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LINEAR OPTIMAL CONTROL IN MULTI-SEGMENT MARKETING [1]

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Summary. We consider a linear optimal control model for the marketing of seasonal products which are produced by the same firm and sold by retailers in different market segments. The horizon is divided in two consecutive non-intersecting intervals, production and selling periods, respectively. The production period state variables are the inventory levels and two kinds of goodwills (for consumers and for retailers) while the selling period state variables are the sales levels and the two kinds of goodwills. In the production interval there are three kinds of controls: on production, quality and advertising, while in the selling one the controls are on communication via advertising, promotion for consumers and incentives for retailers. The optimal control problem is transformed into an equivalent nonlinear programming problem.

Introduction

In 1962 Nerlove and Arrow [6] proposed a dynamic model taking into account the *goodwill* level. Their paper originated a research stream and a great number of publications (see e.g. the review paper [5]).
In this paper we consider a linear optimal control model for the marketing of seasonal products which are produced by the same firm and sold by retailers in different market segments.
More precisely, the firm produces and sells a seasonal product with different attributes in two consecutive time intervals, the first one devoted to production and the second one to the sale of the product. In the production period the firm can control production, quality and can make advertising both to retailers and to the different market segments. In the selling period the firm has a wider communication possibility via promotion for consumers, incentives for retailers and advertising for both. The communication expenditure acts on the consumers' and retailers' behaviours modifying their goodwills. The motion equations of the model refer explicitly to the consumers' and retailers' goodwills as state variables. The firm seeks the maximum profit and has to satisfy some requirement for what concerns the minimum level of goodwills at the end of the selling period. This way we generalize the linear models proposed in [4], [2], [3]. For the sake of simplicity we will consider the case of only one kind of communication for every segment and every retailer (cf. [1]).

1. A multi-segment linear model

Consider n segments, r retailers and introduce the index sets $I = \{1, \dots, n\}$, $J = \{1, \dots, r\}$.

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For i-th segment, $i \in I$, define state variables $m_i(t)$ (inventory level at time t, $t \in [t_0, t_1]$), $x_i(t)$ (sales level at time t, $t \in [t_1, t_2]$), $A_{1i}(t)$ (consumer goodwill at time t, $t \in [t_0, t_2]$) and control $u_i(t)$ (production expenditure rate at time t, $t \in [t_0, t_1]$). For j-th retailer, $j \in J$, define state variable $A_{2j}(t)$ (j-th retailer goodwill at time t, $t \in [t_0, t_2]$). Define control $a(t)$ (communication expenditure rate at time $t \in [t_0, t_2]$ with upper bound $\overline{a}$). Let q be the expenditure rate for improving quality; as in one-segment linear model [3], q will be considered constant and nonnegative with the upper bound $\overline{q}$. For i-th segment, $i \in I$, define the constants $p_i > 0$ (product sale price), $c_i > 0$ (inventory marginal cost per time unit), $A_{1i}^0 = A_{1i}(t_0)$, A_{1i}^2 (lower bound of A_{1i} at t_2), $\overline{u}_i > 0$ (upper bound for $u_i(t)$), $l_{x_i} \geq 0$ and $l_{A_{1i}} \geq 0$ (marginal productivity of the expenditure rate q for sale and for goodwill A_{1i}), $k_{u_i} > 1$ (marginal productivity of production expenditure), $\delta_{1i} > 0$ (decay rate of A_{1i}), $\alpha_i > 0$ (saturation aversion parameter), $\delta_{x_i A_1} > 0$ (goodwill productivity in terms of sale related to A_{1i}), $\beta_i > 0$ (word-of-mouth productivity in terms of A_{1i}), $\epsilon_{A_{1i}}^{(p)} \geq 0$, $\epsilon_{x_i}^{(s)} \geq 0$ and $\epsilon_{A_{1i}}^{(s)} \geq 0$ (marginal productivity of expenditure in communication in the production period via advertising in terms of A_{1i}, and in the sales period via promotion in terms of x_i and via advertising in terms of A_{1i}). For j-th retailer, $j \in J$, define the constants $A_{2j}^0 = A_{2j}(t_0)$), A_{2j}^2 (lower bound of A_{2j} at t_2), $l_{A_{2j}} \geq 0$ (marginal productivity of q for A_{2j}), $\delta_{2j} > 0$ (decay rate of A_{2j}), $\epsilon_{A_{2j}}^{(p)} \geq 0$ and $\epsilon_{A_{2j}}^{(s)} \geq 0$ (marginal productivity of communication expenditure in terms of A_{2j} via advertising in the production period, and via advertising and incentives in the sales period). For every $i \in I$, $j \in J$ define the constant $\delta_{x_i A_{2j}} \geq 0$ (goodwill productivity in terms of sale in i-th segment related to the j-th retailer goodwill A_{2j}).

