Social interactions and heterogeneous agent models. Applications to economics and finance.

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Abstract. Relying on my work in the field of contagion models, based on interacting particle systems, I will discuss some open issues concerning the applicability of complex systems in economics and finance. I will present some applications of a class of Markov models that are in line with recent research in economic theory. In particular I will highlight the importance of modeling social interactions, bounded rationality, heterogeneous agents and random utilities.

Sunto. Prendendo spunto dalla ricerca svolta nell'ambito dei modelli di contagio, basati su sistemi di particelle interagenti, discuterò una serie di questioni aperte sull'applicabilità di sistemi complessi alla teoria economica e alla finanza. In particolare presenterò qualche applicazione di una classe di modelli Markoviani che ben si inseriscono in un nascente filone di ricerca in ambito economico che vuole catturare aspetti non convenzionali quali interazioni sociali, razionalità limitata, eterogeneità degli agenti, casualità nei processi decisionali.

Keywords: intensity-based models, mean-field interactions, non-reversible Markov processes, random utility models, social interactions.

1 Introduction

Do *social interactions* and *agents' heterogeneity* really matter in economic theory? For a long period, the neoclassical (general equilibrium) approach with complete markets has provided a negative answer.

Social interactions and heterogeneity are only two of the aspects analyzed into the range of the so called *complex systems*. The debate concerning the applicability of complex systems in economics goes back to the eighties and the nineties and addresses some open issues left unsolved by the *neoclassical* approach based on the representative (perfectly rational) agent. Just to give the intuition, we could say that the neoclassical approach is

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based on a strong *rationality* paradigm that can be translated into the following modeling assumptions: perfect information shared by all the agents (no learning is needed), no space for errors in the decision process, no influence of the society on the personal beliefs.

On the opposite, the complex approach has been motivated by the desire to explain some empirical phenomena that can not be explained under the strict hypothesis of prefect rationality and efficiency of the markets. We are referring for instance to *i*) the statistical properties of financial time series (power laws with heavy tails in the stock returns, high temporal correlations, scaling laws); *ii*) path dependence and non-ergodicity in the pattern of the adoption of a new technology; *iii*) social behaviors as herding, peer pressure and conformity effects. A more detailed analysis of these issues can be found in [4, 5].

Here we focus on some of these aspects. In particular we shall describe a framework where *bounded rationality*, *social interactions* and *heterogeneity* can be introduced but maintaining tractability.

2 Random utility models

This is the case of the class of problems we intend to analyze: binary choices with interacting agents and random utility functions. The setting is the one posed in [1, 2].

The economy is made up of I agents facing a discrete binary choice problem: $\omega_i \in \{-1;+1\}$ for $i = 1, ..., I$.

Agents' utility function is made up of three components: private utility, social utility and an error term. For agent i we have that

$$(1) \qquad u_i(\omega_i) = v(\omega_i) + J_i \omega_i \bar{m}_i^e + \epsilon(\omega_i).$$

$v(\omega_i)$ is any kind of private utility associated with the binary choice. Being ω a binary variable, we can assume w.l.o.g. $v(\omega) = h\omega + k$, for $h, k \in \mathbb{R}$. Notice that $h > 0$ makes the choice $\omega = 1$ preferable. $S(\omega_i, \bar{m}_i^e) = J_i \omega_i \bar{m}_i^e$ is the social component of the utility where $\bar{m}_i^e$ is the expectation from the point of view of agent i of the behavior of the others. Finally $\epsilon(\omega_i)$ is a random term whose distribution is extreme valued, i.e., $P(\epsilon(-1) - \epsilon(1) \le x) = 1/(1 + e^{-\beta x})$, where $\beta > 0$ is a measure of the impact of the random component in the decision process: high values of β means that the deterministic part plays a relevant role in the maximization of the utility. Instead when β tends to zero the error term dominates and the choice between $\omega = 1$ or $\omega = -1$ is basically a coin tossing. The error component can be interpreted as a *bounded rationality* effect on the behavior of the agents.

We assume that the propensity to conformity of the agents (labeled by J_i for $i = 1, ..., N$) is not the same for all agents and is not constant. Note that $J_i > 0$ means that the agent is conformational (she tends to follow the social behavior). In what follows, we assume that $J \in \{-1;+1\}$; in this way we capture the presence of conformist and non conformist agents.

