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Abstract
This paper compares the performance of a quadratic utility function and discusses how to change its characteristic parameter, ARA, so that rating is consistent with return and risk measurements. In particular, this parameter is modified in such a way that a positive return Fund has always a rating higher than one with a negative yield.
This modification confirms the possibility of building a new ranking procedure which is more coherent with the actual behaviour of investors.

Keywords
Quadratic Utility Function, Positive and Negative Returns, Absolute Risk Aversion, Morningstar Rating, Truncated Normal Distribution, Incomplete Gamma Function, Italian Pension Fund.

JEL Codes
G11, G14, G24.

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1. Introduction

The purpose of this paper is to consider a proper way to rank investment Funds using a quadratic utility function.

The choice of this type of function is motivated by the requirement to link the ratings to risk (*standard deviation*) and return, conditionally to the subjective risk-aversion. The risk-aversion is determined by coefficients of this function.

Irrespective of their standard deviation, it seems counterintuitive that a Fund with positive performance may be ranked below another Fund with negative return.

In addition, the parameter ARA (*Absolute Risk Aversion*), usually defined on the basis of general considerations, in this case is defined only in accordance with the characteristics of return and risk of Funds being analyzed, and it is therefore strongly linked to the fact that we can give an its measure independently from other considerations.

This is a crucial parameter and its link with the characteristics of the Funds makes possible a clear distinction of ranking between Funds with positive and those with negative return.

Conversely, an example is given in which a utility function of the type used by Morningstar, i.e. CRRA (*Constant Relative Risk Aversion*), leads to the possibility that a fund with positive performance has a rating less than one with a negative return.

In this latter case, by following a truncated Gaussian density function, its non-central moments of order n are computed in closed form. The subsequent calculation of the utility function coming from a series expansion highlights that this function provides a rating based essentially on measures of returns and considers the risk in a marginal way. In fact, the level curves in the risk/return plane have a linear behaviour, almost parallel to the risk axis, when standard deviations are less than 20%.

Therefore, the CRRA utility function, focused on the constancy of the RRA parameter, greatly reduces the importance of the risk component.

Morningstar is the most influential rating agency for mutual funds. It is therefore important to analyze the sensitivity of its evaluations with respect the risk and return dimensions.

Our results give a methodological support to other studies that have examined the role of risk measurements in the Morningstar rating under various aspects. For example, Lisi and Caporin (2012) show that ratings obtained with the setting of Morningstar are very similar to those obtained by assuming that the investor is risk-indifferent.

Del Guercio and Tkac (2008) estimate the value of a star in terms of the asset flow it generates for the typical fund and, following rating changes, they find economically and statistically significant abnormal flow in the expected direction, positive for rating upgrades and negative for rating downgrades, ranging from 13 to 30% of normal flows.

In the analysis of Amenc and Le Sourde (2007) it appears that the Morningstar rating (among other rating measures) does not deal adequately with some aspects of fund evaluation. One of these aspects concerns risk, both in terms of measuring the risks that were really taken by the manager and of the necessity of taking into account extreme risks.

Casarin et alias (2005) provide a comprehensive analysis of the relative benchmark performance measure (*Morningstar rating*) applied to Italian equity funds. They find that the

Morningstar rating is highly correlated with the classical performance measures (*Sharpe ratio, Sortino ratio* and *Treynor ratio*) and lowly correlated with the customized benchmark measure (*Information ratio*). This result is important because the information ratio measures the quality of the manager's information discounted by the residual risk in the betting process.

Adkisson and Fraser (2003) perform a statistical analysis which shows an inverse relationship between fund rating and fund age, both for the overall ratings and time-specific ratings, with a tendency for larger funds to dominate the middle star rankings. Similar results are found by Vinod and Morey (2002), that is, young fund ratings contain more estimation risk than middle-aged and/or seasoned funds.

The paper is organised as follows. Section 2 introduces the Quadratic Utility Function (QUF) as objective function on which the Arrow-Pratt measure of absolute risk-aversion (ARA) is applied. The ARA measure depends on parameters which are modified through the definition of the Normalized Quadratic Utility Function (NQUF). In contrast with Morningstar rating, with the NQUF rating measure a positive return Fund has always a rating higher than one with a negative return. Furthermore, following this new NQUF measure, also new ranking classes are suggested. In Section 3 we prove that if we assume the Morningstar utility function, then there exist funds with negative return having better rating of funds with positive returns. Section 4 discusses analytically the lack of dependence of the Morningstar utility function from the risk represented by the standard deviation. In Section 5 a comparison between NQUF and Morningstar rating measures is done by using the actual sample of Italian pension Funds. Section 6 contains the conclusions.

2. Quadratic Utility Function

The evaluation of Funds is based on the Quadratic Utility Function (QUF), where Return and Standard Deviation enter as parameters. Consequently, risk aversion is determined by the values of these parameters and the rating is given without considering other external data to the considered market. The use of Returns and Standard Deviations without other information sources assures a more objective measurement of the Ratings.

Some useful constraints on the ARA (Absolute Risk Aversion) measure related to the QUF are done for avoiding the higher rating assignments to the Funds with negative returns with respect the ones positive.

Consider the following general quadratic utility function (QUF):

$$U(W) = a + bW - cW^2 \qquad b, c > 0 \ , \tag{2.1}$$

where W is the wealth (or a quantity of the uncertain payment):

$$W = W_0(1+R) \ ,$$

with the initial value W_0 and the (monthly) return R.

If (2.1) function has positive first derivative and negative second derivative, it represents a risk-averse person with insatiable appetite, that is:

(2.2)
$$U' = b - 2cW > 0$$
$$U'' = -2c < 0 \rightarrow c > 0$$

(2.3)
$$ARA = -\frac{U''}{U'} = \frac{2c}{b-2cW} > 0$$

Now consider a sequence of returns of K Funds $\left\{R_j; j=1,\ldots,K\right\}$, each with mean $\mu_j = E(R_j)$ and standard deviation $\sigma_j = \left\{E\left[R_j - E(R_j)\right]^2\right\}^{1/2}$. We indicate with μ_M the maximum expected return within all the sequence of returns, i.e. $\mu_M = \max\left\{\mu_j, j=1,\ldots,K\right\}$ and we chose $b = 2cW_0(1+\mu_M)$. The rationale of this latter choice is that the maximum of the expected value of the QUF will be reached in the point $(0,\mu_M,\psi(\sigma,\mu_M))$ on the space $(\sigma,\mu,\psi(\sigma,\mu))$, where σ is the standard deviation, μ is the mean and $\psi(\sigma,\mu)$ is the expected value of the QUF.

The condition that a Fund with $\mu_i > 0$ cannot have an expected value of QUF less than a Fund with $\mu_i < 0$ implies that the ARA of the investors (and consequently the coordinates of the maximum of $\psi(\sigma,\mu)$) must be modified through a transformation of values (σ_i,μ_i) of the Funds. This means that ARA is defined only on the values (σ_i,μ_i) of the Funds rather than by external considerations to these values.

Theorem 2.1: *With the definition* $b = 2cW_0(1+\mu_M)$, *the expected value of QUF in* (2.1) *is a function* ψ *of standard deviation* σ *and expected return* μ *represented by a paraboloid in the risk-return space* (σ,μ,ψ) *with downward concavity, whose vertex is given by the point* $(0,\mu_M,\psi(0,\mu_M))$. *That is:*

$$E[U(W)] = \psi(\sigma,\mu) = U(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu-\mu_M)^2] \ ,$$

where $U(W_0) = a + bW_0 - cW_0^2$.

Proof: Appendix A . □

From (2.3) and from the choice $b = 2cW_0(1+\mu_M)$ ARA expression becomes:

$$ARA(W) = \frac{1}{W_0(1+\mu_M) - W}$$

and

$$ARA(W_0) = \frac{1}{W_0\mu_M} \ .$$

If we compute the same expression in $E(W) = W_0(1+\mu)$, then:

(2.4) $$ARA(\mu) = \frac{1}{W_0(\mu_M - \mu)} \quad .$$

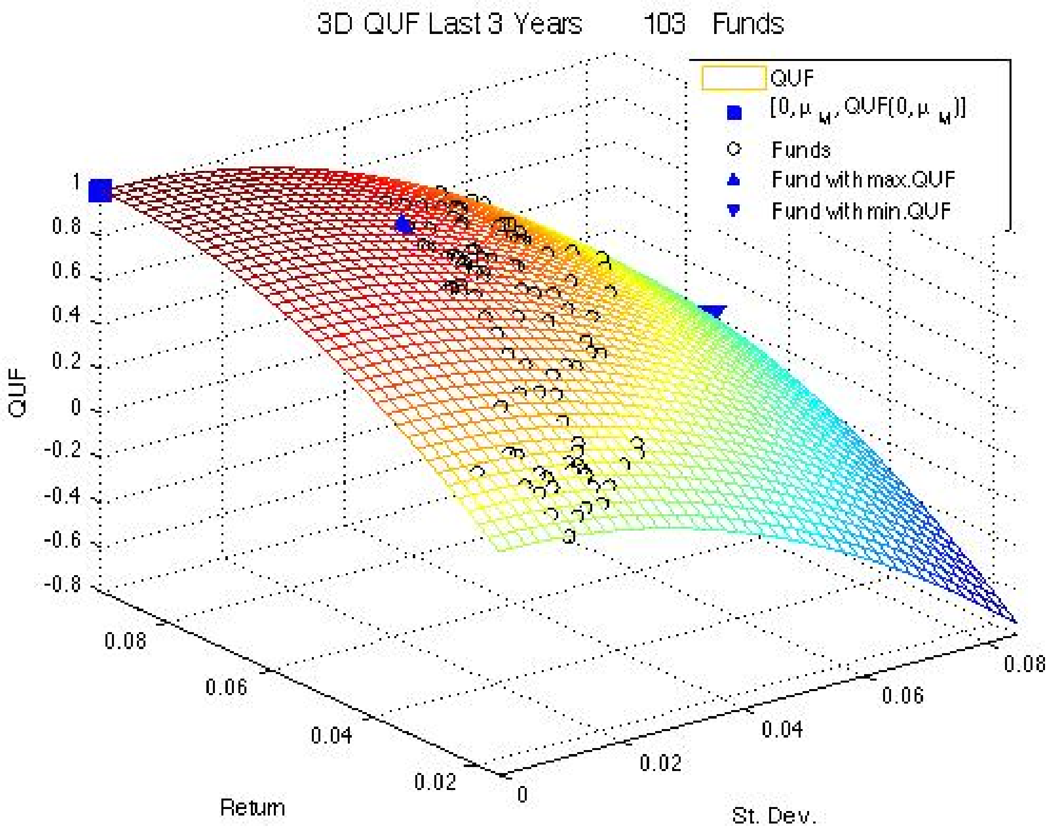
This expression implies that risk aversion (absolute and relative) increases with increment of expected return.

A criticism says that in some cases this can be unrealistic: approaching to a limit level of expected return, it makes no sense to risk. Despite of this, the quadratic function is very useful because it can be seen, according to the Taylor expansion, as the second-order approximation to any utility function, and because a person with Quadratic Utility Function takes its decisions on the basis of both Returns and Standard Deviations parameters.

Figure 2.1 shows the three dimensional paraboloid representing the QUF. The paraboloid is defined on the half-plane (σ, μ) with $\sigma > 0$ and $\mu \le \mu_M$ and, due to the definition of μ_M and the choice of b, contains all the Funds.

QUF is applied to the values of return and standard deviation of a set of Funds and, from geometrical point of view, the *i-th* Fund has a geometrical representation as a point of coordinates $(\sigma_i, \mu_i, \psi(\sigma_i, \mu_i))$.

