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P. Dai Pra, F. Fontini, E. Sartori, and M. Tolotti

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PAOLO DAI PRA

Dept. of Pure and Applied Mathematics

Università di Padova

FULVIO FONTINI

M. Fanno Dept. of Economics and Management

Università di Padova

ELENA SARTORI

Dept. of Management

Università Ca' Foscari Venezia

MARCO TOLOTTI

Dept. of Management

Università Ca' Foscari Venezia

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Abstract. In this paper we propose a simple binary mean field game, where N agents may decide whether to trade or not a share of a risky asset in a liquid market. The asset's returns are endogenously determined taking into account demand and transaction costs. Agents' utility depends on the aggregate demand, which is determined by all agents' observed and forecasted actions. Agents are boundedly rational in the sense that they can go wrong choosing their optimal strategy. The explicit dependence on past actions generates endogenous dynamics of the system. We, firstly, study under a rather general setting (risk attitudes, pricing rules and noises) the aggregate demand for the asset, the emerging returns and the structure of the equilibria of the asymptotic game. It is shown that multiple Nash equilibria may arise. Stability conditions are characterized, in particular boom and crash cycles are detected. Then we precisely analyze properties of equilibria under significant examples, performing comparative statics exercises and showing the stabilizing property of exogenous transaction costs.

Keywords: Endogenous dynamics, Nash equilibria, bounded rationality, transaction costs, mean field games, random utility.

JEL Classification Numbers: D81 - C62 - C72

Correspondence to:

Marco Tolotti: Department of Management, Università Ca' Foscari Venezia
Cannaregio, 873 - 30121 Venice, Italy
phone: [+39] 041-234-6928; e-mail: tolotti@unive.it

1 Introduction

The emergence of the financial crisis of the latest years, posing endogenous risks to the world economic recovery, seriously challenges the standard optimality properties of first-year-course like risk allocation models based on frictionless, timeless, market equilibria with homogeneous, rational and perfectly foresighted agents. Researchers try to explain market's behavior relaxing one or more of the restrictive hypotheses on which models are based. Indeed, agents have limited information and cognitive ability, form their expectations and behavior following habits and trends, over or under react to external signals, adjust progressively according to the time they have and the cost of their choices, just to name some of the most common extensions to basic market equilibrium analysis. Several financial models, in particular, try to explain the rise of market cycles that seem not to be related to the oscillation of the fundamentals, overtaking the agents' perfect rationality assumption or focusing on external driving forces of individual behavior. The former approach points at the vast (and perhaps inhomogeneous) bulk of literature that can be named as behavioral finance[1]; the latter one explains market values' dynamics as a result of the rational decision of agents, who are leaded by some external random variable uncorrelated with fundamentals, i.e., a so-called sunspot[2]. Both fields of research provide interesting, rich insights on market dynamics, yet relying on strong and perhaps unnecessary assumptions that could be relaxed. Behavioral finance interestingly focuses on agents' bounded rationality, but it rests on external ad-hoc distinctions between agents' types (informed and noise-traders) to give rise to market's dynamics. Sunspot equilibria models explicitly consider rational players' interactions, but rely on exogenous properly-introduced signals as the driving forces that lead agents' behavior. In our work, we try to encompass the most interesting features of both approaches in a innovative framework, that seriously takes into account agents bounded rationality and, at the same time, explicitly models assets' returns dynamics and the rise of booms and crashes in financial markets as the result of market features and agents' interplay. The bounded rationality assumption is included in the model considering the social interactions that arise in repeated people interplay as well as the individual possibility of making mistakes.

[1]The literature on behavioral finance is too vast to be reviewed here. We refer to books that categorize and review most common approaches, such as Schleifer (2000) and Thaler (2005), or recent textbooks such as Forbes (2009).

[2]Again, the literature on sunspot equilibria is too vast to be reviewed. For introduction, models and references see for instance Guesnerie (2001), Citanna et al. (2004) and references therein.

Indeed, tracing back to Föllmer and Schelling seminal contributions[3], a recent flow of literature has focused on the explicit dependence of agents' behavior on the other people' s action, i.e., on their social interaction, as a driving force of individual binary choices. Nadal et al. (2005), for instance, consider N agents facing the problem of choosing whether to buy or not a unique share of a risky asset, whose price P is settled by a monopolistic (exogenous) market maker. The utility function of each agent depends on a (random) idiosyncratic willingness to pay, on a social component motivated by an imitation argument and on the exogenous (static) price P. Bouchaud and Cont (2000) and Cont et al. (2005) follow Föllmer and Schweizer (1993) in modeling a linear dependence of the price change of the risky asset as a function of the aggregate demand, emerging from the underlying N dimensional market. The demand for the good is not driven by an optimization process, but rather by a statistical mechanism. In both papers the price (or the return) is extrapolated *a posteriori* as a function of the aggregate demand for the asset.

We adopt here the same binary choice model framework, assuming that people's utility depends on both their own choice and other peoples' ones. However, differently from the previously cited literature, we let social interaction affect the individual behavior through the Nash equilibria of a game representing the market. In other words, we define an endogenous interaction between market's dynamics and agents' expectations. In order to focus on the relationship between social interaction, bounded rationality and asset dynamics, we do not consider change in values (and returns) that depend on external, supply-side aspects (such as production costs, firms' profits, market structure, and similar), but we assume that market prices and returns are driven by the demand-side only. This is tantamount to define a liquid market, one in which the value of the asset does not depend on the probability of closing a trading relationship (as it is, for instance, in the search-and-matching models)[4], but rather on the demand shifts. As for the bounded rationality assumption, we allow agents to go wrong in their process of making their optimal choice. This is encompassed in the model assuming that players have a utility function that has the features of the random utility models for binary choices[5]. In words, we assume that,

[3]See Föllmer (1974) and Schelling (1978).

[4]See Rocheteau and Weill (2011) for a review of applications of search-and-matching models to financial settings.

[5]Binary random utility models have been inspired by Schelling (1978), Manski (1988) and McFadden (1974) and recently formalized in Brock and Durlauf (2001). A dynamic counterpart of this modeling framework can be found in Blume and Durlauf (2003).

at discrete times, N agents may decide whether to trade or not a share of a risky asset. Their utility function depends on the effective return of the asset and there is a random variable that with some (possibly) positive probability makes them following the opposite choice to the one that yields the highest utility, i.e., making the wrong one. We study market's dynamics and individual expectation that mutually reinforce themselves: if agents believe that the market value increases, they rise their demand; this spikes the value, which further rises the expectation and so on. However, at each point in time the value depends on the strategic interactions of agents, which are functions of their expectations on other agents' behaviors. The explicit dependence on past actions generates endogenously the dynamics of the system that are studied in the paper. These are highly non linear and are hard to be evaluated for a fixed, large N. Therefore, we let N go to infinity and focus on the asymptotic model, namely, the one with infinite agents. We first focus on the game played by each agent at each trading date, computing the optimal share q for the asset (i.e., the proportion of agents that own the asset at the equilibrium), and the related return R, where the value of the demand $\tilde{q}$ at the previous trading date enters as a parameter. This gives rise to an *implicit* dynamic equation connecting q (and thus R) to $\tilde{q}$, of the form

$$G(\tilde{q}, q) = 0. \tag{1}$$

This equation does not necessarily produce a well defined dynamical system: given $\tilde{q}$, more than one solution q may exist. This reflects the fact that multiplicity of Nash equilibria for the N-dimensional static game may persist in the $N \to +\infty$ limit. We focus on the steady states for (1). In particular, we identify all equilibria for (1), i.e., those q for which $G(\tilde{q}, q) = 0$; we also give conditions under which the dynamics given by (1) are well defined in a neighborhood of fixed points, and discuss their local stability. Moreover, we show that, under suitable conditions on the parameters of the model, cycles of period 2 may arise, and that they may coexist with stable fixed points.

We are interested also in evaluating to what extent our endogenous dynamics are influenced by exogenous factors, such as transaction costs. We are motivated by the factual consideration that in real life settings transaction costs exist and provide a rationale for bid-ask spreads; we are also inspired by the debate that has recently arisen on the possible stabilizing properties of a transaction tax like the Tobin one, which might lower the speed of adjustment in markets that seem to react abruptly to changes in fundamentals, reducing the possibility of generating or propagating a financial contagion. We evaluate to which extent the same effect might be played by the transaction costs in the dynamics of our model.

Summarizing, we believe that our paper provides significant contributions in at least three different directions.

- From a microeconomic perspective, we enrich the framework of random utility models *à la* Brock and Durlauf (2001), letting the agents update their opinion in a *parallel way*, i.e., their action is the consequence of a game, whose payoffs depend on the expectations on the behavior of the population.

- From a game theoretical perspective, we provide an existence result of pure strategy Nash equilibria for a rather large class of binary N-player non-cooperative games. Indeed, it is a well known result in literature that games in which players have binary strategies not necessarily have pure strategy Nash equilibria (e.g., the matching pennies game). On the other hand, in Cheng et al. (2004) it is shown that binary games admit pure strategy Nash equilibria as long as agents' payoff functions are symmetric. We extend and generalize such a result showing the existence of pure strategy Nash equilibria in a less restrictive setting, defining a class of monotone binary games (see Definition 3.1).

- From a financial economic perspective, we build a model where the returns R of the risky asset are function of both today's (known) and tomorrow's (unknown) aggregate demand. On the other hand, the aggregate demand q depends on R. The dynamics for the aggregate demand and returns are, thus, coupled. We also study the impact of transaction costs on the equilibria characteristics and the efficiency of the market.

The paper is organized as follows. The market model and the structure of the one period game played by the N agents are described in Section 2. In Section 3 we solve the static non-cooperative game and we derive the implicit evolution equation (1). Section 4 contains a detailed discussion of fixed points for (1), the proof of the existence of cycles of period 2 and the study of the stability both of the fixed points and of the 2-cycles. We also provide results on the volume of trades at the equilibrium. In Section 5 we propose some examples and perform comparative static exercises on them. Concluding remarks and the bibliography follow.

2 The model

In this chapter we describe both the market structure and the rules of the non-cooperative game played by the agents, starting with the former.

2.1 The market structure

We consider N agents acting on a market for a risky asset open at discrete dates $\{t_k\}_{k>0}$. At each trading date the agents have to choose whether to trade or not one share of the asset. We denote by $\omega_i(t_k) \in \{0,1\}$, for $i = 1, \ldots, N$, the choice of the i-th agent at time t_k. In this context, $\omega_i(t_k) = 1$ means that at time t_k agent i does own the share, $\omega_i(t_k) = 0$ means that agent i does not.