In this setting it can be shown that

$$(2) \qquad P(\omega_i | \bar{m}_i^e) = \frac{e^{\beta\omega_i\left(h + J_i \bar{m}_i^e\right)}}{e^{\beta\omega_i\left(h + J_i \bar{m}_i^e\right)} + e^{-\beta\omega_i\left(h + J_i \bar{m}_i^e\right)}}.$$

In [2] it is proved that when agents are homogeneous ($J_i = J$ for all i) and share the same expectations on the choice of the others, i.e. when $\bar{m}_i^e = m$ for all i, then the equilibria of the system are described by the following fixed point argument:

$$m = \tanh(\beta h + \beta J m). \tag{3}$$

The main consequence of this fact is that the equilibria of the system can be one or three depending on the value of β and J. In particular, for high values of β and J there are multiple equilibria.

Some questions arise. Is it possible to characterize the equilibria of this system assuming heterogeneity and a dynamic updating of the beliefs of the agents? What is the relationship between the static equilibria provided by (3) and the steady states found via a dynamic approach? We try to give some insights in these directions.

2.1 Dynamic set up

We denote by $\omega_i(t) \in \{-1; 1\}$, where $i : 1, ..., I$ and $t \in [0, T]$, the choice of the $i - th$ agent at time t and by $\underline{\omega}(t) = (\omega_1(t), ..., \omega_I(t))$ the vector of the state variables (agents' decisions at time t).

We assume that agents update their decisions at random Poissonian times characterized by certain *intensities* or *rates* (the inverse of the average waiting times) that depend on the state of the economy at that time and on the information of the agent.

Inspired by [1], we proposed a dynamic version of equation (2):

$$\lambda_i(t) = \lim_{\tau \to 0} \frac{1}{\tau} P(\omega_i(t+\tau) \neq \omega_i(t) | \underline{\omega}(t)) = e^{-\beta\omega_i(t)\left(h + J_i(t)\frac{\sum_j \omega_j(t)}{I}\right)}, \tag{4}$$

where now all the state variables are indexed with time and where $\lambda_i(t)$ denotes the local rate of probability that agents i changes his choice between time t and t^+, given the state of the system at time t.

Having in mind applications where the propensity of the agents to conformity is endogenously varying, we define a similar dynamics for the J_i variables:

$$\mu_i(t) = \lim_{\tau \to 0} \frac{1}{\tau} P(J_i(t+\tau) \neq J_i(t) | \underline{\omega}(t)) = e^{-\gamma J_i(t)\frac{\sum_j \omega_j(t)}{I}}, \tag{5}$$

where γ is a parameter that quantify the dependence of J_i by the state of the world.

We notice that the $i-$th agent's decision depends on the system only through the *aggregate statistic*

$$s_I = \frac{1}{I}\sum_{j=1}^{I} \omega_j.$$

This variable is indeed an *empirical mean* of the system and incorporates only a partial (averaged) information on the state vector $\underline{\omega}$. This simplifying assumption is called *mean field assumption*: the interaction among different agents only depends on the value of s_I. Notice that higher values of s_I imply an higher probability for agent i to choose $\omega_i = 1$

(when $J_i = +1$). On the other hand, high values of s_I make also larger the probability that $J_i = +1$.

Intensities as (4) make the state variables evolve as a continuous-time Markov chain on $\{-1,1\}^{2N}$ with the following infinitesimal generator:

$$(6)\quad \mathcal{L}_I f(\underline{\omega}, \underline{J}) = \sum_{i=1}^{I} e^{-\beta\omega_i(h+J_i s_I)} \left(f(\underline{\omega}^i, \underline{J}) - f(\underline{\omega}, \underline{J})\right) + \sum_{i=1}^{I} e^{-\gamma J_i s_I} \left(f(\underline{\omega}, \underline{J}^i) - f(\underline{\omega}, \underline{J})\right)$$

where $\underline{\omega}^i$ (resp. $\underline{J}^i$) denotes the vector $\underline{\omega}$ (resp. $\underline{J}$) where the i−th component has been switched:

$$\omega_j^i = \begin{cases} \omega_j & \text{for } j \neq i \\ -\omega_i & \text{for } j = i. \end{cases}$$

