DISCRETE MONITORING CORRECTION OF AMERICAN OPTIONS EXERCISE

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ABSTRACT - A convenient two-step Monte Carlo simulation procedure enables to deepen the study of the exercise of American options. In particular, it is possible to analyze the optimal exercise time and the probability that the option is exercised at or before maturity. Nevertheless, the results obtained with a simulation method are affected by a bias due to discrete monitoring. In this contribution we first study how the discrete monitoring affects the estimation of the exercise features of American options. In particular, the optimal exercise times turn out to be heavily affected by the width of the monitoring interval. Furthermore, we propose some extrapolation method that applies the Richardson extrapolation technique in order to accelerate the convergence and reduce the effects of this monitoring bias. A wide simulation analysis is carried out to test the accuracy of the extrapolation procedures proposed.

KEYWORDS - American options, optimal exercise boundary, Monte Carlo simulation, discrete monitoring bias, Richardson extrapolation.

1 INTRODUCTION

The optimal exercise boundary is a time dependent barrier which allows, once computed, to define a stopping rule that can be used to check the convenience of early exercise of American options. This stopping rule can be embodied in a two-step simulation method in order to define a procedure which enables us to determine not only the option price but also the exercise features of American options. Examples of such features are the optimal exercise time and the probability that the option is exercised at or before maturity.

Nevertheless, the estimation of the first passage times obtained with a simulation method is affected by a bias due to discrete monitoring. The reasons are known, and lie in the fact that we monitor the prices at discrete points in time, thus neglecting what happens between two adjacent points; hence, in a Monte Carlo simulation we do not test continuously if the optimal stopping boundary is touched. Obviously, this bias vanishes when the monitoring interval tends to zero.

In this contribution we first study how the discrete monitoring affects the estimation of the exercise features of American options. In particular, the optimal exercise times turn out to be heavily affected by the width of the monitoring interval.

Furthermore, we propose some extrapolation method that apply the Richardson extrapolation technique in order to accelerate the convergence and reduce the effects of this monitoring bias.

In numerical analysis, applied mathematics, and financial engineering one has often to deal with sequences and series which are obtained by iterative methods or approximation procedures depending on some parameter such as the time step. Since often the convergence of numerical schemes is slow, acceleration methods have been proposed. The Richardson extrapolation technique is an acceleration method that can be used when we are valuing financial derivatives using a numerical procedure.

A wide simulation analysis is carried out in order to test the accuracy of the extrapolation procedures proposed. This analysis points out that the improvements in the accuracy of the estimation results seem to concern mainly the exercise features.

2 THE OPTIMAL EXERCISE BOUNDARY

We assume that the price $S = (S_t)_{t\geq 0}$ of the underlying asset is governed by the following risk neutral diffusion process

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t, \qquad S_0 > 0, \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process, δ is the continuous dividend yield and σ is the volatility of the asset returns. By assumption, money can be invested at the risk-free interest rate $r \geq 0$. All these parameters are supposed constant.

Let us consider an American style put option with maturity T and strike price X. Some known symmetry properties between American calls and puts allow to extend the results obtained for the American puts also to American calls; for a comprehensive treatment of the symmetry properties for American options see Detemple (2001).

It is known that, according to the arbitrage pricing theory, the fair value at time $t = 0$ of an American put option can be obtained by solving the following *optimal stopping problem*

$$P_0 = \sup_{t\in[0,T]} \mathbb{E}\left[e^{-rt}(X - S_t)^+\right], \tag{2}$$

where the expectation is computed under the risk neutral probability measure (see e.g. Karatzas and Shreve, 1998, Shiryaev et al., 1994)). Since the holder has the right to exercise the option at any time, the supremum is taken over the class of all possible exercise times.

The solution of this optimal stopping problem would entail the joint calculation of the supremum and the optimal exercise time t^* at which it is reached

$$t^* \quad \text{s.t.} \quad P_0 = \mathbb{E}\left[e^{-rt^*}(X - S_{t^*})^+\right]. \tag{3}$$

Unfortunately, it is not known a computational formula for the time t^* which maximizes the present value of the future cash-flows received by the holder.

Let B_t denote the *critical exercise price* which separates the exercise region from the continuation region at time $t \in [0, T]$. For an American put B_t is the price of the underlying asset below which it is convenient to exercise the option. The function of time t

$$B : [0, T] \to \mathbb{R}^+ \tag{4}$$

which for each $0 \leq t \leq T$ gives the critical exercise price is known as *early exercise boundary* or *optimal exercise boundary.*

B is the set of asset prices in correspondence of which the optimal strategy involves immediate exercise and is the optimal solution of the following *problem of first passage*

$$P_0 = \sup_B \ \mathbb{E}\left[e^{-rt_B}(X - S_{t_B})^+\right], \tag{5}$$

where S_0 is assumed to be greater than B_0 and the *stopping time* t_B is the first passage time of S through the boundary B

$$t_B = \inf \ \{\{ t \in [0, T] : S_t \leq B_t \} \ \cup \ \{ T \}\}. \tag{6}$$

t_B is set equal to T when the price path is always above the boundary B, so that the stopping time t_B is well defined. Of course, t_B is different along the various trajectories of the price process S. and it is the first time t in correspondence to which the option value P_t is equal to its intrinsic value

$$t_B = \inf \ \left\{ t \in [0, T] : P_t = (X - S_t)^+ \right\}. \tag{7}$$

The early exercise boundary divides the time-asset price space $\{(t, S)\}$ into two regions: the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$. For a put option, the continuation and the stopping regions are defined as follows

$$\mathcal{C} = [0, T] \times (B_t, +\infty) \tag{8}$$
$$\mathcal{S} = [0, T] \times [0, B_t]. \tag{9}$$

In the stopping region $\mathcal{S}$ the value of the put option is given by its intrinsic value, i.e.

$$(t, S_t) \in \mathcal{S} \quad \Leftrightarrow \quad P_t = (X - S_t)^+. \tag{10}$$

The main properties of the optimal exercise boundary B for the case $r > 0$ can be summarized as follows (for a detailed discussion see Basso, Nardon and Pianca, 2002b).