For each trading date t_k, the agents are facing a static non-cooperative game subject to their information at time t_{k-1}. Let us denote $t_{k-1} = \tilde{t}$ and $t_k = t$. We assume that all the agents know the past price of the share, denoted by $\tilde{P} = P(\tilde{t})$, but they can only forecast the price $P = P(t)$ prevailing at the end of the trading day on the market. They, clearly, know their own choice at time $\tilde{t}$, denoted by $\tilde{\omega} =: \omega(\tilde{t})$, and the aggregate number of shares owned by the participants at time $\tilde{t}$, denoted by $\tilde{Q} = \sum_{j=1}^{N} \tilde{\omega}_j$. When deciding their action, agents take into account the unknown return R of the asset that arises at the end of the trading period. This implies that agents are myopic, in the sense that they only look at one period returns, neglecting future $\{t_{k+1}, t_{k+2}, ...\}$ values. In other words, it is as if agents are assumed to hold a specific version of variable rate of time preferences[6], namely, they exhibit extreme impatience, in the sense that they dislike so much future values that attach to them an infinitesimal weight in their utility function, which can be neglected. This implies that the game, being a sequence of static, repeated interactions, cannot be solved by means of backward induction. Such an approach, assuming agents that are not perfectly rational in a game theoretic sense, i.e., perfectly foresighted expected utility maximizers, is coherent with our framework of boundedly rational players.

As already said, a peculiarity of this model is that R is a function of $Q = \sum_{j=1}^{N} \omega_j(t)$, the number of shares owned by participants as resulting from the game at time t. We normalize quantities,

[6] On the implausibility of the standard constant discount rate of time preferences assumption and consequences for time-consistency of the choices, see, for instance, Thaler (1981) and Loewenstein and Prelec (1992).

defining $q = \frac{1}{N}\sum_{j=1}^{N}\omega_j$ and assuming that

$$R = g\left(\tilde{q}, q\right), \tag{2}$$

where $g : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous map, increasing in q and decreasing in $\tilde{q}$. In particular, this specification reflects the stylized fact that prices are positively influenced by an increasing demand[7]. Equation (2) is extremely stylized. It does not take into account exogenous supply-side factors, which could be introduced through an exogenous random shock ξ influencing returns, so that $R = g(\tilde{q}, q) + \xi$. Coherently with the scope of the paper we focus on the relationship between returns and demand and thus leave such an extension to further researches.

We adopt the following important assumptions about market design.

A.1 All the transactions are closed with a *Market Maker* (MM), who guarantees that all the buy/sell orders are fulfilled. As a consequence, we are assuming an *infinite supply.* Such an assumption is justified by our intent to focus on the relationship between individual behavior and collective one, i.e., market demand, leaving aside exogenous quantity constraints to market supply which would affect the asset's price. As said, we propose a different framework compared, for instance, to the search-and-matching one. In that context it is assumed that supply is fixed, the result of market interaction is the amount of trades that occurs and prices derive from it[8]. We follow the opposite rationale, namely, supply is infinite, the output of the model are prices and trade volumes follow. Thus, our stylized model can be useful to explain asset dynamics in all those markets, where liquidity is large enough (or individuals are so infinitesimal with respect to the market), so that aggregate demand is the driving force of the market. Notice, moreover, that we are considering a market with N participants, where each agent may hold at most one asset. This means that the infinite supply assumption can be relaxed in assuming that there are N shares of the risky asset offered to the market by an offering entity (a firm, a sovereign or a private investment bank). The shares not bought in the first round remain to the offering entity, that consequently acts as a market maker for this good in the next periods.

A.2 On the market there are frictions, in particular *proportional transaction costs.* The market maker offers a *[bid, ask] spread* such that $\tilde{P}_b \le \tilde{P} \le \tilde{P}_a$, where we denote by $\tilde{P}_b$ the price at

[7] As in Föllmer et al. (1994), for instance.

[8] See Rocheteau and Weill (2011).

which the share is sold on the market, and by $\tilde{P}_a$ the price at which the share is bought on it. In our simplified market, where only one share of the asset can be traded, a proportional transaction cost results in a constant proportion $\mu \geq 0$ subtracted to the realized return of trading. In particular we can define the *effective return* R^{eff} made on trade as

$$
\begin{aligned}
R^{eff} &= R - \mu \ , && when\ buying\ from\ the\ MM; \\
R^{eff} &= -\mu \ , && when\ selling\ to\ the\ MM; \\
R^{eff} &= R \ , && when\ keeping\ the\ asset.
\end{aligned}
$$

2.2 The one period non-cooperative game

Let us focus now on the one period non-cooperative game. At any trading date, each agent has to decide whether to trade or not. Given their information, the N agents simultaneously choose their actions, according to their utility function. Following the random utility models approach[9], we assume that the utility has two components. The first (the rational part) is based on the forecasted profit, the second is an error term representing the bounded rationality of the agents: they may fail in choosing correctly their action. We define, for $i = 1, \ldots, N$,

$$U_i(\omega_i) = f(R^{eff}) + \epsilon_i \, \omega_i, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing map such that $f(0) = 0$ and it represents the agents' *risk attitude*. The terms $(\epsilon_i)_{i=1,\ldots,N}$ are i.i.d. random variables with common distribution η.

It is easy to see that, in our setting, the choice is made according to the scheme described in Table 1.

$\tilde{\omega} \to \omega$	event	U_i
$0 \to 1$	if $f(R-\mu) + \epsilon_i > 0$	$f(R-\mu) + \epsilon_i$
$0 \to 0$	if $f(R-\mu) + \epsilon_i \leq 0$	0
$1 \to 1$	if $f(R) + \epsilon_i > f(-\mu)$	$f(R) + \epsilon_i$
$1 \to 0$	if $f(R) + \epsilon_i \leq f(-\mu)$	$f(-\mu)$

Table 1: Scheme of the four possible scenarios (actions) and related payoffs.

[9]See Brock and Durlauf (2001) for a detailed discussion on Random Utility Models.

It follows that the utility function U_i for each agent can be written as

$$\begin{aligned} U_i(\omega_i) &= \omega_i(1-\tilde{\omega}_i)f(R-\mu) + \omega_i\tilde{\omega}_i f(R) + \tilde{\omega}_i(1-\omega_i)f(-\mu) + \epsilon_i\,\omega_i \\ &= \omega_i\left[(1-\tilde{\omega}_i)f(R-\mu) + \tilde{\omega}_i f(R) - \tilde{\omega}_i f(-\mu) + \epsilon_i\right] + \tilde{\omega}_i f(-\mu)\,. \end{aligned} \tag{4}$$

Since, by equation (2), $R = g(\tilde{q}, q)$, utility U_i explicitly depends on the $N-$dimensional vector $\boldsymbol{\omega}$. Therefore, with a slight abuse of notations, we often write $U_i(\omega_i, \boldsymbol{\omega}_{-i})$, where $\boldsymbol{\omega}_{-i}$ denotes the vector of actions $\boldsymbol{\omega}$ deprived of ω_i. Note that the dependence in $\boldsymbol{\omega}_{-i}$ is only through the normalized aggregate demand $q_N = \frac{1}{N}\sum_i \omega_i$. Therefore, our N dimensional game can be considered as a finite dimensional version of a *mean field game* in the sense of Lasry and Lions[10]. In fact, we have a non-cooperative game with a very large number of similar agents, each of them subject to some noise, ϵ_i, and coupled with each other through their utility function, which depends on an aggregate quantity, q_N, of the state variables.

3 Solution to the static non-cooperative game

In this section we turn our attention to the solution of the static non-cooperative game: we study existence and characterization of Nash equilibria. In particular, we first show existence of a pure strategy Nash equilibrium in the game played by the N agents. Then we describe the *expected behavior* of the population when the number of players goes to infinity. This leads to a characterization of the steady states of a sequence of repeated static games.

We look for an action $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) = (\omega_i, \boldsymbol{\omega}_{-i})$ such that

$$U_i(\omega_i, \boldsymbol{\omega}_{-i}) \leq U_i(\hat{\omega}_i, \boldsymbol{\omega}_{-i})\,, \qquad i = 1, \ldots, N, \tag{5}$$

where $\hat{\omega}_i = 1 - \omega_i$ denotes a switch in agent i's action.

We, firstly, prove existence of pure strategy Nash equilibria in a rather general class of N-player non-cooperative games that encompasses the problem we are facing. To this aim we define the class of *binary monotone mean field games* as follows.

Definition 3.1 (Binary monotone mean field game)**.** *A N player non-cooperative game is called binary monotone mean field when:*

- *the N players face binary actions: $\omega_i \in \Omega = \{0; 1\}$, for $i = 1\ldots, N$;*

[10]See Lasry and Lions (2007) and Cardaliaguet (2010).

- *their payoffs are of the kind*

$$U_i(\omega_1, \ldots, \omega_N) = \omega_i \left[G_i^{(1)} \left(\sum_{j=1}^{N} \omega_j \right) + \xi_i \right] + G_i^{(2)}(\boldsymbol{\omega}_{-i}), \tag{6}$$

  where $G_i^{(1)}$ is monotone, for $i = 1 \ldots, N$.

In the next lemma, we prove the existence of pure strategy Nash equilibria for binary monotone mean field games. As already said, the game is mean field in the sense that the payoff depends on $\boldsymbol{\omega}$ only through an aggregate variable. The monotonicity is related to the functions $G_i^{(1)}(\cdot)$. This result extends and generalizes previous works on existence of pure strategy Nash equilibria in the context of binary decisions (see, for instance, Cheng et al. (2004)) to games that are not necessarily symmetric.

Lemma 3.2. *A binary monotone mean field game as defined in Definition 3.1 admits pure strategy Nash equilibria.*

Proof. We give the proof for the increasing case. The proof for the decreasing one follows the same lines.
For $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \{0,1\}^N$, set $A(\boldsymbol{\omega}) := \{i : \omega_i = 1\}$. Thus vectors of strategies can be identified with subsets of $\{1, 2, \ldots, N\}$. Note that the following statements are equivalent:

N1. $\boldsymbol{\omega}$ is a Nash equilibrium;

N2. Setting $A := A(\boldsymbol{\omega})$,

  1. for every $i \in A$, we have $G_i^{(1)}(|A|) + \xi_i \geq 0$;
  2. for every $i \notin A$, we have $G_i^{(1)}(|A| + 1) + \xi_i \leq 0$.

