The Theory of Fuzz Logic and its Application to Real Estate Valuation

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Abstract. Fuzzy logic is based on the central idea that in fuzzy sets each element in the set can assume a value from 0 to 1, not just 0 or 1, as in classic set theory. Thus, qualitative characteristics and numerically scaled measures can exhibit gradations in the extent to which they belong to the relevant sets for evaluation. This degree of membership of each element is a measure of the element's "belonging" to the set, and thus of the precision with which it explains the phenomenon being evaluated. Fuzzy sets can be combined to produce meaningful conclusions, and inferences can be made, given a specified fuzzy input function. The article demonstrates the application of fuzzy logic to an income-producing property, with a resulting fuzzy set output.

Introduction

The pricing function (or valuation) is always a problem in free market economies. Even in well-organized, relatively efficient markets, like securities markets, participants typically lack precise information, and in setting prices they consider a variety of factors and different relationships among factors. In the field of real estate the lack of precise information associated with property investments is usually greater than that involving securities because: (1) data are usually not readily available in consistent form; (2) the diversity among properties often resists analysis and interpretation of general trends; and (3) properties are place-bound, producing even greater uncertainty about the future prospects of a property than for other economic goods that can be moved.

Thus, analysts use a great deal of judgment to identify the characteristics (attributes) of properties that contribute to returns and values and the relationships among these characteristics so as to derive estimates of investment and market values. Additionally, they usually have to consider qualitative characteristics such as structural quality, architectural attractiveness and desirability of the neighborhood and locational convenience. Therefore, they use a great deal of judgment even in specifying the inputs to investment and valuation models. This is a major problem and the Achilles heel of the analytical process: many judgments are made, but analysts have no formal way of specifying their natural lack of precision. Hence, they cannot work with

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investment and valuation models that are shareable by a team of analysts, without simplifying the contexts.

Dubois and Prade (1980) contend that "...fuzziness seems to pervade most human perception and thinking processes." It is the contention of this article, therefore, that the Theory of Fuzzy Sets and specifically fuzzy logic is highly appropriate to the tasks of real estate valuation and decision making. Fuzzy logic provides the means by which judgments that characterize our method of reasoning can be formalized without resorting to an artificial process of making these judgments precise. The purpose of this study is to examine the Theory of Fuzzy Sets to demonstrate the applicability of fuzzy logic for expressing the inherent imprecision in the way that people think and make decisions about the pricing of real estate.

The Theory of Fuzzy Sets and the Central Idea of the Membership Function

The central idea that differentiates Fuzzy Set Theory from Classic Set Theory is the generalization of the characteristic function so that it can assume values between 0 and 1, not just 0 or 1, depending on the degree of membership of the element in the set. Given the universal set U, a fuzzy set A^* can thus be expressed by the following equation:

$$A^* = (x, m_{A*}(x) | x \in U),$$

where $m_{A*}(x)$ is the membership function that expresses the degree of belonging of the general element x to the fuzzy set A^*, assuming the following values:

$$m_{A*}(x) = 1 \text{ if } x \in A,$$

$$0 < m_{A*}(x) < 1 \text{ if } x \text{ partially belongs to } A,$$

$$m_{A*}(x) = 0 \text{ if } x \notin A.$$

Since the membership function is a generalization of the characteristic function, we may define a fuzzy set as a generalization of classic sets.

For example, the classic set A of "tall men" defined by the following characteristic function produces the graphic presentation shown in Exhibit 1, where $f_A(x) = 1$ if $x > 150$ cm or $f_A(x) = 0$ if $x \leq 150$ cm.

By contrast, the membership function of the corresponding fuzzy set A^* having the following values would produce the graphic presentation shown in Exhibit 2,[1] where $m_{A*}(x) = 1$ if $x \geq 200$ cm, $m_{A*}(x) = [(x - 100)/100]$ if 100 cm $< x < 200$ cm and $m_{A*}(x) = 0$ if $x \leq 100$ cm.

In the classic set, the characteristic function would determine that a height of 151 cm is "tall," while a height of 150 cm would be regarded as "not tall." Instead, in the

Exhibit 1
Classic Set A
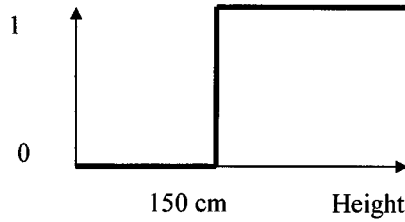
fuzzy set, because the membership function is a sloped line, the result would be a less discriminating definition of the two categories. It is evident, therefore, that the greater the slope of the line representing the membership function, the greater is the degree of fuzziness in defining the membership function of fuzzy set A^*. The degree of fuzziness can also assume much more complex forms (Novak, 1987). Consider, for example, the degree of fuzziness in the fuzzy set A^* defined by the following membership function and represented by the curve in Exhibit 3.

$$m_{A*}(x) = 1/(1 - e^{-(6.\overline{6}e+297)(x-150)}) \text{ with } 0 \text{ cm} < x < 300 \text{ cm}.$$

A curve such as the one shown in Exhibit 3 would describe a membership function for which every element of the universal set would belong, and at the same time would not belong, in some measure to the fuzzy set A^* tall men. Fuzzy Set Theory, therefore, resolves the traditional problem of whether a glass of water is half-full or half-empty. Instead of requiring that the glass be counted either in the set of "full glasses" or the set of "empty glasses," as in Classical Set Theory, Fuzzy Set Theory measures the extent to which the partially filled glass belongs to both sets.

From the examples presented we can see that the concept of the characteristic function evolves into that of the membership function, which is the essence of Fuzzy Set Theory. But what is the real significance of the membership function? Additionally,
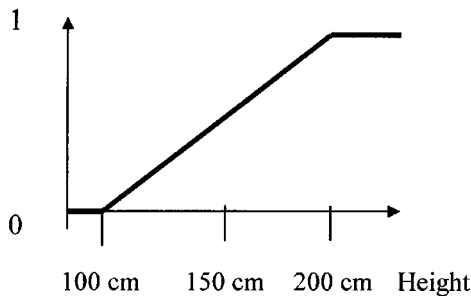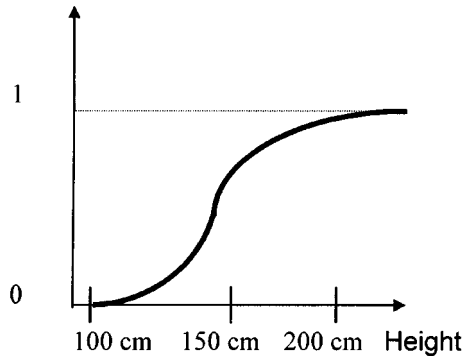
Exhibit 2
Fuzzy Set *A*

Exhibit 3
Fuzzy Set *A**

what are the opportunities that arise from the ability to measure the belonging of the partially filled glass to the set of full glasses or to the set of empty glasses?

The Membership Function as a Fuzzy Measure

To understand the full meaning of the membership function it is helpful to consider that there is a substantial difference between a measure expressed in centimeters and a measure expressed in degrees of membership. This is especially true when the membership function is a continuous curve that avoids the possibility of over-simplifying the relationships among the objects represented, such as a characteristic function in which all heights are classified as tall or not tall. The first measurement (centimeters) is an objective measurement that we could call a "reference measurement." Such a measurement, being independent from the system of values of a person or a group of persons, precludes our understanding of how people judge, for example, whether a height of 156 cm is tall or not tall.

The second measurement (degrees of membership), instead, depends on the system of values of a person or a group of persons. It is used to express a judgment based on a reference measurement; it is a measure of valuation, therefore, or simply a valuation. The membership function assumes, therefore, as does the characteristic function, a meaning of measurement for the formal expression of a judgment, or, in other words, of the rules to express a judgment. The transition from a classical set to a fuzzy set requires a more analytical specification of the system of values that we believe define the characteristic function. It is not just an increase in the subjectivity that characterizes the system of values. The membership function therefore differs from the characteristic function in that it permits the formalization of concepts that we can define imprecisely or vaguely.

Also, we must distinguish between fuzzy logic and probability theory; they are not substitutes for each other. As Thomas (1995) points out, fuzzy logic is a system for managing imprecision of the present, while probability theory is a system for

managing uncertainty about the future. This is an important distinction because some decisions must be made about the present (for example, whether to buy or sell a property at a given price). In contrast, other decisions must be based on uncertain future events (such as the future income and expenses for inclusion in a cash flow forecast).

Since precision is a relative concept, when we speak about precise measures or judgments, we should consider the objectives for which these measures or judgments are made. For example, if the objective is to judge whether an element belongs or does not belong to a set, it is evident that using the membership function instead of the characteristic function would enable us to formalize the degree of imprecision of the judgment. But if the objective is to decide on the degree of membership of an element (x), it is also true that a judgment that the degree of membership is equal to 0.5 is itself a precise measure.

Without considering the merit of each objective, however, one can note how also with the second objective (determining membership degree) one could use a membership function to fuzzify the expressed judgment, if this fuzzification were consistent with the objective. If, in fact, one were not able to measure with a precise number the degree of membership of an element to a given set, a fuzzy number could measure this degree of membership. Such a fuzzy number is represented by a fuzzy set determined by a membership function (f) in which R is the universal set of "real numbers." For example, if we wanted to fuzzify the judgment of 0.5 previously expressed, its membership function could assume the following values, and its form would be represented by the curve shown in Exhibit 4.[2] While this form is usually triangular, it could assume other shapes as well (Novak, 1987). Where $m_{5*}(x) = 0$ if $x \leq 0$, $m_{5*}(x) = 0.20x$ if $0 < x \leq 0.5$, $m_{5*}(x) = 2 - (0.20x)$ if $0.5 < x < 1$ and $m_{5*}(x) = 0$ if $x \geq 1$.