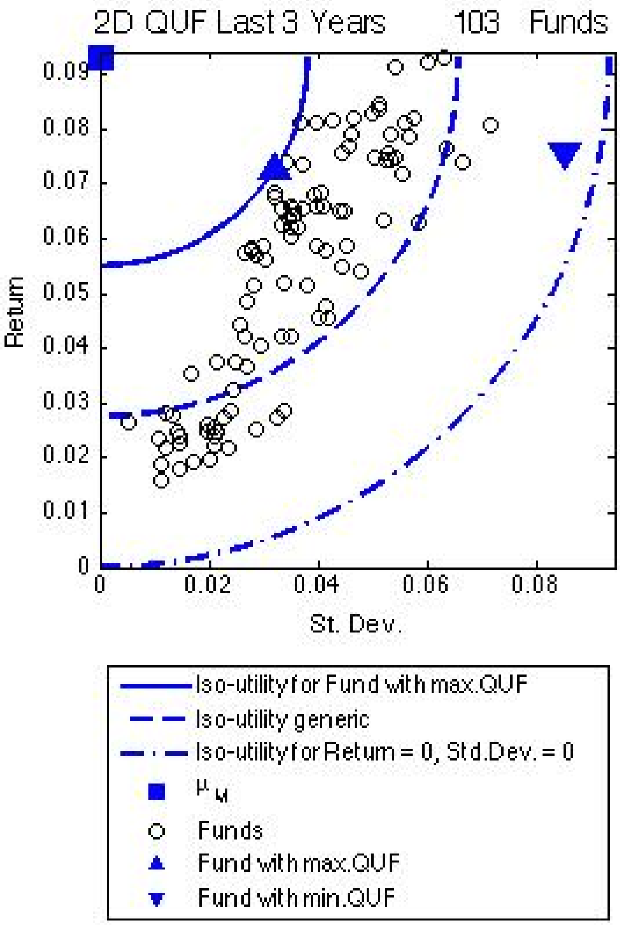
Figure 2.1: 3D Quadratic Utility Function



The iso-utility curves can be obtained from the equality $\psi(\sigma, \mu) = K$, where K is a constant. In this way we have a sheaf of circumferences on the plane (σ, μ) with centre *(0, μ_M)*.

The *i-th* Fund has a geometrical representation as a point of coordinates (σ_i, μ_i).

Figure 2.2: Quadratic Utility Function, Iso-utility Curves

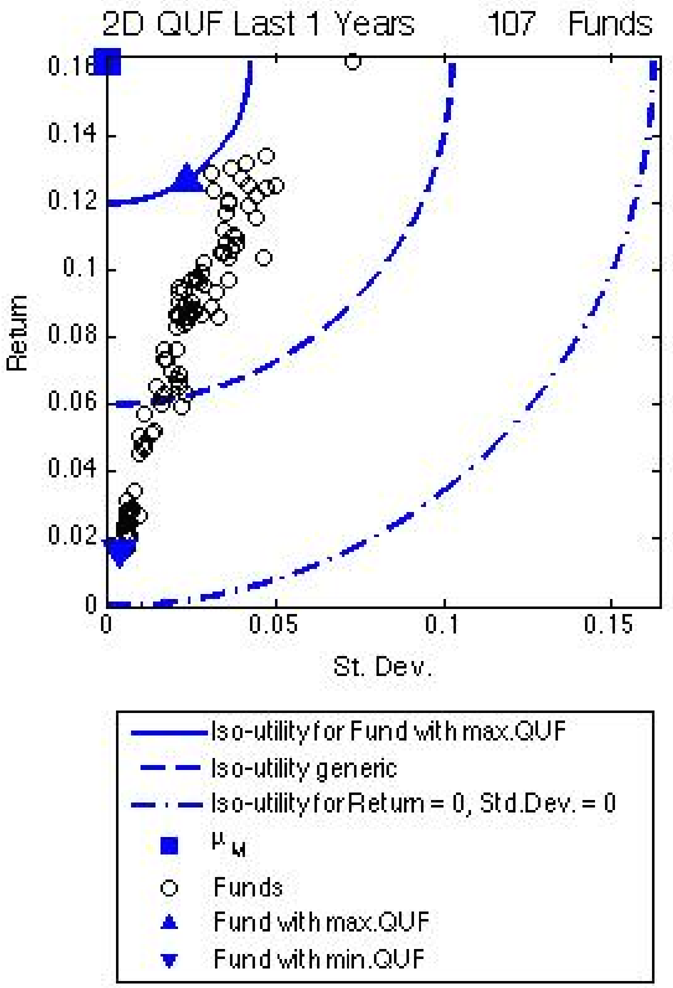


For the evaluation of the iso-utility values we assume that the maximum utility value corresponds to the maximum value of return, i.e. for $\mu = \mu_M$. In this point, where the radius of the sheaf of circumferences is equal to zero, the utility is maximum; assumes decreasing values for the points of coordinates (σ, μ) which have an increasing radius, i.e. it moves away from the vertex. In the following we consider $\mu_M > 0$.

If (σ_e, μ_e) are the co-ordinates of the most distant point from the centre of the sheaf of circumferences, we can distinguish 2 cases.

a) Funds with positive returns only, i.e.: $\sigma_e^2 + (\mu_e - \mu_M)^2 \le \mu_M^2$

Figure 2.3 –QUF - Funds with positive returns only



The Fund with greater distance from the centre is in any case inside a circle with centre $(0, \mu_M)$ and radius μ_M

The distance of a generic point of coordinates (σ, μ) from the centre of the sheaf of circumferences is given by:

(2.5) $$D = \sqrt{\sigma^2 + (\mu - \mu_M)^2}$$

If D_e is the maximum value of these distances, then we can divide all the distances of the points of generic (σ, μ) coordinates from the point of coordinates $(0, \mu_M)$ by D_e obtaining the normalized distance:

(2.6) $$D_N(\sigma, \mu) = \frac{\sqrt{\sigma^2 + (\mu - \mu_M)^2}}{D_e}$$

By imposing the boundary condition[1]:

$$\psi(\sigma_e, \mu_e) = 0$$

we obtain the following value of $U(W_0)$:

(2.6) $$U(W_0) = cW_0^2(D_e^2 - \mu_M^2) < 0$$

We define the Normalized Quadratic Utility Function (NQUF) as follows:

(2.7) $$\psi_N(\sigma,\mu) = \frac{U(W_0) + cW_0^2\left\{\mu_M^2 - [\sigma^2 + (\mu-\mu_M)^2]\right\}}{cW_0^2 D_e^2} \ .$$

Consequently, for any coordinate (σ, μ) the NQUF is:

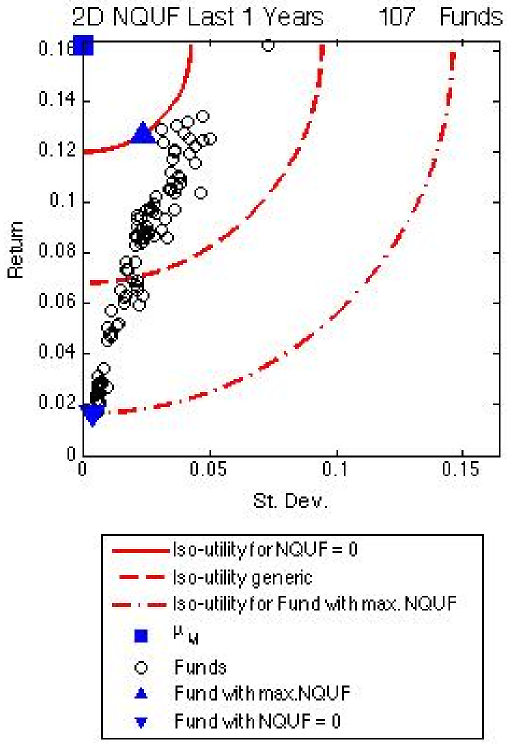
(2.8) $$\begin{aligned}\psi_N(\sigma,\mu) &= \frac{U(W_0) + cW_0^2\left\{\mu_M^2 - [\sigma^2 + (\mu-\mu_M)^2]\right\}}{cW_0^2 D_e^2} \\ &= \frac{cW_0^2(D_e^2 - \mu_M^2) + cW_0^2\left\{\mu_M^2 - [\sigma^2 + (\mu-\mu_M)^2]\right\}}{cW_0^2 D_e^2} \\ &= \frac{cW_0^2 D_e^2 - cW_0^2\left[\sigma^2 + (\mu-\mu_M)^2\right]}{cW_0^2 D_e^2} \\ &= 1 - D_N^2(\sigma,\mu)\end{aligned}$$

The NQUF reaches its maximum value 1 at the point $(0, \mu_M)$ and the minimum at the point (σ_e, μ_e). The representation in three dimension indicates a paraboloid with vertex at the point $(0,\ \mu_M, 1)$.

Considering two dimensions, *Figure 2.3* becomes:

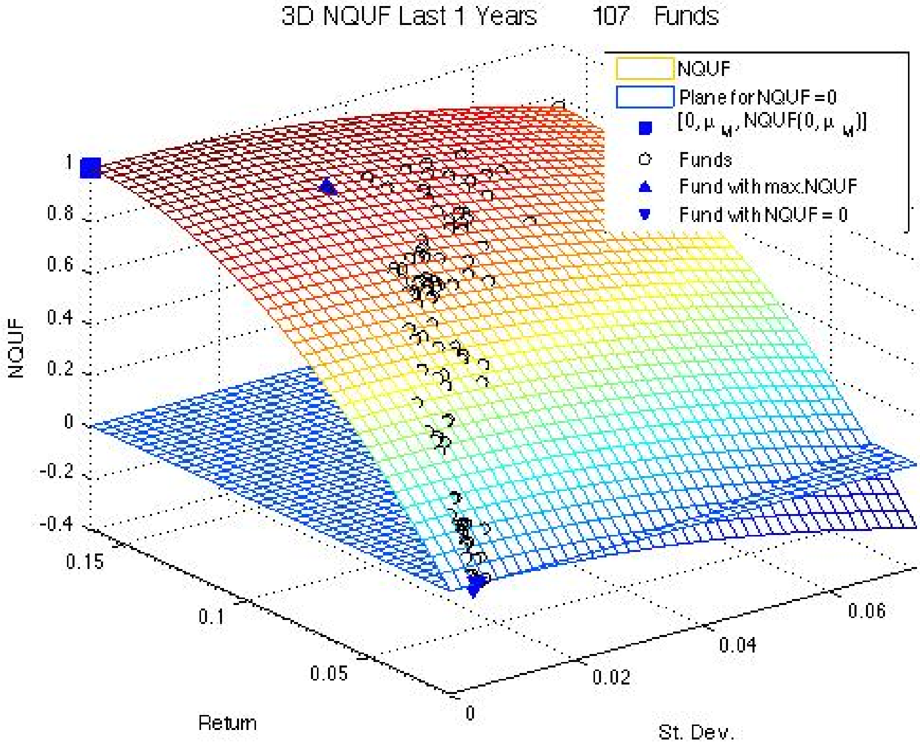
[1] This condition is given simply for the purpose of ordering. Any alternative constant could be used with the effect of translating the paraboloid if the same constant is added to the vertex.

Figure 2.4 –NQUF- Funds with positive returns



and in three dimensions is:

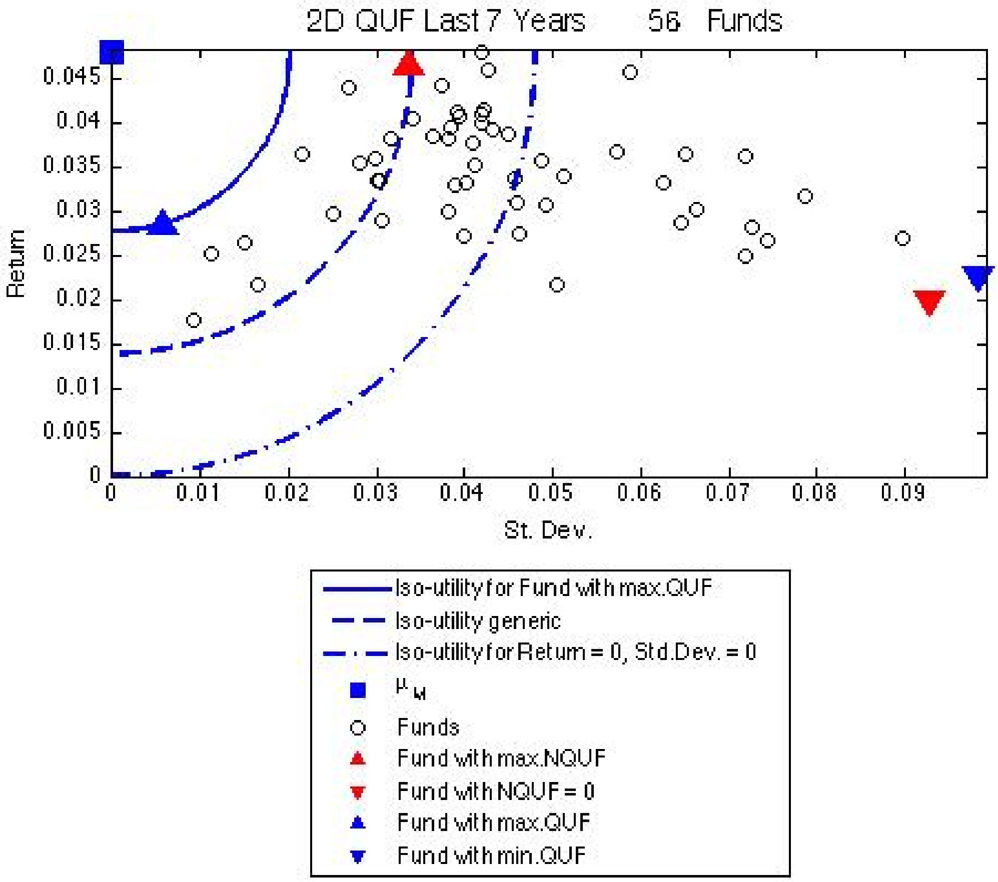
Figure 2.5 –3D NQUF - Funds with positive returns



b) Funds with positive and negative returns, i.e.: $\sigma_e^2 + (\mu_e - \mu_M)^2 > \mu_M^2$

In this case the application of the methodology here proposed gives NQUF with rating higher for Funds with positive returns with respect to any other Fund with negative return.
The appropriate representation needs to determine the centre of coordinates $(0, \mu_{Me})$ such that the related circle crosses the point $(0,0)$ and includes all the Funds with positive returns, that is to say, in the Theorem 2.1 the coordinate μ_{Me} substitutes μ_M. The unknown value μ_{Me} can be determined by imposing the condition that the value of NQUF is equal zero to the Fund with positive return and maximum distance from this new centre.
This means that Funds with negative returns have negative value of NQUF.

