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A model for communication in a multi-segment market

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Abstract. *We consider a linear optimal control model for the marketing of seasonal products which are produced by the same firm and sold by retailers in different market segments The horizon is divided in two consecutive non-intersecting intervals, called production and selling periods, respectively. The production period state variables are the inventory levels and two kinds of goodwills (consumers' and retailers' goodwill, respectively) while the selling period state variables are the sales levels and the two kinds of goodwills. In the production interval there are three kinds of controls: on production, quality and advertising, while in the selling one the controls are on communication via advertising, promotion addressed to consumers and incentives given to retailers. We consider the case of several kinds of communications. The optimal control problem is transformed into an equivalent nonlinear programming problem.*

Keywords. *Optimal control, advertising, seasonal products, segmentation.*

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1. Introduction

In this paper we consider a linear optimal control model for the marketing of seasonal products which are produced by the same firm and sold by retailers in different market segments. More precisely, the firm produces and sells a seasonal product with different attributes in two consecutive time intervals, the first one devoted to production and the second one to the sale of the product. In the production period the firm can control production, product quality and advertise the product both to retailers and to the different market segments. In the selling period the firm has a wider communication activity via promotion to consumers, incentives to retailers and advertising to both. We assume that the communication efforts of the firm act on the goodwill that consumers and retailers assign to the product. The motion equations of the model refer explicitly to the consumers' and retailers' goodwills as state variables. The firm seeks the maximum profit but has also to keep a minimum level of goodwills at the end of the selling period. This way we generalize the linear models proposed in [1]-[3] (see also [4]-[6]).

2. A multi-segment linear model

We consider a model with n market segments, r retailers and d kinds of communications. As in [2], we assume that production and sales take place in two distinct intervals $[t_0,t_1]$ (production period) and $[t_1,t_2]$ selling period). We will use the index sets

$$I=\{1,\dots,n\},\ J=\{1,\dots r\},\ K=\{1,\dots d\}.$$

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2.1. State variables and controls

Through the production period we consider n state variables

$m_i(t)=$ inventory level in the i-th segment, $i\in I$, at time $t\in[t_0,t_1]$,

and $n+1$ controls

$u_i(t)=$ production expenditure rate in the i-th segment, $i\in I$, at time $t\in[t_0,t_1]$,

$q=$ expenditure rate for improving quality; as in the one-segment linear model [2], q will be considered constant and nonnegative.

Through the selling period the state variables are

$x_i(t)=$ sales level in the i-th segment, $i\in I$, at time $t\in[t_1,t_2]$.

Moreover, through both production and selling periods we consider the state variables

$C_i(t)=$ consumer goodwill in the i-th segment, $i\in I$, at time $t\in[t_0,t_2]$,

$R_j(t)=$ goodwill of the j-th retailer, $j\in J$, at time $t\in[t_0,t_2]$,

and d controls

$a_k(t)=$ expenditure rate in the k-th type of communication, $k\in K$, at time $t\in[t_0,t_2]$.

The consumer goodwills $C_i(t)$ explain how easily the consumers in the i-th segment choose to buy the product, while the goodwills $R_j(t)$ take into account how favorably the j-th retailer keeps the product in stock (see [2]). For what concerns controls $a_k(t)$, we consider that during the production period the communication coincides with advertising, while in the selling period advertising, promotion and incentives are all available.

2.2. Dynamics

The dynamics of the model is rather different in the production period $[t_0, t_1]$ and in the selling period $[t_1, t_2]$.

The dynamics in the production period $[t_0, t_1]$ is described by $2\mathrm{n}+\mathrm{r}$ motion equations: n equations for the inventory levels $m_i(t)$, n equations describing the goodwills $\mathrm{C}_\mathrm{i}(\mathrm{t})$ of the consumers' segments and r equations for the goodwills $\mathrm{R}_\mathrm{j}(\mathrm{t})$ of the retailers.

For the i-th segment, $i \in I$, the inventory level dynamics is the following

$$\dot{m}_i(t) = \mu_i u_i(t),$$

where

$\mu_i =$ marginal productivity of production expenditure, $\mu_i > 1$.