It is evident that the membership function, independently from the objective of the measurement, assumes a meaning of measurement that enables us to formalize imprecise or fuzzy judgments, or to provide a meaning of fuzzy measurement. As stated by Dubois and Prade (1980), "...the maximum membership value of a fuzzy

Exhibit 4
Fuzzy Number 0.5
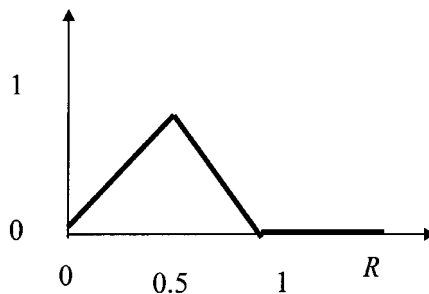
number is interpreted as a degree of reliability, and its spreads model the imprecision of a measurement.''

After having defined the full meaning of the membership function, we must now return to the second question we asked: what are the opportunities created by recourse to this function? In other words, why is it useful to formalize ''gray'' valuations that are not expressible in the two extreme alternatives of ''yes or no'' that characterize the Theory of Classic Sets?

Fuzzy Logic to Formalize the Incorporation of Experience in Valuation Processes

The advantages and opportunities provided by expressing imprecise or vague judgments in formal terms derive from the fact that our method of reasoning is not based, or only rarely, on a dual or bivalent logic. It is not based, in other words, on a logic that permits only two possible answers such as yes/no, black/white or 0/1 but rather on what Zadeh (1973:28) defines as fuzzy logic:

> The key elements in human thinking are not numbers, but labels of fuzzy sets, that is, classes of objects in which the transition from membership to non-membership is gradual rather than abrupt. Indeed the pervasiveness of fuzziness in human thought processes suggests that much of the logic behind human reasoning is not the traditional two-valued or even multi-valued logic, but a logic with fuzzy truths, fuzzy connectives, and fuzzy rules of inference. In our view, it is this fuzzy, and as yet not well understood, logic that plays a basic role in what may well be one of the most important facts of human thinking, namely the ability to summarize information, to extract from the collections of masses of data impinging upon the human brain those and only those subcollections which are relevant to the performance of the task at hand.

The advantage of the membership function is, therefore, that it enables us to formalize the judgments that characterize our method of reasoning without resorting to an artificial process of making these judgments precise. Since this advantage is a necessary but insufficient condition for the rigorous processing of the judgments in traditional systems for the treatment of information,[3] it assumes relevance when we want to incorporate into formal valuation methods the effective process from which the judgments themselves are derived. For example, the judgment that an organizer of a reception who must decide on the most appropriate number of people to invite to a party, given the capacity of the hall, might be ''not fewer than fifteen so that the hall is not too empty, and not more than thirty-five so that everyone can be comfortable.'' Thus, the appropriate number can be:

1. Expressed by a fuzzy number or, more generally, by a fuzzy set;
2. Correctly interpreted as the result of the application of such rules as ''if the hall is little/normal/spacious/huge, then the number of persons to be invited should be small/medium/large/enormous,'' rules that relate

fuzzy concepts to each other, whose representation is made possible by fuzzy sets.

It is the possibility of interpreting the human ability to reason with fuzzy concepts, to reprocess information as Zadeh explains it, as the ability to relate in vague but meaningful terms two or more fuzzy sets, that enables us to incorporate into formal valuation methods the processes from which human judgments are derived.

As an example, we might represent the fuzzy concepts used by someone who is in the business of organizing parties and receptions with the fuzzy sets shown in Exhibit 5. The manner in which the organizer decides on the appropriate number of people to invite, given the capacity of the hall, could be expressed by the following four rules (Kosko, 1993; and Kickert, 1978):

1. If the hall is little, then the number of people to invite must be small.
2. If the hall is normal, then the number of people to invite must be medium.
3. If the hall is spacious, then the number of people to invite must be large.
4. If the hall is huge, then the number of people to invite must be enormous

These rules can be geometrically represented in the form shown in Exhibit 6. Kosko (1993:194) refers to a similar geometric representation of the rules represented by rectangles, explaining,

> I designed the rule as a rectangle. A rectangle is derived as the mathematical product of two segments...Above all the true rule is a tridimensional figure that is difficult to draw on the page. It resembles a tent with four pallets

Exhibit 5
Fuzzy Sets for a Reception

Panel A: Capacity of the Hall
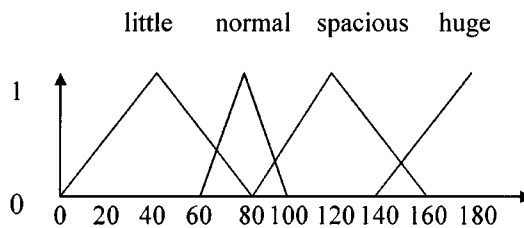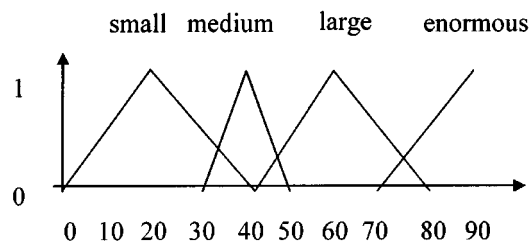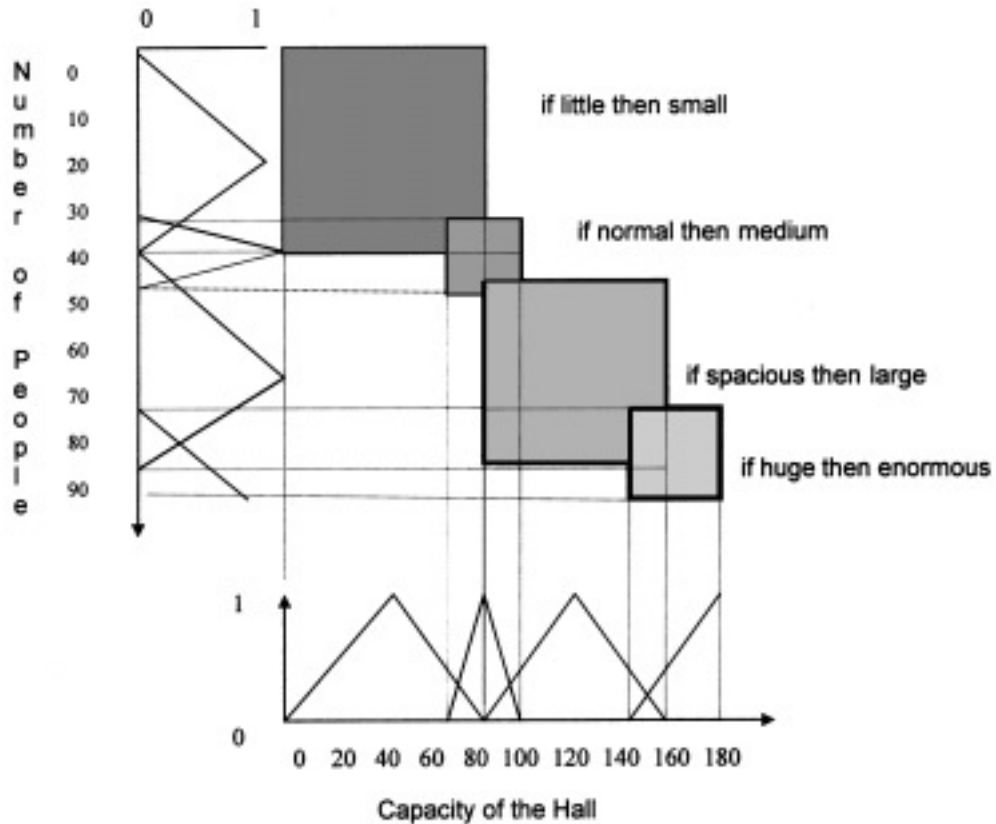

Panel B: Number of Invitees

**Exhibit 6**
Geometric Representation of the Fuzzy Rules

turned on their sides and a single pallet placed high directly in the center. Each point of the floor is a couple of numbers X − Y...The height of the tent tells us the measure in which the couple of numbers belong to the rule. Only the central point of the tent belongs 100%, while the other points belong to lesser degrees. All of the points outside of the floor of the tent belong 0.0%.

Therefore, the formalization by fuzzy logic of the effective ways in which the organizer develops the valutative processes, leads to a representation by rectangular, shaded areas of the relationships that exist between the capacity of the hall and the number of people to invite to the party. One can see that the dimensions of a shaded area (and thus the precision with which the rule is expressed) depend on the size of the connected triangles. The sizes of the triangles are determined, in turn, by the degree of fuzziness that the organizer must accommodate in order to represent meaningfully the concept employed in performing the job.

The smaller the base of the triangle, the less vague is the rule that results. At the limit, in the case in which all triangles become single vertical lines, the relationship would be defined precisely as a continuous curve and would be expressed as an equation of the type $y = f(x)$ with y being the ''number of invitees'' and x being the ''capacity of the hall in square meters.'' The curve representing the application of the general equation to Exhibit 6 would appear as shown in Exhibit 7.