Figure 2.6 –QUF - Funds with positive and negative returns



Therefore, the boundary condition is:

$$\psi(0,0) = 0 \ ,$$

which implies:

$$U(W_0) = 0$$

Now, considering all the generic points (σ_i, μ_i) which satisfy the inequality b) with $\mu_i > 0$, we impose the condition that the paraboloid with vertex $(0, \mu_{Mi}, \psi_N(0, \mu_{Mi}))$, where μ_{Mi} is the unknown centre correspondent to each Fund, has value of zero for the point (σ_i, μ_i):

$$\psi(\sigma_i, \mu_i) = 0 \ ,$$

that is, it is coplanar with the point (0, 0).

Consequently, for the *i-th* Fund we have:

$$\psi(\sigma_i, \mu_i) = cW_0^2 \left\{ \mu_{Mi}^2 - [\sigma_i^2 + (\mu_i - \mu_{Mi})^2] \right\} = 0 \ ,$$

from which:

$$\mu_{Mi} = \frac{\sigma_i^2 + \mu_i^2}{2\mu_i} \ .$$

The new coordinate μ_{Me}, vertex of the paraboloid, and also centre of the sheaf of iso-utility circumferences, is given by the choice:

(2.9) $$\mu_{Me} = \underset{i}{Max}(\mu_{Mi}) \ ,$$

and the NQUF becomes:

$$\psi_N = 1 - \frac{[\sigma^2 + (\mu - \mu_{Me})^2]}{\mu_{Me}^2}$$

Taking in account that the *i-th* Fund satisfies the inequality b), we can write:

$$\sigma_i^2 + \mu_i^2 - 2\mu_i\mu_M + \mu_M^2 > \mu_M^2$$

$$\sigma_i^2 + \mu_i^2 - 2\mu_i\mu_M > 0 \ ,$$

and being $\mu_i > 0$, we have:

$$\mu_{Mi} = \frac{\sigma_i^2 + \mu_i^2}{2\mu_i} > \mu_M \ ,$$

consequently:

$$\mu_{Me} > \mu_M$$

The centre μ_{Me} of the sheaf of circumferences shifts towards superior values of μ_M along the vertical axis. This means that to obtain a greater rating for positive return Funds with respect to the negative, the ARA parameter given in (2.4), i.e.:

$$ARA(W) = \frac{1}{W_0(\mu_M - \mu)}$$

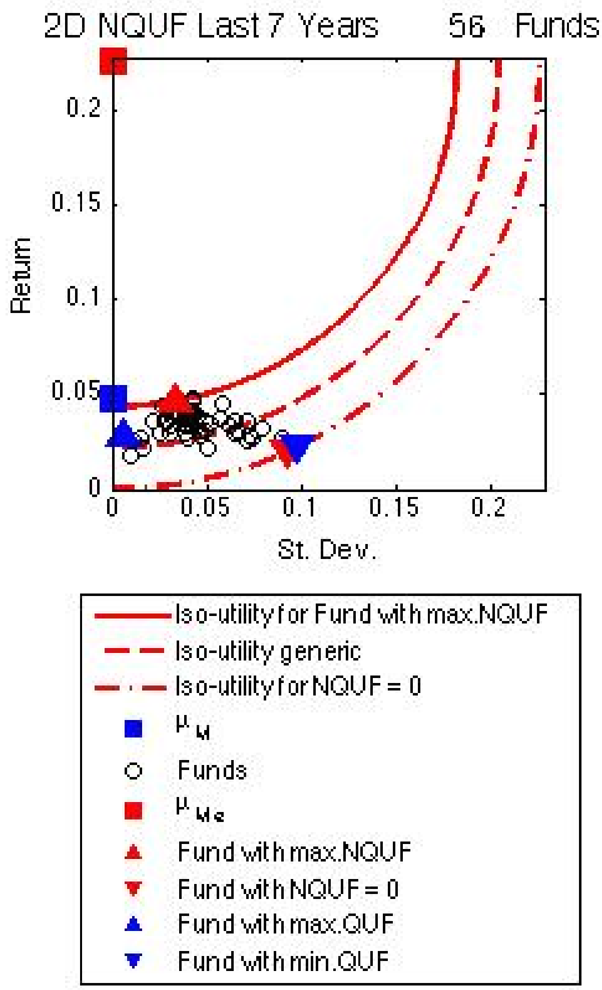
diminishes assuming the value:

(2.10) $$ARA_N(W) = \frac{1}{W_0(\mu_{Me} - \mu)}$$

In this case the NQUF assumes value 1 in $(0, \mu_{Me})$ and 0 in (σ_e, μ_e); furthermore, it has value 0 also in (0, 0).
Graphically, the representation of *Figure 2.6* changes to the representation given by the *Figure 2.7.*

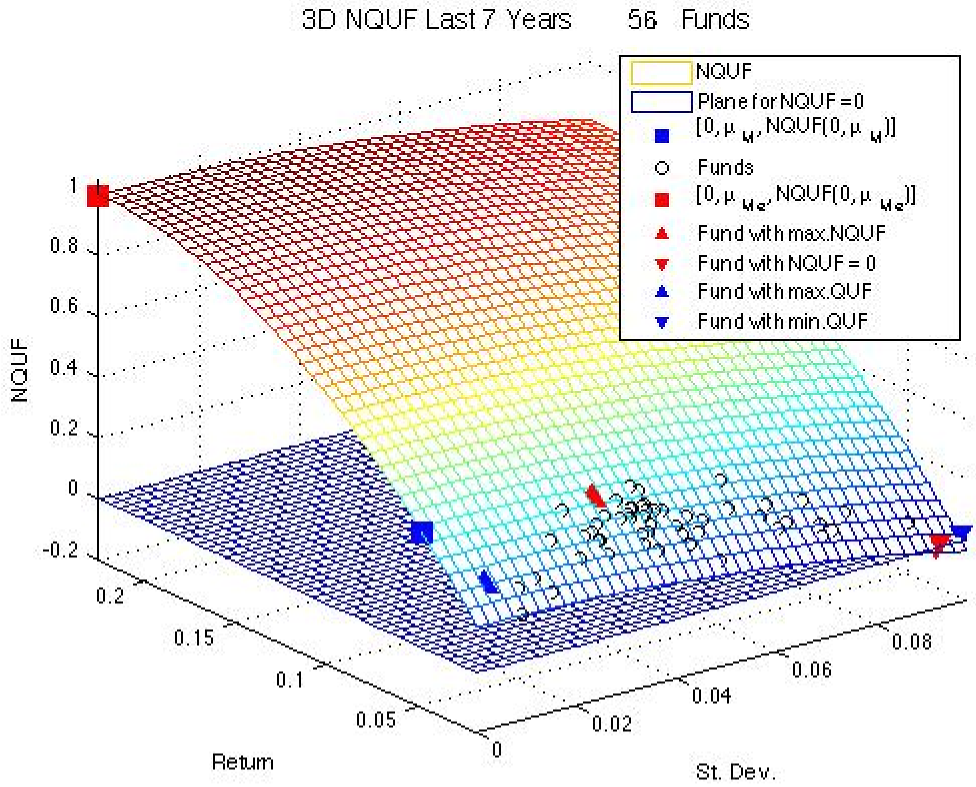
In conclusion, in case a), which represents Funds with only positive returns, the centre of the sheaf of the circumferences is given by μ_M and the ARA by the relation (2.4); in the case b), which considers Funds with positive and negative returns, the centre of the sheaf of circumferences is given by (2.9) and the ARA by (2.10).

Figure 2.7 –NQUF – Iso-utility curves



and in three dimensions:

Figure 2.8 –NQUF 3D



The comparison among the different Funds is done by computing the NQUF after the choice of the appropriate window, for example 10, 7, 3 years or 1 year. The values of NQUF are rescaled between (0, 10) as follows:

(2.11) $$Rescaled\ \ NQUF = 10\frac{NQUF - Min(NQUF)}{Max(NQUF) - Min(NQUF)}$$

The ranking can be done assigning a particular class, only for the Funds that convey their returns since at least 10 years. This period is consistent with financial assets which need to show their quality in the long-run.

Here we propose in *Table 2.1* a possible ranking using 7 classes[2] formed on the basis of the values obtained in (2.11), in decreasing order.

Table 2.1 – Ranking classes (7-th is the best)

Class 7	(0%, 5.0%]
Class 6	(5.0%, 17.5%]
Class 5	(17.5, 37.5%]
Class 4	(37.5%, 62.5%]
Class 3	(62.5%, 82.5%]
Class 2	(82.5%, 95.0%]
Class 1	(95.0%, 100.0%]

[2] It is understandable that 7 classes give more detailed information in comparison to 5 classes, therefore they are more sensitive to the change of the parameters.

To calculate the outlook, i.e. the tendency of the ranking values over time, the period of 10 years is divided into 5 windows of 2 years length.
Within each window NQUF is calculated and the obtained values are rescaled between 0 and 10 using the (2.11).
Therefore, the outlook is determined by using a regression on the re-scaled numbers. Minimum deviations from the value zero are set to zero to avoid frequent changes in sign with the inclusion of new data, that is we avoid the formation of a band of hysteresis.
A graphical representation of the outlook can be done by using upward arrows if the sign of the regression coefficient is positive, and downward arrows if negative.

3. The Morningstar Utility Function

Morningstar uses a utility function of exponential type with CRRA (Constant Relative Risk Aversion) with the risk aversion parameter γ.

