilde{x}_c) \in \mathcal{X}(k_0)$ such that the associated trajectory $\tilde{k}(t) \equiv k(t; k_0, \tilde{x}, \tilde{x}_c)$ reaches $\bar{k}$ at time $T(k_0)$, that is $\tilde{k}(T(k_0)) = \bar{k}$.*

Proof. We first consider the case in which k_0 lies on $\{\gamma\bar{k} \,:\, \gamma \in \mathbb{R}^+\}$, the ray through 0 and $\bar{k}$, that is, $k_0 = \gamma_0\bar{k}$, for a $\gamma_0 > 0$. If $\gamma_0 = 1$, there is nothing to prove. If $\gamma_0 > 1$, we choose

$$\tilde{x}(t) = 0, \quad \tilde{x}_c(t) = 0, \quad \forall t \geq 0,$$

so that the constraints are trivially satisfied, and $\tilde{k}(t) = \gamma_0\bar{k}e^{-\delta t}$, for all $t \geq 0$. With the choice $T(k_0) = \delta^{-1}\ln\gamma_0$, we obtain $\tilde{k}(T) = \bar{k}$. If instead $\gamma_0 < 1$, we choose

$$\tilde{x}_c(t) = 0, \quad \text{and} \quad \tilde{x}(t) = g\tilde{k}(t), \quad \forall t \geq 0,$$

and choose the constants g and T so that the associated pair is admissible, as it is shown next. Since $\tilde{k}$ solves $\dot{k}(t) = (g - \delta)k(t)$,then $\tilde{k}(t) = \gamma_0\bar{k}e^{(g-\delta)t}$, for all t in $[0, T]$. We choose then

$$T(k_0) = (g - \delta)^{-1}\ln(1/\gamma_0),$$

so that $\tilde{k}(T(k_0)) = \bar{k}$. We show now that we can choose $g > \delta$ so that the previous expression is meaningful and strictly positive. Indeed, in order for $(\tilde{k}, \tilde{x}, \tilde{x}_c)$ to be admissible, the following inequalities need be satisfied for all $t \in [0, T(k_0)]$

$$gA\tilde{k}(t) \leq \hat{k}(t), \qquad g\left\langle \ell, \tilde{k}(t) \right\rangle \leq 1. \tag{41}$$

Note that the second inequality is satisfied for all t in $[0, T(k_0)]$ if and only if $ge^{(g-\delta)T}\gamma_0 \left\langle \ell, \bar{k} \right\rangle \leq 1$, that is $g \leq \left\langle \ell, \bar{k} \right\rangle^{-1}$, so that (41) becomes

$$gA\bar{k} \leq \bar{k}, \quad \text{and} \quad g \leq \left\langle \ell, \bar{k} \right\rangle^{-1}$$

If we set $g_1 = \arg\max_{g \geq 0}\{(I - gA)\bar{k}\}$, and $g_2 = \left\langle \ell, \bar{k} \right\rangle^{-1}$, then any suitable g need satisfy

$$\delta < g \leq \min\{g_1, g_2\}$$

provided the interval is nonempty. In the case of the example one has

$$\delta = \frac{2}{3} = 0.67, \quad g_1 = \frac{23}{31} = 0.74, \quad g_2 = \frac{9}{7} = 1.28$$

so that g may be chosen as follows

$$\frac{2}{3} < g \leq \frac{23}{31}.$$

We now treat the case when $k_0 \notin \{\gamma\bar{k} : \gamma_0 \in \mathbb{R}^+\}$. Assume for instance that $k_{02}/k_{01} > \bar{k}_2/\bar{k}_1$, and choose a control (y, y_c), with associated trajectory k^y as follows

$$y_c(t) = 0, \quad y_1(t) = gk_1^y(t), \quad \text{and} \quad y_2(t) = 0, \quad \forall t \geq 0,$$

with g a positive constant, so that the trajectory is

$$k_1^y(t) = e^{(g-\delta)t}k_{10}, \quad k_2^y(t) = e^{-\delta t}k_{20}, \quad \forall t \geq 0,$$

which reaches $\{\gamma\bar{k} : \gamma_0 \in \mathbb{R}^+\}$ when $k_2^y(t)/k_1^y(t) = \bar{k}_2/\bar{k}_1$, that is, at time

$$T_0 = \frac{1}{g}\ln\left(\frac{k_{02}}{k_{01}}\frac{\bar{k}_1}{\bar{k}_2}\right) > 0.$$