The opposite limiting case would be a representation of the relationship between the capacity of the hall and the number of invitees that is a single, large, rectangular, shaded area expressing the rule ''the larger the capacity of the hall, the higher is the number of invitees.'' The rule, although correct, would be too approximate to be useful for formal valuation methods.

The use of fuzzy logic, however, is not undertaken to express mathematically the relationships characterizing the represented phenomena. In fact, fuzzy measurement becomes relevant only when the intrinsic complexity of the phenomena makes it impossible to define the nature of the relationships precisely and meaningfully (because of what Zadeh (1973:28–44) calls the ''principle of incompatibility.'' He summarizes this principle in the following words: ''...as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics.'' Therefore, it is evident that recourse to fuzzy measurement is even more important in complex contexts, in

Exhibit 7
Curve Produced by the General Equation
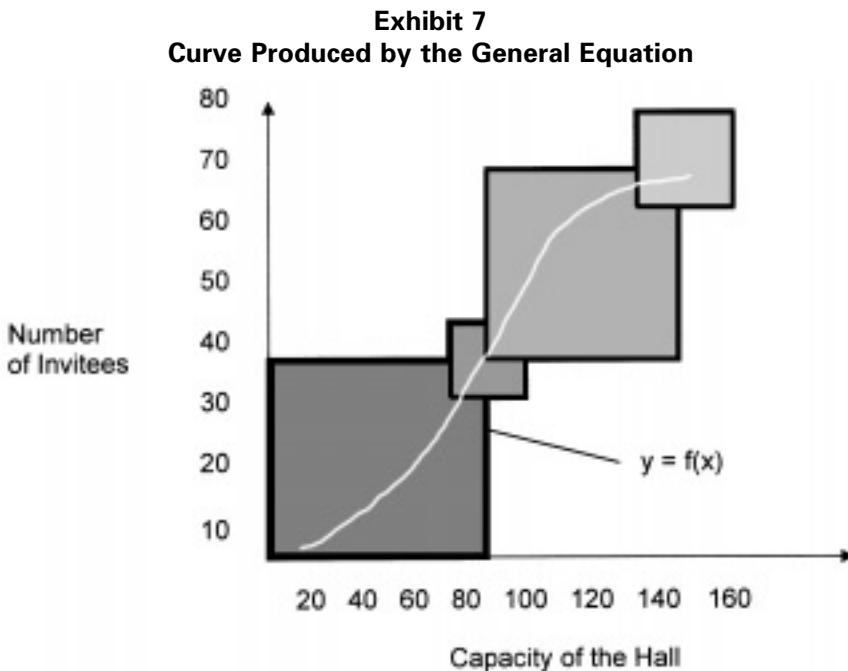
which one variable depends on a number of independent variables, not just one independent variable as in the simple example of the reception hall.

The reason for this is that even if the result that can be obtained by using fuzzy logic is only an approximation of the relationships characterizing real phenomena, this approximation enables us to represent the phenomena without resorting to the reduction of the intrinsic complexity—a different result than would be obtained by expressing such relationships by mathematical equations (especially when such equations are linear in form).[4] Kosko (1993:196–98) states:

> Fuzzy systems allow us to make conjectures that conform to the nonlinear world but without a mathematical model of the world....The technical term that indicates this character is valuation or approximation by models...The key is the absence of mathematical models, valuation without models, freedom from models. If one has a mathematical model, good. But where can we find it? We find good mathematical models only in books and mathematics courses. They are simplistic responses to simple problems. They are not at all important to the real world.

It is therefore the desire to work with methods for comprehending real phenomena that are shareable by a team of persons, but which do not require the simplification of the contexts, that provides the motivation to incorporate into formal valuation methods the effective processes from which human judgments are derived. The formal treatment of the environmental complexity by ''summaries'' depends on the human ability to gather in aggregate terms the informative components of real phenomena. When these summaries are sufficiently robust to be meaningful, even in their imprecision, the use of fuzzy logic is justified. Olivotto (1995) states:

> These (fuzzy methods) enable us to review the ways in which the subjects ''summarize'' complex relationships that, at the limit, are explicit in vague and imprecise terms within the decision-making processes. These summaries are the basis of approximate reasoning that seems to be used frequently for the effective solution of problems. The availability of computational methods having the ability to reprocess ''complex phenomena'' as fuzzy events and relationships is an interesting alternative to their simplification.

The Problem of Pricing Real Estate Investments: The Ratcliffian System

Three models have traditionally dominated the pricing (or valuation) of real estate by professional real estate appraisers—the sales comparison approach, the cost approach and the income capitalization approach. These models, in fact, are well known and used almost exclusively by professional valuers, although each of them has been severely criticized by professional appraisers, researchers and members of related professions.

Another set of models proposed by Ratcliff (1972) received little attention by the appraisal profession, although Dilmore (1991) cites Ratcliff's work as extremely

important. He states that, "The resulting quality rating/price regression model, though it attracted little attention at the time, was a crucial step in development of an entirely new way of approaching the extraction of probable price from the market." The stated goal of the Ratcliffian system is to enable appraisers to "replicate the buyer calculus," that is, to follow the same thinking process of a property's potential buyers. Thus, to the extent that traditional methods do not emulate the system of thinking of buyers, the Ratcliffian models may overcome this presumed defect.

Ratcliff proposed two different models: the "statistical inference" model and the "market simulation" model. We concentrate on the first, because it is a standard type of model, long recognized in other fields as a legitimate and appropriate method of performance and price evaluation.

The statistical inference model of Ratcliff is, in reality, a multi-attribute model. That is, it identifies a number of characteristics (attributes) of properties and provides a method for appraisers to estimate the property's value based on the relative influence of each attribute. Ratcliff claims that this model more closely replicates the buyer calculus because buyers consider a number of different attributes simultaneously. They do not look at several comparable properties and make price additions and subtractions to each comparable for ways in which it differs from the subject property.

An important technical assumption in the statistical inference model of Ratcliff is that the attributes are independent variables. This is probably one of the major defects of the model in terms of replicating the buyer calculus, because buyers naturally take into consideration any joint influences, as well as the separate effects, of the attributes when evaluating information about a property. However, demonstration of the application of fuzzy logic to this standard type of model can proceed within the constraints of this assumption.

The Ratcliff model also has an application drawback that appraisers have probably regarded as more intractable than the problems associated with the traditional three approaches. Ratings of the attributes and their weights, that represent the relative influence of each attribute in determining differences among property values, are not readily observable in the market. Rather, they result from a judgment process that cannot be formalized in an effective way. In other words, the process cannot be formalized without resorting to an artificial convention for making the information precise. This drawback, however, may be overcome by incorporating the Ratcliff statistical inference (multi-attribute) model within a fuzzy logic system.[5] The next section demonstrates by an example how fuzzy logic could enable analysts to recognize formally the imprecision in deriving the weights and ratings from their experience.[6]

Fuzzy Logic to Price an Income-Producing Property with the "Statistical Inference" Model

The statistical inference method of Ratcliff requires the appraiser to identify the principal characteristics of a property that cause it to vary in value from other, similar

properties. For an income-producing property, Ratcliff suggests the following six characteristics, although he recognizes that this list could differ among properties and market situations: (1) location; (2) space arrangements; (3) physical condition; (4) operating ratio; (5) mechanical equipment; and (6) extra services and facilities.

Weights must be assigned to the attributes to reflect their relative importance in causing variances in values. Then, each comparable and the subject property is rated to reflect its relative desirability according to a predetermined scale. The ratings are then multiplied by the weight assigned to that characteristic, and the weighted ratings are summed for each comparable property and the subject property to find a synthetic degree of desirability. These features, their weights, the ratings, the synthetic degree of desirability for each comparable property and the subject property, and the selling prices of the comparable properties are shown in Exhibit 8.[7]

In this approach, the judgment of the analyst is fundamental in determining (by emulating the system of thinking of buyers) both the weights and the ratings of each attribute for the properties under analysis. Therefore, given the traditional method of human reasoning, the expression of a judgment by a crisp number can occur only after having made the judgment precise in an artificial way. This artificial process and the consequent simplification of the contexts represented by the human ''summaries,'' however, can be avoided by the rigorous processing of vague judgments with fuzzy logic. Fuzzy logic also permits the formalization of the rules from which the judgments are derived, makes possible their incorporation into formal investment and valuation methods, and makes the judgment-making process shareable by a valuation team.

To demonstrate how fuzzy logic makes possible the formalization of the rules from which the judgments are derived, we have chosen as an example one of the attributes—location. We could, of course, have selected any other attribute or its connected weight. We interpret the fuzzy rating number as a result of fuzzy rules of the type: if the distance is ''near'' then the rating number is ''low.'' We hypothesize

Exhibit 8
Weights and Ratings for an Income Property

Productivity Feature	Weight	A	B	C	D	E	Subject
Location	30	1	2	2	1	4	3
Space Arrangements	20	8	4	4	8	1	6
Physical Condition	10	8	6	10	3	1	2
Operating Ratio	10	3	2	1	1	5	4
Mechanical Equipment	10	1	3	1	2	3	1
Extra Services and Facilities	20	2	3	3	1	3	2
Degree of Desirability		350	310	320	270	290	320
Selling Price ($000)		860	875	895	925	910	?