Given a period of months T, the Morningstar Risk-Adjusted Return is defined as follows:

(3.1)
$$MRAR(\gamma)=\begin{cases}\left[\dfrac{1}{T}\sum_{t=1}^{T}(1+ER_t)^{-\gamma}\right]^{-\frac{12}{\gamma}}-1, & \gamma\neq 0\\ \left[\prod_{t=1}^{T}(1+ER_t)\right]^{\frac{12}{T}}-1, & \gamma=0\end{cases},$$

with:

(3.2)
$$ER_t=\frac{1+LR_t}{1+Rf_t}-1 \quad ,$$

where LR_t is the monthly return including the commissions and Rf_t is the risk free rate.
$MRAR(\gamma)$ are annualized values.
For $\gamma>0$, the investor is risk adverse and calls a premium against his choice of a risky asset.
The value $\gamma=2$ is chosen by Morningstar, considered consistent with the risk aversion of the typical retail customers.
As a measure of performance Morningstar uses the annualized geometric return

(3.3)
$$MRAR(0)=\left[\prod_{t=1}^{T}(1+ER_t)\right]^{\frac{12}{T}}-1$$

That, considering the definition $a_t=(1+ER_t)^2$ becomes:

$$MRAR(0) = \left[\prod_{t=1}^{T} a_t \right]^{\frac{6}{T}} - 1.$$

This represents a geometrical mean G_a of the series a_t raised to the sixth to which 1 is subtracted. Therefore:

(3.4) $$MRAR(0) = G_a^6 - 1$$

As for the risk-adjusted performance Morningstar uses the Morningstar Risk-Adjusted Returns (2), or MRAR (2):

$$\begin{aligned} MRAR(2) &= \left[\frac{1}{T} \sum_{t=1}^{T} (1 + ER_t)^{-2} \right]^{-\frac{12}{2}} - 1 \\ &= \left[\frac{1}{T} \sum_{t=1}^{T} \frac{1}{(1+ER_t)^2} \right]^{-6} - 1 \\ &= \frac{1}{\left[\frac{1}{T} \sum_{t=1}^{T} \frac{1}{(1+ER_t)^2} \right]^{6}} - 1 \end{aligned}$$

Which is the harmonic mean H_a of the series a_t raised to the sixth to which 1 is subtracted:

(3.5) $$MRAR(2) = H_a^6 - 1$$

Consequently, the well known Morningstar Risk, *MRisk*, becomes the difference between the geometric and harmonic mean of the series a_t as follows:

$$\begin{aligned} MR &= MRAR(0) - MRAR(2) \\ &= G_a^6 - H_a^6 \end{aligned}$$

Consider two Funds, the Fund A has MRAR (0)> 0 and the Fund B has MRAR (0) <0. We want to see if there are conditions for which the Fund B is evaluated in a better way than the Fund A, in other words if a Fund with a negative return can have a better rating than a Fund with positive return. The monthly returns are defined with ER_{At} and ER_{Bt} for the Fund A and B respectively. We define also:

$$\begin{aligned} a_t &= (1 + ER_{At})^2 \\ b_t &= (1 + ER_{Bt})^2 \end{aligned}$$

We consider two series a_t and b_t whose characteristics satisfy MRAR (0)> 0 for Fund A and MRAR (0) <0 for Fund B and we build the series b_t such that:

$$\begin{cases} b_1 = \varepsilon_1 a_1 \\ b_2 = \varepsilon_2 a_2 \\ b_t = a_t, \quad t > 2 \end{cases},$$

that is the series b_t differs from the series a_t for two elements and ε_1, ε_2 are two positive parameters subjected to some constraints.

We can prove the following

Theorem 3.1*: Given two Funds, Fund A with MRAR (0)> 0 and Fund B with MRAR (0) <0, which differ for two positive elements* ε_1 and ε_2*, under the conditions* $\varepsilon_1 \varepsilon_2 > \frac{4a_1 a_2}{(a_1 + a_2)^2}$ *Fund B can have a Morningstar rating better than Fund A.*

Proof*:* Appendix B. □

The series b_t can differ from the series a_t for more than two elements. In this case other constraints need to be found, but in any case remains shown the existence of Funds with negative return for which the Morningstar rating can be greater than other Funds with positive return.

4. Morningstar's Utility Function and Standard Deviation

The starting utility function utilized by Morningstar is:

(4.1) $$U = z + w(1 + ER)^{-\gamma}$$

where:
U represents the model investor's utility or satisfaction from each monthly return, *z* is any number, *w* is any negative number, ER indicates the monthly geometric excess return for the Fund (adjusted for commissions and the risk-free rate) and γ is a parameter that expresses an investor's sensitivity to risk.
Without loss of generality, we can substitute in (4.1) the excess return ER simply with the return R .

As stated in the previous chapter, Morningstar sets γ = 2 to illustrate a model investor's sensitivity to risk[3]. Furthermore, Morningstar states z = 2 and w = -1, so that the Morningstar Utility Function (MRUF) becomes:

(4.2) $$MRUF(2) = 2 - \frac{1}{W^2} \quad ,$$

where

[3] The value 2 of the parameter γ comes from Morningstar's Fund analysts who have concluded that γ=2 results in Fund rankings that are consistent with the risk tolerances of typical retail investors (Cf. Morningstar (2009), p. 12).

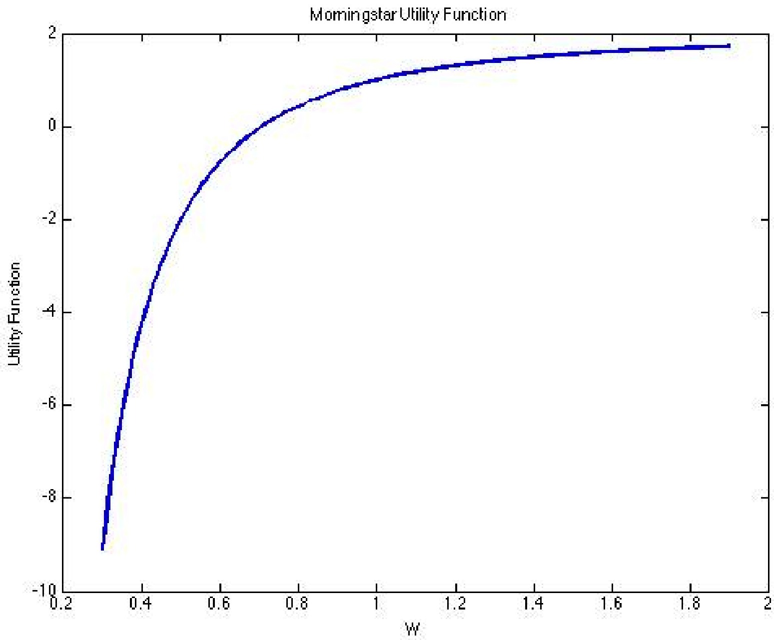
$$W = W_0(1+R)$$

with the initial amount W_0 and the monthly return R.

The ARA has the expression:

$$ARA = \frac{3}{W}$$

Figure 4.1: Morningstar's Utility Function



The expression (4.2) becomes:

$$MRUF(2) = 2 - \frac{1}{W_0^2(1+R)^2}$$

Now, we consider the expression:

(4.3) $$\frac{1}{(1+R)^2}$$

It is easily shown that (4.3) can be written as:

(4.4) $$\frac{1}{(1+R)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)R^n$$

In fact, under the condition $|R|<1$, we know that:

$$\frac{1}{1+R} = \sum_{n=0}^{\infty}(-1)^n R^n \quad ,$$

therefore:

$$\frac{1}{(1+R)^2} = -\frac{d}{dR}\left[\frac{1}{1+R}\right] = -\frac{d}{dR}\left[\sum_{n=0}^{\infty}(-1)^n R^n\right] = \sum_{n=0}^{\infty}(-1)^n (n+1)R^n$$

Computing the expected value of (4.4) we have:

(4.5) $$E\left[\frac{1}{(1+R)^2}\right] = E\left[\sum_{n=0}^{\infty}(-1)^n (n+1)R^n\right] = \sum_{n=0}^{\infty}(-1)^n (n+1)\mu_n \quad ,$$

where μ_n is the nth non central moment of the random variable R.

Consequently, the expected value of *MRUF(2)* is:

(4.6) $$\begin{aligned} E\left[MRUF(2)\right] &= 2 - E\left[\frac{1}{W_0^2(1+R)^2}\right] = 2 - \frac{1}{W_0^2}\sum_{n=0}^{\infty}(-1)^n (n+1)\mu_n \\ &= 2 - \frac{1}{W_0^2} + \frac{2}{W_0^2}\mu - \frac{3}{W_0^2}(\sigma^2+\mu^2) + \frac{4}{W_0^2}\mu_3 - \cdots \end{aligned}$$

In Appendix C we show that the non central moment of a normal distribution can increase with the order increment.

For the convergence of (4.4) we need to constrain the returns.

Theorem 4.1: *If the returns are normally distributed in the range (-1, 1), then the non central moments are decreasing.*

Proof: Appendix D. □

Consider the truncated Normal $u \sim N(\mu,\sigma^2)$ constrained to assume values only in the interval $K=(k_1,k_2)$, with $-1<k_1<0\le k_2<1$ and $k_1<\mu<k_2$

From computational point of view, it is well known[4] that the non central moment μ_n of a truncated normal distribution in the interval $K=(k_1,k_2)$, with $-1<k_1<0\le k_2<1$ and $k_1<\mu<k_2$, has a recursive representation as follows:

$$\mu_n = \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k I_k \quad ,$$

with:

$$I_k = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2}\tau^k\phi(\tau)d\tau \quad , h_1=(k_1-\mu)/\sigma, \;\; h_2=(k_2-\mu)/\sigma$$

[4] Cf for example Dhrymes (2005) or Burkardt (2014), pg. 25.

where $\Phi(h)$ represents the cumulative distribution function of the standardized random variable h, $\phi(h)$ the related density function, and $\Delta\Phi_H$ the probability that $u \in K$:

$$\Delta\Phi_H = \Phi(h_2) - \Phi(h_1)$$

Therefore:

$$\begin{aligned}
I_0 &= 1 \\
I_1 &= \frac{\phi(h_1) - \phi(h_2)}{\Delta\Phi_H} \\
&\dots \\
I_k &= \frac{h_1^{(k-1)}\phi(h_1) - h_2^{(k-1)}\phi(h_2)}{\Delta\Phi_H} + (k-1)I_{k-2}, \qquad \mathit{for}\ k \geq 2
\end{aligned}.$$

Alternatively, in Appendix E we give the closed form expression for the non central moments of the Truncated Normal Distribution.

The calculation of μ_n is possible also with the use of the incomplete gamma function (see the Appendix F). For $n \geq 0$ we have:

$$\mu_n = \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2} \left[(-1)^k \gamma\left(\upsilon, \frac{h_1^2}{2}\right) + \gamma\left(\upsilon, \frac{h_2^2}{2}\right)\right] \tag{4.7}$$

where $\upsilon = (k+1)/2$ and

$$\gamma(\upsilon, x) = \int_0^x t^{\upsilon-1} e^{-t} dt ..$$

is the lower incomplete gamma function.

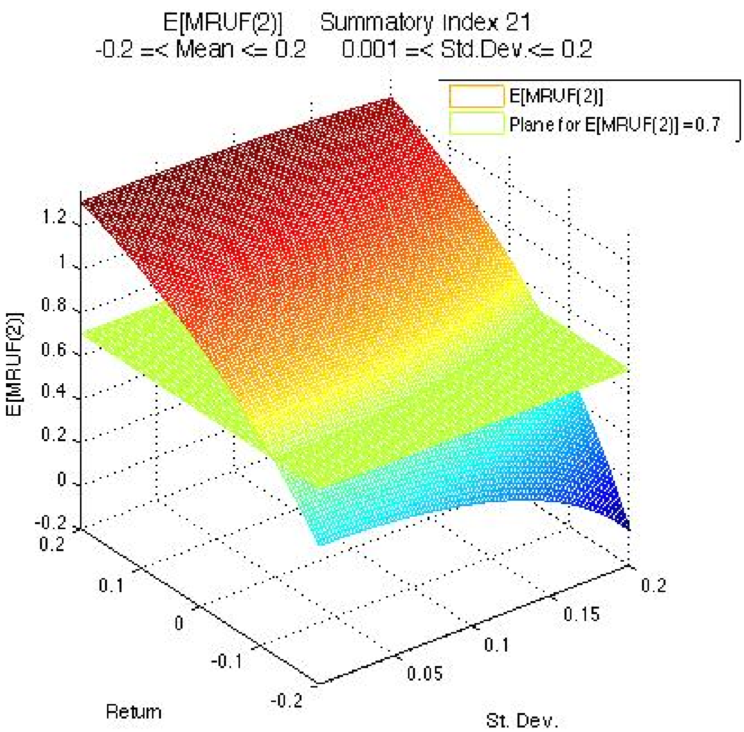
Now we consider again the relation (4.6) in which, without any loss of generality, we state $W_0^2 = 1$:

(4.8)

$$\begin{aligned}
E[MRUF(2)] &= 2 - E[1/(1+R)^2] \\
&= 2 - \sum_{n=0}^{\infty} (-1)^n (n+1)\mu_n \\
&= 2 - \sum_{n=0}^{\infty} (-1)^n (n+1) \left\{ \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2} \left[(-1)^k \gamma\left(\upsilon, \frac{h_1^2}{2}\right) + \gamma\left(\upsilon, \frac{h_2^2}{2}\right)\right] \right\}
\end{aligned}.$$

It is possible to give the three-dimensional representation of iso-utility curves considering their intersection with a general plane.