Further, in the i-th segment, $i \in I$, the goodwill level dynamics is given by the motion equations

$$\dot{C}_i(t) = -\delta_{C_i} C_i(t) + \sum_{k\in K} \varepsilon^{(p)}_{C_i k} a_k(t),$$

where

$\delta_{C_i} =$ decay rate of goodwill C_i, $\delta_{C_i} > 0$,

$\varepsilon^{(p)}_{C_i k} =$ marginal productivity of expenditure in communication a_k, via advertising, in terms of goodwill C_i in the production period, $\varepsilon^{(p)}_{C_i k} \geq 0$.

Remark that $\varepsilon^{(p)}_{C_i k} > 0$ means that, in the production period, communication expenditure a_k for advertising increases the goodwill C_i.

Finally, for the j-th retailer, $j \in J$, the goodwill level dynamics is given by the motion equations

$$\dot{R}_j(t) = -\delta_{R_j} R_j(t) + \sum_{k\in K} \varepsilon^{(p)}_{R_j k} a_k(t),$$

where

$\delta_{R_j} =$ decay rate of goodwill R_j, $\delta_{R_j} > 0$,

$\varepsilon^{(p)}_{R_j k} =$ marginal productivity of expenditure in communication a_k via advertising, in terms of goodwill R_j in the production period, $\varepsilon^{(p)}_{R_j k} \geq 0$.

The dynamics in the selling period $[t_1, t_2]$ is also described by $2\mathrm{n}+\mathrm{r}$ motion equations: n equations for sales levels $\mathrm{x}_\mathrm{i}(\mathrm{t})$ in i-th segment and, as in the production period, n equations for goodwill C_i in i-th segment and r equations for goodwill R_j of j-th retailer.

For the i-th segment, $i \in I$, the inventory level dynamics is the following

$$\dot{x}_i(t) = -\alpha_i x_i(t) + \gamma_{C_i} C_i(t) + \sum_{j\in J} \gamma_{x_i R_j} R_j(t) + \\ + \sum_{k\in K} \varepsilon^{(s)}_{x_i k} a_k(t) + l_{x_i} q_1,$$

where

$\alpha_i =$ saturation aversion parameter, $\alpha_i > 0$,

$\gamma_{C_i} =$ goodwill productivity in terms of sales originated by goodwill C_i, $\gamma_{C_i} > 0$,

$\gamma_{x_i R_j} =$ goodwill productivity in terms of sales in i-th segment originated by the j-th retailer goodwill R_j, $\gamma_{x_i R_j} \geq 0$,

$\varepsilon^{(s)}_{x_i k} =$ marginal productivity of expenditure in communication a_k via promotion in terms of sales level x_i during the selling period, $\varepsilon^{(s)}_{x_i k} \geq 0$,

$l_{x_i} =$ marginal sale productivity of the expenditure rate q for improving quality, $l_{x_i} \geq 0$,

$q_1 = q(t_1 - t_0) =$ the total expenditure for improving quality in the production period.

Further, for i-th segment, $i \in I$, the goodwill level dynamics is given by the motion equation

$$\dot{C}_i(t) = \beta_i x_i(t) - \delta_{C_i} C_i(t) + \sum_{k\in K} \varepsilon^{(s)}_{C_i k} a_k(t) + l_{C_i} q_1,$$

where

$\beta_i =$ word-of-mouth in terms of goodwill C_i, $\beta_i \geq 0$,

$\delta_{C_i} =$ decay rate of goodwill C_i, $\delta_{C_i} > 0$,

$\varepsilon^{(s)}_{C_i k} =$ marginal productivity of expenditure in communication a_k via advertising in terms of goodwill C_i in the sales period, $\varepsilon^{(s)}_{C_i k} \geq 0$,

$l_{C_i} =$ marginal productivity of quality expenditure rate q for goodwill C_i, $l_{C_i} \geq 0$.

Finally, for the j-th retailer, $j \in J$, the goodwill level dynamics is given by the motion equation

$$\dot{R}_j(t) = -\delta_{R_j} R_j(t) + \sum_{k\in K} \varepsilon^{(s)}_{R_j k} a_k(t) + l_{R_j} q_1,$$

where

$\delta_{R_j} =$ decay rate of goodwill R_j, $\delta_{R_j} > 0$,

$\varepsilon^{(s)}_{R_j k} =$ marginal productivity of expenditure in communication a_k, via advertising and incentives, in terms of goodwill R_j in the sales period, $\varepsilon^{(s)}_{R_j k} \geq 0$,

$l_{R_j} =$ marginal productivity of q for goodwill R_j, $l_{R_j} \geq 0$.