1. B is continuously differentiable on the interval $[0, T)$;

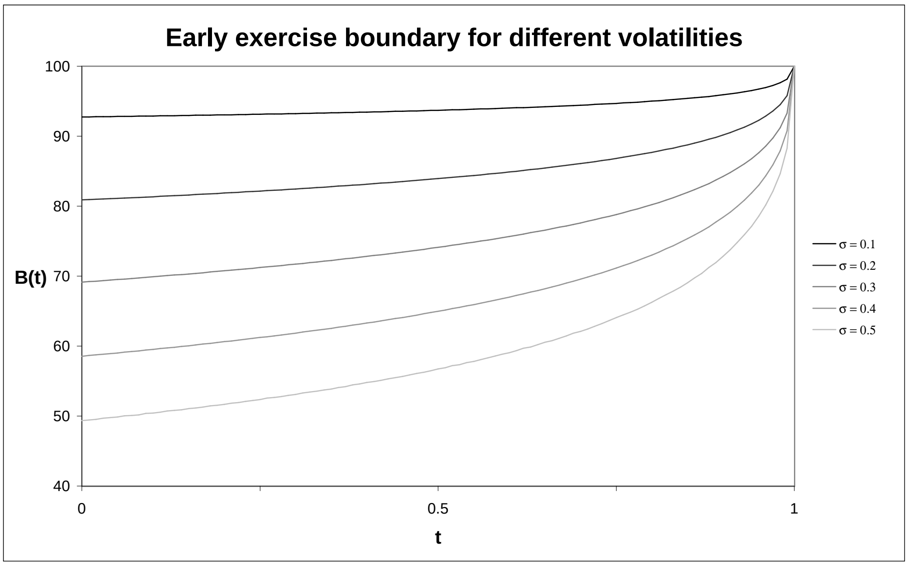

Figure 1: Optimal exercise boundary of an American put option for different volatilities; the parameter values are $X = 100$, $T = 1$, $r = 0.05$, $\delta = 0$.

2. B is nondecreasing in t; in time to maturity $\tau = T - t$);

3. $B_T = X$; near expiration we have

$$\lim_{t \to T} B_t = \begin{cases} X & \text{if} \quad \delta \leq r \\ \frac{r}{\delta} X & \text{if} \quad \delta > r; \end{cases} \tag{11}$$

4. B does not depend on the current price of the underlying asset, S_0;

5. B is linearly homogeneous in X;

6. let B^∞ denote the constant optimal exercise boundary of a perpetual put option; for a finitely-lived American put option the following bounds hold

$$B^\infty \leq B_t \leq X \qquad\qquad t \in [0, T]. \tag{12}$$

Figures 1 to 3 show the behavior of the optimal exercise boundary as the model parameters σ, r and δ vary. As we can see, the exercise boundary decreases when σ rises, increases with r and diminishes with δ.

3 A TWO-STEP SIMULATION PROCEDURE TO COMPUTE THE EXERCISE FEATURES

The knowledge of the early exercise boundary would allow to determine the optimal exercise strategy: indeed, the holder could decide whether the option value exceeds the intrinsic value, and hence if immediate exercise is convenient. Obviously, it is optimal to exercise an American put option as soon as $S_t \leq B_t$ while if $S_t > B_t$ one should continue with the option.

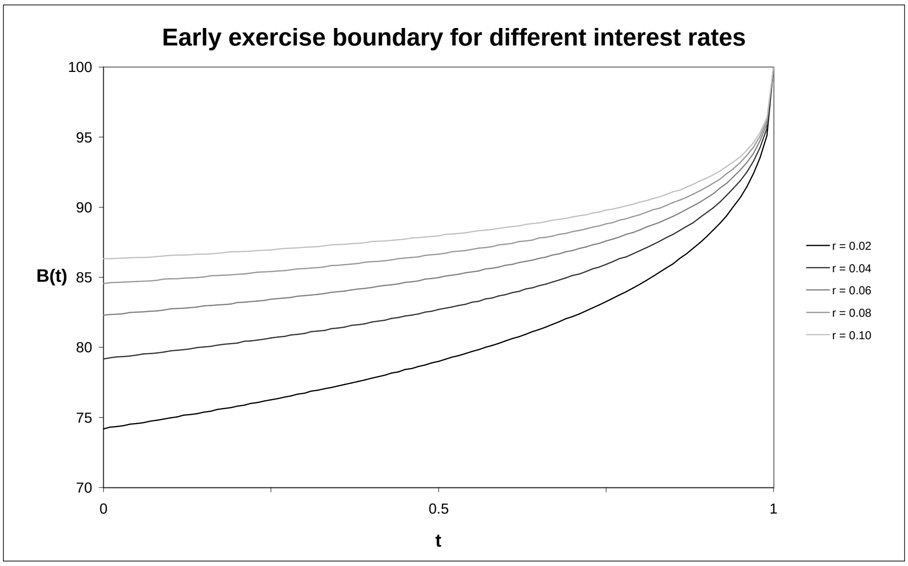

Figure 2: Optimal exercise boundary of an American put option for different interest rate values; the parameter values are $X = 100$, $T = 1$, $\sigma = 0.2$, $\delta = 0$.