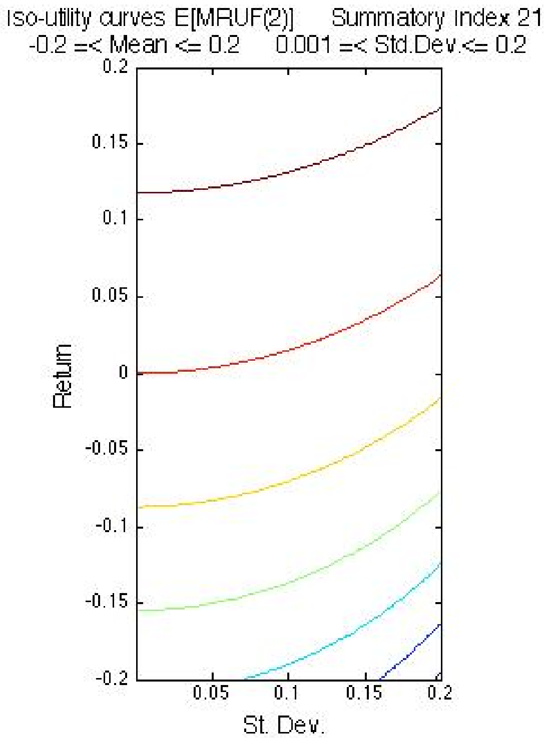
Figure 4.2: 3D E[MRUF(2)]



It is noteworthy the difference with *Figure 2.1* where the intersection with a plane parallel to the plane (σ, μ) has a very different behavior. In this latter case the curves appear only very slightly linked to the standard deviation because the concavity of *E[MRUF(2)]* is not very accentuated.

This lack of dependence of the Morning Star Utility Function from the Standard Deviation is better shown looking at the two-dimensional representation of the iso-utility curves

Figure 4.3: 2D Iso-utility curves of E[MRUF(2)]



Looking at these results, a question can arise about the persistence of the concavity in the utility function. If the range of mean and standard deviation is such that the mean assumes values near -0.9, then we have a counterintuitive behaviour of the contour lines. Near the value -1 for the mean, which represents a discontinuity point for the utility function, the change of the slope of the function $\mu(\sigma)$ are relevant. We graph this case in *Figure 4.4*, but again the anomalous behavior is more evident in *Figure 4.5*.

This interesting question is not faced here because is outside the scope of the present paper and is discussed in Corradin-Sartore (2014).

Figure 4.4: 3D E[MRUF(2)]

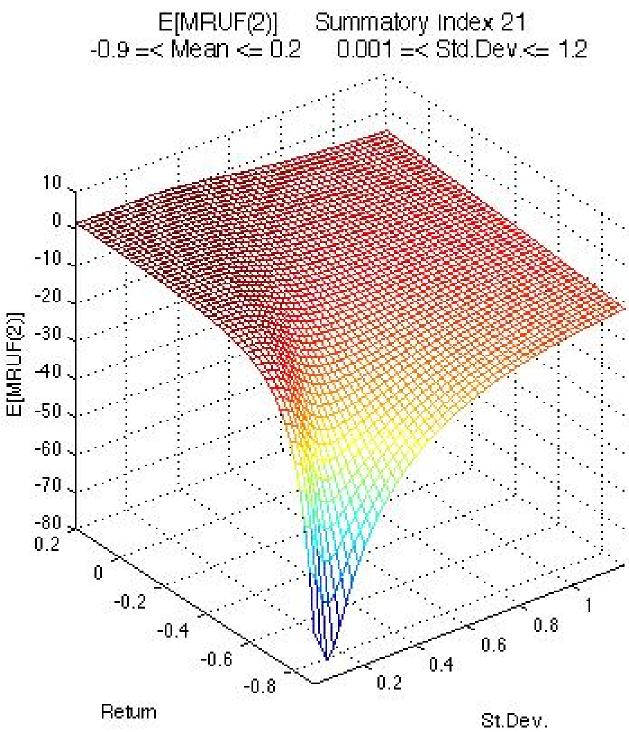
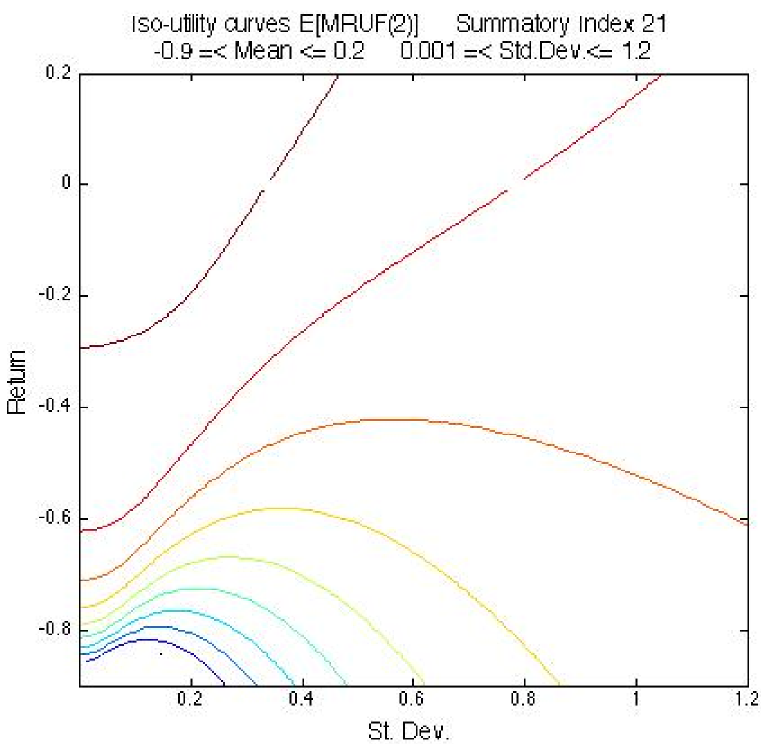


Figure 4.5: 2D Iso-utility curves of E[MRUF(2)]

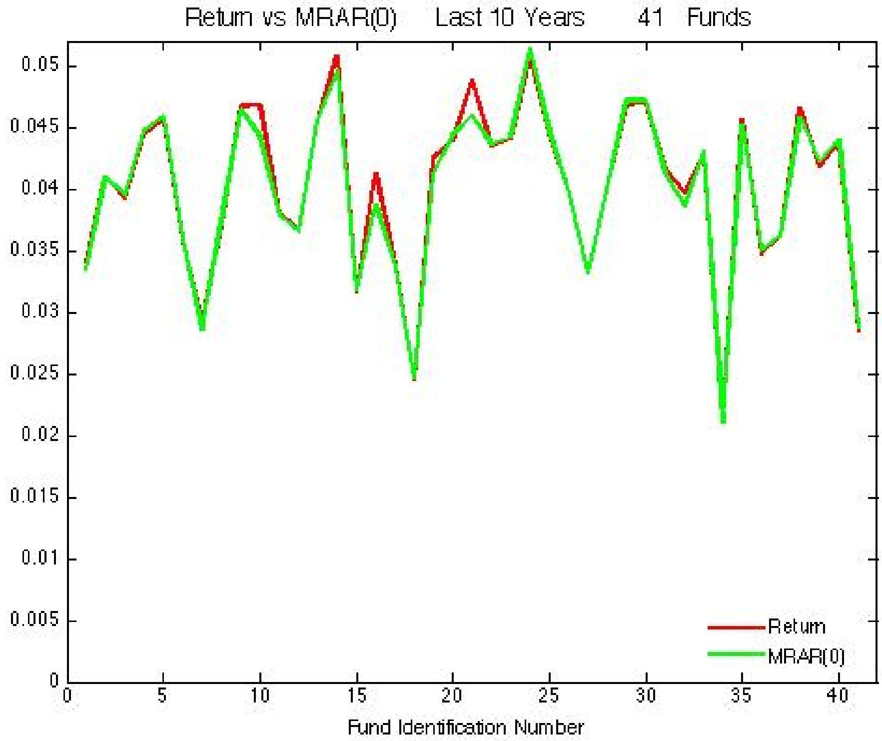


5. Comparison between NQUF e MRAR(2) using an actual sample

The discrepancies between NQUF and MRAR(2) could be shown also in the applied measurements. We had already mentioned Lisi and Caporin (2012) paper in which evidence is given on the fact that the Morning star rating system is mainly influenced by profitability, and only marginally by risk. Here, we give a new application with the computation of the two mentioned approaches, NQUF and MRAR(2), performed on the actual sample given by the monthly return values of the Italian Pension Funds. The sample collects all the Funds with not less than 10 years of activity from July 2004 until June 2014[5].

The first step is to compare the parameter values used. With the annualized geometric mean of returns, formula (3.3), we compute MRAR(0), that is the returns defined by Morningstar, and compare them with the average annualized returns.

Figure 5.1: Average annualized returns vs MRAR(0)



The graphical comparison shows that the two curves are approximately equal, confirmed by the calculation of the correlation coefficient 0.99290.

The Morningstar Risk (MRisk) is computed with the relation:

$$MRisk = MRAR(0) - MRAR(2)$$

In this case the comparison is done with the Standard Deviation.

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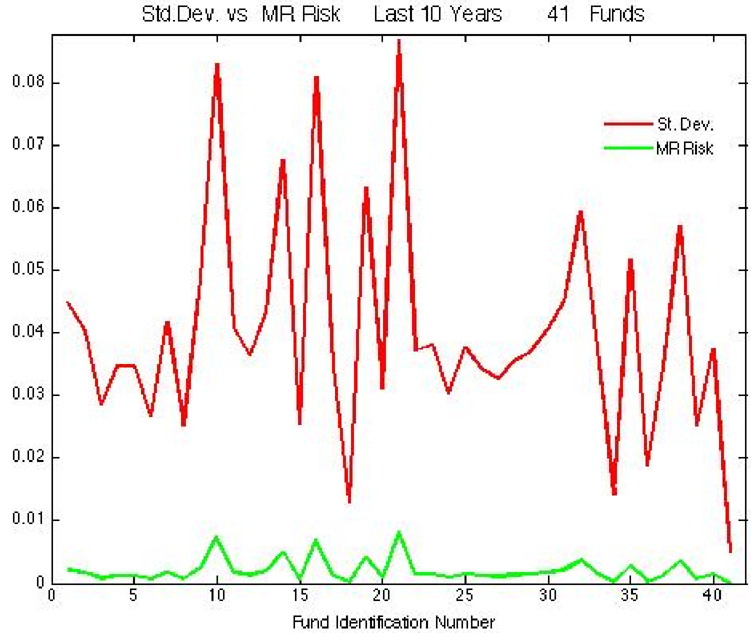
[5] Morningstar does not compute the MRAR(2) for the Italian Pension Funds, therefore the computations are done by using the formulas given in Morningstar (2009).

Figure 5.2: Standard Deviation vs Morningstar Risk



We can see that there is a large discrepancy in the measurement scale even though the behaviour looks very similar, in fact the correlation coefficient is 0.96322.

Hence, the MRAR(0) values are equivalent to the average returns but MR Risk is largely lower with respect to the standard deviation measure.

We can compare graphically the NQUF, NQUF Rating, MRAR(2), MRAR(2) Rating versus some other computed parameters. For this purpose, their values are been rescaled between 0 and 1 and are renamed with additional suffix "_N" in the graphs.