Exhibit 9
Conversion of Distance to Fuzzy Sets

Average distance in Km
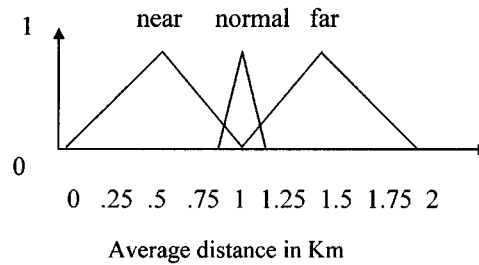
or, in analytical notation:

A1: Near = {(0, 0); (0.25, 0.5); (0.5, 1); (0.75, 0.5); (1, 0)}
A2: Normal = {(0.75, 0); (1, 1); (1.25, 0)}
A3: Far = {(1, 0); (1.25, 0.5); (1.50, 1); (1.75; 0.5); (2, 0)}

that the reference variable to judge the location is the average distance of the property from principal destinations, such as a major highway, a major industrial zone, a shopping center, etc.

Specifically, we hypothesize that the fuzzy concepts used by the appraiser to express a judgment about the location of the property are represented by the fuzzy sets shown in Exhibits 9 and 10. Exhibit 9 shows the conversion of reference measures of distance in kilometers to fuzzy sets, while Exhibit 10 demonstrates the conversion of locational ratings (judgments) to fuzzy sets.

Exhibit 10
Conversion of Location Ratings to Fuzzy Sets

Rating of Location Convenience
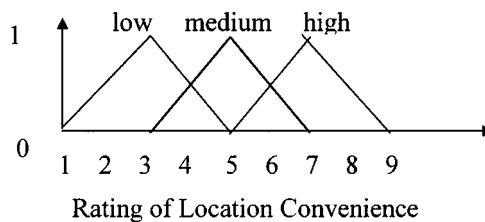
or, in analytical notation:

B1: Low = {(1, 0); (2, 0.5); (3, 1); (4; 0.5); (5, 0)}
B2: Medium = {(3, 0); (4, 0.5); (5, 1); (6, 0.5); (7, 0)}
B2: High = {(5, 0); (6, 0.5); (7, 1); (8, 0.5); (9, 0)}.

The procedures used to rate the location of the property could be expressed in formal terms by the following fuzzy rules:

1. If the distance is near, then the rating number must be low.
2. If the distance is normal, then the rating number must be medium.
3. If the distance is far, then the rating number must be high.

The impact of the rules on the fuzzy sets can be geometrically represented as shown in Exhibit 11.

To demonstrate the process by which the location judgment is derived, we must:

1. Formalize the single fuzzy rules by means of the Cartesian product that produces the results shown in Exhibit 12.[8]
2. Formalize the union of all the fuzzy rules by the union operation shown in Exhibit 13.

Exhibit 11
Geometric Representation of the Fuzzy Rules on the Fuzzy Sets for Distance and Location Rating
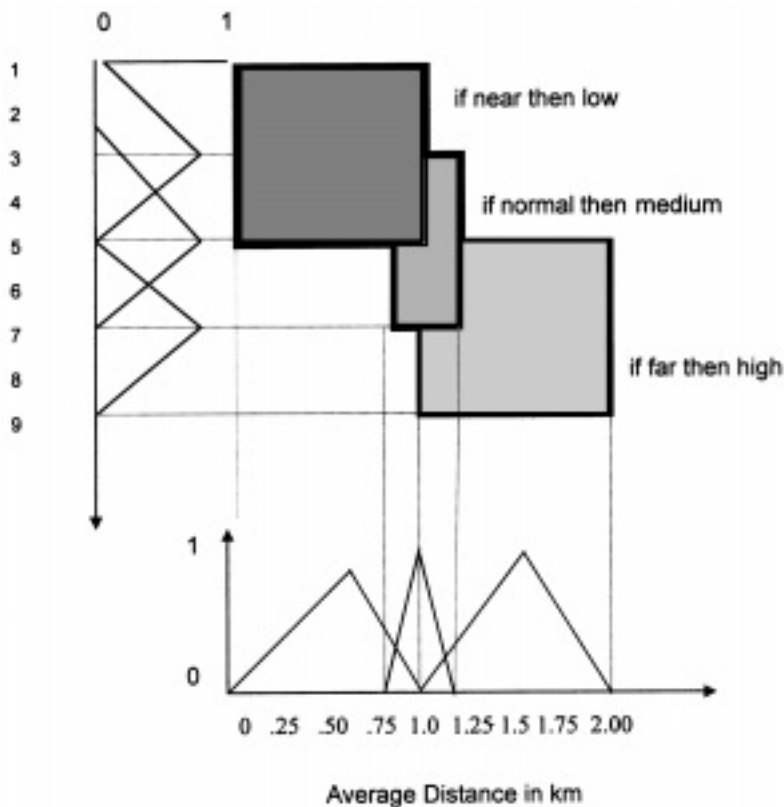
Exhibit 12
Application of the Single Fuzzy Rules

$x(m_{A1}(x))$ / $y(m_{B1}(y))$	0 (0)	0.25 (0.5)	0.50 (1)	0.75 (0.5)	1.00 (0)
1 (0)	0	0	0	0	0
2 (0.5)	0	0.5	0.5	0.5	0
3 (1)	0	0.5	1.0	0.5	0
4 (0.5)	0	0.5	0.5	0.5	0
5 (0)	0	0	0	0	0

R1: if $A1$ then B1 Cartesian Product $= A1 \times B1$
where: $m_{a1 \times B1}(x,y) = m_{R1}(x,y) = min[m_{A1}(x); m_{B1}(y)]$.

$x(m_{A2}(x))$ / $y(m_{B2}(y))$	0.75 (0)	1.00 (1)	1.25 (0)
3 (0)	0	0	0
4 (0.5)	0	0.5	0
5 (1)	0	1.0	0
6 (0.5)	0	0.5	0
7 (0)	0	0.	0

R2: if $A2$ then $B2$ Cartesian Product $= A2 \times B2$
where: $m_{A2 \times B2}(x,y) = m_{R2}(x,y) = min[m_{A2}(x); m_{B2}(y))]$.

$x(m_{A1}(x))$ / $y(m_{B1}(y))$	1.00 (0)	1.25 (0.5)	1.50 (1)	1.75 (0.5)	2.00 (0)
5 (0)	0	0	0	0	0
6 (0.5)	0	0.5	0.5	0.5	0
7 (1)	0	0.5	1.0	0.5	0
8 (0.5)	0	0.5	0.5	0.5	0
9 (0)	0	0	0	0	0

R3: if $A3$ then $B3$ Cartesian Product $= A3 \times B3$
where: $m_{A3 \times B3}(x,y) = m_{R3}(x,y) = min[m_{A3}(x); m_{B3}(y)]$.

In the construction of a fuzzy system it is necessary to go beyond formalization of the single fuzzy rules. Also required is the formalization of their union, that is, the ways that permit the joint consideration of the single rules. This is necessary so that when two or more rules are acting simultaneously (geometrically, when parts of different shaded areas are overlain on each other), it is clear as to which rule prevails. In other words, pairs of values are examined (*e.g.*, (0.75,3) that are present in more than one rule but with different degrees of membership (*e.g.*, $m_{A1 \times B1}(0.75,3) = 0.5$

Exhibit 13
Union of All Fuzzy Sets

X Y	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1	0	0	0	0	0	0	0	0	0
2	0	0.5	0.5	0.5	0	0	0	0	0
3	0	0.5	1.0	0.5	0	0	0	0	0
4	0	0.5	0.5	0.5	0.5	0	0	0	0
5	0	0	0	0	1.0	0	0	0	0
6	0	0	0	0	0.5	0.5	0.5	0.5	0
7	0	0	0	0	0	0.5	1.0	0.5	0
8	0	0	0	0	0	0.5	0.5	0.5	0
9	0	0	0	0	0	0	0	0	0

R: if $A1$ then $B1$ or if $A2$ then $B2$ or if $A3$ then $B3$
where: $m_R(x,y) = max_i[m_{AixBi}(x,y)]$ or $m_R(x,y) = max[m_{A1xB1}(x,y); m_{A2xB2}(x,y); m_{A3xB3}(x,y)]]$.

while $m_{A2xB2}(0.75,3) = 0$). The union rule establishes which degree of membership must be recognized within the system. Also in this case, as we noted previously, there are multiple possibilities for defining the operation of union. For this article we adopt an operation of union that selects the maximum degrees of membership from among the previously selected minimums (*e.g.*, $m_R(0.75,3) = 0.5$): $m_R(x,y) = max_i[m_{AixBi}(x,y)]$. The advantages of this operation of union are presented in Kickert (1978:122ff).

3. Apply the rule of inference by means of the following definition: $m_{B\text{-}output}$ $(y) = max_x \min[m_{A\text{-}input}(x); m_R(x,y)]$. Hypothesizing, for example, a precise input as $A_{input} = (1, 1)$ produces the results shown in Exhibit 14.[9] First, the lower degree of membership for each combination of input and the union set is obtained, and then the maximum degree of membership for each row (Y value) is determined. These maximums are shown in bold.

Therefore, in terms of the final output, the result would be as follows:

$B_{output} = \{(1, 0); (2, 0); (3, 0); (4; 0.5); (5, 1); (6, 0.5); (7; 0); (8, 0); (9, 0)\}$.