Define the state variables vectors $m(t)$, $x(t)$, $A_1(t)$ with elements $m_i(t)$, $x_i(t)$, $A_{1i}(t)$, $i \in I$; and $A_2(t)$ with elements $A_{2j}(t)$, $j \in J$; control vector $u(t)$ with elements $u_i(t)$, $i \in I$; constant vectors p, c, A_1^0, l_x, l_{A_1}, A_1^2, $\overline{u}$, $\epsilon_{A_1}^{(p)}$, $\epsilon_x^{(s)}$, $\epsilon_{A_1}^{(s)}$ with elements p_i, c_i, A_{1i}^0, l_{x_i}, $l_{A_{1i}}$, A_{1i}^2, $\overline{u}_i$, $\epsilon_{A_{1i}}^{(p)}$, $\epsilon_{x_i}^{(s)}$, $\epsilon_{A_{1i}}^{(s)}$, $i \in I$; and A_2^0, l_{A_2}, A_2^2, $\epsilon_{A_2}^{(p)}$, $\epsilon_{A_2}^{(s)}$ with elements A_{2j}^0, $l_{A_{2j}}$, A_{2j}^2, $\epsilon_{A_{2j}a_k}^{(p)}$, $\epsilon_{A_{2j}a_k}^{(s)}$, $j \in J$; diagonal matrices k_u, δ_1, α, δ_{xA_1}, β with diagonal elements k_{u_i}, δ_{1i}, α_i, $\delta_{x_i A_1}$, β_i, $i \in I$; and δ_2 with diagonal elements δ_{2j}, $j \in J$; matrix δ_{xA_2} with elements $\delta_{x_i A_{2j}}$, $i \in I$, $j \in J$. Define

$$A(t) = \begin{pmatrix} A_1(t) \\ \\ A_2(t) \end{pmatrix}, \ t \in [t_0, t_2], \ X(t) = \begin{pmatrix} x(t) \\ \\ A(t) \end{pmatrix}, \ t \in [t_1, t_2], \ A^0 = \begin{pmatrix} A_1^0 \\ \\ A_2^0 \end{pmatrix}, \ A^2 = \begin{pmatrix} A_1^2 \\ \\ A_2^2 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ & \\ 0 & \delta_2 \end{pmatrix}, \ E_p = \begin{pmatrix} \epsilon_{A_1}^{(p)} \\ \\ \epsilon_{A_2}^{(p)} \end{pmatrix}, \ Q = \begin{pmatrix} \alpha & -\delta_{xA_1} & -\delta_{xA_2} \\ & & \\ -\beta & \delta_1 & 0 \\ & & \\ 0 & 0 & \delta_2 \end{pmatrix}, \ E_s = \begin{pmatrix} \epsilon_x^{(s)} \\ \\ \epsilon_{A_1}^{(s)} \\ \\ \epsilon_{A_2}^{(s)} \end{pmatrix}, \ L = \begin{pmatrix} l_x \\ \\ l_{A_1} \\ \\ l_{A_2} \end{pmatrix}.$$

This way the linear marketing model with n segments and r retailers is

$$\begin{aligned} P: \text{maximize} \quad & px(t_2) - \int_{t_0}^{t_1} (cm(t) + \sum_{i \in I} u_i(t) + q)\,\mathrm{d}t - \int_{t_0}^{t_2} a(t)\,\mathrm{d}t, \\ \text{subject to} \quad & \dot{m}(t) = k_u u(t), \ \dot{A}(t) = -\Delta A(t) + a(t) E_p\ , \ t \in [t_0, t_1], \ m(t_0) = 0, \ A(t_0) = A^0, \\ & \dot{X}(t) = -QX(t) + a(t)E_s + q(t_1 - t_0)L, \ t \in [t_1, t_2], \\ & x(t_1) = 0, \ x(t_2) \leq m(t_1), \ A(t_2) \geq A^2, \ 0 \leq u(t) \leq \overline{u}, \ a(t) \in [0, \overline{a}], \ q \in [0, \overline{q}]. \end{aligned}$$