Figure 5.3: Comparison NQUF and MRAR(2) vs Average Returns

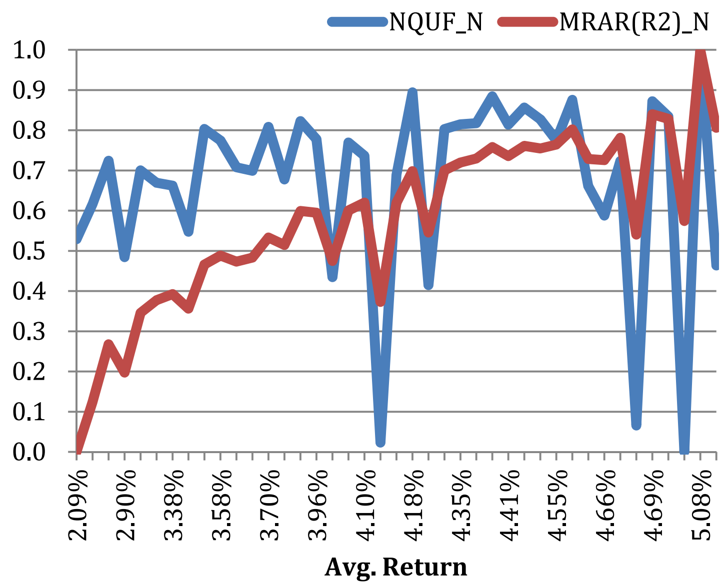
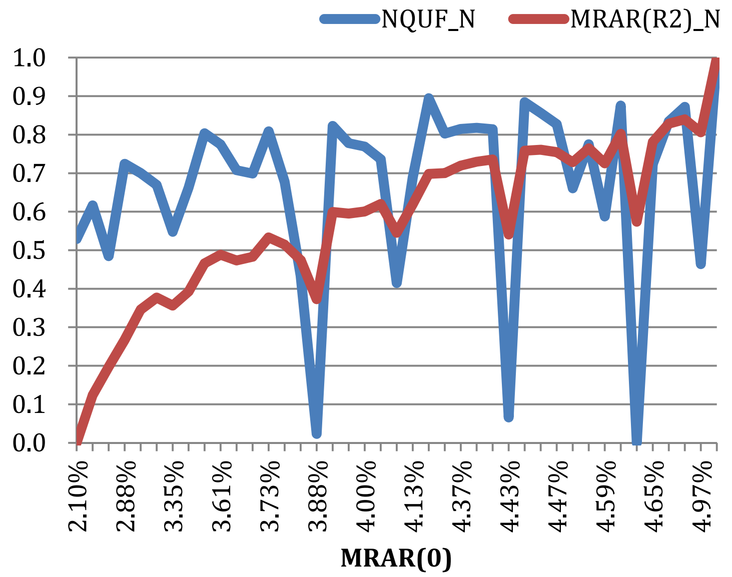


Figure 5.4: Comparison NQUF and MRAR(2) vs MRAR(0)



Figures 5.3 and 5.4 show big downturns for NQUF. MRAR(2) has very weak downturns at the same points. This different behaviour is attributable to the different sensitivity of the two measures regarding the variance.

This different behaviour translates directly into the rating scale. In Figure 5.5 and Figure 5.6 we compare the rating of the Italian Pension Funds for the NQUF Rating, which follows the rating classes of the Table 2.1 and the MRAR(2) Rating, obtained with the application of the method suggested by Morningstar.

Figure 5.5: Comparison NQUF Rating and MRAR(2) Rating vs Average Returns

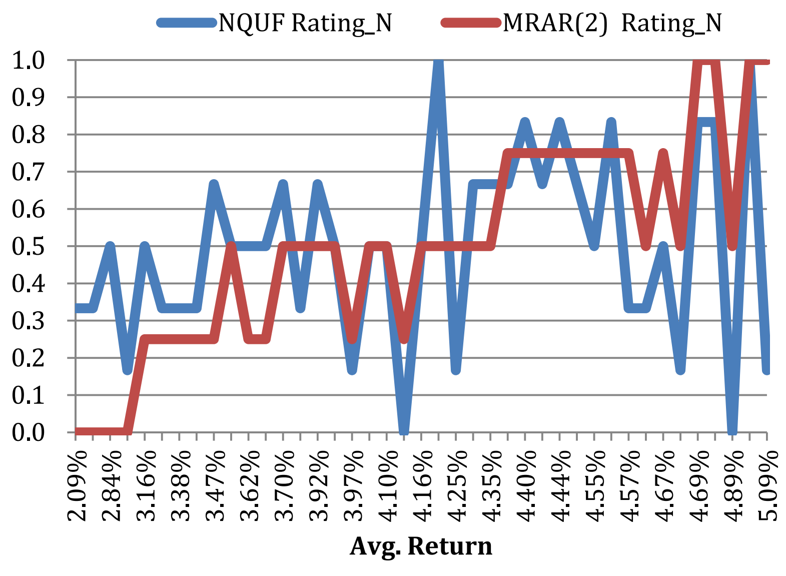
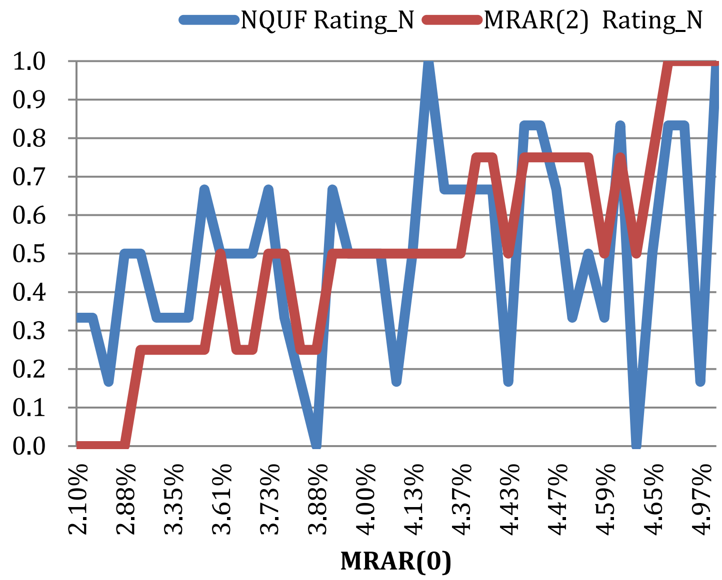


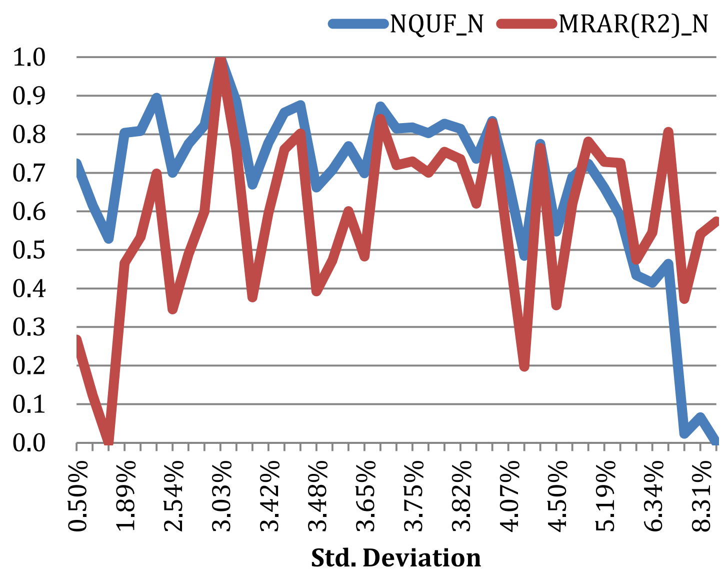
Figure 5.6: Comparison NQUF Rating and MRAR(2) Rating vs MRAR(0)



It is graphically evident that MRAR(2) Rating has a smoother behaviour and is increasing trend with respect the increasing Average Return or MRAR(0).

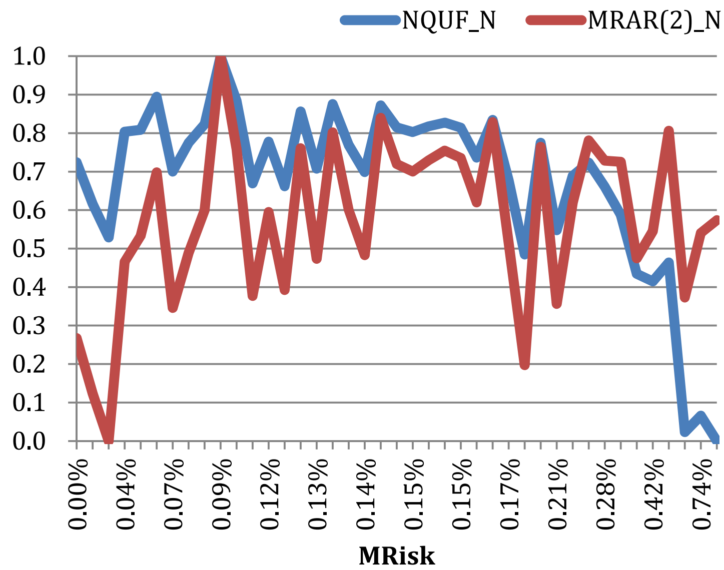
Similarly, we can see the following graphs:

Figure 5.7: Comparison NQUF and MRAR(2) vs Standard Deviation



and

Figure 5.8: Comparison NQUF and MRAR(2) vs MRisk



Again, both the Figures 5.7 and 5.8 show that for high values of the Standard Deviation or MRisk, MRAR(2) is clearly less sensitive comparing with NQUF. Furthermore, looking at the Figure 5.9 and Figure 5.10 there is graphical evidence that the MRAR(2) Rating does not decrease for high levels of Standard Deviations or MRisk.

Figure 5.9: Comparison NQUF Rating and MRAR(2) Rating vs Standard Deviation

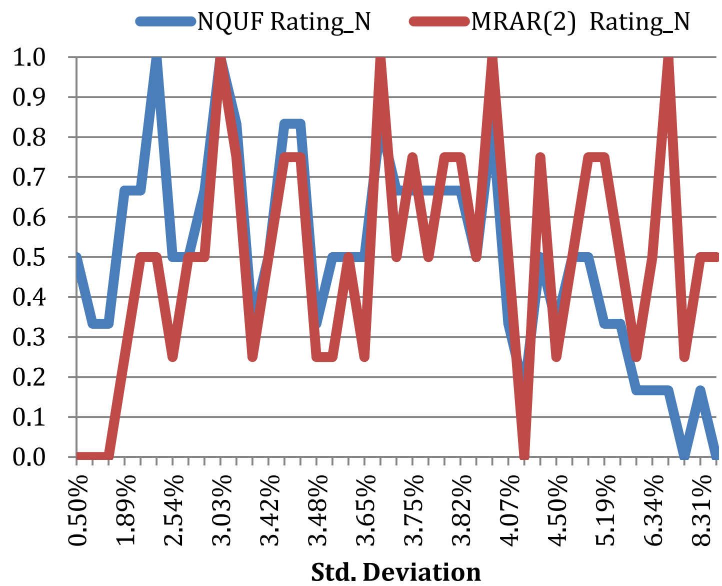
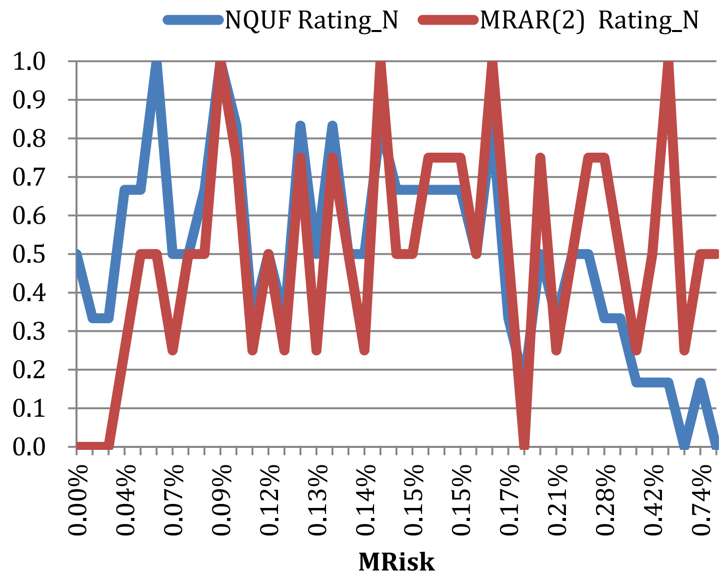


Figure 5.10: Comparison NQUF Rating and MRAR(2) rating vs MRisk



Beyond the graphical evidence, we can look at the correlation coefficients between the different measures.

Table 5.1: Correlations

Avg. Returns	MRAR(0)	Std. Deviation	Mrisk	NQUF	MRAR(2)	NQUF Rating	MRAR(2) Rating
1.00000	0.99290	0.58313	0.47550	0.03478	0.92160	0.23381	0.87236
	1.00000	0.48579	0.36757	0.15312	0.96121	0.33624	0.90654
		1.00000	0.96322	-0.75742	0.23422	-0.59791	0.24449
			1.00000	-0.86212	0.09679	-0.65718	0.12128
				1.00000	0.41957	0.88118	0.35776
					1.00000	0.55477	0.93423
						1.00000	0.48656
							1.00000

In Table 5.1 we can notice the strong dependence of MRAR(2) and MRAR(2) Rating to the Average Returns, along with their scarce sensitivity to the risk. On the contrary, NQUF and NQUF Rating exhibit negative correlation coefficients, as expected, instead of positive values of MRAR(2) and MRAR(2) Rating.