From the geometric representation in Exhibit 15, one can see that $R2$ is operating totally, while $R1$ and $R3$ are inoperative.
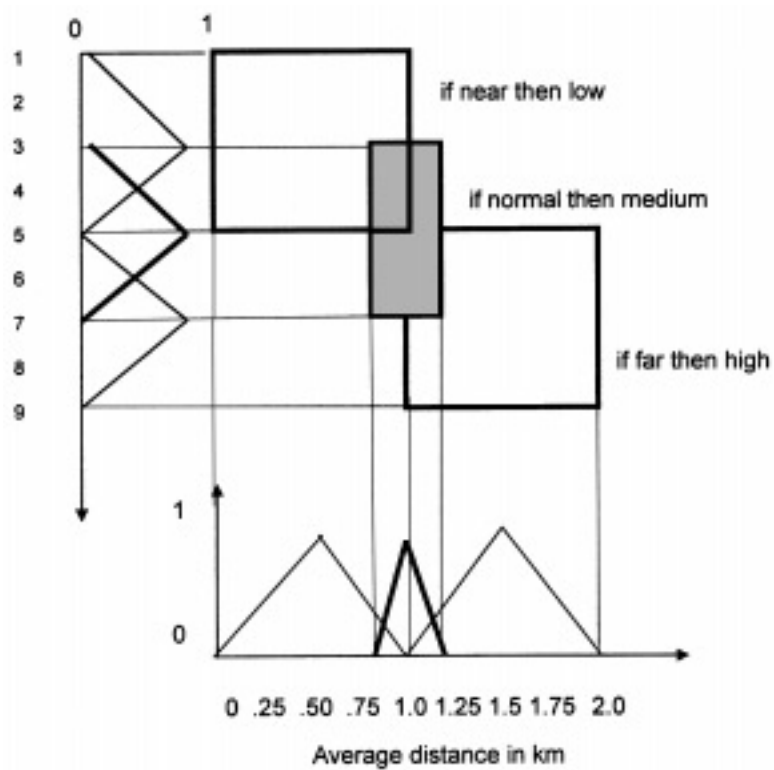
Fuzzy logic, however, makes possible not only the formalization of the rules from which judgments such as location ratings or location weight are derived, it also permits formalization of the procedures by which complex judgments (such as the degree of desirability of a property investment) are derived. In this case, the formalization of these rules is performed, as in the traditional multi-attribute valuation model, by means of the weighted rating operation.

Exhibit 14
Application of Rule of Inference

X Y	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0.5	0	0	0	0
5	0	0	0	0	1.0	0	0	0	0
6	0	0	0	0	0.5	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Note: $min[m_{A\ input}(x); m_R(y,x)]$ and $m_{B\text{-}output}(y) = max_x\ min[m_{A\text{-}input}(x); m_R(x,y)]$.

Exhibit 15
Geometric Representation of the Rule of Inference

if near then low

if normal then medium

if far then high

0 .25 .50 .75 1.0 1.25 1.5 1.75 2.0

Average distance in km

Using comparable A as an example, we hypothesize that this operation using triangular fuzzy numbers produces the fuzzified degrees of desirability as shown in Exhibit 16. The triangular form is, of course, only one of the possible forms that the fuzzy concepts or numbers can assume. Other basic forms could be the trapezoid, camelback (double hump), tent or spire. We use the triangular form for two reasons: First, it is the easiest to demonstrate computationally. Second, given the propensity for real estate characteristics and values to exhibit a central tendency we believe the triangle is the most intuitive form for explaining the functioning of a fuzzy system. This operation would produce the results shown in Exhibit 17.

These results are represented by triangular fuzzy numbers: the first and the last value have a degree of membership equal to 0, while the middle value has a degree of membership equal 1. They are, however, didactically derived from the results shown in Exhibit 8. In fact, the middle values are the same as the original values shown in Exhibit 8, while the first and the last values are equal to the original values $\pm$ 1 for the attributes and $\pm$ 5 for the weights. The results for the comparable properties could be represented geometrically as shown in Exhibit 18:
Additionally, fuzzy logic supports the second step of the statistical inference model—pricing of the subject property based on its degree of desirability. In fact, it is possible to interpret the causal relationships among the degrees of desirability and the selling prices of the comparable properties as fuzzy rules for pricing the subject property. In other words, instead of finding a linear regression relationship that would be valid between the degree of desirability of a generic property investment and its market price, we can interpret the relationships characterizing the comparable properties as five fuzzy rules of the type (see Exhibit 19):[10]

R1: if the degree of desirability is *A* **(165, 350, 595)** then the market price is **860**.

R2: if the degree of desirability is *B* **(140, 310, 540)** then the market price is **875**.

R3: if the degree of desirability is *C* **(145, 320, 390)** then the market price is **895**.

R4: if the degree of desirability is *D* **(120, 270, 480)** then the market price is **925**.

R5: if the degree of desirability is *E* **(135, 290, 455)** then the market price is **910**.

Exhibit 16
Fuzzification of the Weighted Ratings for Comparable A

		A		
Weighted rating of location	=	25 × 0	30 × 1	35 × 2
		+	+	+
Weighted rating of space arrangements	=	15 × 7	20 × 8	25 × 9
		+	+	+
....	=	......	......	......
		=	=	=
Degree of desirability		165	350	595

Exhibit 17
Weighted Ratings of Properties

Productivity Feature	Weight			A			B			C			D			E			Subject		
Location	25	30	35	0	1	2	1	2	3	1	2	3	0	1	2	3	4	5	2	3	4
Space Arrangements	15	20	25	7	8	9	3	4	5	3	4	5	7	8	9	0	1	2	5	6	7
Physical Condition	5	10	15	7	8	9	5	6	7	9	10	0	2	3	4	0	1	2	1	2	3
Operating Ratio	5	10	15	2	3	4	1	2	3	0	1	2	0	1	2	4	5	6	3	4	5
Mechanical Equip.	5	10	15	0	1	2	2	3	4	0	1	2	1	2	3	2	3	4	0	1	2
Extra Services and Facilities	15	20	25	1	2	3	2	3	4	2	3	4	0	1	2	2	3	4	1	2	3
Degree of desirability				165	350	595	140	310	540	145	320	390	120	270	480	135	290	455	160	320	540
Selling Price ($000)					860			875			895			925			910			?	

Exhibit 18
Representation by Fuzzy Sets of the Comparables' Weighted Ratings
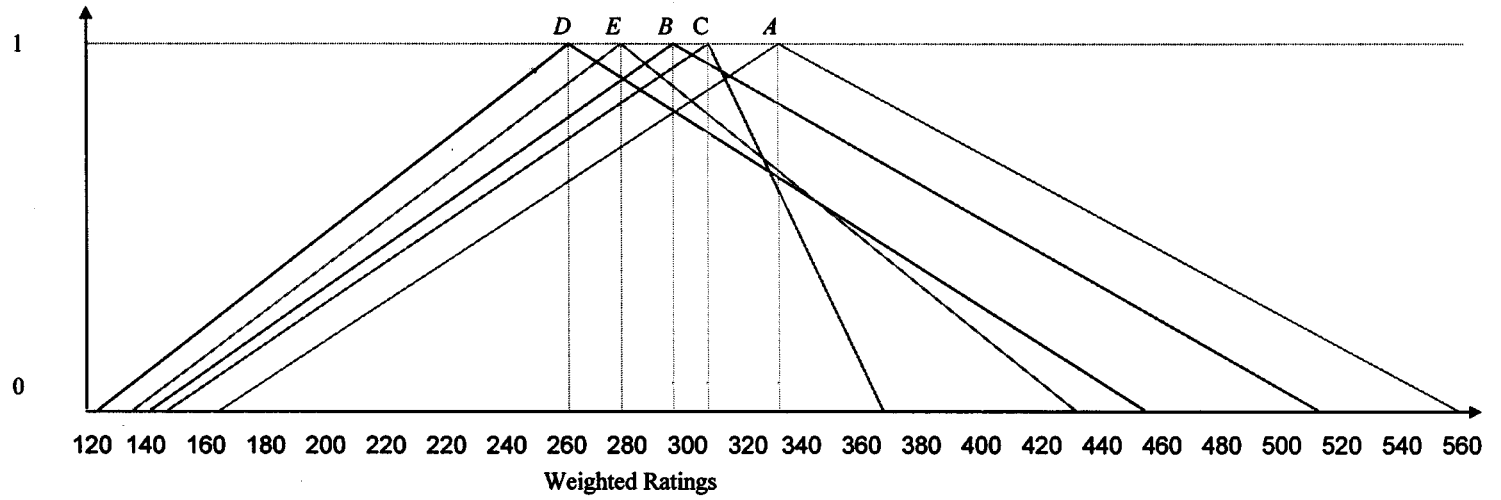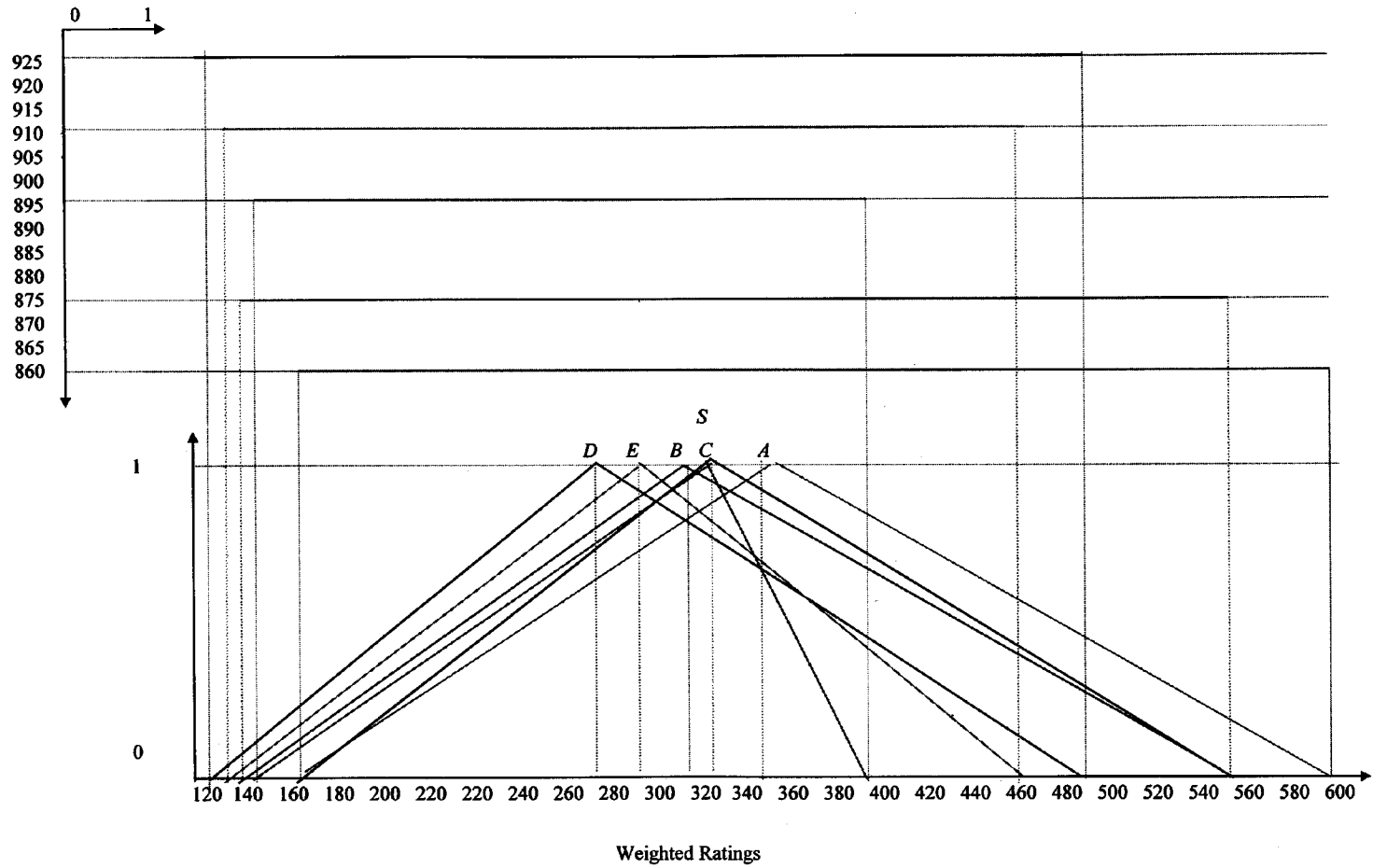