As argued in [1], heterogeneity leads to what is called *non reversibility* of the dynamical system. Reversibility is related to the shape of the generator (6) that describes the time evolution of the state variables of the system. It can be shown that when the system is reversible (homogeneous) there are standard techniques useful to describe the stationary distributions and hence the steady states (equilibria) of the system. In our case it is not possible to rely on such methodologies; in particular it has not yet been addressed in the literature whether more complex (non reversible) models might exhibit a behavior in line with the findings as in [1, 2]. One of our aims is to explore exactly this conjecture: is it possible to describe general Markov models that exhibit this kind of equilibria linked to the paper by Brock and Durlauf? We shall provide some insights in this directions showing in particular how to take advantage of the Markovian approach.

In order to characterize the equilibria we shall look at the *limiting behavior* of the system, i.e., the dynamics and the steady states of the $I \to \infty$ case. Our findings are in line with (3): for low values of β and γ (low interaction) there exists a unique (stationary) equilibrium; for high values there are instead three equilibria.

3 Results: Dynamics and equilibria

Inspired by [3], we now state a law of large number on a family of probability measures that will help us in determining the dynamics of the population behavior of the system. In what follows we shall denote with $(\omega_i[0,T], J_i[0,T])$ the trajectory on $[0,T]$ of the state indicators of the i-th agent. We also denote with $\mathcal{D}([0,T])$ the Skorohod space of (discontinuous) trajectories on $[0,T]$ endowed with the weak topology. With the notation $\mathcal{M}_1(X)$ we denote the space of probability measures on X.

Let $(\omega_i[0,T], J_i[0,T])_{i=1}^I \in \mathcal{D}([0,T])^{2I}$ denote a path of the system process in the time-interval $[0,T]$ for a generic $T > 0$. We define the so called *empirical measure* of the I−dimensional system as

$$(7)\qquad \rho_I = \frac{1}{I}\sum_{i=1}^{I} \delta_{(\omega_i[0,T], J_i[0,T])}.$$

We may think of ρ_I as a (random) element of $\mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$, the space of probability measures on $\mathcal{D}([0,T]) \times \mathcal{D}([0,T])$ endowed with the weak convergence topology.

Let now q be any probability measure on $\{-1;1\}^2$. Define

$$m_q^\omega := \sum_{\omega,J=\pm 1} \omega\, q(\omega, J),$$

that can be interpreted as the average choice under q.

Theorem 3.1 *Suppose that the distribution at time $t = 0$ of the Markov process $(\underline{\omega}(t), \underline{J}(t))_{t\geq 0}$ with generator (6) is such that the random variables $(\omega_i(0), J_i(0))$, for $i = 1, \dots, I$, are independent and identically distributed with law λ. Then there exists a probability $Q^* \in \mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$ such that*

$$\rho_I \rightarrow Q^* \quad \textit{almost surely}$$

in the weak topology. Moreover, if $q_t \in \mathcal{M}_1(\{-1;1\}^2)$ denotes the marginal distribution of Q^* at time t, then q_t is the unique solution of the nonlinear (McKean-Vlasov) equation

$$\begin{cases} \frac{\partial q_t}{\partial t} = \mathcal{L} q_t, t \in [0,T] \\ q_0 = \lambda \end{cases} \tag{8}$$

where

$$\mathcal{L} q(\omega, J) = \nabla^\omega \left[e^{-\beta\omega(h + J m_q^\omega)} q(\omega, J) \right] + \nabla^J \left[e^{-\gamma J m_q^\omega} q(\omega, J) \right] \tag{9}$$

with $(\omega, J) \in \{-1,;1\}^2$ and where $\nabla^x f(x,y) = f(-x,y) - f(x,y)$. ∎

Equation (8) describes the dynamics of the system with generator (6) in the limit as $I \rightarrow +\infty$. In what follows we characterize the equilibrium points, or stationary (in t) solutions of equation (8), i.e. solutions of $\mathcal{L} q_t = 0$ and, more generally, the large time behavior of its solutions.