6. Conclusions

In this paper we compare the performance of a quadratic utility function and discusses how to change its characteristic parameter, ARA, so that rating is consistent with return and risk measurements. In particular, this parameter is modified in such a way that a positive return Fund has always a rating higher than one with a negative return.

The parameters that contribute to this assessment are only the return and risk; the considerations on how to change the NQUF are all within the market of reference, not influenced by measurements taken outside.

Here, the risk aversion is considered as a parameter characteristic only of the considered market and therefore does not require analysis or surveys on a variety of agents.

The preference is given to the analysis of the actual dependence of the rating by the return and risk, considering these parameters as necessary for any score; ARA is modified accordingly in order to obtain results that are not counterintuitive.

A counter example is the utility function used by Morningstar, which makes it possible reward a Fund with a negative yield with respect to a positive return. This function also shows to be dependent in a weak form by the standard deviation in a range which can include a good percentage of the actual cases observed.

A new application with the computation of NQUF and Morningstar rating measures are performed on the actual sample given by the monthly return values of the Italian Pension Funds. Through the graphical analysis and the correlation analysis, the rating results give empirical evidence to the scarce sensitivity of Morningstar to the risk comparing with the NQUF rating measure.

References

Adkisson J.A., Fraser R.D., (2003), Reading the Stars: Age Bias in Morningstar Ratings, *Financial Analysts Journal*, 59, 5, 24-27.

Amenc, N. and Le Sourd, V., (2007), Rating the ratings. A critical analysis of fund rating systems. *EDHEC Working paper*.

Burkardt, J. (2014), The Truncated Normal Distribution, http://people.sc.fsu.edu/~jburkardt/presentations/truncated_normal.pdf

Casarin, R., Lazzarin, M., Pelizzon, L. and Sartore, D., (2005), Relative Benchmark Rating and Persistence Analysis: Evidence from Italian Equity Funds, *The European Journal of Finance*, 11, 4, 297-308.

Corradin, F. and Sartore, D. (2014), Risk Aversion: Differential Conditions for the Concavity in Transformed Normal Distributions, *mimeo*, Department of Economics, Ca' Foscari University of Venice.

Del Guercio, D. and Tkac, P.A., (2008), Star power: The effect of Morningstar ratings on mutual fund flow, *Journal of Financial and Quantitative Analysis*, 43, 907–936.

Dhrymes, P. J. (2005), Moments of Truncated (Normal) Distributions, http://www.columbia.edu/~pjd1/mypapers/mycurrentpapers/dummytruncated.pdf

Lisi, F. and M. Caporin (2012): On the role of risk in the Morningstar rating for mutual Funds, *Quantitative Finance*, 12:10, 1477-1486.

Morningstar (2009), The Morningstar Rating™ Methodology, June 30, 2009.

Vinod, H.D. and Morey, M.R., (2002), Estimation risk in Morningstar fund ratings, *Journal of Investing*, 2002, 11, 67–75.

Appendix A. Proof of Theorem 2.1

Theorem 2.1: *With the definition* $b = 2cW_0(1+\mu_M)$*, the expected value of QUF in* (2.1) *is a function* ψ *of standard deviation* σ *and return* μ *represented by a paraboloid in the the risk-return space* (σ, μ, ψ) *with downward concavity, whose vertex is given by the point* $(0, \mu_M, \psi(0, \mu_M))$*. That is:*

$$E[U(W)] = \psi(\sigma,\mu) = U(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu-\mu_M)^2] \ ,$$

where $U(W_0) = a + bW_0 - cW_0^2$.

Proof: Consider the expected value of the utility function (2.1):

$$\begin{aligned} E[U(W)] &= E[a + bW - cW^2] = E[a + bW_0(1+R) - cW_0^2(1+R)^2] \\ &= a + bW_0(1+E[R]) - cW_0^2(1+2E[R]+E[R^2]) \\ &= a + bW_0 + bW_0\mu - cW_0^2 - 2cW_0^2\mu - cW_0^2(\sigma^2+\mu^2) \\ &= U(W_0) + W_0\mu(b - 2cW_0) - cW_0^2(\sigma^2+\mu^2) \end{aligned} \ ,$$

where $\mu = E(R)$ and $\sigma^2 = E(R-\mu)^2$.

Substituting the parameter b with its expression, we have:

$$\begin{aligned} E[U(W)] &= U(W_0) + W_0\mu(2cW_0 + 2c\mu_M W_0 - 2cW_0) - cW_0^2(\sigma^2+\mu^2) \\ &= U(W_0) + 2cW_0^2\mu\mu_M - cW_0^2(\sigma^2+\mu^2) \end{aligned}$$

Adding and subtracting the same quantity $cW_0^2\mu_M^2$ and considering the expectation of $U(W)$ in function of σ and μ we obtain:

$$\text{(A.1)} \qquad \begin{aligned} \psi(\sigma,\mu) &= U(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu-\mu_M)^2] \\ &= U(W_0) + cW_0^2\left\{\mu_M^2 - [\sigma^2 + (\mu-\mu_M)^2]\right\} \end{aligned}$$

The expression (A.1) represents a paraboloid in the space (σ, μ, ψ) with downward concavity, whose vertex is $(0, \mu_M, \psi(0, \mu_M))$. □

Appendix B. Proof of Theorem 3.1

Theorem 3.1*: Given two Funds, Fund A with MRAR (0)> 0 and Fund B with MRAR (0) <0, which differ for two positive elements* ε_1 and ε_2*, under the conditions* $\varepsilon_1\varepsilon_2 > \frac{4a_1a_2}{(a_1+a_2)^2}$ *Fund B can have a Morningstar rating better than Fund A.*

Proof*:* For Fund A, the hypothesis MRAR (0)>0 implies

$$G_a^6 - 1 > 0 \ \rightarrow \ G_a^6 > 1 \ ,$$

while for Fund B, the hypothesis MRAR (0)<0 implies

$$G_b^6 - 1 < 0 \;\; \rightarrow \;\; G_b^6 < 1 \;\rightarrow\; \varepsilon_1\varepsilon_2 < \frac{1}{G_a^T}$$

The last inequality is easily obtained by considering the relationship $G_b = (\varepsilon_1\varepsilon_2)^{1/T} G_a$.

Consider now the harmonic mean of the Fund B:

$$MRAR(2) = H_b^6 - 1 = \frac{1}{\left[\dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\frac{1}{b_t}\right]^6} - 1 \quad ,$$

taking in account that

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t} = \frac{1}{T}\frac{\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1 + a_2)}{\varepsilon_1\varepsilon_2 a_1 a_2} + \frac{1}{T}\sum_{t=1}^{T}\frac{1}{a_t} \quad ,$$

and

$$H_a = \frac{1}{\dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\frac{1}{a_t}} \quad ,$$

then

$$\frac{1}{\left[\dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\frac{1}{b_t}\right]^6} = \frac{1}{\left[\dfrac{1}{T}\dfrac{\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1 + a_2)}{\varepsilon_1\varepsilon_2 a_1 a_2} + \dfrac{1}{H_a}\right]^6} \; .$$

Therefore

$$\begin{aligned} H_b^6 - 1 &= \frac{1}{\left[\dfrac{1}{T}\dfrac{\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1 + a_2)}{\varepsilon_1\varepsilon_2 a_1 a_2} + \dfrac{1}{H_a}\right]^6} - 1 \\ &= \frac{H_a^6 (T\varepsilon_1\varepsilon_2 a_1 a_2)^6}{\left\{T\varepsilon_1\varepsilon_2 a_1 a_2 + H_a\left[\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1 + a_2)\right]\right\}^6} - 1 \end{aligned}$$

The condition that the rating of the Fund B is greater than the rating of Fund A can be expressed as:

$$MRAR(2)_B > MRAR(2)_A \rightarrow \; H_b^6 - 1 > H_a^6 - 1 \rightarrow H_b^6 > H_a^6$$

This means that:

$$\frac{H_a^6 (T\varepsilon_1\varepsilon_2 a_1 a_2)^6}{\left\{T\varepsilon_1\varepsilon_2 a_1 a_2 + H_a\left[\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1+a_2)\right]\right\}^6} > H_a^6$$
$$\frac{(T\varepsilon_1\varepsilon_2 a_1 a_2)^6}{\left\{T\varepsilon_1\varepsilon_2 a_1 a_2 + H_a\left[\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1+a_2)\right]\right\}^6} > 1 \quad ,$$
$$(T\varepsilon_1\varepsilon_2 a_1 a_2)^6 > \left\{T\varepsilon_1\varepsilon_2 a_1 a_2 + H_a\left[\varepsilon_1 a_1 + \varepsilon_2 a_2 - \varepsilon_1\varepsilon_2(a_1+a_2)\right]\right\}^6$$

and being $H_a > 0$, we can simplify as:

$$\varepsilon_1\varepsilon_2 > \frac{\varepsilon_1 a_1 + \varepsilon_2 a_2}{a_1 + a_2}$$

In conclusion, the constraint on the product $\varepsilon_1\varepsilon_2$ is:

$$\frac{\varepsilon_1 a_1 + \varepsilon_2 a_2}{a_1 + a_2} < \varepsilon_1\varepsilon_2 < \frac{1}{G_a^T}.$$

If we define $K = \varepsilon_1\varepsilon_2 = \frac{1}{G_a^T} - \delta$, where δ is any arbitrary small positive real number such that $K > 0$, then $\varepsilon_1 = \frac{1}{K\varepsilon_2}$ and

(B.1) $$K < \frac{1}{G_a^T} \quad .$$

Furthermore, given the positive values of K, a_1, a_2, we have:

(B.1) $$\frac{\varepsilon_1 a_1 + \varepsilon_2 a_2}{a_1 + a_2} < K \rightarrow \varepsilon_2^2 a_2 - K\varepsilon_2(a_1 + a_2) + \frac{a_1}{K} < 0.$$

The inequality (B.1) is satisfied if

(B.2) $$K\left[K(a_1+a_2)^2 - 4a_1 a_2\right] > 0 \rightarrow K > \frac{4a_1 a_2}{(a_1+a_2)^2}$$

In this way, taking in consideration series that have:

$$\frac{1}{G_a^T} > \frac{4a_1 a_2}{(a_1+a_2)^2} \rightarrow \delta > 0$$

will be possible fixing δ, if the original series respects the conditions (B.2), then the solutions can be obtained from (B.1) and (B.1). □

Appendix C. Moments of Normal Distribution

Given a standard Normal $Z \sim N(0,1)$, the central moments of order *n* are the following:

$$E(Z^n)=\begin{cases}(n-1)!! & for\ even\ n\\ 0 & for\ \ odd\ \ n\end{cases}$$

As well known, the standard Normal can be defined from any non standard Normal $X \sim N(\mu,\sigma^2)$ considering the transformation: $Z=(X-\mu)/\sigma$.

Considering that $X=\sigma Z+\mu$, its non central moments are given by:

$$\begin{aligned}E(\sigma Z+\mu)^n &= \sum_{k=0}^{n}\binom{n}{k}\sigma^k E(Z^k)\mu^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}\sigma^k (k-1)!!\,\mu^{n-k}\\ &= \mu^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k (k-1)!! \quad for\ \ even\ k\end{aligned}.$$

If $\mu>0$, then $E(\sigma Z+\mu)^n$ increases when *n* increases.