Weighted Ratings

Exhibit 19
Representation by Fuzzy Sets of the Subject's and Comparables' Weighted Ratings

Weighted Ratings

We can, therefore, deal with the pricing problem of the subject property in the following terms: if the degree of desirability of the subject property **S** is **(160, 320, 540)** then the market price is ?.[11]

In this particular case the union set of all the rules is nothing more than a simple layering of the single fuzzy rules.[12] Therefore, we could apply the composition rule of inference, comparing separately the input fuzzy set S, that represents the desirability degree of the subject property, to each single fuzzy rule:

$$m_{B\text{-}output} (860) = max_x \, min[m_S(x); \, m_{R1}(x,y)]$$

$$m_{B\text{-}output} (875) = max_x \, min[m_S(x); \, m_{R2}(x,y)]$$

$$m_{B\text{-}output} (895) = max_x \, min[m_S(x); \, m_{R3}(x,y)]$$

$$m_{B\text{-}output} (925) = max_x \, min[m_S(x); \, m_{R4}(x,y)]$$

$$m_{B\text{-}output} (910) = max_x \, min[m_S(x); \, m_{R5}(x,y)]$$

Since $m_{R1}(x,y)$, $m_{R2}(x,y)$, etc. are always equal to the fuzzy sets that represent the degree of desirability of the comparable properties and, therefore, to $m_{A1}(x)$, $m_{A2}(x)$,[13] etc., it is possible to make a geometric interpretation of this alternative method of applying the composition rule of inference by determining: (1) the $min[m_S(x); m_{Rn}(x,y)]$ (shown in bold) that represents the superimposition of the graphs of the input fuzzy set S and the single fuzzy rule considered; and (2) inside the superimposition obtained (that could also be considered a fuzzy set) the element having the highest degree of membership to this set (see Exhibits 19–24).

Thus, the results that an analyst would obtain from this series of comparisons are five precise outputs with different degrees of membership to a hypothetical fuzzy set "market value of subject property." However, the fuzzy set "market value of subject property" could be expressed in terms of a single precise number using a mathematical method of approximation such as linear regression. Such a procedure would allow the analyst to identify the degree of membership of a price to the fuzzy set "market price of subject property," even when the price differs from the five precise outputs.

It should be pointed out that the degree of membership represents, ultimately, the possibility (not the probability) that the value obtained would be the selling price of the property.[14] Nevertheless, a range of possible future selling prices with different degrees of membership to the hypothetical fuzzy set "market value of subject property," seems to approximate better the thinking process of a property's potential buyers. Thus, the final result produced by a fuzzy system should be more realistic than a single number with confidence intervals produced by linear regression (see Exhibit 25).

It should also be noted that the final result of the fuzzy analysis will not produce Ratcliff's "probable price," since the mathematical processes differ. The result should be similar, but more information is provided in terms of several "possible" prices

Exhibit 20
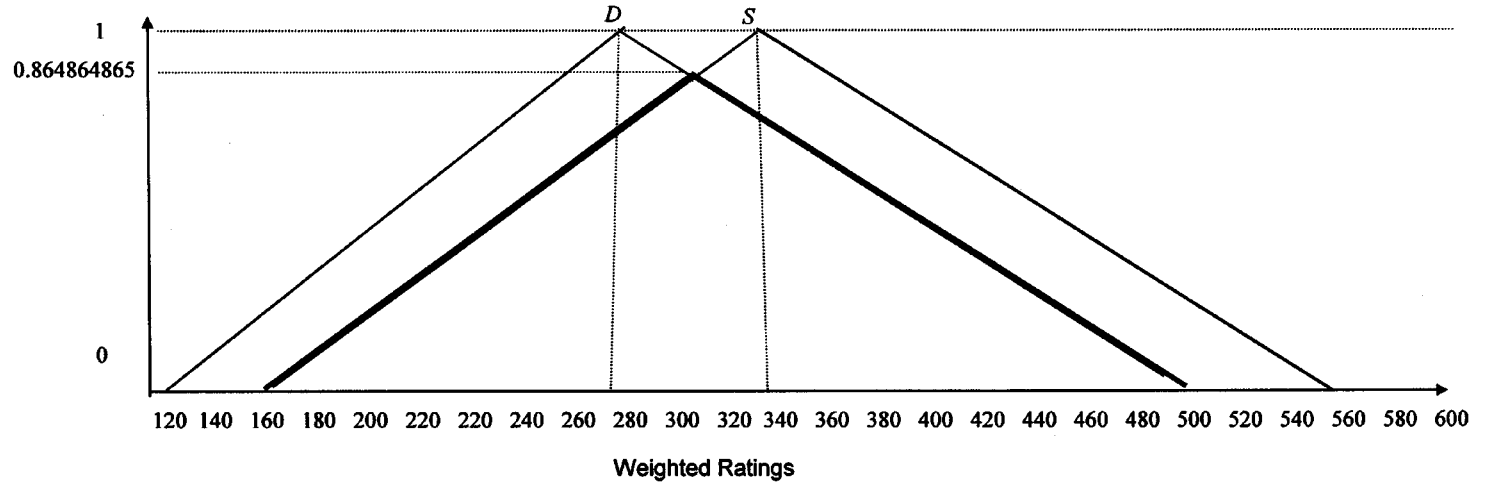**Representation of the Comparison between "*D*" and the Subject—Max(*x*) Min[$M_s(x)$; $M_D(x,925)$]**

Weighted Ratings

Exhibit 21
Representation of the Comparison between "*E*" and the Subject—Max(*x*) Min[*M_s*(*x*); *M_E*(*x*,910)]
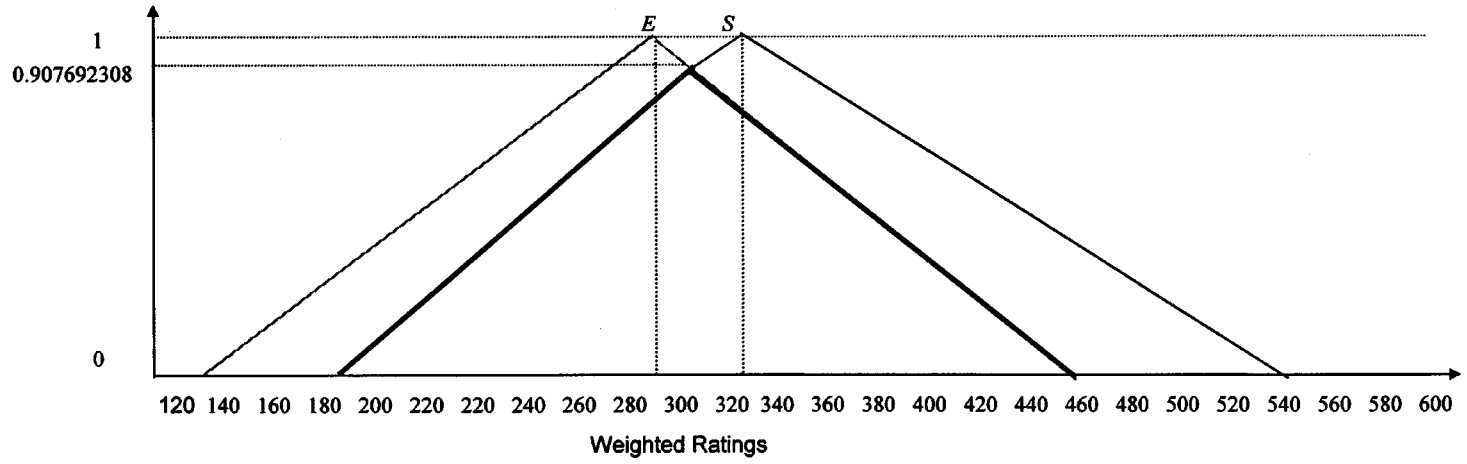
Exhibit 22
Representation of the Comparison between "C" and the Subject—Max(x) Min[$M_s(x)$; $M_c(x,895)$]