2. Decomposition into parametric subproblems

Problem P can be solved by first solving the subproblems, depending on parameters and then solving nonlinear programming problem in which the parameters are the decision variables. Define parameters vectors $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_2)$, $\tilde{A}$, $\bar{A}$. Let $\tilde{m} = m(t_1)$, $\tilde{A} = A(t_1)$, $\bar{A} \geq A^2$. Parametric subproblems $P_1^{(i)} = P_1^{(i)}(\tilde{m}_i)$, $i \in I$, $P_2 = P_2(\tilde{A})$, $P_3 = P_3(\tilde{m}, q, \tilde{A}, \bar{A})$ are

$$P_1^{(i)} : \text{maximize} \quad -\int_{t_0}^{t_1} (c_i m_i(t) + u_i(t))\, dt,$$
$$\text{subject to} \quad \dot{m}_i(t) = k_{u_i} u_i(t), \quad m_i(t_0) = 0, \quad m_i(t_1) = \tilde{m}_i, \quad u_i(t) \in [0, \overline{u}_i].$$

$$P_2 : \text{maximize} \quad -\int_{t_0}^{t_1} a(t)\, dt,$$
$$\text{subject to} \quad \dot{A}(t) = -\Delta A(t) + a(t) E_p,\ A(t_0) = A^0,\ A(t_1) = \tilde{A},\ a(t) \in [0, \overline{a}].$$

$$P_3 : \text{maximize} \quad -\int_{t_1}^{t_2} a(t)\, dt,$$
$$\text{subject to} \quad \dot{X}(t) = -QX(t) + a(t) E_s + q_1 L,\ X(t_1) = \tilde{X},\ X(t_2) = \bar{X},\ a(t) \in [0, \overline{a}],$$

where $q_1 = (t_1 - t_0) q$, $\tilde{X} = \begin{pmatrix} 0 \\ \tilde{A} \end{pmatrix}$, $\bar{X} = \begin{pmatrix} \tilde{m} \\ \bar{A} \end{pmatrix}$. Let $\tilde{F}_1^{(i)} = \tilde{F}_1^{(i)}(\tilde{m}_i)$, $i \in I$, $\tilde{F}_2 = \tilde{F}_2(\tilde{A})$, $\tilde{F}_3 = \tilde{F}_3(\tilde{m}, q, \tilde{A}, \bar{A})$ be the optimal values of problems $P_1^{(i)}$, $i \in I$, P_2, P_3 respectively. Then problem P is equivalent to a nonlinear programming problem, where the objective function

$$p\tilde{m} - (t_1 - t_0) q + \sum_{i=1}^{n} \tilde{F}_1^{(i)} + \tilde{F}_2 + \tilde{F}_3 \tag{1}$$

must be maximized. In the following we will assume that the general position condition (GPC) holds for each parametric optimal control subproblems.

In problems $P_1^{(i)}$, $i \in I$, the optimum value is

$$\tilde{F}_1^{(i)} = -\frac{c_i (\tilde{m}_i)^2}{2 k_{u_i} \overline{u}_i} - \frac{\tilde{m}_i}{k_{u_i}}, \tag{2}$$

where

$$\tilde{m}_i \in [0,\ k_{u_i} \overline{u}_i (t_1 - t_0)],\ i \in I. \tag{3}$$

In problem P_2, due to GPC, the number of switches in the optimal control is no more than $n + r$ ([7], p.166). Denote them via $\tau_1, \ldots, \tau_{n+r}$ and let $\tau_1 \leq \ldots \leq \tau_{n+r} \leq \tau_{n+r+1} = t_1$. Then switch times satisfy

$$e^{t_1 \Delta} A(t_1) - e^{t_0 \Delta} A(t_0) = \begin{cases} \overline{a} \sum\limits_{j=1}^{n+r} (-1)^j\, e^{\tau_j \Delta}\, \Delta^{-1} E_p\,, & \text{if } n+r \text{ is even} \\ \overline{a} \sum\limits_{j=1}^{n+r+1} (-1)^j\, e^{\tau_j \Delta}\, \Delta^{-1} E_p\,, & \text{if } n+r \text{ is odd} \end{cases} \tag{4}$$

while the optimum value is

$$\tilde{F}_2 = \begin{cases} \bar{a} \sum\limits_{j=1}^{n+r} (-1)^{j-1}\tau_j, & \text{if } n+r \text{ is even} \\ \bar{a} \sum\limits_{j=1}^{n+r+1} (-1)^{j-1}\tau_j, & \text{if } n+r \text{ is odd} \end{cases} \tag{5}$$