Thus, all we need is to find $A \subseteq \{1, 2, \ldots, N\}$ for which N2 holds. For $A \subseteq \{1, 2, \ldots, N\}$, define the transformation

$$TA := \{i \in A : \xi_i \geq -G_i^{(1)}(|A|)\}.$$

Now set $A_0 := \{1, 2, \ldots, N\}$. If $TA_0 = A_0$, then N2 holds for A_0, in other words $(1, 1, \ldots, 1)$ is a Nash equilibrium. Otherwise, set $A_1 := TA_0 \subsetneq A_0$. Note that, if $i \notin A_1$, then

$$\xi_i < -G_i^{(1)}(|A_0|) \leq -G_i^{(1)}(|A_1| + 1).$$

So, if $TA_1 = A_1$, then N2 holds for A_1. Otherwise we iterate. By induction on k we show that, whenever $A_k := TA_{k-1} \subsetneq A_{k-1}$,

$$i \notin A_k \;\Rightarrow\; \xi_i \leq -G_i^{(1)}(|A_k|+1).$$

We have shown it for $k=1$. Suppose it is true for $k-1$. If $i \in A_{k-1} \setminus A_k$, by definition of T we have that

$$\xi_i < -G_i^{(1)}(|A_{k-1}|) \leq -G_i^{(1)}(|A_k|+1).$$

On the other hand, if $i \notin A_{k-1}$, then, by the inductive assumption,

$$\xi_i \leq -G_i^{(1)}(|A_{k-1}|+1) \leq -G_i^{(1)}(|A_k|+1).$$

Now, as soon as we have $TA_k = A_k$, then N2 holds for A_k. But, since T maps a finite set to a subset of itself, it must eventually reach a fixed point, which is possibly the empty set. ∎

In the following theorem we apply the existence result shown in Lemma 3.2 to the market setting defined in the previous section:

Theorem 3.3. *There exists a pure strategy Nash equilibrium $\boldsymbol{\omega}^*$ for the non-cooperative game played at time t by the N agents with utilities as in equation (4). Moreover, $\boldsymbol{\omega}^*$ is such that*

$$\omega_i^* = H\Bigg((1-\tilde{\omega}_i) f\left(g\left(\tilde{q}_N, q_N^* + \frac{1-\omega_i^*}{N}\right) - \mu\right) + \tilde{\omega}_i f\left(g\left(\tilde{q}_N, q_N^* + \frac{1-\omega_i^*}{N}\right)\right) - \tilde{\omega}_i f(-\mu) + \epsilon_i\Bigg), \tag{7}$$

where $H(u) = \mathbb{I}_{[0,+\infty)}(u)$ and where $\tilde{q}_N = \frac{1}{N}\sum_{i=1}^N \tilde{\omega}_i$ and $q_N^* = \frac{1}{N}\sum_{i=1}^N \omega_i^*$.

Proof. Existence follows directly from Lemma 3.2. It is enough to put

$$\begin{cases} G_i^{(1)}(x) = (1-\tilde{\omega}_i) f(g(\tilde{q}, x/N) - \mu) + \tilde{\omega} f(g(\tilde{q}, x/N)) \\ G_i^{(2)}(y) = \tilde{\omega}_i f(-\mu) \\ \xi_i = -\tilde{\omega}_i f(-\mu) + \epsilon_i \end{cases}$$

We are left to show that any pure strategy Nash equilibrium satisfies equation (7).
First of all, notice that the term $\tilde{\omega}_i f(-\mu)$ in equation (4) does not play any role in the maximization, so that we can ignore it. Fix now a pure strategy $\boldsymbol{\omega}^*$; suppose, firstly, that $\omega_i^* = 0$. This means that $U_i(0, \boldsymbol{\omega}_{-i}^*) \geq U_i(1, \boldsymbol{\omega}_{-i}^*)$. $U_i(0, \boldsymbol{\omega}_{-i}^*) = 0$, whereas it is not difficult to see that

$$U_i(1, \boldsymbol{\omega}_{-i}^*) = (1-\tilde{\omega}_i) f\left(g\left(\tilde{q}_N, q_N^* + \frac{1}{N}\right) - \mu\right) + \tilde{\omega}_i f\left(g\left(\tilde{q}_N, q_N^* + \frac{1}{N}\right)\right) + \epsilon_i.$$

Thus $\omega_i^* = 0$ implies $(1-\tilde{\omega}_i) f\left(g\left(\tilde{q}_N, q_N^* + \frac{1}{N}\right) - \mu\right) + \tilde{\omega}_i f\left(g\left(\tilde{q}_N, q_N^* + \frac{1}{N}\right)\right) + \epsilon_i \leq 0$.
Consider now $\omega_i^* = 1$. We have $U_i(1, \boldsymbol{\omega}_{-i}^*) \geq U_i(0, \boldsymbol{\omega}_{-i}^*)$. Note that in this case

$$U_i(1, \boldsymbol{\omega}_{-i}^*) = (1-\tilde{\omega}_i) f\left(g\left(\tilde{q}_N, q_N^*\right) - \mu\right) + \tilde{\omega}_i f\left(g\left(\tilde{q}_N, q_N^*\right)\right) + \epsilon_i \geq 0.$$

Summarizing, ω_i^* is characterized by equation (7). ∎

Equation (7) characterizes the pure strategies of the game. In order to study the evolution of returns in the market, an important statistics is q_N^*, the optimal proportion of agents choosing $\omega = 1$ in the game with N players. Equation (7) can be rewritten in terms of q_N^* via the so called *auto consistency equation*, obtained summing over i and dividing by N both terms in equation (7). We also allow the return R_N to depend on the number of agents: these dependencies are functional to the $N \to +\infty$ limit that will be taken later.

$$\begin{cases} q_N^* &= \frac{1}{N}\sum_{i=1}^N H\Big(f\left(g\left(\tilde{q}_N, q_N^* + \frac{1-\omega_i^*}{N}\right) - \mu\right) \\ & \quad -\tilde{\omega}_i\left(f\left(g\left(\tilde{q}_N, q_N^* + \frac{1-\omega_i^*}{N}\right) - \mu\right) - f\left(g\left(\tilde{q}_N, q_N^* + \frac{1-\omega_i^*}{N}\right)\right) + f(-\mu)\right) + \epsilon_i\Big) . \\ R_N &= g(\tilde{q}_N, q_N^*) \end{cases} \tag{8}$$

Note that Theorem 3.3 guarantees that, for $\tilde{q}_N$ given, *at least* one solution (q_N^*, R_N) to (8) exists. On the other hand uniqueness may fail; in other words, the dynamical system described by (8) can be ill defined. Nevertheless, one can, at a heuristic level, consider the limit of (8) as $N \to +\infty$, at least along convergent subsequences. We now prove the main theorem of this paper.

Theorem 3.4. *Assume that*

$$\lim_{N\to+\infty} \tilde{q}_N = \tilde{q},$$

and, for each N, let q_N be a solution of equation (8). Then $(q_N)_{N\geq 1}$ is a tight *sequence of random variables. Each weak limit point q is, with probability one, solution of the limiting equation*

$$q = \tilde{q}\eta\left(f(g(\tilde{q}, q)) - f(-\mu)\right) + (1-\tilde{q})\eta\left(f(g(\tilde{q}, q) - \mu)\right). \tag{9}$$

Proof. For simplicity of notations, we set $f = id$; no change is needed in the proof for the general case.
To begin with, tightness of (q_N) is obvious, since q_N is $[0,1]$-valued. We still denote by (q_N) a subsequence converging to q. By no loss of generality, possibly enlarging the probability space

where the ϵ_i's are defined (see the Skorohod Representation Theorem in Billingsley (1999), Theorem 1.6.7), we can assume $q_N \to q$ almost surely. Now, let ρ_N be the empirical measure

$$\rho_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{(\tilde{\omega}_i,\epsilon_i)}.$$

Its weak limit comes from the assumptions: for h bounded and continuous,

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} h(\tilde{\omega}_i,\epsilon_i) = \tilde{q}\int h(1,\epsilon)\eta(d\epsilon) + (1-\tilde{q})\int h(0,\epsilon)\eta(d\epsilon) \tag{10}$$

almost surely.

Now, let H^δ, with $\delta > 0$, be a family of Lipschitz functions with $\inf_\delta H^\delta = H$. For δ fixed, by (8), (10) and the fact that

$$g\left(\tilde{q}_N, q_N + \frac{1-\omega_i}{N}\right) \to g(\tilde{q}, q)$$

almost surely, we have

$$\begin{aligned} q = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} H\left(g\left(\tilde{q}_N, q_N + \frac{1-\omega_i}{N}\right) + \mu(2\tilde{\omega}_i - 1) + \epsilon_i\right) \\ \leq \limsup_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} H^\delta\left(g\left(\tilde{q}_N, q_N + \frac{1-\omega_i}{N}\right) + \mu(2\tilde{\omega}_i - 1) + \epsilon_i\right) \\ = \tilde{q}\int H^\delta(g(\tilde{q},q)+\mu+\epsilon)\eta(d\epsilon) + (1-\tilde{q})\int H^\delta(g(\tilde{q},q)-\mu+\epsilon)\eta(d\epsilon). \end{aligned}$$

Taking the limit over a sequence $\delta_n \downarrow 0$, by Dominated Convergence we obtain

$$\begin{aligned} q \leq \tilde{q}\int H(g(\tilde{q},q)+\mu+\epsilon)\eta(d\epsilon) + (1-\tilde{q})\int H(g(\tilde{q},q)-\mu+\epsilon)\eta(d\epsilon) \\ = \tilde{q}\eta\left(g(\tilde{q},q)+\mu\right) + (1-\tilde{q})\eta\left(g(\tilde{q},q)-\mu\right). \end{aligned}$$

The corresponding lower bound is obtained similarly: one takes a Lipschitz lower bound H_δ for H, such that $H_\delta \uparrow H^-$ as $\delta \downarrow 0$, where H^- differs from H only for $H^-(0) = 0 < H(0) = 1$. We obtain

$$q \geq \tilde{q}\int H^-(g(\tilde{q},q)+\mu+\epsilon)\eta(d\epsilon) + (1-\tilde{q})\int H^-(g(\tilde{q},q)-\mu+\epsilon)\eta(d\epsilon).$$

Since, by assumption, η is a continuous distribution, we can replace H^- with H in this last formula, and the conclusion follows. ■

Whenever the limit equation (9), for a given $\tilde{q}$, has a unique solution q, Theorem 3.4 gives a law of large numbers with a deterministic limit. For most examples, multiplicity of solutions is actually exceptional, occurring only for few values of $\tilde{q}$.

Remark 3.5. *The static game has been solved for* given $\epsilon_1, \epsilon_2, \dots, \epsilon_N$*, as a game with* total information. *One could argue that a more realistic model is obtained assuming that each player i can only observe its own noise ϵ_i, while only the distribution of the other players' noise is known to him. In rigorous terms this means:*

- *strategies are measurable functions from the noise space $\mathbb{R}$ to $\{0,1\}$: in other words $\omega_i = \omega_i(\epsilon_i)$;*
- *the utility of the i^{th} player, which is of the form $U_i(\omega_i, \omega_{-i}, \epsilon_i)$, has to be replaced by*

$$\mathcal{U}_i := \mathbb{E}_{\epsilon_{-i}} U_i(\omega_i, \omega_{-i}, \epsilon_i),$$

  where $\mathbb{E}_{\epsilon_{-i}}$ denotes average with respect to the noise variables ϵ_j, $j \neq i$.