The chosen control (y, y_c) is then admissible when for all $t \in [0, T_0]$

$$\begin{cases} a_{11}g \leq 1 \\ a_{21}ge^{gt}k_{01} \leq k_{02} \\ \ell_1 ge^{(g-\delta)t}k_{10} \leq 1 \end{cases}$$

If $g \in (0, \delta)$, the previous is satisfied if and only if

$$\begin{cases} g \leq 1/a_{11} \\ g \leq \bar{k}_2/(\bar{k}_1 a_{21}) \\ g \leq 1/(\ell_1 k_{10}) \end{cases}$$

so that we may choose

$$0 < g \leq \min\left\{\delta, \frac{1}{a_{11}}, \frac{\bar{k}_2}{\bar{k}_1 a_{21}}, \frac{1}{\ell_1 k_{10}}\right\}.$$

Once on $\{\gamma\bar{k} : \gamma_0 \in \mathbb{R}^+\}$, we may stir the trajectory on $\bar{k}$ by making use of the control $(\hat{x}, \hat{x}_c)$ built in the first part of the proof, pulled back of the time T_0, and reach the steady state in time $T(k_0) = T_0 + T(k^y(T_0))$. □

5.1 The undiscounted case

We use the previous example to show that the golden rule given by Definition 3.3 may fail to be optimal when the discount ρ is null. More in general, we will prove that when $\rho = 0$ the cycles described by (38)(40) are maximal but fail to be optimal for all k_0 close enough to $\bar{k}$, and derive as a particular case that the golden rule $(\bar{k}, \bar{x}, \bar{x}_c)$ is maximal and not optimal at $\bar{k}$.

Remark 5.3 Regardless the initial condition, when $\rho = 0$ the utility yielded by the control $\hat{x}_c(t)$ described in (40) in a time interval of a period length equals the utility yielded by $\hat{x}_c$ in the same time span. Note that the period of the cycle is

$$P = 3\pi/\sqrt{7}.$$

Then

$$\int_\sigma^{\sigma+P} \hat{x}_c(t)dt = \int_\sigma^{\sigma+\frac{3\pi}{\sqrt{7}}} \bar{x}_c dt = \frac{2\sqrt{7}}{21}\pi, \quad \forall\sigma \geq 0,$$

as one can check by direct computation. □

Lemma 5.4 *For all initial capital stocks $k_0 \in \mathbb{R}^n_+$, there exists a maximal couple starting at k_0. In particular, the cycles described by* (38)(40) *and the golden rule* $(\bar{k}, \bar{x}, \bar{x}_c)$ *are maximal.*

Proof. In order to apply Theorem 4.7 it is enough to show that cycles described by (38) are minimizers of the integral of losses described in (30). Note that

$$\begin{aligned}\theta(\hat{k}(t), \dot{\hat{k}}(t)) &= \mathcal{L}(\bar{k}, 0) - \mathcal{L}(\hat{k}(t), \hat{x}(t) - \delta\hat{k}(t)) - \left\langle \bar{\lambda}, \hat{x}(t) - \delta\hat{k}(t) \right\rangle \\ &= \bar{x}_c - \hat{x}_c(t) - \left\langle \bar{\lambda}, \hat{x}(t) - \delta\hat{k}(t) \right\rangle.\end{aligned}$$

Moreover, since $(\hat{k}, \hat{x}, \hat{x}_c)$ are supported by the same prices $\bar{\lambda}, \bar{q}, \bar{w}$ of the golden rule, then (12) implies

$$\bar{x}_c + \langle \bar{\lambda}, \bar{x} \rangle - \left[\langle \bar{k}, \bar{q} \rangle + \bar{w}\right] = 0 = \hat{x}_c(t) + \langle \bar{\lambda}, \hat{x}(t) \rangle - \left[\left\langle \hat{k}(t), \bar{q} \right\rangle + \bar{w}\right]$$

Recalling that $\bar{q} = \delta\bar{\lambda}$ and $\bar{x} - \delta\bar{k} = 0$, one has

$$\begin{aligned}0 &= \left\{\bar{x}_c + \langle \bar{\lambda}, \bar{x} \rangle - \left[\langle \bar{k}, \bar{q} \rangle + \bar{w}\right]\right\} - \left\{\hat{x}_c(t) + \langle \bar{\lambda}, \hat{x}(t) \rangle - \left[\left\langle \hat{k}(t), \bar{q} \right\rangle + \bar{w}\right]\right\} \\ &= \bar{x}_c - \hat{x}_c(t) - \left\langle \bar{\lambda}, \hat{x}(t) - \delta\hat{k}(t) \right\rangle = \theta(\hat{k}(t), \dot{\hat{k}}(t)) \quad (42)\end{aligned}$$