Weighted Ratings

Exhibit 23
Representation of the Comparison between "B" and the Subject—Max(x) min[$M_s(x)$; $M_B(x, 875)$]

Weighted Ratings
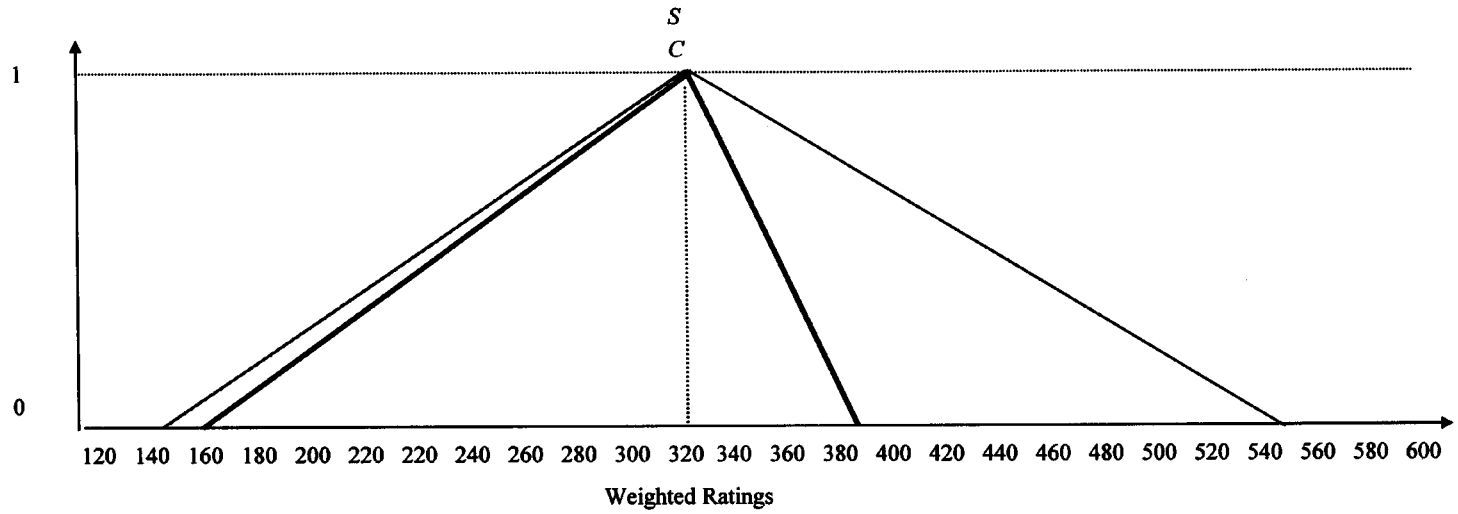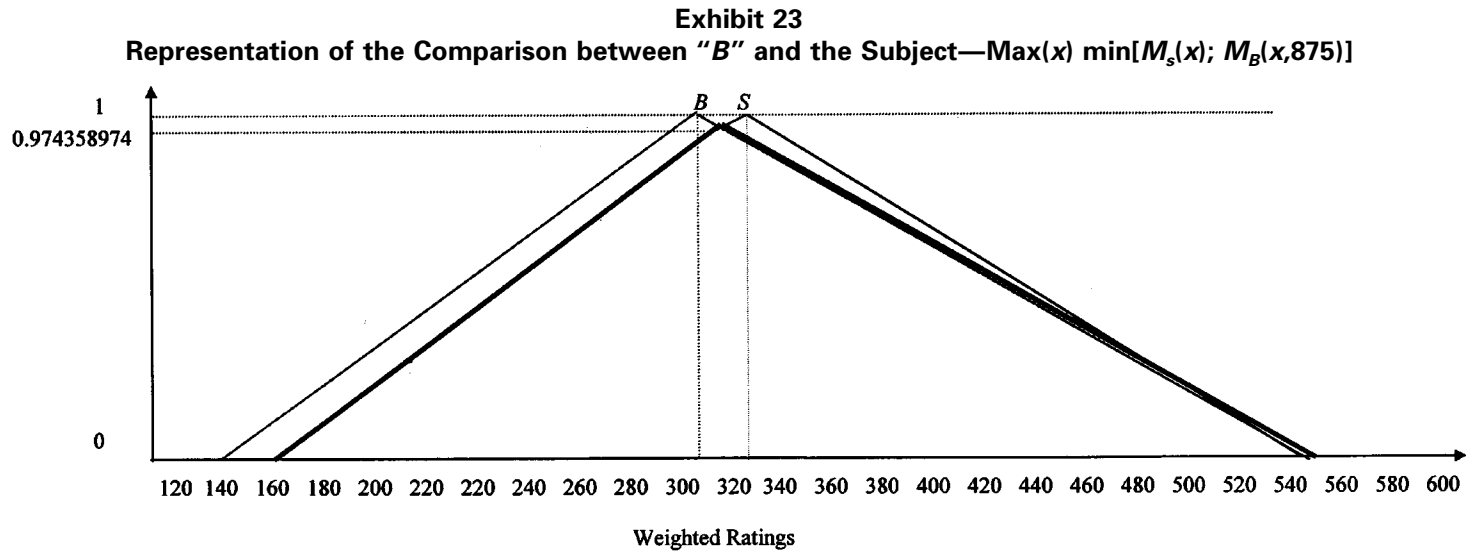
Exhibit 24
Representation of the Comparison between "*A*" and the Subject—Max(*x*) Min[*M*ₛ(*x*); *M*ₛ(*x*,860)]
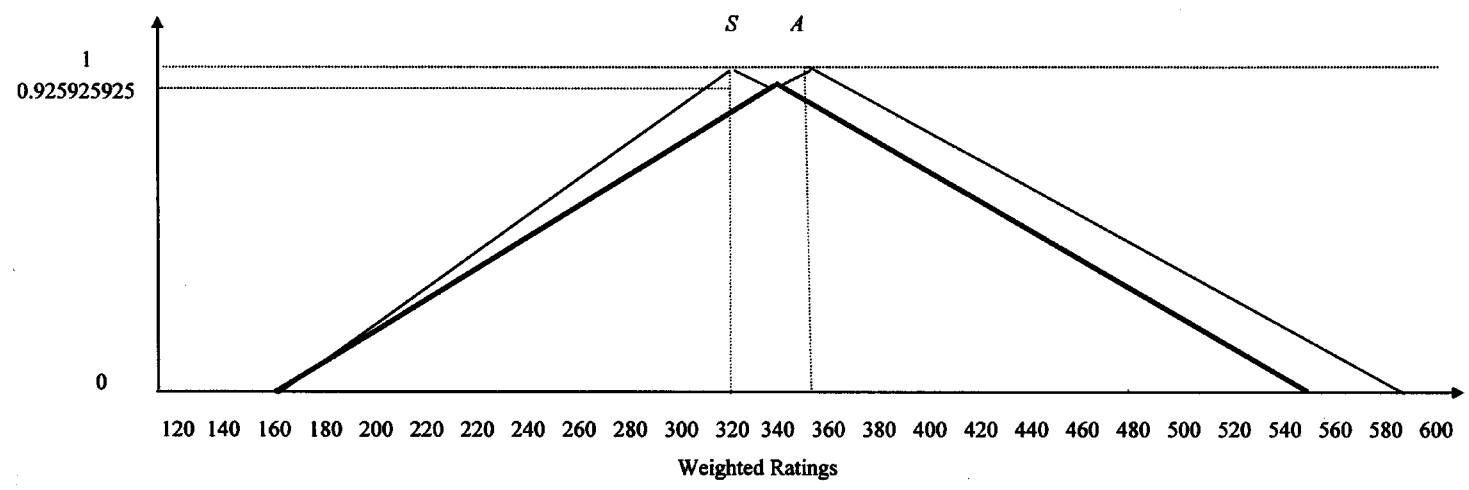