Appendix D. Proof of Theorem 4.1

Theorem 4.1: *If the returns are normally distributed in the range (-1, 1), then the non central moments are decreasing.*

Proof: Consider the truncated Normal $u \sim N(\mu,\sigma^2)$ constrained to assume values only in the interval $K=(k_1,k_2)$, with $-1<k_1<0\le k_2<1$ and $k_1<\mu<k_2$

If we define the standardized variable $h=(u-\mu)/\sigma$, and $\Phi(h)$ represents its standardized cumulative distribution, then the probability that $h\in K$ is:

(D.1) $$\Delta\Phi_H=\Phi(h_2)-\Phi(h_1) \ ,$$

where $h_1=(k_1-\mu)/\sigma$ and $h_2=(k_2-\mu)/\sigma$

The truncated density of the random variable *u* is given by:

$$f_k(u)=\begin{cases}\dfrac{\phi\left(\dfrac{u-\mu}{\sigma}\right)}{\sigma\Delta\Phi_H} & u\in K\\ 0 & u\notin K\end{cases}.$$

The non central moments of order *n* of the random variable *u* are:

(D.2) $$E(u^n\,|\,u\in K)=\mu_n=\frac{1}{\sigma\Delta\Phi_H}\int_{k_1}^{k_2}\xi^n\phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi \ .$$

This moments can be written as:

$$\mu_n \le |\mu_n| \le \frac{1}{\sigma\Delta\Phi_H}\left|\int_{k_1}^{k_2} \xi^n \phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi\right| \ ,$$

and using the Schwarz's inequality we have:

$$\frac{1}{\sigma\Delta\Phi_H}\left|\int_{k_1}^{k_2} \xi^n \phi\left(\frac{\xi-\mu}{\sigma}\right)d\xi\right| \le \frac{1}{\Delta\Phi_H}\left(\int_{k_1}^{k_2}|\xi^n|^2 d\xi\right)^{1/2}\left(\int_{k_1}^{k_2}\left|\frac{\phi\left(\frac{\xi-\mu}{\sigma}\right)}{\sigma}\right|^2 \phi\, d\xi\right)^{1/2} .$$

The calculation of the integrals on the right-hand side of the inequality gives:

$$\left(\int_{k_1}^{k_2}|\xi^n|^2 d\xi\right)^{1/2} = \left(\int_0^{k_2}\xi^{2n}d\xi + \int_0^{-k_1}\xi^n d\xi\right)^{1/2} = \left[\frac{k_2^{2n+1}+(-k_1)^{2n+1}}{2n+1}\right]^{1/2} \ ,$$

and

$$\left(\int_{k_1}^{k_2}\left|\frac{\phi\left(\frac{\xi-\mu}{\sigma}\right)}{\sigma}\right|^2 \phi\, d\xi\right)^{1/2} = \left[\int_{k_1}^{k_2}\frac{\exp(-2(\xi-\mu)^2/2\sigma^2)}{2\pi\sigma^2}\, d\xi\right]^{1/2}$$

This last integral can be solved by substitution with the change of variable $t = \dfrac{\sqrt{2}(\xi-\mu)}{\sigma}$.

We have:

$$\begin{aligned}\left[\int_{k_1}^{k_2}\frac{\exp(-2(\xi-\mu)^2/2\sigma^2)}{2\pi\sigma^2}\, d\xi\right]^{1/2} &= \left[\int_{\sqrt{2}(k_1-\mu)/\sigma}^{\sqrt{2}(k_2-\mu)/\sigma}\frac{\exp(-t^2/2)}{2\sqrt{2}\pi\sigma}\, dt\right]^{1/2} \\ &\left[\frac{\Phi(\sqrt{2}(k_2-\mu)/\sigma)-\Phi(\sqrt{2}(k_1-\mu)/\sigma)}{2\sigma\sqrt{\pi}}\right]^{1/2} \ , \\ &= \left[\frac{\Delta\Phi_{H\sqrt{2}}}{2\sigma\sqrt{\pi}}\right]^{1/2}\end{aligned}$$

with $\Delta\Phi_{H\sqrt{2}} = \Phi(\sqrt{2}(k_2-\mu)/\sigma) - \Phi(\sqrt{2}(k_1-\mu)/\sigma)$ according with the notation in (D.1).

In conclusion:

(D.3)
$$\mu_n \le \frac{\left[\frac{k_2^{2n+1}+(-k_1)^{2n+1}}{2n+1}\right]^{1/2}\left[\Delta\Phi_{H\sqrt{2}}\right]^{1/2}}{\sqrt{2\sigma}\sqrt[4]{\pi}\Delta\Phi_H} .$$

The application of the ratio test shows that the series (4.5) is convergent because the modulus values of k_1, k_2 in (D.3) are less than one. □

Appendix E. Closed form expression for the non central moments of Truncated Normal Distribution

Consider the expression (D.2):

$$\mu_n = \frac{1}{\sigma\Delta\Phi_H}\int_{k_1}^{k_2} \xi^n \phi\left(\frac{\xi-\mu}{\sigma}\right) d\xi = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} (\sigma\tau+\mu)^n \phi(\tau) d\tau \quad ,$$

and define:

$$I_k = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^k \phi(\tau) d\tau \quad .$$

If we substitute in (D.2) the following term:

$$(\sigma\tau+\mu)^n = \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k\tau^k \quad ,$$

we obtain:

$$\mu_n = \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k I_k$$

We can use the following result:

$$\frac{d\phi(\tau)}{d\tau} = -\tau\phi(\tau) \quad ,$$

for the integration by parts of I_k :

$$\begin{aligned} I_k &= \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^k \phi(\tau) d\tau = \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2} \tau^{k-1}\left\{\frac{d}{d\tau}\left[-\phi(\tau)\right]\right\} d\tau \\ &= \frac{1}{\Delta\Phi_H}\left\{\left[-\tau^{k-1}\phi(\tau)\right]_{h_1}^{h_2} + (k-1)\int_{h_1}^{h_2} \tau^{k-2}\phi(\tau) d\tau\right\} \\ &= -h_2^{k-1}\frac{\phi(h_2)}{\Delta\Phi_H} + h_1^{k-1}\frac{\phi(h_1)}{\Delta\Phi_H} + (k-1) I_{k-2} \end{aligned} \quad .$$

With the notation:

$$\rho_2 = \frac{\phi(h_2)}{\Delta\Phi_H}, \ \ \rho_1 = \frac{\phi(h_1)}{\Delta\Phi_H}, \ \ \Delta\rho_{k-1} = -\rho_2 h_2^{k-1} + \rho_1 h_1^{k-1} \quad ,$$

the previous expression can be simplified in:

$$I_k = \Delta\rho_{k-1} + (k-1) I_{k-2} \quad ,$$

which is a non-autonomous non-homogeneous difference equation of second order.

The solution can be found recursively given the initial condition for I_0 and I_1. That is:

$$
\begin{aligned}
I_0 &= \frac{1}{\Delta\Phi_H}\int_{h_1}^{h_2}\phi(\tau)d\tau = 1 \\
I_1 &= \frac{1}{\Delta\Phi_H} = \int_{h_1}^{h_2}\phi(\tau)d\tau = \frac{\phi(h_1)-\phi(h_2)}{\Delta\Phi_H} = \Delta\rho_0
\end{aligned}
.
$$

Alternatively, we can find the closed form solution for every I_k as function of the solely $\Delta\rho_k$. For this purpose, we consider the behaviour of I_k for higher values of k, by following its recursive formula, that is:

$$
\begin{aligned}
&I_2 = \Delta\rho_1 + 1 \\
&I_3 = \Delta\rho_2 + \frac{2!!}{0!!}\Delta\rho_0 = \Delta\rho_2 + 2\Delta\rho_0, \\
&I_4 = \Delta\rho_3 + \frac{3!!}{1!!}\Delta\rho_1 + 3!! = \Delta\rho_3 + 3(\Delta\rho_1 + 1), \\
&I_5 = \Delta\rho_4 + \frac{4!!}{2!!}\Delta\rho_2 + \frac{4!!}{0!!}\Delta\rho_0 = \Delta\rho_4 + 4\Delta\rho_2 + 4\cdot 2\cdot\Delta\rho_0, \\
&I_6 = \Delta\rho_5 + \frac{5!!}{3!!}\Delta\rho_3 + \frac{5!!}{1!!}\Delta\rho_1 + 5!! = \Delta\rho_5 + 5\Delta\rho_3 + 5\cdot 3\cdot(\Delta\rho_1 + 1), \\
&etc.
\end{aligned}
$$

therefore:

$$
\begin{aligned}
I_k &= \sum_{r=0}^{\underline{\underline{(k-1)/2}}} \frac{(k-1)!!}{(k-1-2r)!!}\Delta\rho_{k-1-2r} + \frac{\left[1+(-1)^k\right]}{2}(k-1)!! \\
&= (k-1)!!\left\{\sum_{r=0}^{\underline{\underline{(k-1)/2}}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{\left[1+(-1)^k\right]}{2}\right\}, \qquad k = 1, 2, \ldots
\end{aligned}
,
$$

where $\underline{\underline{(k-1)/2}}$ is the whole number preceding the value $(k-1)/2$ and $0!! = 1$, $1!! = 1$.

In conclusion:

$$
\begin{aligned}
\mu_n &= \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k (k-1)!!\left\{\sum_{r=0}^{\underline{\underline{(k-1)/2}}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{\left[1+(-1)^k\right]}{2}\right\} \\
&= \mu^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k (k-1)!!\left\{\sum_{r=0}^{\underline{\underline{(k-1)/2}}} \frac{\Delta\rho_{k-1-2r}}{(k-1-2r)!!} + \frac{\left[1+(-1)^k\right]}{2}\right\}
\end{aligned}
.
$$

Appendix F. Computation of Non Central Moments of a Standard Normal with the Incomplete Gamma Function

Consider (D.2), here reported for brevity:

$$E(u^n \mid u \in K) = \mu_n = \frac{1}{\sigma \Delta\Phi_H} \int_{k_1}^{k_2} \xi^n \phi\left(\frac{\xi-\mu}{\sigma}\right) d\xi \quad .$$

With the definition of the following function

$$I_k = \frac{1}{\Delta\Phi_H} \int_{h_1}^{h_2} \tau^k \phi(\tau) d\tau \quad ,$$

we have:

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k I_k \quad ,$$

and with the definition of

$$G_k = \int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} d\tau \quad ,$$

we have:

$$\mu_n = \frac{1}{\sqrt{2\pi}\Delta\Phi_H} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k G_k \quad .$$

From the constraints $-1 < k_1 < 0 \le k_2 < 1$ it follows $h_1 < 0,\ h_2 > 0$.

If $k > 0$ is even, then:

$$\int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} d\tau = \int_0^{-h_1} \tau^k e^{-\tau^2/2} d\tau + \int_0^{h_2} \tau^k e^{-\tau^2/2} d\tau \quad .$$

If $k > 0$ is odd, then:

$$\int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} d\tau = -\int_0^{-h_1} \tau^k e^{-\tau^2/2} d\tau + \int_0^{h_2} \tau^k e^{-\tau^2/2} d\tau \, .$$

Therefore:

$$\int_{h_1}^{h_2} \tau^k e^{-\tau^2/2} d\tau = (-1)^k \int_0^{-h_1} \tau^k e^{-\tau^2/2} d\tau + \int_0^{h_2} \tau^k e^{-\tau^2/2} d\tau \quad .$$

We can use the definition of the Lower Incomplete Gamma Distribution:

$$\gamma(\upsilon, x) = \int_0^x t^{\upsilon-1} e^{-t} dt$$

to obtain the following relation:

$$\int_0^x \tau^k e^{-\beta\tau^p} d\tau = \frac{\gamma(\upsilon, \beta x^p)}{p \beta^\upsilon}, \qquad \upsilon = \frac{k+1}{p},\ \beta > 0,\ p > 0 \quad .$$

Now we can write the expression:

(F.1) $$\mu_n = \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2}\left[(-1)^k\gamma\left(\upsilon,\frac{h_1^2}{2}\right)+\gamma\left(\upsilon,\frac{h_2^2}{2}\right)\right]$$

If $k=0$, the Gamma functions inside the square brackets become becomes:

$$\sqrt{2}\int_{h_1}^{h_2} e^{-\tau^2/2}\,d\tau = \sqrt{2}\sqrt{2\pi}\Delta\Phi_H \ ,$$

and taking in consideration that $\binom{0}{0}=1$ the (F.1) is true for $n \geq 0$.

We can rewrite the expression as:

$$\begin{aligned}\mu_n &= \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H}\left\{\sqrt{2\pi}\Delta\Phi_H + \sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2}\left[(-1)^k\gamma\left(\upsilon,\frac{h_1^2}{2}\right)+\gamma\left(\upsilon,\frac{h_2^2}{2}\right)\right]\right\} \\ &= \mu^n + \frac{\mu^n}{\sqrt{2\pi}\Delta\Phi_H}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k 2^{(k-1)/2}\left[(-1)^k\gamma\left(\upsilon,\frac{h_1^2}{2}\right)+\gamma\left(\upsilon,\frac{h_2^2}{2}\right)\right]\end{aligned}.$$