Then cycles $\hat{k}$ are minimizers for the ALP and, as a consequence, $(\hat{k}, \hat{x}, \hat{x}_c)$ is a maximal couple. □

Remark 5.5 The control described in (40) is good. Indeed, since cycles realize a null loss, one has

$$\mathcal{L}(k(t), \dot{k}(t)) - \mathcal{L}(\bar{k}, 0) = -\left\langle \bar{\lambda}, \dot{\hat{k}} \right\rangle$$

and since by direct proof one has

$$\int_0^P \dot{\hat{k}}_1(t)dt = \int_0^P \dot{\hat{k}}_2(t)dt = 0$$

then (31) is implied by

$$\int_0^T \left| \left\langle \bar{\lambda}, \dot{\hat{k}}(t) \right\rangle \right| dt \leq \frac{3\pi}{\sqrt{7}} \left\| \bar{\lambda} \right\| \max_{t \in [0,P]} \left| \dot{\hat{k}}(t) \right|, \quad \forall T \geq 0.$$

Theorem 5.6 *There exists a neighborhood U of $\bar{k}$ in $\mathbb{R}^n_+$, such that for all k_0 in U the cycles $(\hat{k}, \hat{x}, \hat{x}_c)$ starting at k_0 and described by* (38)(40) *are not optimal at k_0. More in general, with data* (33)*, there is no admissible control which is optimal at such k_0.*

From the previous Theorem, we derive the following corollary.

Corollary 5.7 *The golden rule $(\bar{k}, \bar{x}, \bar{x}_c)$ is not optimal at $\bar{k}$,and there is no admissible control which is optimal at $\bar{k}$.*

Proof of Theorem 5.6. We start by proving that the cycles $(\hat{k}, \hat{x}, \hat{x}_c)$ starting at k_0 are not optimal. We proceed as follows: we build an admissible couple $(k^y, (y, y_c))$ starting at k_0 and such that, in a (small) interval $[0, \tau]$, the control (y, y_c) yields a utility which positively exceeds that of the cycle $(\hat{x}, \hat{x}_c)$, and in $[\tau, +\infty)$ it stirs the trajectory along the cycle starting at $k^y(\tau)$ and described by (38)(40). Doing so that the overall control (y, y_c) yields a utility that periodically overtakes that of $(\hat{x}, \hat{x}_c)$, implying that $(\hat{x}, \hat{x}_c)$ is not optimal.

Assume $\theta > 0$ is such that for all $k_0 \in B(\bar{k}, 2\theta)$ the cycles described by (38)(40) starting at k_0 are supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$. Then select $k_0 \in B(\bar{k}, \theta)$, so that (39) imply

$$|c_1| < \theta, \quad |c_2| < 9\theta/(3\sqrt{7})$$

and (40) implies for all $t \geq 0$

$$|\hat{x}_i(t) - \bar{x}_i| < 6\theta, \text{ and } |\hat{x}_c(t) - \bar{x}_c| < 7\theta. \tag{43}$$

For an arbitrarily chosen $\tau > 0$, we set

$$y(t) = 0, \quad y_c(t) = \bar{x}_c + 8\theta, \text{ for all } t \in [0, \tau]$$

By explicit calculations, one may see that there exist positive θ_1 and τ_1 such that for all $0 < \theta < \theta_1$ and $0 < \tau < \tau_1$ the constraints (2) (3) (4) are satisfied in $[0, \tau]$ (for instance, $\tau_1 = 3/2$, $\varepsilon_1 = (26 - 7e)[42\,(1 + 6e)]^{-1}$). We assume also $\tau < \tau_2$ where $\tau_2 > 0$ is such that $\|k^y(\tau_2) - k_0\| < \theta$, so that the cycle starting at $k^y(\tau)$ and described by (38)(40) is supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$. Instead for $t \in (\tau, +\infty)$ we set

$$y(t) = \hat{x}(t - \tau), \quad y_c(t) = \hat{x}_c(t - \tau), \text{ for } t \in (\tau, +\infty)$$

Note that (43) implies

$$y_c(t) > \hat{x}_c(t) + \theta, \text{ for all } t \in [0, \tau]$$

so that

$$\int_0^\tau y_c(t)dt > \int_0^\tau \hat{x}_c(t)dt + \tau\theta$$

By direct clculation one obtains (see also Remark 5.3, where P is defined) that for all $n \in \mathbb{N}$

$$\int_0^{\tau+nP} (y_c(t) - \hat{x}_c(t))dt \geq \tau\theta > 0, \tag{44}$$

and $(\hat{x}, \hat{x}_c)$ cannot be optimal.