The equation for the (pure) Nash equilibrium becomes

$$\omega_i = H\Bigg((1-\tilde{\omega}_i)\mathbb{E}_{\epsilon_{-i}}\left[f\left(g\left(\tilde{q}_N, q_N + \frac{1-\omega_i}{N}\right) - \mu\right)\right] + \tilde{\omega}_i \mathbb{E}_{\epsilon_{-i}}\left[f\left(g\left(\tilde{q}_N, q_N + \frac{1-\omega_i}{N}\right)\right)\right] - \tilde{\omega}_i f(-\mu) + \epsilon_i\Bigg). \quad (11)$$

The existence of a Nash equilibrium now becomes more problematic, since strategies belong to a larger space. However, ignoring this problem, and assuming further that a law of large numbers holds, i.e., q_N converge to a constant, in the limit as $N \to +\infty$, the q_N obtained from (11) converge to a solution of (9)*, as for the total information case. In other words, we expect that the total and the partial information model yield the same macroscopic behavior.*

4 Long time behavior of the limit evolution equation

Equation (9) describes the behavior of the system with utility function (4) in the infinite volume limit. We are interested in detecting the t-stationary solution(s) of this equation and studying its (their) stability properties. We refer to (9) as the *limit evolution equation* of our system. Notice that at some points it will be more convenient to rewrite the system (9) as the implicit equation $G(\tilde{q}, q) = 0$, where

$$G(\tilde{q}, q) := q - \tilde{q}\eta(f(g(\tilde{q}, q)) - f(-\mu)) - (1-\tilde{q})\eta(f(g(\tilde{q}, q) - \mu)). \quad (12)$$

Non linear equations of the type of (9) have already appeared in literature; they have been analyzed, for instance, in the context of heterogeneous agent based models (see Brock and Hommes (1998) or Hommes (2006) for a survey). A recent application of agent based models to a stylized financial market can be found in Chang (2007). In those papers, non linearity emerges as a consequence of *ex ante* assumptions on agents' heterogeneities or non linear price dynamics. What makes our approach different is the derivation of (9) as a limit of a N-player game, where agents are not heterogeneous in their expectations and where pricing rules are not forced to be non linear.

The main results of this section are Theorem 4.3 and Theorem 4.4, in which we show, respectively, existence and local (linear) stability of attractors for the limit evolution equation. In Theorem 4.3 we consider two specific kinds of attractors for the dynamics induced by (12): *fixed points* q such that $G(q,q)=0$ and cycles of period 2 (briefly called *2-cycles*). We define a 2-cycle as a pair $(\tilde{q},q)\in(0,1)^2$, $q\neq\tilde{q}$ such that $G(\tilde{q},q)=G(q,\tilde{q})$.
Before stating the main theorems, we give some basic assumptions needed for the analysis of the model:

Assumption 4.1. *The model we are considering is such that*

A.1 $g:[0,1]\times[0,1]\to\mathbb{R}$, $g:(x,y)\mapsto r$, *introduced in* (2), *is of class* $\mathcal{C}^{1,1}$, *decreasing in* x *and increasing in* y *and such that* $g(x,x)\equiv 0$.

A.2 There exists a $\mathcal{C}^1$ *map*

$$i: Im(g)\to Im(g),$$

where $Im(g)$ *is the image of* $g(\cdot,\cdot)$, *which is strictly decreasing, and*

$$g(y,x)=i(g(x,y))$$

for all $x,y\in(0,1)$.

A.3 The distribution function η *of random terms, as introduced in* (3), *is absolutely continuous, with continuous strictly positive and symmetric density.*

Assumption A.1 states that returns are increasing in demand. Concerning Assumption A.2, the map $i(\cdot)$ describes the way the returns are transformed for time reversal and depends on the specific models; some examples are presented in Section 5. Decreasingness guarantees consistency with Assumption A.1. Finally, A.3 is a technical assumption and could be possibly relaxed. We firstly prove a technical lemma:

Lemma 4.2. *Define*

$$A(R) := \eta(f(R) - f(-\mu)) - \eta(f(R-\mu)) \quad \textit{and} \quad B(R) := \eta(f(R-\mu)).$$

Under Assumption 4.1, the map τ defined as

$$\tau(R) := \frac{A(R)B(i(R)) + B(R)}{1 - A(i(R))A(R)}, \tag{13}$$

is well defined and $Im(\tau) \in (0,1)$.

Proof. Since η has, by assumption, a strictly positive derivative, $\eta(x) < 1$ for every $x \in \mathbb{R}$. In particular $0 < B(R) < 1$, $|A(R)| < 1$, $A(R)+B(R) = \eta(f(R)-f(-\mu)) \in (0,1)$. In particular, $A(i(R))A(R) < 1$, so that τ is well defined. We now show that $\tau(R) \in (0,1)$.

- $\tau(R) > 0$ amounts to $A(R)B(i(R)) + B(R) > 0$. This is clearly true whenever $A(R) \geq 0$. If $A(R) < 0$, then
$$A(R)B(i(R)) + B(R) > A(R) + B(R) > 0.$$
- $\tau(R) < 1$ amounts to
$$A(R)[A(i(R)) + B(i(R))] + B(R) < 1.$$
  This is clearly true when $A(R) \leq 0$. If $A(R) > 0$, then
$$A(R)[A(i(R)) + B(i(R))] + B(R) < A(R) + B(R) < 1.$$

■

Next theorem shows that fixed point and 2-cycle can be described in terms of returns as fixed points of a suitable map.

Theorem 4.3. *Define the map*

$$\phi(R) := g(\tau(i(R)), \tau(R)), \tag{14}$$

where $\tau(\cdot)$ and $i(\cdot)$ are as in Lemma 4.2. Then, under Assumption 4.1, the set of fixed points and 2-cycles for (12) *is characterized as the set of fixed points of the map ϕ. In particular,*

i) $q \in (0,1)$ is a fixed point for G if and only if $q = \frac{1}{2}$. Moreover $\phi(R)_{|R=0} = 0$.

ii) G admits a 2*-cycle if and only if the map* ϕ *has a nonzero fixed point.*

Proof. A fixed point for (9) is such that

$$\begin{aligned} q &= qA(R) + B(R) \\ R &= g(q, q). \end{aligned}$$

By assumption $g(q, q) = 0$, so that $R = 0$ and $q = \frac{B(0)}{1-A(0)} = \frac{1}{2}$. Note, moreover, that $i(0) = 0$ and $\tau(0) = \frac{1}{2}$, so that $\phi(0) = 0$. This proves that $q = \frac{1}{2}$ is the only fixed point for (9) and, since $R = 0$, it also corresponds to the zero value fixed point for the map ϕ.
We are now left with point (*ii*). To this aim, note that the system (9) has a 2-cycle when there exists a pair $(\tilde{q}, q) \in (0, 1)^2$, with $\tilde{q} \neq q$, such that

$$\begin{aligned} q &= \tilde{q}A(R) + B(R) \\ R &= g(\tilde{q}, q) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \tilde{q} &= qA(\tilde{R}) + B(\tilde{R}) \\ \tilde{R} &= g(q, \tilde{q}) \end{aligned} \tag{16}$$

both hold. Suppose (15) and (16) hold. Note that $\tilde{R} = i(R)$. Plugging in (15) the expression for $\tilde{q}$ in (16), we get $q = \tau(R)$ and $\tilde{q} = \tau(i(R))$. This, inserted in the second equation of (15), gives $R = \phi(R)$.
Conversely, if $R \neq 0$ is such that $R = \phi(R)$, then $q = \tau(R)$ and $\tilde{q} = \tau(i(R))$ solve (15) and (16). This concludes the proof of point (*ii*). ∎

Theorem 4.3 shows that the only steady state equilibrium corresponds to half of the agents owning the asset at each point in time. Moreover, it provides conditions for the existence of a 2-cycle. Even if Theorem 4.3 does not rule out more complex attractors such as longer periodic orbits or chaotic behaviors, however, all these situations have not been detected in simulations corresponding to the examples given in Section 5.

In the next result we discuss the local (linear) stability of the fixed point and of the 2-cycle.

Theorem 4.4. *Let* R_* *be any fixed point (possibly* $R_* = 0$*) of* ϕ*,* $\tilde{R}_* := i(R_*)$ *and* $(\tilde{q}_*, q_*) := (\tau(i(R_*)), \tau(R_*))$ *the associated* 2*-cycle, possibly the fixed point* $\left(\frac{1}{2}, \frac{1}{2}\right)$*. Then, under Assumption 4.1, the following statements hold true.*

1. *Well definiteness. Assume*

$$\tilde{q}_* A'(R_*)\partial_2 g(\tilde{q}_*, q_*) + B'(R_*)\partial_2 g(\tilde{q}_*, q_*) \neq 1 \tag{17}$$

and

$$q_* A'(i(R_*))\partial_2 g(q_*, \tilde{q}_*) + B'(i(R_*))\partial_2 g(q_*, \tilde{q}_*) \neq 1. \tag{18}$$

The implicit map (9) is well defined both in a neighborhood of $(\tilde{q}_*, q_*)$ *and of* $(q_*, \tilde{q}_*)$.

2. *Local stability. Set*

$$\rho := \left| \frac{A(R_*) + \tilde{q}_* A'(R_*)\partial_1 g(\tilde{q}_*, q_*) + B'(R_*)\partial_1 g(\tilde{q}_*, q_*)}{1 - \tilde{q}_* A'(R_*)\partial_2 g(\tilde{q}_*, q_*) - B'(R_*)\partial_2 g(\tilde{q}_*, q_*)} \right| \times \left| \frac{A(i(R_*)) + \tilde{q}_* A'(i(R_*))\partial_1 g(q_*, \tilde{q}_*) + B'(i(R_*))\partial_1 g(q_*, \tilde{q}_*)}{1 - \tilde{q}_* A'(i(R_*))\partial_2 g(q_*, \tilde{q}_*) - B'(i(R_*))\partial_2 g(q_*, \tilde{q}_*)} \right|. \tag{19}$$

Then the 2*-cycle* $(\tilde{q}_*, q_*)$ *is linearly stable for* $\rho < 1$*, and linearly unstable for* $\rho > 1$.

Proof. We only give a sketch of this rather straightforward proof. Note that we are considering the fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$ as a particular case of (degenerate) 2-cycle.
Statement 1. follows from a standard application of the Implicit Function Theorem to the implicit function (9). Concerning the stability, we apply twice the map (9) to a small perturbation of $\tilde{q}_*$:

$$\begin{aligned} q^\varepsilon &:= (\tilde{q}_* + \varepsilon)A(g(\tilde{q}_* + \varepsilon, q^\varepsilon) + B(g(\tilde{q}_* + \varepsilon, q^\varepsilon) \\ \tilde{q}^\varepsilon &:= q^\varepsilon A(g(q^\varepsilon, \tilde{q}^\varepsilon)) + B(g(q^\varepsilon, \tilde{q}^\varepsilon)). \end{aligned}$$

We then compute, by implicit differentiation, $q' := \frac{d}{d\varepsilon} q^\varepsilon|_{\varepsilon=0}$ first, and $\tilde{q}' := \frac{d}{d\varepsilon}\tilde{q}^\varepsilon|_{\varepsilon=0}$ then. We obtain $|\tilde{q}'| = \rho$, from which our statement of linear stability follows. ∎

All conditions given in Theorem 4.4 simplify for the fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$, since equations (17) and (18) coincide as well as the two factors in the right hand side of (19).