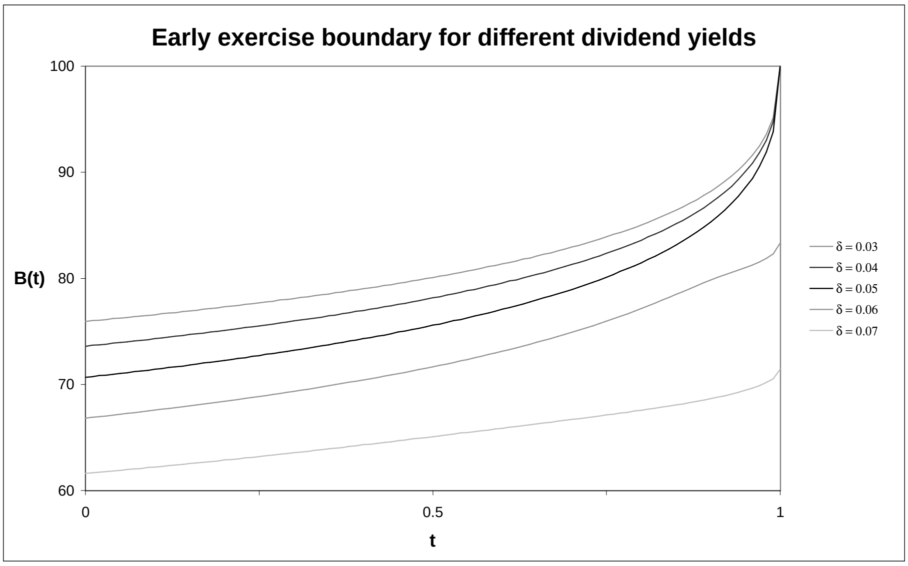

Figure 3: Optimal exercise boundary of an American put option for different values of the dividend yield; the parameter values are $X = 100$, $T = 1$, $\sigma = 0.2$, $r = 0.05$.

This optimal stopping rule can be embodied in a Monte Carlo simulation framework in order to analyze not only the American option value but also its exercise features. In particular, the forward-looking procedure implicit in a simulation method allows to determine an estimate of the first passage times in the various simulated paths.

Starting from these observations a two-step simulation procedure can be proposed which enables to calculate the relevant exercise features:

1. in the first step an estimation of the optimal exercise boundary is computed by means of a convenient numerical algorithm;

2. in the second step this boundary approximation is included in a Monte Carlo simulation procedure and used in order to compute the optimal exercise rule.

The two-step procedure proposed allows to study not only the option price but also other interesting features such as the optimal exercise time t^*, the probability p^e that the option is exercised and the probability p^a of early exercise.

An estimate of the optimal exercise time can be obtained by averaging the first passage times over the simulated trajectories in which the option has been exercised. Analogously, the relative frequency of exercise (of early exercise) gives an estimate of the probability to exercise the option (to exercise before maturity).

As regards the numerical algorithm to be used in the first step of the procedure proposed in order to compute an approximation of the optimal exercise boundary, several alternative approaches can be used. A number of procedures have been proposed in the literature to approximate the boundary with fast computations. These procedures approximate the boundary using different numerical techniques; the main approaches are the following:

1. The randomization approach proposed by Carr (1998) uses a staircase approximation of the boundary and provides a good approximation of the initial critical stock price; in Basso, Nardon and Pianca (2002a) this approach is exploited to build the whole boundary.

2. Ju (1998) approximates the early exercise boundary as a multipiece exponential function and then uses Richardson extrapolation to obtain better estimates of the option price.

3. AitShalia and Lai (1999, 2001) approximate the boundary by linear splines.

4. Allegretto et al. (1995) and Bunch and Johnson (2000) propose a few numerical algorithms that compute an approximation of the boundary by solving algebraic equations.

5. Broadie and Detemple (1996) derive lower bounds for the critical stock price and use them to obtain bounds for the American call option price.

6. Huang, Subrahmanyam and Yu (1996), Little, Pant and Hou (2000) and Sullivan (2000) study numerical schemes which solve the integral equation which implicitly defines the early exercise boundary.

7. Basso, Nardon and Pianca (2002a) present an improved binomial method which compute an approximation of the boundary using a lattice approach with a large number of steps. A convenient interpolation procedure of the asset prices around the critical nodes improves the precision of the optimal exercise boundary and reduces its fluctuations.

We have to point out that the approximation of the boundary used in the first step of the two-step procedure must be precise and computationally robust. For instance, Carr's randomization approach is very fast and precise; see Basso, Nardon and Pianca (2002a) for a comparison with other numerical approaches and Basso, Nardon and Pianca (2002b) for the extension to the dividend paying case. On the other hand, the optimal exercise boundary computed with the improved binomial method is much slower, compared to the other numerical approximations mentioned above, but if a sufficiently high number of steps is used it gives a very accurate boundary approximation. Actually, the boundary obtained with a lattice technique is considered in the literature as the most accurate, so that it is usually adopted as a benchmark in the comparison tests.

In the procedure proposed the optimal exercise time t^* is estimated as the average passage time through the boundary B of the simulated trajectories, computed over the paths which lead to the option exercise

$$\hat{t}^* = \frac{1}{|\mathcal{E}|} \sum_{k \in \mathcal{E}} t_B^k, \tag{13}$$

where $\mathcal{E}$ denotes the set of simulated paths in which the option has been exercised and t_B^k is the first passage time through B for the k-th trajectory. If we restrict the computation of the first passage time average to the paths in the set $\mathcal{A}$ of the trajectories in which the option has been exercised before maturity ($\mathcal{A} = \{\, k : t_B^k < T\,\}$), we find the mean early exercise time $\hat{t}^a = \sum_{k\in\mathcal{A}} t_B^k/|\mathcal{A}|$.

Figure 4 shows the behavior of both the average exercise time $\hat{t}^*$ and the average early exercise time $\hat{t}^a$ for an American put option with $T = 1$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, for different moneyness ratios; the exercise times are estimated using Monte Carlo simulation with 100 000 paths and $n = 250$ time steps. As can be observed, both $\hat{t}^*$ and $\hat{t}^a$ diminish as the moneyness increases.

Figure 5 presents the relative frequency of exercise $\hat{p}^e = |\mathcal{E}|/N$, where N denotes the number of simulated paths, and the relative frequency of early exercise $\hat{p}^a = |\mathcal{A}|/N$. As can be expected, both relative frequencies increase with the moneyness. Moreover, the frequency of early exercise tends to coincide with the frequency of exercise for either the deep-out-of-the money or deep-in-the money options.