Now we show that no admissible strategy can be optimal at k_0. Assume by contradiction that there exists a strategy $(\tilde{x}, \tilde{x}_c) \in \mathcal{X}(k_0)$ optimal at k_0. Then for $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt \geq -\varepsilon, \text{ and } \int_0^T (\tilde{x}_c(t) - y_c(t))dt \geq -\varepsilon, \text{ for all } T \geq T_\varepsilon. \tag{45}$$

Since $T \mapsto \int_0^T (y_c(t) - \hat{x}_c(t))dt$, is continuous and (44) holds, there exists a (small) $\upsilon > 0$ such that,

$$\int_0^T (y_c(t) - \hat{x}_c(t))dt \geq \frac{\tau\theta}{2}, \text{ for all } T \in [\tau, \tau + \upsilon].$$

Set $T_n = \tau + nP$, and $n_\varepsilon = \min\{n \in \mathbb{N} : T_n > T_\varepsilon\}$. Then, by periodicity

$$\begin{aligned} \int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt &= \int_0^T (y_c(t) - \hat{x}_c(t))dt + \int_0^T (\tilde{x}_c(t) - y_c(t))dt \\ &\geq \frac{\tau\theta}{2} - \varepsilon, \text{ for any } T \in [T_n, T_n + \upsilon], \ n \geq n_\varepsilon \end{aligned} \tag{46}$$

We show first that

$$\liminf_{n\to\infty} \frac{1}{T_n + \upsilon} \int_0^{T_n+\upsilon} \left(\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt \right) dT \geq \frac{\theta\tau\upsilon}{4P} \tag{47}$$

Note that, if $* = \int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt$, one may split the previous integral as follows, and use (45) and (46) to derive

$$\begin{aligned} \int_0^{T_n+\upsilon} (*)\, dT &= \int_0^{T_{n_\varepsilon}} (*)\, dT + \sum_{i=n_\varepsilon}^{n} \int_{T_i}^{T_i+\upsilon} (*)\, dT + \sum_{i=n_\varepsilon}^{n-1} \int_{T_i+\upsilon}^{T_{i+1}} (*)\, dT \\ &\geq \int_0^{T_{n_\varepsilon}} (*)\, dT + (n - n_\varepsilon + 1) \left(\frac{\tau\theta}{2} - \varepsilon \right) \upsilon - (P - \upsilon)\varepsilon(n - n_\varepsilon) \end{aligned} \tag{48}$$

so that

$$\frac{1}{T_n + \upsilon} \int_0^{T_n+\upsilon} (*)\, dT \geq \frac{\theta\tau\upsilon}{2P} - \varepsilon + o\left(\frac{1}{n}\right)$$

with $o(1/n)$ tending to 0, as n tends to $+\infty$, and consequently (47) holds when $\varepsilon \leq (\theta\tau\upsilon)/(4P)$.

On the other hand, (29) and (42) imply

$$\tilde{x}_c(t) - \hat{x}_c(t) \leq \langle \bar{\lambda}, \dot{\hat{k}}(t) - \dot{\tilde{k}}(t) \rangle$$

so that

$$\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))\, dt \leq \langle \bar{\lambda}, \hat{k}(T) - \tilde{k}(T) \rangle$$

and

$$\frac{1}{S}\int_0^S \left(\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))\, dt \right) dT \leq \langle \bar{\lambda}, \frac{1}{S}\int_0^S \left(\hat{k}(T) - \tilde{k}(T) \right) dT \rangle \tag{49}$$

Note that both $(\hat{x}, \hat{x}_c)$ and $(\tilde{x}, \tilde{x}_c)$ are good controls in the sense of Definition 4.9 (the first by Remark 5.5, the second by Lemma 4.10). Hence, by Lemma 4.8 one has

$$\lim_{S\to\infty} \int_0^S \hat{k}(T)dT = \lim_{S\to\infty} \int_0^S \tilde{k}(T)dT = \bar{k},$$

so that passing to limits in (49) and comparing with (47) we derive a contraddiction. □

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