Lemma 3.2 *Let q_t be as defined in (8) and define the expectations:*

$$m_t^\omega := \sum_{\omega,J=\pm 1} \omega q_t(\omega, J), \quad m_t^J := \sum_{\omega,J=\pm 1} J q_t(\omega, J), \quad m_t^{\omega J} := \sum_{\omega,J=\pm 1} \omega J q_t(\omega, J). \tag{10}$$

Then equation (8) can be rewritten in the following form:

$$\begin{cases} \dot{m}_t^\omega &= 2S(\beta h) S(\beta m_t^\omega) m_t^{\omega J} - 2S(\beta h) C(\beta m_t^\omega) + \\ & \quad +2C(\beta h) S(\beta m_t^\omega) m_t^J - 2C(\beta h) C(\beta m_t^\omega) m_t^\omega \\ \dot{m}_t^J &= 2S(\gamma m_t^\omega) - 2C(\gamma m_t^\omega) m_t^J \\ \dot{m}_t^{\omega J} &= 2[S(\beta h) S(\beta m_t^\omega) + S(\gamma m_t^\omega)] m_t^\omega - 2S(\beta h) C(\beta m_t^\omega) m_t^J \\ & \quad +2C(\beta h) S(\beta m_t^\omega) - 2[C(\beta h) C(\beta m_t^\omega) + C(\gamma m_t^\omega)] m_t^{\omega J}. \end{cases} \tag{11}$$

where $C(x) = \cosh(x)$ and $S(x) = \sinh(x)$. ∎

A numerical inspection of these equations shows that, depending on the values of the parameters, we can have either one unique solution (for low values of β and γ) or three different solutions (for high values of the parameters).

To better understand the qualitative representation of the steady states, we analyze the simplified system where $h = 0$. In this case (11) reduces to the following:

$$(12) \qquad \begin{cases} \dot{m}_t^\omega &= 2S(\beta m_t^\omega)m_t^J - 2C(\beta m_t^\omega)m_t^\omega \\ \dot{m}_t^J &= 2S(\gamma m_t^\omega) - 2C(\gamma m_t^\omega)m_t^J \\ \dot{m}_t^{\omega J} &= 2S(\gamma m_t^\omega)m_t^\omega + 2S(\beta m_t^\omega) - 2[C(\beta m_t^\omega) + C(\gamma m_t^\omega)]m_t^{\omega J}. \end{cases}$$

Here the dynamics of $(\dot{m}_t^\omega, \dot{m}_t^J)$ does not depend on $\dot{m}_t^{\omega J}$. This means that the differential system (12) is essentially two-dimensional: first one solves the two-dimensional system (on $[-1,1]^2$)

$$(13) \qquad (\dot{m}_t^\omega, \dot{m}_t^J) = V(m_t^\omega, m_t^J),$$

with $V(x,y) = (2\sinh(\beta x)y - 2\cosh(\beta x)x, 2\sinh(\gamma x) - 2\cosh(\gamma x)y)$, and then one solves the third equation in (12), which is linear in $m_t^{\omega J}$.

In particular the solutions of the system $V(x,y) = \underline{0}$ are

$$x = \tanh(\beta x)\tanh(\gamma x)\ ; \quad y = \tanh(\gamma x).$$

Notice that x plays exactly the role of m as in equation (3). Indeed, $x = \tanh(\beta x)\tanh(\gamma x)$ is a fixed point argument for which $x = 0$ is always solution. Moreover, for values of γ and β large enough there are three solutions: $(0, x_1, x_2)$ such that $0 < x_1 < x_2$. These findings are qualitatively in line with the equilibria found in [2] for the static and homogeneous model (see equation (3)).

4 Conclusions

We have presented a dynamic version of a random utility model. The main novelty of our analysis with respect to the literature is that we have introduced *agents' heterogeneity* in the social interaction attitude. Agents are heterogeneous in their degree of conformity/complementarity and this feature changes over time endogenously as a function of the behavior of the agents. We have shown how to introduce this source of complexity maintaining tractability.

A first application of these results to finance can be found in [3], it concerns the analysis of credit portfolio losses. We believe that this dynamic formulation can be fruitfully applied also to the study of complex credit derivatives and to the micro-foundation of prices in financial markets. In the context of social studies we are now applying it to the study of crimes rates and in the context of technology adoption patterns.

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