The rather awkward stability condition we have obtained, can in principle be checked numerically in specific examples. We perform such an analysis for specific examples in the next section. However, before moving forward, let us point out that a simpler geometrical condition for the stability of the 2-cycle, when $\mu = 0$ (i.e., no transaction costs), is shown in the next result.

Corollary 4.5. *Assume there are no transaction costs. Then, under Assumption 4.1,*

1. *the fixed point is linearly stable if and only if* $\phi'(0) < 1$*,*

2. *a* 2*-cycle* $(\tilde{q}_*, q_*) := (\tau(i(R_*)), \tau(R_*))$ *is linearly stable if* $\phi'(R_*) < 1$*.*

Proof. Note that, when $\mu = 0$, we have $A(R) \equiv 0$. So, by Theorem 4.4, we get

$$\rho = \left| \frac{B'(R_*)\partial_1 g(\tilde{q}_*, q_*)}{1 - B'(R_*)\partial_2 g(\tilde{q}_*, q_*)} \right| \left| \frac{B'(i(R_*))\partial_1 g(q_*, \tilde{q}_*)}{1 - B'(i(R_*))\partial_2 g(q_*, \tilde{q}_*)} \right|.$$

Moreover,

$$\phi(R) = g(B(i(R)), B(R)),$$

so that

$$\phi'(R) = \partial_1 g(B(i(R)), B(R))B'(i(R))i'(R) + \partial_2 g(B(i(R)), B(R))B'(R).$$

In what follows we use the fact that, since $g(x, y) = i(g(y, x))$, we have

$$\partial_1 g(x, y) = i'(g(y, x))\partial_2 g(y, x)\,, \qquad \partial_2 g(x, y) = i'(g(y, x))\partial_1 g(y, x). \tag{20}$$

Assume that $\phi'(R_*) < 1$, i.e.,

$$\partial_1 g(\tilde{q}_*, q_*)B'(i(R_*))i'(R_*) + \partial_2 g(\tilde{q}_*, q_*)B'(R_*) < 1. \tag{21}$$

Note that, by (20), we can rewrite (21) in the alternative form

$$\partial_2 g(q_*, \tilde{q}_*)B'(i(R_*)) + \partial_1 g(q_*, \tilde{q}_*)i'(i(R_*))B'(R_*) < 1. \tag{22}$$

Note that, since $B(R) = \eta(f(R))$, we have $B' > 0$. Moreover $\partial_1 g < 0$, $\partial_2 g > 0$, and $i' < 0$. Finally $i'(i(R_*)) = (i^{-1})'(i(R_*)) = \frac{1}{i'(R_*)}$. By (21) we obtain

$$1 - \partial_2 g(\tilde{q}_*, q_*)B'(R_*) > \partial_1 g(\tilde{q}_*, q_*)B'(i(R_*))\frac{1}{i'(R_*)} > 0$$

or, equivalently,

$$\left| \frac{\partial_1 g(\tilde{q}_*, q_*)B'(i(R_*))}{i'(R_*)[1 - \partial_2 g(\tilde{q}_*, q_*)B'(R_*)]} \right| < 1. \tag{23}$$

Similarly, by (22),

$$1 - \partial_2 g(q_*, \tilde{q}_*)B'(i(R_*)) > \partial_1 g(q_*, \tilde{q}_*)i'(R_*)B'(R_*) > 0$$

or, equivalently,

$$\left| \frac{\partial_1 g(q_*, \tilde{q}_*)i'(R_*)B'(R_*)}{1 - \partial_2 g(q_*, \tilde{q}_*)B'(i(R_*))} \right| < 1. \tag{24}$$

By multiplying (23) and (24) we get the stability condition $\rho < 1$. ∎

In the next section we characterize the volume of trades, i.e., the number of shares exchanged on the market at any trading date. We see that, even at the equilibrium, where $q = \frac{1}{2}$, some trade may still be in place. The volume of trades depends on the parameters of the model.

4.1 Volume of trades

The total volume percentage of trades in the market with N players is given by the sum of the trades of players who do not hold the asset and want to acquire it and those who own it and want to sell it:

$$\begin{aligned} S_N &= \frac{1}{N}\sum_{i=1}^{N}\omega_i(1-\tilde{\omega}_i) + \frac{1}{N}\sum_{i=1}^{N}\tilde{\omega}_i(1-\omega_i) \\ &= \frac{1}{N}\sum_{i=1}^{N}\omega_i - \frac{2}{N}\sum_{i=1}^{N}\omega_i\tilde{\omega}_i + \frac{1}{N}\sum_{i=1}^{N}\tilde{\omega}_i \\ &= q_N - 2s_N + \tilde{q}_N\,, \end{aligned} \tag{25}$$

where $s_N = \frac{1}{N}\sum_{i=1}^{N}\omega_i\tilde{\omega}_i$.

Proposition 4.6. *Assume that* $\lim_{N\to+\infty}\tilde{\rho}_N = \tilde{\rho}$. *Then,* $\lim_{N\to+\infty} S_N = S$, *where* S *is the limiting trading volume percentage, in the game with infinite agents. Moreover,* S *solves the following equation*

$$S = q - 2s + \tilde{q}\,, \text{ where } s = \tilde{q}\eta(f(R) - f(-\mu))\,. \tag{26}$$

Proof.

$$s_N = \frac{1}{N}\sum_{i=1}^{N}\omega_i\tilde{\omega}_i = \bar{\rho}_N(1,1)\,, \tag{27}$$

where $\bar{\rho}_N(\tilde{\omega},\omega)$ is the joint distribution of each couple $(\tilde{\omega}_i,\omega_i)$. To find the $N\to+\infty$ limit of equation (27), we, first, need to prove there exists $\lim_{N\to+\infty}\bar{\rho}_N = \bar{\rho}$; but $(\bar{\rho}_N)$ is a sequence on a compact set, then it admits limit points. So we can consider the limit on convergent subsequences, and, if the dynamics are well defined, there exists only one limit point. Thus a law of large numbers apply and, in particular, it yields

$$s_N \to s = \bar{\rho}(1,1)\,.$$

Now

$$
\begin{aligned}
\bar{\rho}(1,1) &= \mathbb{P}(\tilde{\omega}=1, \omega=1) \\
&= \mathbb{P}(\tilde{\omega}=1, H((1-\tilde{\omega})f(R-\mu)+\tilde{\omega}(f(R)-f(-\mu))+\epsilon)=1) \\
&= \mathbb{P}(\tilde{\omega}=1)\cdot\mathbb{P}(H((1-\tilde{\omega})f(R-\mu)+\tilde{\omega}(f(R)-f(-\mu))+\epsilon)=1|\tilde{\omega}=1) \\
&= \tilde{q}\cdot\mathbb{P}(f(R)-f(-\mu)+\epsilon>0)=\tilde{q}\eta(f(R)-f(-\mu))\,,
\end{aligned}
$$

where $H(x)=\mathbb{I}_{[0,+\infty)}(x)$. ∎

Recall that, when $q=\tilde{q}=\frac{1}{2}$, $R=0$. In this case, $S=S(\mu)=1-\eta(-f(-\mu))=\eta(f(-\mu))$, is a decreasing function of μ. This confirms the intuition that a transaction tax lowers the volume of trades. Moreover, $S\in\left[0,\frac{1}{2}\right]$, since $S(0)=\frac{1}{2}$ and $\lim_{\mu\to+\infty}S(\mu)=0$.
Note that $S(0)=\frac{1}{2}$ means that, under no transaction costs, half of the agents are trading the asset at the equilibrium, even though the returns are zero. The reason being that in average, half of the population believes in a decreasing market and half believes in the opposite (due to the idiosyncratic error terms ϵ). As long as μ increases, the cost makes the transaction unworthy for a larger fraction of agents. In this sense, the transaction tax μ has a regularizing effect, since it mitigates the volume of transactions driven by boudedly rational traders.

5 Examples and comparative statics

All results derived so far have been given in a rather general form. In this section we discuss in details the meaning and implication of our results, constraining the analysis to examples, in which we adopt specific functional forms for the utility function f, for the distribution η of the error parameter introduced in (3) and for the relationship between asset's demand and its return given by the function g defined in (2).

5.1 Examples

Demands and returns.
From now on, we rely on a common definition of returns: $R=\frac{P-\tilde{P}}{\tilde{P}}$.

Case 1. Linear evolution of prices. We consider linearity in the relationship between demand changes and asset's returns:

$$
R=k\cdot(q-\tilde{q})\,,\quad k>0. \tag{28}
$$

This implies the following price dynamics:

$$P - \tilde{P} = \tilde{P} \cdot k\,(q - \tilde{q}). \tag{29}$$

This specification is inspired by Bouchaud and Cont (2000) and Nadal et al. (2005). $k \geq 0$ is sometimes referred as the *market depth*. It measures how deeply the price is related to the excess demand on the market. In this very stylized context it can be considered as a simplified form of elasticity of price in demand.

Case 2. Loglinear evolution of prices. Simple as it might be, a linear relationship between demand and returns can hide some interesting behavior of the system. In order to test such a working hypothesis we allow for a non linear relationship between q and R. A simple way to introduce it is assuming that returns follow an exponential shape:

$$R = \left(\frac{q}{\tilde{q}}\right)^k - 1\,, \quad k > 0, \tag{30}$$

which yields a loglinear price dynamics:

$$\ln P - \ln \tilde{P} = k \cdot (\ln q - \ln \tilde{q}). \tag{31}$$

Risk attitude.

We constraint the agents to be either risk neutral or risk-averse, two common features of portfolio selection models.

Case A. Risk neutral agents; f is linear (e.g. the identity function). In this case agents' utility function becomes:

$$U_i(\boldsymbol{\omega}) = \omega_i\,[R + \mu(2\tilde{\omega}_i - 1) + \epsilon_i] - \tilde{\omega}_i \mu\,. \tag{32}$$

Case B. Risk averse agents; let f be a CARA utility function ($f(x) = 1 - e^{-Ax}$). In this case we have:

$$U_i(\boldsymbol{\omega}) = \omega_i \left[(1 - \tilde{\omega}_i)(1 - e^{-A\,(R-\mu)}) + \tilde{\omega}_i(1 - e^{-A\,R}) - \tilde{\omega}_i(1 - e^{A\mu}) + \epsilon_i\right] + \tilde{\omega}_i(1 - e^{A\mu})\,, \tag{33}$$

where $A > 0$ is the Arrow Pratt coefficient of absolute risk aversion.

Distribution of errors.

We only consider *logistically distributed* error terms:

$$\eta(x) = \mathbb{P}(\epsilon \leq x) = \frac{1}{1 + e^{-\beta x}}\,, \quad \beta > 0. \tag{34}$$

The choice of logistic error terms is a rather common choice in many models of evolving social systems[11]. Note that the logistic distribution is absolutely continuous and symmetric, as it is needed in Sections 3 and 4. The parameter β measures the impact of the random component in the decision process. When β is high, the deterministic part plays a major role in the maximization process; when β is close to zero, the error term dominates and the choice between $\omega = 1$ and $\omega = -1$ approximates a coin tossing.