In order to test the goodness of the employment of the early exercise boundary in a simulation context, first of all we have carried out a large number of experiments which make a comparison between the price obtained for an American put option in the binomial model and the estimate of this price obtained with the two-step Monte Carlo procedure proposed. The simulation analysis takes into consideration 5 760 randomly generated option valuation problems. The parameter ranges are: $r \in [0.005, 0.12]$, $\delta \in [0, 0.12]$, $\sigma \in [0.1, 0.5]$, $X/S_0 \in [0.7, 1.3]$, with $S_0 = 100$ and

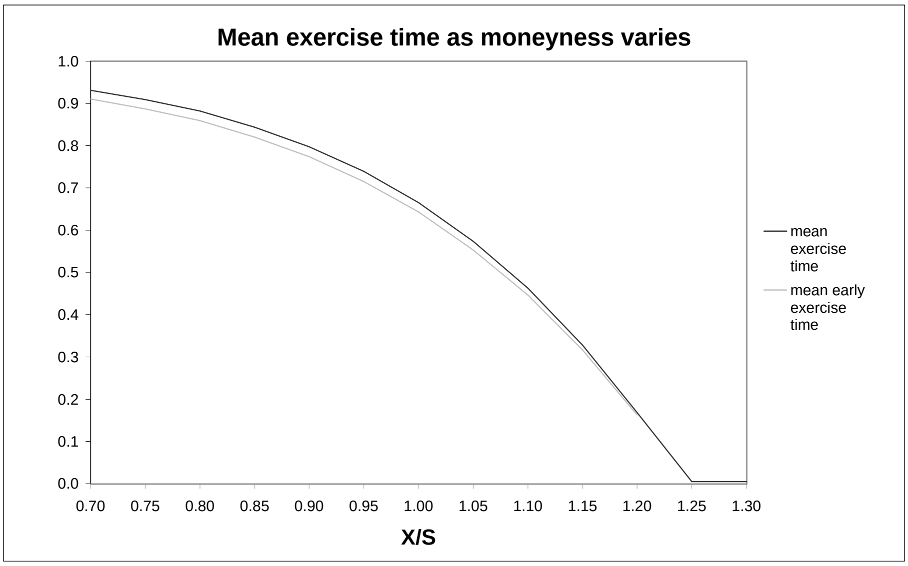

Figure 4: Average exercise time for an American put option with $T = 1$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, for different moneyness ratios; the exercise times are estimated using the two-step simulation procedure with 100 000 paths and $n = 250$ time steps.

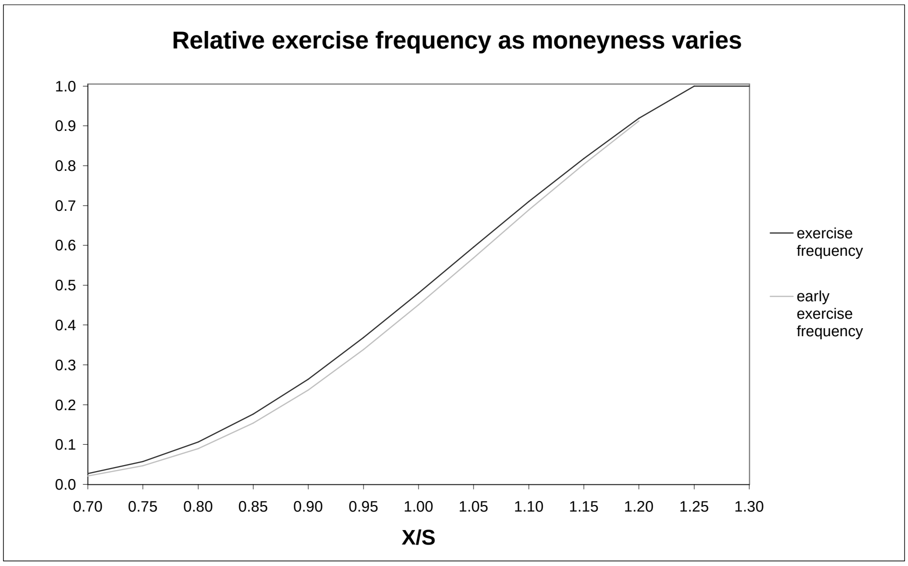

Figure 5: Relative frequencies of exercise and early exercise for an American put option with $T = 1$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, for different moneyness ratios; the relative frequencies are computed using the two-step simulation procedure with 100 000 paths and $n = 250$ time steps.

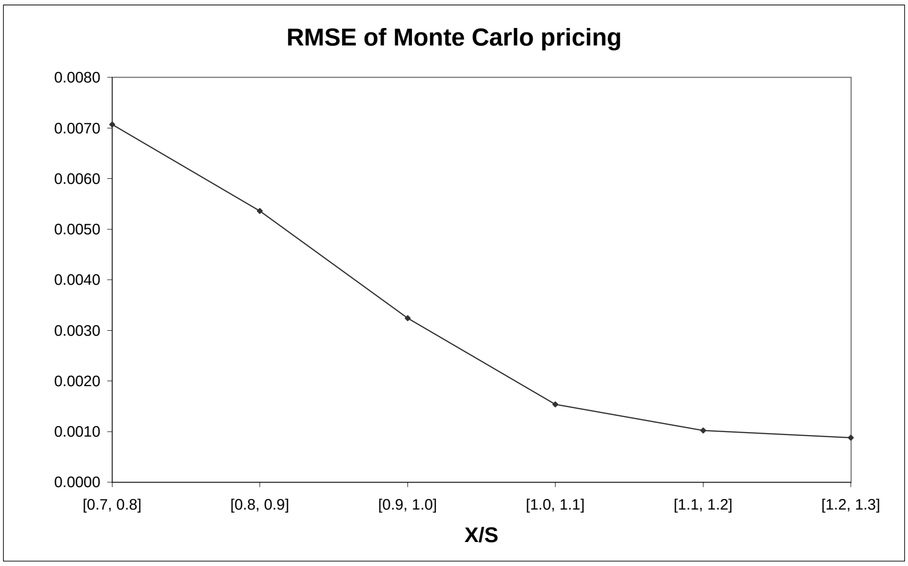

Figure 6: Root mean square errors of Monte Carlo estimation of American put option prices obtained with the two-step procedure as moneyness varies; the option maturity is $T = 1$; the price estimates are based on 100 000 simulated paths with $n = 250$ time steps.