Weighted Ratings
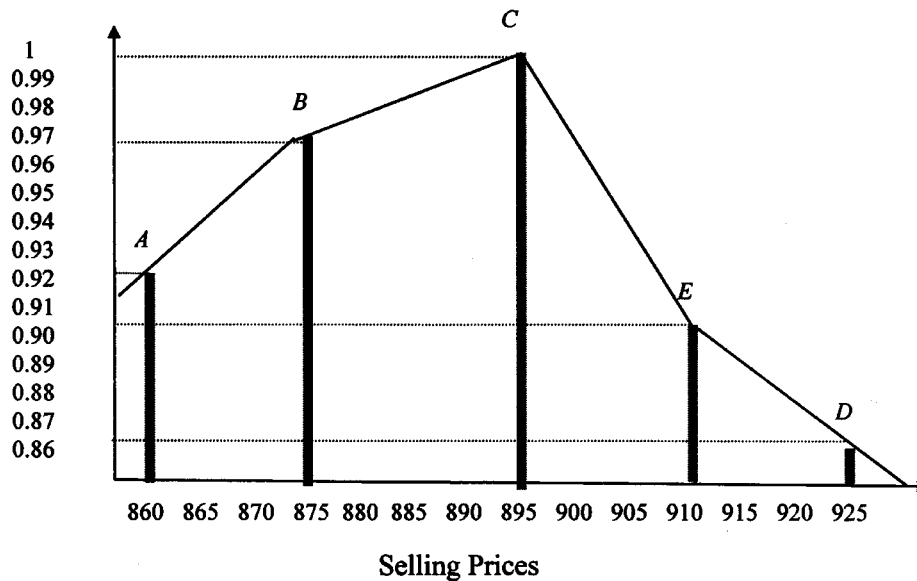
Exhibit 25
Membership Degrees of the Comparables' Selling Prices to the Fuzzy Set of
the Subject's Market Value

Selling Prices

having different degrees of membership with the fuzzy set "market value of the subject property." In other words, the fuzzy number having the highest degree of membership to the fuzzy set will equal only by happenstance the "most probable price" produced by Ratcliff's weighted rating system.

Conclusion

Dilmore was perhaps the first appraisal practitioner and scholar to recognize the potential applications of fuzzy logic to real estate. He makes several important points about fuzzy logic (Dilmore, 1993). First, fuzzy logic is not sloppy thinking. Rather it is a way of dealing with the lack of precision that we make in most of our daily decisions and evaluations (*e.g.,* something is nice, or pretty, or well located). Second, reality is not precise. Dilmore quotes Einstein, who said, "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality." McNeill and Freiberger (1993) cite many existing and potential applications of fuzzy logic. Among applications that have existed for several years are the system that controls subway trains in Sendai, Japan; computers that are able to recognize a person's handwriting; and "intelligent" washing machines, microwaves, cameras, camcorders and automobiles. For example, a washing machine equipped with a fuzzy logic system of control can automatically detect which cycle is best for the particular load of clothes.

The potential uses of fuzzy logic are truly staggering. These include expert decision systems that can decipher and interpret documents, intelligent cars that can drive

themselves, machines that can write novels in a particular style, sex robots with a humanlike repertoire of behavior and substances that can be injected into the body that will kill only cancerous cells and that will slow the aging process.

It seems likely that the potential for applications of fuzzy logic is equal or even greater in the social sciences than in the relatively "hard" sciences. The Japanese are developing expert decision support systems for foreign exchange trading, language and image understanding, intelligent robots and fuzzy neural networks. Certainly, the development of real estate decision support systems for property management, evaluation of property performance, property valuation and portfolio construction with continuous monitoring can be among the important advances provided by fuzzy logic. And current GIS systems can be fuzzified to provide more meaningful analytic conclusions. As McNeill and Freiberger (1993:283) point out, "... fuzzy logic is practical in the highest sense: direct, inexpensive, and bountiful. It forsakes not precision, but pointless precision. It abandons an either/or hairline that never existed and brightens technology at the cost of a tiny blur."

Notes

[1] Regarding the convention for the definition of the membership function, see Zimmerman (1986: 204–8).

[2] It can be noted that a fuzzy number can be interpreted as a generalization of the classic number or, perhaps more correctly, that a classic number is a limited case of a fuzzy number that can be represented by a triangle having a base constrained so that it is reduced to a single vertical line.

[3] Another necessary condition is that of extending to fuzzy sets the operations generally applied to classic sets. The reason for this condition is that to be able to extract the judgment that a person would express, for example, about the degree of membership of an element x in a fuzzy set C^* of "handsome and intelligent men," we must know the definitions not only of the fuzzy sets H^* "handsome men" and I^* "intelligent men," but also of the operation of fuzzy intersection.

Another major difference from classic sets is that in fuzzy sets there is not a unique definition for the operation of fuzzy intersection, fuzzy union, fuzzy complement, etc. To determine the degree of membership of the element x to the fuzzy set C^* in the following three cases:

$$M_{H^*}(x) = 1; \; m_{I^*}(x) = 1,$$

$$M_{H^*}(x) = 0; \; m_{I^*}(x) = 0,$$

$$M_{H^*}(x) = 0.9; \; m_{I^*}(x) = 0.1.$$

It is evident that the first equation would be equal to $m_{C^*}(x) = 1$, and the second equation would be equal to $m_{C^*}(x) = 0$. The value of the third equation, however, is not intuitive without the operation of fuzzy intersection on the degree of membership.

Theory defines two principal operations that restrict the degree of membership in the interval (0,1), acting in the following terms: $m_{C^*}(x) = (m_{H^*}(x) \cap m_{I^*}(x)) = min(m_{H^*}(x), m_{I^*}(x))$, that applied in the example would determine that $m_{C^*}(x) = 0.1$; $m_{C^*}(x) = (m_{H^*}(x) \cap m_{I^*}(x)) = m_{H^*}(x) \cdot m_{I^*}(x)$, that applied in the example would determine that $m_{C^*}(x) = 0.09$.

However, there are more definitions of operations of fuzzy intersection than those presented here, such as fuzzy union, etc.; the appropriate definition must be selected based upon the context of the problem (Schmucker, 1980:5–17).

[4] In reality, the capacity of the hall in the proposed example was valued only on the basis of its size in square meters, not including other potentially important variables such as its arrangement with dividing walls, the placement of tables, etc. Although such a summary measure does not include precisely all of the characteristics of the phenomenon represented, it is a synthesis of the characteristics that is workable because of the ability of human beings to function on the basis of summary measures. It seems evident that acceptable results can be obtained even by a single measure if it represents a basis that is adequate for making considered judgments.

[5] Dilmore (1994) demonstrates the application of fuzzy numbers to the traditional approaches.

[6] For some fuzzy logic applications and simulation software, see McNeill and Thro (1994).

[7] We use the same example provided by Ratcliff, changing only the prices of the comparables to reflect deflation in the value of the dollar by multiplying the price of each comparable by five.

[8] Geometrically a fuzzy rule assumes a rectangular form, or more correctly, a tridimensional pyramidal form in which the base is the rectangular shaded area derived from the product of the bases of the two triangles characterizing the rule (mathematically it is the result of the product of two segments that form pairs of values (x,y), while the height of the single points forming the base of each triangle express their degree of membership to the rule. The principal problem, therefore, in the formalization of a fuzzy rule is the assignment of a degree of membership to the rule at every point of the shaded area, or to pairs of values (x,y). Although there are a number of possibilities for formalizing the assignment of degrees of membership, we have chosen the solution of the Cartesian Product of the two fuzzy sets (A x B) comprising the rule. The reasons for preferring this solution can be summarized in the fact that it allows us to define the fuzzy rules (such as if An then Bn) so that an input of the fuzzy set An produces an output equal to that which would be obtained from the application of nonfuzzy rules, that is, exactly equal to Bn. See Kickert (1978:124–6).

[9] The final step for defining a fuzzy set is formalization of the composition rule of inference. This rule permits us to obtain a final output (B_{output}) given an initial input (A_{input}) using the union set of all of the fuzzy rules. While there are several ways for defining a composition rule of inference. We prefer the definition shown here for reasons advocated by Kickert (1978:122ff).

[10] This application of fuzzy logic is different from the example used previously. Since the selling prices are observed from market transactions, the fuzzy rules are only in part derived from the experience of analysts. In this kind of application the induction of the rules through the observation of input-output pairs is usually accomplished through the use of neural network techniques. The results are "Neuro Fuzzy" systems that possess the capability to learn and adapt from experience. Regarding Neuro Fuzzy applications see Lin and Lee (1996).

[11] The geometric representation of these rules will not be a rectangular shaded area, but a segment. (Mathematically, it is the result of the product of one segment times another segment which is a single point; the result is pairs of values (x,y) with y being fixed). This is a special case in which the fuzzy sets that constitute the fuzzy rules are "degenerated." This means that there is only one element in the fuzzy set with a degree of membership of one. These sets can also be interpreted as classic sets characterized by only one element, or more simply, as classic numbers.

[12] Every rule is in fact characterized by a different degenerated fuzzy set in output, so it is not possible that the same pair of values exists in more than one rule (geometrically, the fact that the segments are parallel makes it impossible to superimpose one segment on others).

[13] In the case under analysis, $m_{Bn}(y)$ will always be equal to one, and for this reason min[$m_{An}(x)$; $m_{Bn}(y)$] will always be equal to $m_{An}(x)$. Therefore, the degree of membership to the specific fuzzy rule characterizing the element x will always be assigned to every pair of values.

[14] According to Lin and Lee (1996), a possibility measure differs from a probability measure by not requiring the axiom of additivity.

In Italy, in order for a coauthor of a publication to receive credit for professional advancement, the publication must identify the sections prepared primarily by each of the coauthors. In this article, sections two, three four and six were authored primarily by Dr. Bagnoli, while sections one, five and seven were authored primarily by Dr. Smith. Integrational work on all sections was done by both authors.

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