The general results on existence and (local) stability of fixed point and 2-cycle of Theorems 4.3 and 4.4 can be rewritten in terms of the parameters of the model, depending on the cases described above, as follows.

Theorem 5.1.

Case 1A (linear price evolution; risk neutral agents)

There exists a critical value $k^c(\beta,\mu)$ *for* k*, where* $k^c(\beta,\mu) := \frac{1}{\beta}\left(1+e^{\beta\mu}\right)$*, such that:*

- *for* $k < k^c(\beta,\mu)$*,* $q^* = \frac{1}{2}$ *is locally stable; moreover, there are no locally stable 2-cycles;*
- *for* $k > k^c(\beta,\mu)$ *and* $k \neq \frac{1}{\beta}\left(2+e^{\beta\mu}+e^{-\beta\mu}\right)$*,* $q^* = \frac{1}{2}$ *is unstable; moreover, there exists a unique 2-cycle* $(\tilde{q}, q)$*, which is locally stable.*

Case 2A (loglinear price evolution; risk neutral agents)

There exist two critical values $k_l^c(\beta,\mu) \leq k_u^c(\beta,\mu)$ *for* k*, where* $k_u^c(\beta,\mu) := \frac{1}{2\beta}\left(1+e^{\beta\mu}\right)$*, such that:*

- *for* $k < k_l^c(\beta,\mu)$*,* $q^* = \frac{1}{2}$ *is locally stable; moreover, there are no locally stable 2-cycles;*
- *there exists a non-zero measure set* $\mathcal{H} \in (\, k_l^c(\beta,\mu);\; k_u^c(\beta,\mu)\,)$ *such that, for* $h \in \mathcal{H}$*, the locally stable fixed point* $q^* = \frac{1}{2}$ *and a locally stable 2-cycle coexist;*
- *for* $k > k_u^c(\beta,\mu)$ *and* $k \neq \frac{1}{2\beta}\left(2+e^{\beta\mu}+e^{-\beta\mu}\right)$*,* $q^* = \frac{1}{2}$ *is unstable; moreover, there exists a unique locally stable 2-cycle.*

Case 1B (linear price evolution; risk averse agents)

There exists a critical value $k^c(\beta,\mu,A)$ *for* k*, where* $k^c(\beta,\mu,A) := \frac{2}{A\beta(e^{A\mu}+1)}\left(1+\exp\{\beta(e^{A\mu}-1)\}\right)$*, such that:*

[11] See, for instance, Brock and Durlauf (2001).

- *for* $k < k^c(\beta,\mu,A)$*,* $q^* = \frac{1}{2}$ *is locally stable; moreover, there are no locally stable 2-cycles;*

- *for* $k > k^c(\beta,\mu,A)$ *and* $k \neq \frac{2}{A\beta(e^{A\mu}+1)}\left(\exp\{\beta(e^{A\mu}-1)\} + \exp\{-\beta(e^{A\mu}-1)\} + 2\right)$*,* $q^* = \frac{1}{2}$ *is unstable; moreover, there exists a unique* 2*-cycle* $(\tilde{q}, q)$*, which is locally stable.*

Case 2B (loglinear price evolution; risk averse agents)

There exist two critical values $k_l^c(\beta,\mu,A) \leq k_u^c(\beta,\mu,A)$ *for* k*,*
where $k_u^c(\beta,\mu,A) := \frac{1}{A\beta(e^{A\mu}+1)}\left(1 + \exp\{\beta(e^{A\mu}-1)\}\right)$*, such that:*

- *for* $k < k_l^c(\beta,\mu,A)$*,* $q^* = \frac{1}{2}$ *is locally stable; moreover, there are no locally stable 2-cycles;*

- *there exists a non-zero measure set* $\mathcal{H} \in (\, k_l^c(\beta,\mu,A);\ k_u^c(\beta,\mu,A)\,)$ *such that, for* $h \in \mathcal{H}$*, the locally stable fixed point* $q^* = \frac{1}{2}$ *and a locally stable 2-cycle coexist;*

- *for* $k > k_u^c(\beta,\mu,A)$ *and* $k \neq \frac{1}{A\beta(e^{A\mu}+1)}\left(\exp\{\beta(e^{A\mu}-1)\} + \exp\{-\beta(e^{A\mu}-1)\} + 2\right)$*,* $q^* = \frac{1}{2}$ *is unstable; moreover, there exists a unique locally stable* 2*-cycle.*

The lower thresholds $k_l^c(\beta,\mu)$ *and* $k_l^c(\beta,\mu,A)$ *in cases 2A and 2B, cannot be determined explicitly.*

Remark 5.2. *We conjecture that the statement of this theorem can be strengthened. In particular, we believe that* $\mathcal{H} = (\, k_l^c;\ k_u^c)$*, meaning that coexistence happens for all values of* k *in the interval. Up to now, we are only able to provide specific examples, where coexistence certainly holds.*

Proof.

We only concentrate on cases 1A and 2A. The corresponding risk bearing situations (1B and 2B) can be treated in a similar way.

Case 1A. The value $k^c(\beta,\mu) = \frac{1}{\beta}\left(1 + e^{\beta\mu}\right)$ is determined by computing ρ, as defined in equation (19). To this aim, note that for $g(x,y) = k(y-x)$ we have $i(r) = -r$ (where $i(\cdot)$ has been defined in Assumption 4.1). Concerning the fixed point $\left(\frac{1}{2},\frac{1}{2}\right)$, after some manipulations, one finds that

$$\rho_{|\left(\frac{1}{2},\frac{1}{2}\right)} = \left(\frac{1 - 2\eta(-\mu) - k\eta'(\mu)}{1 - k\eta'(\mu)}\right)^2,$$

where $\rho_{|\left(\frac{1}{2},\frac{1}{2}\right)}$ denotes the value of ρ computed in $(\tilde{q}_*, q_*) = \left(\frac{1}{2},\frac{1}{2}\right)$. The stability condition follows solving $\rho\left(\frac{1}{2},\frac{1}{2}\right) < 1$, and it is easy to show it holds true if and only if $k < \frac{\eta(\mu)}{\eta'(\mu)}$. Substituting the logit expression for the distribution η, it yields

$$k < \frac{1}{\beta}\left(1 + e^{-\beta\mu}\right).$$

Concerning the 2-cycle, on the other hand, let firstly assume $\mu = 0$. A 2-cycle corresponds to the non zero fixed point of the map ϕ, which, in case 1A, gives

$$\phi(R) = k\frac{B(R) - B(-R)}{1 + A(R)} = 2k\eta(R) - 1. \tag{35}$$

It is easy to see that ϕ is an increasing concave map, clearly $\phi(0) = 0$. Then a 2-cycle exists if and only if the equation $\phi(R) = R$ has a positive solution if and only if $\phi'(0) > 1$. This, in turns, implies that the graph of ϕ in $R^* \neq 0$ (corresponding to the 2-cycle) intersects the identity with slope less than 1, giving also the stability condition (see Corollary 4.5). Now, $\phi'(0) > 1$, in this case, reads $2k\eta'(0) > 1$, i.e., $k > \frac{2}{\beta}$.

If $\mu \neq 0$, the map ϕ can be written as

$$\phi(R) = k\frac{\eta(R+\mu) + \eta(R-\mu) - 1}{\eta(R+\mu) - \eta(R-\mu) + 1},$$

which is an increasing function, since

$$\phi'(R) = 2k\frac{\eta'(R+\mu)(1-\eta(R-\mu)) + \eta(R+\mu)\eta'(R-\mu)}{[\eta(R+\mu) - \eta(R-\mu) + 1]^2} > 0.$$

Moreover, it is concave, since

$$\begin{aligned}\phi''(R) =& \frac{2k}{[\eta(R+\mu) - \eta(R-\mu) + 1]^3} \cdot \\ & \cdot [\eta''(R+\mu)(1-\eta(R-\mu)) + \eta(R+\mu)\eta''(R-\mu)][\eta(R+\mu) - \eta(R-\mu) + 1] \\ & - 2[\eta'(R+\mu)(1-\eta(R-\mu)) + \eta(R+\mu)\eta'(R-\mu)][\eta'(R+\mu) - \eta'(R-\mu)] < 0.\end{aligned}$$

Then, as before, the 2-cycle exists if and only if $\phi'(0) > 1$, which, in this case, gives $k\frac{\eta(\mu)}{\eta'(\mu)} > 1$, i.e., with logit distribution, $k > \frac{1}{\beta}\left(1 + e^{-\beta\mu}\right)$. Then the 2-cycle exists if and only if the fixed point is unstable.

Concerning the condition $k \neq \frac{1}{\beta}\left(2 + e^{\beta\mu} + e^{-\beta\mu}\right)$, it follows from condition *1.* of Theorem 4.4, just computing the l.h.s. of equations (17) and (18) (the two conditions coincide in the case of linear returns).

Case 2A. The value $k_u^c(\beta, \mu) = \frac{1}{2\beta}\left(1 + e^{\beta\mu}\right)$ follows arguing similarly as before; in this case, since $g(x, y) = \left(\frac{y}{x}\right)^k - 1$, we have $i(r) = \frac{-r}{1+r}$. Thus in this case, it is not difficult to see that $\rho_{|\left(\frac{1}{2},\frac{1}{2}\right)} = \left(\frac{1-2\eta(-\mu)-2k\eta'(\mu)}{1-2k\eta'(\mu)}\right)^2$. Hence $\rho\left(\frac{1}{2}, \frac{1}{2}\right) < 1$ if and only if $k < \frac{1}{2\beta}\left(1 + e^{-\beta\mu}\right)$. This is enough to discuss the (local) stability of the fixed point. Concerning the condition $k \neq \frac{1}{2\beta}\left(2 + e^{\beta\mu} + e^{-\beta\mu}\right)$, it follows from condition *1.* of Theorem 4.4 arguing as before.