$T = 1$. The parameter space has been partitioned with a $3 \times 4 \times 4 \times 6$ grid into 288 rectangular subsets and we have randomly generated 20 instances from each subset.

For each generated problem we have applied the two-step procedure by computing the boundary approximation in the first step with the highly accurate improved binomial method. More precisely, the time interval $[0, T]$ between current time and the option maturity has been divided into $m = 20\,000$ sub-intervals of length T/m; in addition, we have made the binomial algorithm start 5 000 steps before time $t = 0$ (so that the total number of steps in the lattice is $n = 25\,000$) in order to have a wide range of prices defined for all time steps related to the option life. In this way in the time interval $[0, T]$ the early exercise boundary is usually well defined.

The simulation analysis at the second step of the procedure have been carried out by randomly generating 100 000 paths of the underlying asset price for each option valuation problem. In order to reduce the variance of the simulation, an antithetic variate technique has been applied. Of course, we can only monitor the prices at discrete points in time; the monitoring interval chosen is daily, so that we have $n = 250$ price observations for each path.

The results of the simulation experiments carried out with respect to the option price are summarized in figure 6 which shows the relative root mean square error (RMSE) of the simulation results with respect to the binomial price as moneyness varies. To avoid emphasizing the round off errors in the computation of the RMSE, the option prices lower than 0.5 are omitted from the calculations. The RMSE ranges from 0.1 percent to 0.7 percent. Moreover, we can notice that the estimation accuracy highly depends on the option moneyness X/S_0. Actually, the higher the moneyness of the put option is, the more accurate the simulation estimate is; moreover, the accuracy seems quite different according to the nature, in or out of-the-money, of the option.

4 THE EFFECTS OF DISCRETE MONITORING

The forward procedure of a simulation method allows to determine an estimate of the first passage times t_B in the various simulated paths. However, the estimate of the first passage times thus obtained could be affected by a bias due to discrete monitoring.

Actually, let us observe how the path of the underlying asset price is generated in simulation approaches: the asset prices are simulated at discretely sampled points. However, by proceeding in discrete time we neglect what happens between two adjacent points; in particular, we do not test continuously the optimal stopping rule. In particular, by discretely monitoring the underlying asset price we could ignore the fulfillment of the optimal stopping rule, if it occurs between two sampled points.

Hence, discrete monitoring introduces a bias in the option valuation. This bias vanishes when the time interval Δt between two adjacent sampled point tends to zero but it can cause substantial errors even for close sampled points.

In order to investigate how much the discrete monitoring bias affects the estimation of the price and the exercise features of the option we have performed a wide empirical analysis for different monitoring intervals. We have randomly generated 1 500 option valuation problems with $S_0 = 100$, $T = 1$, $r \in [0.005, 0.12]$, $\delta \in [0, 0.12]$, $\sigma \in [0.1, 0.5]$, $X/S_0 \in [0.7, 1.3]$. In order to see if this bias changes with the moneyness, the moneyness parameter space has been split into 6 subintervals of equal amplitude, with 250 instances generated in each subinterval. As in the previous section, the simulation analysis is based on the generation of 100 000 paths of the underlying asset price using an antithetic variate technique. The optimal stopping rule is monitored at n equally spaced time steps, with $n \in \{ 10, 50, 250, 1250, 5000 \}$. The monitoring intervals analyzed correspond roughly to monthly, weekly, daily, 5 time-a-day and hourly time steps.

The main results of the experiments are summarized in table 1 and figures 7-10, which report the RMSE of the estimation of the price and exercise features of American put options obtained with the two-step procedure for different monitoring intervals. The exercise features analyzed are the exercise time t^*, the early exercise time t^a, the exercise probability p^e and the probability p^a of early exercise. The RMSEs are computed using as estimate of the “true” value the option price obtained with the CRR binomial method with 25 000 steps and the value of the exercise features obtained using the two-step procedure with 5 000 monitoring intervals.

From table 1 it is clear that discrete monitoring introduces a bias in the estimation of both the option value and the exercise features. In particular, the exercise time of the option turns out to be heavily affected by the discrete monitoring bias. Figures 7-10 show how this bias varies with the option moneyness.

Actually, discrete monitoring biases have been observed for other options, too. In particular, this problem is known to occur for at least three other classes of options and precisely for lookback options, which depend on the extreme values