In problem P_3, due to GPC, the number of switches in the optimal control is no more than $2n+r$ [7]. Denote them via $\rho_1, \ldots, \rho_{2n+r}$ and let $t_1 = \rho_0 \leq \rho_1 \leq \ldots \leq \rho_{2n+r} \leq \rho_{2n+r+1} = t_2$. Since matrix Q is similar to the matrix with block-triangular structure, its eigenvalues are real and can be easily calculated (cf. [1]). For the sake of simplicity let us consider the case where all eigenvalues $\lambda_i, i = 1, \ldots, 2n+r$, of matrix Q are distinct (cf. [3]). Denote

$$\Lambda = diag\ \{\lambda_1, \ldots, \lambda_{2n+r}\} \ = S^{-1}TS,$$

$$G = e^{t_2\Lambda}S^{-1}X(t_2) - e^{t_1\Lambda}S^{-1}X(t_1) + (e^{t_1\Lambda} - e^{t_2\Lambda})\Lambda^{-1}S^{-1}L,$$

where S is the matrix of eigenvalues of Q, nonsingular due to our assumption. Let $M \in I$ be the number of negative eigenvalues of matrix Q. If M is even, then

$$G = \begin{cases} \bar{a} \sum\limits_{j=1}^{2n+r} (-1)^j\ e^{\rho_j\Lambda}\ \Lambda^{-1}S^{-1}E_s\ , & \text{if } r \text{ is even} \\ \bar{a} \sum\limits_{j=1}^{2n+r+1} (-1)^j\ e^{\rho_j\Lambda}\ \Lambda^{-1}S^{-1}E_s\ , & \text{if } r \text{ is odd} \end{cases} \tag{6}$$

$$\tilde{F}_3 = \begin{cases} \bar{a} \sum\limits_{j=1}^{2n+r} (-1)^{j-1}\rho_j, & \text{if } r \text{ is even} \\ \bar{a} \sum\limits_{j=1}^{2n+r+1} (-1)^{j-1}\rho_j, & \text{if } r \text{ is odd} \end{cases} \tag{7}$$

If M is odd, then

$$G = \begin{cases} \bar{a} \sum\limits_{j=0}^{2n+r+1} (-1)^{j+1}\ e^{\rho_j\Lambda}\ \Lambda^{-1}S^{-1}E_s\ , & \text{if } r \text{ is even} \\ \bar{a} \sum\limits_{j=0}^{2n+r} (-1)^{j+1}\ e^{\rho_j\Lambda}\ \Lambda^{-1}S^{-1}E_s\ , & \text{if } r \text{ is odd} \end{cases} \tag{8}$$

$$\tilde{F}_3 = \begin{cases} \bar{a} \sum\limits_{j=1}^{2n+r+1} (-1)^{j-1}\rho_j, & \text{if } r \text{ is even} \\ \bar{a} \sum\limits_{j=1}^{2n+r} (-1)^{j-1}\rho_j, & \text{if } r \text{ is odd} \end{cases} \tag{9}$$

Remark that to receive the explicit form of $\tilde{F}_1^{(i)}$, $\tilde{F}_2$ and $\tilde{F}_3$ and the conditions for switch times we use the following fact establishes the sign of the determinant of a generalization of Vandermonde matrix (the proof see in [1]): if $0 < \mu_1 < \ldots < \mu_{l-1}$ and $0 < \xi_1 < \ldots < \xi_l$, then

$$det \begin{pmatrix} 1 & e^{\mu_1\xi_1} & \ldots & e^{\mu_{l-1}\xi_1} \\ \vdots & \vdots & & \vdots \\ 1 & e^{\mu_1\xi_l} & \ldots & e^{\mu_{l-1}\xi_l} \end{pmatrix} > 0.$$

3. The parametric optimization problem

The explicit form of $\tilde{F}_1^{(i)}$, $\tilde{F}_2$ and $\tilde{F}_3$ and obtaining conditions for switch times allow to formulate the parametric optimization problem. Recall that the conditions

$$\bar{A} \geq A^2, \quad q \in [0, \bar{q}] \tag{10}$$

hold and, moreover, switch times must satisfy

$$t_0 \leq \tau_1 \leq \ldots \leq \tau_{n+r} \leq t_1 \leq \rho_1 \leq \ldots \leq \rho_{2n+r} \leq t_2 \,. \tag{11}$$

If M is even, then the parametric optimization problem which has to be solved to obtain the solution of problem P is the following

P_+ : *maximize (1) subject to (3), (10), (11), (4), (6), where* $\tilde{F}_1^{(i)}$ *is (2)* $\forall i \in I$, $\tilde{F}_2$ *is (5),* $\tilde{F}_3$ *is (7).*

If M is odd, then the parametric optimization problem to be considered becomes

P_- : *maximize (1) subject to (3), (10), (11), (4), (8), where* $\tilde{F}_1^{(i)}$ *is (2)* $\forall i \in I$, $\tilde{F}_2$ *is (5),* $\tilde{F}_3$ *is (9).*

It can be shown that, due to GPC for each subproblem, the necessary optimality conditions for the parametric optimization problem P_+ (or P_-) can be held only on the (relative) boundary of feasible set. Moreover, it can be found conditions (rather natural), under which in the optimal solution the number of switches is strictly less than $3n + 2r$.

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