As far as the 2-cycle is concerned, we are not able to explicitly find the threshold $k_l^c(\beta,\mu)$ for k above, from which a stable 2-cycle starts to exist. Nevertheless, at least for $\mu = 0$, we can rely on Corollary 4.5 in order to provide a closed form expression for the threshold level for k. Indeed, after some algebra, one finds that

$$\phi'(R) = k\beta \cdot \left(\frac{1 + e^{\frac{\beta R}{1+R}}}{1 + e^{-\beta R}} \right)^k \cdot \left(\frac{\frac{1}{(1+R)^2} \cdot e^{\frac{\beta R}{1+R}}}{1 + e^{\frac{\beta R}{1+R}}} + \frac{e^{\beta R}}{1 + e^{-\beta R}} \right). \tag{36}$$

Recall that the threshold for k is determined by $\phi'(R) = 1$. In order to discuss existence and stability of the 2-cycle, we start from the case $\mu = 0$ and we rely on numerical tools. Indeed, plotting the function $\phi(R)$ for different values of the parameters, one sees that, for $k > 0$ but small, the function $\phi(R)$ is such that $\phi'(0) < 1$ and $\phi(R) < R$ for all $R > 0$. Thus, there are no (locally) stable 2-cycles (see left panel of Figure 1 for an example). For k large enough, in particular, for $k > k_u^c(\beta, 0)$, a locally stable 2-cycle exists, since $\phi'(0) > 0$ and $\phi(R) < R$ for R large enough. The last fact is obvious, because, for $R \to \infty$, $\phi(R)$ has finite value $(1 + e^\beta)^k - 1$. Thus ϕ must cross the bisector in some positive R^*, where $\phi(R^*) < 1$.
The last argument to be discussed is coexistence of attractors. We show, in particular, that there are values of β, k and h such that $\phi'(0) < 1$ and $\phi(R^*) < 1$ for $R^* \neq 0$. To this aim, it is sufficient to show that, for specific values of the parameters, $\phi'(0) < 1$, $\phi(R') > R$ and $\phi(R'') < R$ for $R'' > R' > 0$. Let us take $k = 3$, $\beta = 0.3$ and $\mu = 0$. We have $\phi'(0) = 0.9 < 1$. Moreover, $\phi(4) > 4$ and $\phi(10) < 10$. Thus, necessarily, $k = 3$ belongs to the set $\mathcal{H} \in (\,k_l^c(0.3, 0);\; k_u^c(0.3, 0)\,)$. We plot $\phi(R)$ with $k = 3$, $\beta = 0.3$ and $\mu = 0$ in the right panel of Figure 1. In this case $R^* = 9.35$ is the point marked with $(*)$. Therefore, there can be situations in which $k_l^c(\beta,\mu) < k_u^c(\beta,\mu)$ and coexistence takes place. Note, finally, that, for certain values of the parameters, we can also have $k_l^c(\beta,\mu) = k_u^c(\beta,\mu)$.
All these arguments can be extended to the case $\mu > 0$ by means of numerical tools.

■

We discuss the main implications of this corollary in more details. In particular, we concentrate on the coexistence of fixed point and 2-cycle.
Look, firstly, at the case of linear evolution of prices: $g(x, y) = k(y - x)$ (cases 1A and 1B). To ease the notations, we put $\mu = 0$. Under these assumptions, it is not difficult to show that $\phi(x) = g(1 - \eta(x), \eta(x)) = k[2\eta(x) - 1]$, which is concave, since $\phi'(x) = 2k\eta'(x)$ is decreasing

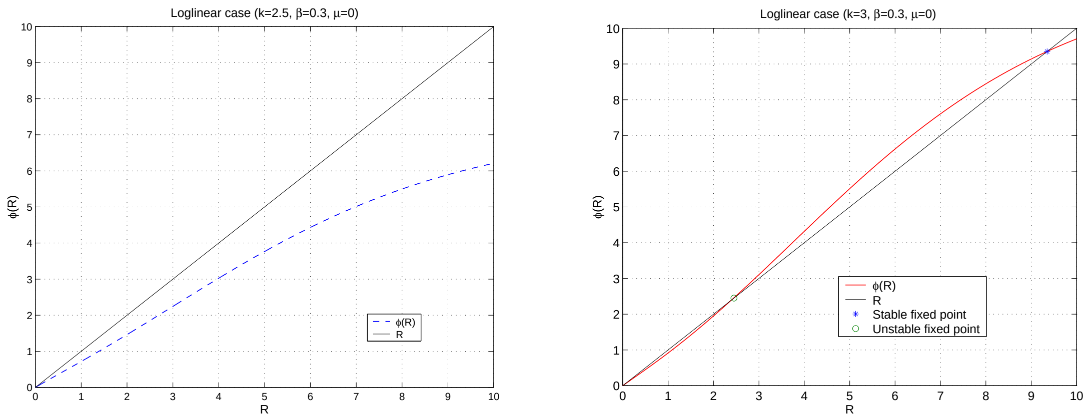

Figure 1: Two plots of the function $\phi(R)$ under the case 2A. In the left panel the only fixed point is $R = 0$. In the right panel we see a strictly positive fixed point of $\phi(R)$, where $\phi'(R) < 1$ (marked with $(*)$).

for $x \geq 0$. The equation $x = \phi(x)$ admits positive solution if and only if $\phi'(0) > 1$ if and only if $k\eta'(0) > \frac{1}{2}$. In this case the 2-cycle is stable, since the graph of ϕ intersects the identity with slope less than 1. As a result, there can be no coexistence of stable fixed point and stable 2-cycle.

On the other hand, under loglinear evolution of prices (i.e. in cases 2A and 2B), coexistence is possible as shown in the right panel of Figure 1. Note that, looking at that figure, we also have an unstable fixed point (denoted by (o) in the graph).

Coexistence has important consequences at the level of the finite dimensional system. We perform some agent-based simulations in order to capture this aspect. More in details, we simulate a large but finite population of N agents. At any time step, we let the N agents play (sequentially) their best response to the (fixed) actions of the other agents. We continue letting the algorithm work unless a fixed profile $\boldsymbol{\omega}$ is reached. In doing this, we are numerically identifying a Nash equilibrium as a strategy profile $\boldsymbol{\omega}$, that is a fixed point of the *best response map*. In the case of multiple Nash equilibria, the algorithm always identifies the nearest to the old equilibrium found in the previous time step (where the nearest means the one where less agents have switched their choice).

In the case of coexistence, it can be shown that the finite dimensional system oscillates between the two different attractors: it remains for rather long time windows near to the fixed point and for rather long times near to the limiting 2-cycle. See Figure 2, where we use the same parameters as in the right panel of Figure 1. This situation, rephrased in terms of returns, mimics a time series, where we have periods of relatively stable returns and periods with very volatile returns. Therefore, we can say that in our simple model we are able to endogenously generate market volatility regime switching.

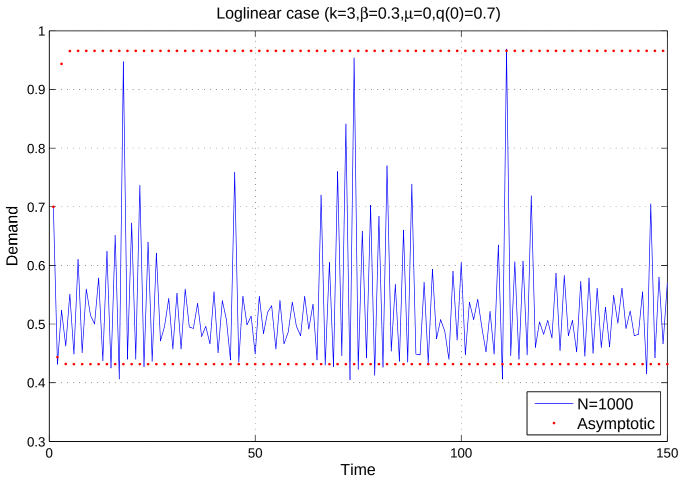

Figure 2: Asymptotic regime (2-cycle) for demand (dotted line) and finite dimensional simulation with $N = 1000$ agents (continuous line).

5.2 Comparative statics

In this section we analyze how the picture of the stationary regime changes depending on the parameters of the model. In fact, depending on the parameters k, β, μ, we discuss how vary the domains of attractions towards the fixed point and the 2-cycle and the size of Δq (and consequently the jump of R), in the case of the limiting stable 2-cycle. For the sake of simplicity we concentrate on the comparative statics only for the risk neutral cases 1A and 2A. In what follows, we consider, without loss of generality, the case when $\tilde{q} < q$, i.e., when $\Delta q > 0$.

Domains of attractions

Concerning the linear case 1A, we analyze how the threshold $k^c(\beta, \mu)$, which separates the regions of local stability of the two attractors (fixed point and 2-cycle), varies as a function of the transaction cost μ, first, and of the error parameter β, then. Recall that $k^c(\beta, \mu) := \frac{1}{\beta}\left(1 + e^{\beta\mu}\right)$. It is clear that k^c is an increasing function of μ, $k^c \geq \frac{2}{\beta}$. Then, when μ increases, the region of local stability of the fixed point enlarges, meaning that the transaction cost stabilizes the market. Concerning the dependence of k^c on β, we prefer to distinguish the case $\mu = 0$ from the case $\mu \neq 0$.

If $\mu = 0$, $k^c = \frac{2}{\beta}$, which is a decreasing function of β. What happens is that, when $\beta \to 0^+$, i.e., the error dominates, only the region of fixed point survives. This is due to the fact that, when $\beta \to 0^+$, the agents are completely randomizing their choice. Hence, in the asymptotic market with infinite agents, half of them chooses $\omega = 1$ and half $\omega = 0$. The demand is, therefore, always equal to $\frac{1}{2}$. Note moreover that since also the vector of past actions $\tilde{\boldsymbol{\omega}}$ is completely randomized, there will be half of the people owning the asset that are willing to sell it and viceversa. This is the reason why, at the equilibrium with $\mu = 0$, the volume of trade is $S = \frac{1}{2}$ (see Proposition 4.6).

Conversely, when $\beta \to +\infty$, i.e., the error vanishes, it seems that only the region of the 2-cycle survives; but, when $\mu = 0$ the amplitude of the 2-cycle is given, as we prove later, by the implicit equation

$$\begin{aligned} q - \tilde{q} =: \Delta q &= 2\eta(k\Delta q) - 1 \\ &= \frac{|e^{\beta k \Delta q} - 1|}{e^{\beta k \Delta q} + 1}, \end{aligned} \tag{37}$$

which gives solutions $\Delta q = 0$ or $\Delta q > 0$. $\Delta q = 0$ is a trivial 2-cycle, that collapses in the fixed point $q = \frac{1}{2}$; instead $\Delta q > 0$ is such that, when $\beta \to +\infty$, $\Delta q \to 1$. The two extreme degenerate situations are explained by the fact that with zero costs and perfectly rational agents, either nobody trades or everybody trades.

If $\mu \neq 0$, when $\beta \to 0^+$ and when $\beta \to +\infty$, $k^c \to +\infty$; moreover, there exists a value $\bar{\beta} \in (1.27\mu, 1.28\mu)$ such that, for $\beta \leq \bar{\beta}$, k^c decreases , while, for $\beta > \bar{\beta}$, k^c increases. When the error dominates or it vanishes, only the region of fixed point survives. The existence of a positive taxation excludes the previous situation in which everyone is trading the share, again confirming the stability property of transaction costs. However, in intermediate situations there is a sort of balancing effect between β and μ. In particular, there are values of k for which the market is highly volatile and values for which the market converges towards the fixed point. The value $\bar{\beta}$ denotes the value of β where k^c reaches its minimum, i.e., the domain of attraction of the 2-cycle is larger.

Concerning the loglinear case 2A, we have to discuss the values of $k^c_l(\beta, \mu)$ and $k^c_u(\beta, \mu)$.