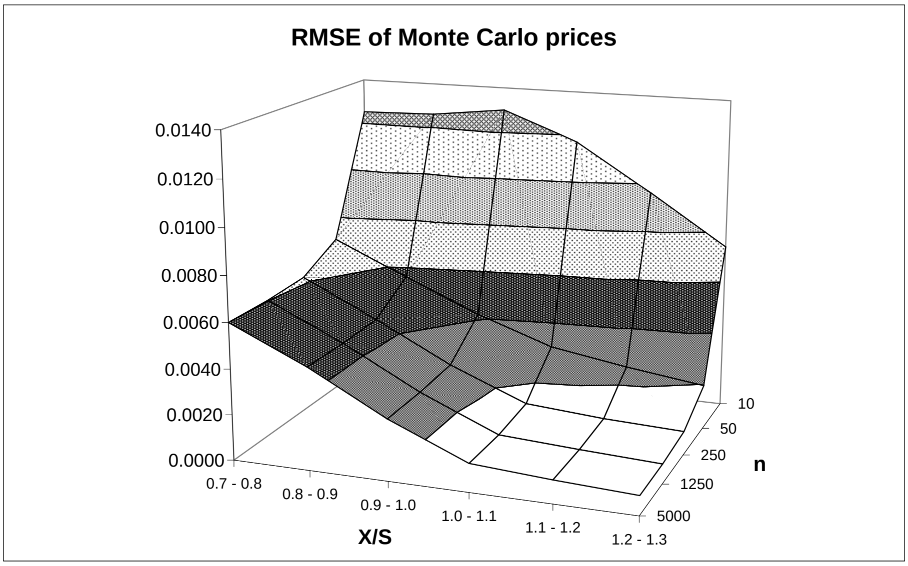

Figure 7: Root mean square errors of the estimation of American put option prices obtained with the two-step procedure as the moneyness and the number of time steps vary; the option maturity is $T = 1$; the price estimates are based on 100 000 simulated paths.

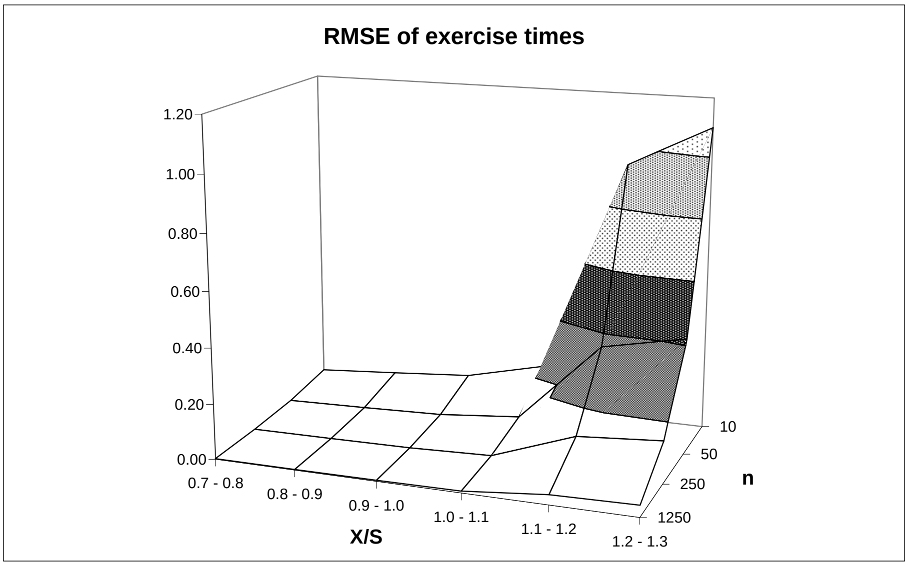

Figure 8: Root mean square errors of the estimation of the exercise time of American put options obtained with the two-step procedure as the moneyness and the number of time steps vary; the option maturity is $T = 1$; the price estimates are based on 100 000 simulated paths.

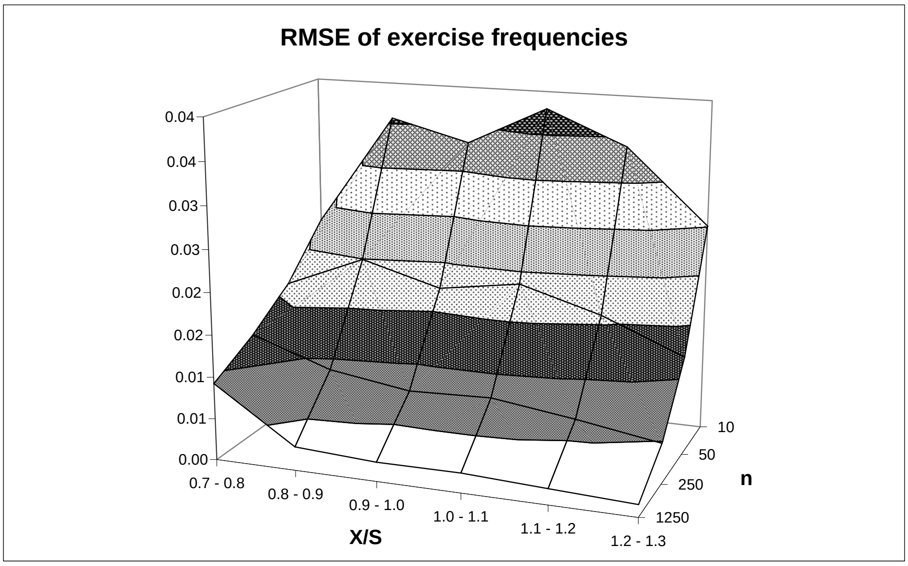

Figure 9: Root mean square errors of the estimation of the exercise probability of American put options obtained with the two-step procedure as the moneyness and the number of time steps vary; the option maturity is $T = 1$; the price estimates are based on 100 000 simulated paths.