The shape of $k_u^c(\beta,\mu)$ is very close to the $k^c(\beta,\mu)$ discussed in the previous case. As already mentioned in Theorem 5.1, existence of the two attractors is possible under the loglinear scenario. Nevertheless, it appears to be quite hard to make comparative statics about $k_l^c(\beta,\mu)$. Looking at numerical simulations, it seems that the region of coexistence is large for very small values of β, but it shrinks very fast increasing β. To show it, look at Table 2, where we collect some values of $k_l^c(\beta,\mu)$ and $k_u^c(\beta,\mu)$ varying β, for $\mu = 0$.

β	$k_l^c(\beta,0)$	$k_u^c(\beta,0)$
0.02	7.66	50
0.1	4.91	10
0.3	2.89	3.33
0.5	1.96	2
0.7	1.42	1.42

Table 2: How $k_l^c(\beta,\mu)$ and $k_u^c(\beta,\mu)$ vary with β. In this simulation it is $\mu = 0$.

Size of Δq

Case 1A. Before analyzing the variation in demand and in R in the region of the 2-cycle, we have to find a relation, which describes the amplitude $|\Delta q|$ of the 2-cycle. We can derive an explicit solution for the equation linking Δq with the parameters of the model only in case 1A. In particular, to do it, let rewrite in this case (15) and (16):

$$q = \tilde{q}\eta(k\Delta q + \mu) + (1-\tilde{q})\eta(k\Delta q - \mu) \tag{38}$$

and

$$\tilde{q} = q\eta(-k\Delta q + \mu) + (1-q)\eta(-k\Delta q - \mu)\,. \tag{39}$$

Now, summing and subtracting (38) and (39), it yields

$$q + \tilde{q} = 1 \tag{40}$$

and

$$\begin{aligned} q - \tilde{q} = \Delta q &= \frac{\eta(k\Delta q + \mu) + \eta(k\Delta q - \mu) - 1}{\eta(k\Delta q + \mu) - \eta(k\Delta q - \mu) + 1} \\ &= \frac{e^{2\beta k\Delta q} - 1}{e^{2\beta k\Delta q} + 2e^{\beta(k\Delta q+\mu)} + 1}\,, \end{aligned} \tag{41}$$

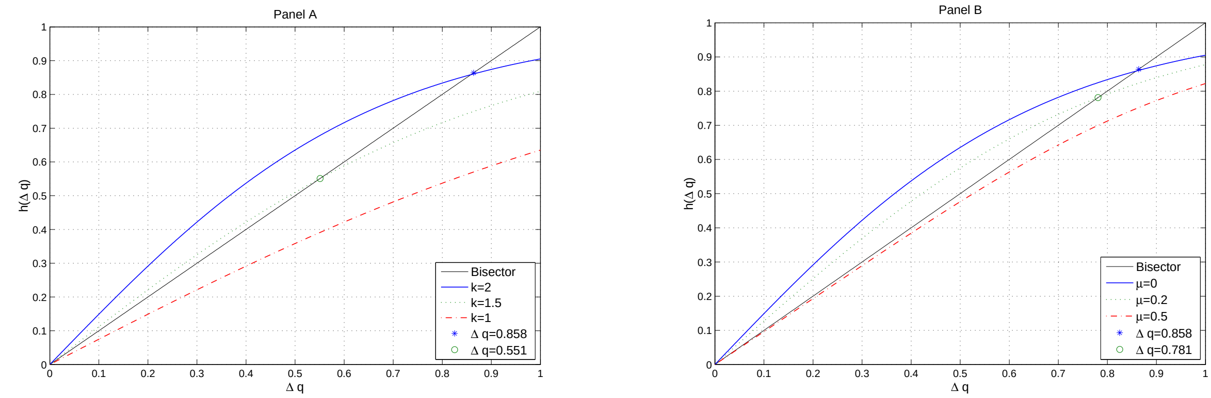

Figure 3: Different solutions of equation (41) varying k (Panel A) and μ (Panel B), where $h(\Delta q)$ denotes the r.h.s. of (41). Note that Δq increases with k and it decreases with μ.

where the last equality follows, after some manipulations, when we substitute the expression of the logit distribution to η.

The fixed points of equation (41) characterize the attractors of our model. Note that $\Delta q = 0$ is always solution for (41) and it corresponds to the fixed point $q = \frac{1}{2}$. Moreover, since the r.h.s. of equation (41) is increasing and concave in Δq, there might be situations in which another solution $\bar{\Delta q} > 0$ exists. It shows the size of the limiting 2-cycle. Figure 3 reports the shape of equation (41), highlighting its fixed points. Note that, for low values of k and/or β, the solution $\Delta q = 0$ is unique. If we increase k and/or β, we can see the positive solution $\bar{\Delta q}$, whose value increases with k and/or β. Eventually, for k and/or β approaching infinity, $\bar{\Delta q}$ approaches 1. In particular, in Panel A of Figure 3 we plot the r.h.s. of (41) finding the (eventual) non zero solution of (41) under increasing values of k. In Panel B we perform the same exercise varying μ. It is interestingly enough Panel B of Figure 3 confirms that transaction costs regularize the market in the sense that the jumps of demand (hence of returns) boil down with an increasing taxation on trades. In order to explicit how the variation in demand at the equilibrium depends on the parameters, we perform numerical simulations for both Cases 1A and 2A described in Figure 4. In Panels A and B we plot the size of the 2-cycle in case 1A of linear returns. In particular, in Panel A we keep k fixed and we let β change in the interval $(0, 2]$. Moreover, we choose three different levels for μ: 0, 0.06 and 0.1. In Panel B we keep β fixed and we let k change in the interval $(0, 2]$ (same levels for μ as in Panel A). In Panels C and D we perform the same analysis in case 2A. The graphs confirm the fact that the size of the 2-cycle decreases with

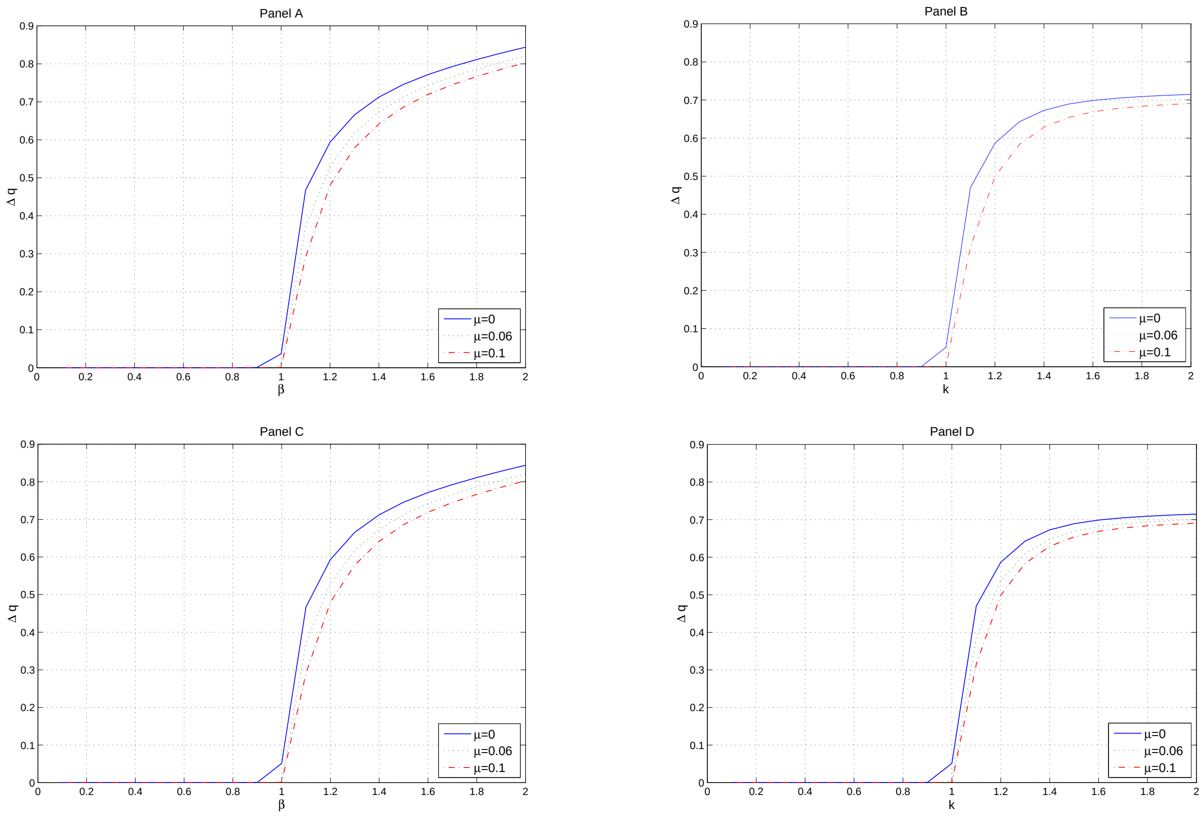

Figure 4: Different levels of $\bar{\Delta q}$ (i.e., the size of the 2-cycle at the equilibrium), varying the parameters of the model in the linear case (Panel A and B) and loglinear case (Panel C and D). Note that in Panels A and C, we keep $k = 1$ fixed and in Panels B and D we keep $\beta = 1$ fixed.

μ and increases with β and k. Note that, for small values of the parameters (β and k), the size of the 2-cycle is zero, meaning that the dynamics for the demand are converging towards the fixed point $q = \frac{1}{2}$. A second important remark is that the size of the 2-cycle does not depend on the model we choose. Indeed, Panels A and C and Panels B and D are exactly equal. This is an interesting result. The size of the 2-cycle does not depend on the specific model adopted, even though the domains of attraction do (as shown in Theorem 5.1).

6 Concluding remarks.

We have built a simple model for a liquid market of a risky asset, where a large number of similar interacting agents can trade a share of the asset. We have also taken transaction costs into account.

We have studied the dynamics of the demand function and the related returns emerging by the trading mechanism. Compared to related literature, we have suggested a framework in which a rather general form of dependence between aggregate demand for the asset and the asset returns is in place. In particular, the dynamics of these two fundamental state variables cannot be separately studied. The joint evolution of these quantities gives rise to interesting stylized facts such as multiplicity of equilibria, non linear dynamics of returns, boom and crash cycles, all facts that have been discussed in details.

Agents interactions have been modeled through a static (one period) non-cooperative game. Agents face social interactions and bounded rationality, the latter introduced following a well known strand of research: random utility models inspired by Brock and Durlauf (2001). From this point of view, we have enriched this framework, letting the agents update their opinion in a *parallel way*, i.e., their action is the consequence of a game whose payoffs depend on the expectations on the behavior of the population.

We have detected the mitigating effects of transaction costs that may help in regularizing the market. Moreover, we have observed that, for some values of the parameters, booms and crashes can arise as the result of agents' interplay and of market features.

Finally, notice that, owing to the assumption of agents' simultaneous updating, at the equilibrium the limiting dynamics converge either to a fixed point or to a 2-cycle. This is a novelty in probabilistic models that describe social interactions and, in case, contagion; see, for instance, Blume and Durlauf (2003) or Dai Pra et al. (2009) in which the stable attractors can be only fixed points, and where agents update their actions sequentially.

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