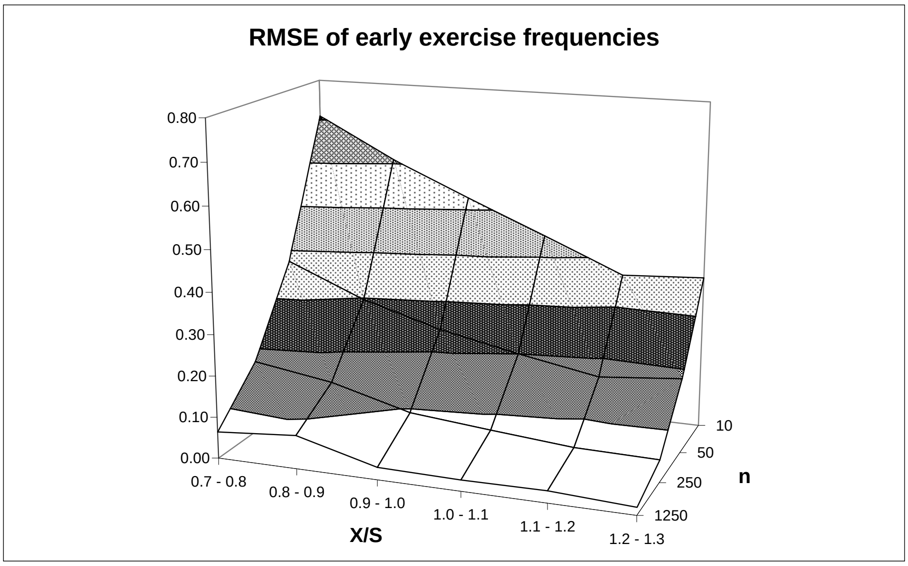

Figure 10: Root mean square errors of the estimation of the probability of early exercise of American put options obtained with the two-step procedure as the moneyness and the number of time steps vary; the option maturity is $T = 1$; the price estimates are based on 100 000 simulated paths.

Table 1: RMSE of the estimation of the price, the exercise time, the early exercise time, the exercise probability and the probability of early exercise of American put options obtained with the two-step procedure for different monitoring intervals; the option maturity is $T = 1$; the estimates are based on the simulation of 1 500 options and 100 000 paths for each option. The RMSEs are computed with respect to the option price obtained with the CRR binomial method with 25 000 steps and the value of the exercise features obtained using the two-step procedure with 5 000 monitoring intervals.

n	RMSE($\hat{P}$)	RMSE($\hat{t}^*$)	RMSE($\hat{t}^a$)	RMSE($\hat{p}^e$)	RMSE($\hat{p}^a$)
10	0.0112	0.5675	0.4944	0.0318	0.5169
50	0.0043	0.2154	0.2008	0.0168	0.2525
250	0.0032	0.0786	0.0761	0.0083	0.1087
1250	0.0031	0.0218	0.0217	0.0043	0.0468
5000	0.0031	–	–	–	–

of the underlying asset price, for barrier options, which depend on whether the underlying asset price touches a predetermined level or not, and for Russian options, a special kind of exotic American style perpetual options. On this subject see Broadie, Glasserman and Kuo (1997, 1999), Levy and Mantion (1997), Beaglehole, Dybvig and Zhou (1997), El Babsiri and Noel (1998), Baldi, Caramellino and Iovino (1999), Basso and Pianca (2000). Using a simulation technique with a non zero sampling time interval Δt in order to price one of these path-dependent options, we cannot obtain an unbiased estimate of a continuously monitored option.

Some methods have been suggested in the literature to give a more precise estimation of a continuously monitored option using a discrete sampling technique such as simulation.

A first approach is proposed by Levy and Mantion (1997) who use Richardson's extrapolation to connect the values of a lookback or barrier option obtained in discrete and continuous time settings.

A second approach is used by Beaglehole, Dybvig and Zhou (1997) and El Babsiri and Noel (1998) to price continuous time lookback, barrier and some other exotic options. Both contributions use a special Monte Carlo simulation technique which, at each sampled time interval, first generates the "observed" final price of the underlying security and then generates the extremal value for this price which has occurred within the interval. In such a way, the appearance of new extremal values between two successive sample dates is explicitly taken into consideration.

Another numerical method is proposed by Baldi, Caramellino and Iovino (1999) to evaluate single and double barrier options with general features; this method uses Sharp Large Deviations estimates to improve the usual Monte Carlo procedure.

5 AN EXTRAPOLATION TECHNIQUE TO CORRECT THE SIMULATION BIAS

In this section we apply a Richardson extrapolation technique in order to improve the estimation of the exercise features of American put options.

Richardson extrapolation is a particular acceleration method which is frequently employed to generate results of high accuracy by using low order formulae. Richardson called this technique a *limit method*; afterwards, this approach was indicated as Richardson extrapolation (see Gautschi, 1997). This technique uses the powerful idea of extrapolating a computed result to the value that would have been obtained if the step size Δt in the approximation scheme had been very much smaller than it actually was. In particular, the desired goal is extrapolation to a zero step size.

In the financial literature the Richardson extrapolation method has been applied to accelerate some valuation schemes for American options. Richardson extrapolation has been first used in finance in a well-known paper by Geske and Johnson (1984) to evaluate an American put option. The value of the put option is determined by first calculating the value of analogous options which can be exercised only at maturity T, at $T/2$ and T, at $T/3$, $2T/3$ and T, and then extrapolating the value of the continuous exercise option through the Richardson technique. Along this line, Levy and Mantion (1997) use the continuous exercise time formula for lookback and barrier options, together with the value of the corresponding discrete time options with only one or two monitoring dates, to give an approximation for the value of the analogous option with any number n of monitoring dates. Moreover, Carr (1998) uses Richardson extrapolation to improve his randomization approach while other contributions apply this extrapolation procedure to accelerate the convergence of a binomial method.

As proposed by Geske and Johnson, let us denote by $F(\Delta t)$ a real function of the time step Δt and let us assume that F can be written as follows

$$F(\Delta t) = F(0) + a_1(\Delta t)^p + a_2(\Delta t)^r + o((\Delta t)^s), \tag{14}$$

where $p < r < s$ and $a_1, a_2 \in \mathbb{R}$. By neglecting the term $o((\Delta t)^s)$, formula (14) can be used either to approximate the continuous time formula $F(0)$, as in Geske and Johnson's approach, or to evaluate the discrete time value $F(\Delta t)$ if the continuous time value is known, as in the approach of Levy and Mantion. Note that the function $F(\Delta t)$ can represent an option value but also other interesting quantities related to discrete monitoring option pricing, such as the optimal exercise time or the exercise probability.

In order to determine the value of the coefficients a_1, a_2, we can follow Geske and Johnson's approach which suggests to compute formula (14) in correspondence with two different time steps $k\Delta t$ and $q\Delta t$ with $q > k > 1$. By omitting the term $o((\Delta t)^s)$, substituting for a_1, a_2 and solving for $F(0)$ we obtain

$$F(0) \simeq F(\Delta t) + \frac{A}{C}[F(\Delta t) - F(k\Delta t)] - \frac{B}{C}[F(k\Delta t) - F(q\Delta t)], \tag{15}$$

Table 2: RMSE of the estimation of the price, the exercise time, the early exercise time, the exercise probability and the probability of early exercise of American put options obtained with the two-point Richardson extrapolation procedure for different monitoring intervals; the option maturity is $T = 1$; the estimates are based on the simulation of 1 500 options and 100 000 paths for each option. The RMSEs are computed with respect to the option price obtained with the CRR binomial method with 25 000 steps and the value of the exercise features obtained using the two-step procedure with 5 000 monitoring intervals.

$n_1 - n_2$	RMSE($\hat{P}$)	RMSE($\hat{t}^*$)	RMSE($\hat{p}^e$)	RMSE($\hat{p}^a$)
		$p = 1$		
10 – 50	0.0033	0.1275	0.0132	0.1941
50 – 250	0.0031	0.0445	0.0065	0.0833
250 – 1250	0.0031	0.0077	0.0039	0.0389
1250 – 5000	0.0031	–	–	–
		$p = 1/2$		
10 – 50	0.0049	0.0702	0.0066	0.1237
50 – 250	0.0033	0.0322	0.0049	0.0880
250 – 1250	0.0031	0.0242	0.0047	0.0499
1250 – 5000	0.0031	–	–	–

where:

$$A = q^r - q^p + k^p - k^r \tag{16}$$

$$B = k^r - k^p \tag{17}$$

$$C = q^r(k^p - 1) - q^p(k^r - 1) + k^r - k^p. \tag{18}$$

As concerns the values of the parameters p and r, Geske and Johnson choose $p = 1$ and $r = 2$; this choice gives the expansion in a Taylor series around $F(0)$ where the terms above the second order are omitted. Levy and Mantion (1997), instead, choose $p = 1/2$ and $r = 1$ on the basis of graphical inspections and the results by Broadie, Glasserman and Kuo (1997, 1999). Hence, the choice of the values of the parameters p and r is not obvious: the values $p = 1$ and $r = 2$ are appealing for the correspondence with the Taylor series expansion, but on the other hand the values $p = 1/2$ and $r = 1$ seem to be suggested by probabilistic considerations (Siegmund and Yuh, 1982).

In order to use this approach for the evaluation of the exercise features of a continuously monitored American option, we need a formula for the discrete stopping time for three different time steps. As such discrete time formula is not known, we approximate the value of $F(\Delta t), F(k\Delta t), F(q\Delta t)$ using the two-step simulation procedure. Of course, by using simulation to evaluate $F(\Delta t), F(k\Delta t), F(q\Delta t)$ we obtain for $F(0)$ an extrapolated value which is affected by another source of errors, besides the exclusion of the higher order terms of the expansion.

Table 3: RMSE of the estimation of the price, the exercise time, the early exercise time, the exercise probability and the probability of early exercise of American put options obtained with the three-point Richardson extrapolation procedure for different monitoring intervals; the option maturity is $T = 1$; the estimates are based on the simulation of 1 500 options and 100 000 paths for each option. The RMSEs are computed with respect to the option price obtained with the CRR binomial method with 25 000 steps and the value of the exercise features obtained using the two-step procedure with 5 000 monitoring intervals.

$n_1 - n_2 - n_3$	RMSE($\hat{P}$)	RMSE($\hat{t}^*$)	RMSE($\hat{p}^e$)	RMSE($\hat{p}^a$)
		$p = 1,\ r = 2$		
10 – 50 – 250	0.0032	0.0410	0.0063	0.0812
50 – 250 – 1250	0.0031	0.0061	0.0039	0.0387
250 – 1250 – 5000	0.0031	–	–	–
		$p = 1/2,\ r = 1$		
10 – 50 – 250	0.0033	0.0229	0.0054	0.1187
50 – 250 – 1250	0.0032	0.0222	0.0054	0.0652
250 – 1250 – 5000	0.0032	–	–	–

Similar extrapolation approaches can be applied with a greater or smaller number of terms than equation (14). For example, if we consider an expansion truncated to the first power term

$$F(\Delta t) = F(0) + a(\Delta t)^p + o((\Delta t)^s), \tag{19}$$

where $p < s$ and $a \in \mathbb{R}$, we obtain the following two-point Richardson extrapolation approximation

$$F(0) \simeq F(\Delta t) + \frac{k}{1-k}[F(\Delta t) - F(k\Delta t)]. \tag{20}$$

It has to be noted that Richardson extrapolation is computationally very sensitive to the behavior of F, so that it can safely be used mainly with monotone functions.

In order to test the goodness of the correction technique proposed, we have carried out a number of simulation experiments, using the same framework as the experiments carried out to study the effect of the monitoring interval on the exercise features. So, we have randomly generated 1 500 option valuation problems and carried out a simulation analysis based on the generation of 100 000 paths of the underlying asset price in which we have monitored the optimal exercise rule at various equally spaced time steps. In order to reduce the variance an antithetic variate technique has been used.

The results are presented in tables 2 and 3. As can be seen by comparing the results reported in these tables, none of the Richardson extrapolation procedures applied always performs better than the others. For few monitoring dates, the Richardson extrapolation methods with $p = 1/2$ seem to be more precise than the methods with $p = 1$, but the opposite happens when the number of monitoring dates is high. On the other hand, the three-point extrapolations are usually more precise than the two-point ones, but the improvements are not so remarkable, if we take into consideration the fact that the three-point extrapolations are more time-consuming.

In any case, the improvements in the accuracy of the estimation results seem to concern the exercise features but not the option price. This is probably due to the monotone convergence exhibited by the exercise features while the convergence of the option price is not often monotone.

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