2.3. Objective function

The total profit of the firm is determined as the difference between the total income and the total expenditures (for production, product quality and communication) and is given by the following objective function:

$$\sum_{i\in I} p_i x_i(t_2) - \int_{t_0}^{t_1} \left(\sum_{i\in I} c_i m_i(t) + \sum_{i\in I} u_i(t) + q \right) dt -$$

$$- \int_{t_0}^{t_2} \sum_{k\in K} a_k(t) dt,$$

where

$p_i =$ product sale price in i-th segment, $p_i > 0$,

$c_i =$ inventory marginal cost per time unit in i-th segment, $c_i > 0$.

2.4. Boundary conditions

The natural boundary conditions for the inventory and sales levels are

$$m_i(t_0) = x_i(t_1) = 0, \quad x_i(t_2) \leq m_i(t_1), \quad i \in I.$$

The boundary conditions for the consumer goodwills are

$$C_i(t_0) = C_i^0, \quad C_i(t_2) \geq C_i^2, \quad i \in I,$$

where

$C_i^0 =$ initial value of goodwill C_i,

$C_i^2 =$ lower bound of goodwill C_i at final time t_2.

The boundary conditions for the retailers' goodwills are

$$R_j(t_0) = R_j^0, \quad R_j(t_2) \geq R_j^2, \quad j \in J,$$

where

$R_j^0 =$ initial value of goodwill R_j,

$R_j^2 =$ lower bound of goodwill R_j at final time t_2.

The boundary conditions for the control variables are

$$u_i(t) \in [0, \bar{u}_i], \quad i \in I, \quad t \in [t_0, t_1],$$

$$a_k(t) \in [0, \bar{a}_k], \quad k \in K, \quad t \in [t_0, t_2],$$

$$q \in [0, \bar{q}].$$

where

$\bar{u}_i =$ upper bound for $u_i(t)$, $\bar{u}_i > 0$,

$\bar{a}_k =$ upper bound for $a_k(t)$, $\bar{a}_k > 0$,

$\bar{q} =$ upper bound for q, $\bar{q} > 0$.

2.5. Formulation of the model

In order to formulate the linear model we consider

n-dimensional state variables vectors $m(t), x(t), C(t)$ with elements $m_i(t), x_i(t), C_i(t), i \in I$, respectively;

r-dimensional state variables vector $R(t)$ with elements $R_j(t), j \in J$; n-dimensional control vector $u(t)$ with elements $u_i(t), i \in I$;

r-dimensional control vector $a(t)$ with elements $a_k(t), k \in K$;

n-dimensional constant vectors $p, c, C^0, l_x, l_C, C^2, \bar{u}$ with elements $p_i, c_i, C_i^0, l_{x_i}, l_{C_i}, C_i^2, \bar{u}_i, i \in I$, respectively;

r-dimensional constant vectors R^0, l_R, R^2 with elements $R_j^0, l_{R_j}, R_j^2, j \in J$, respectively;

d-dimensional constant vector $\bar{a}$ with elements $\bar{a}_k, k \in K$;

diagonal constant matrices $\mu, \delta_C, \alpha, \gamma_C, \beta$ of order n with diagonal elements $\mu_i, \delta_{C_i}, \alpha, \gamma_{C_i}, \beta_i$, $i \in I$, respectively;

diagonal constant matrix δ_R of order r with diagonal elements $\delta_{R_j}, j \in J$;

constant $n \times d$ matrices $\varepsilon_C^{(p)}, \varepsilon_x^{(s)}, \varepsilon_C^{(s)}$ with elements $\varepsilon_{C_ik}^{(p)}, \varepsilon_{x_ik}^{(s)}, \varepsilon_{C_ik}^{(s)}, i \in I, k \in K$, respectively;

constant $r \times d$ matrices $\varepsilon_R^{(p)}, \varepsilon_R^{(s)}$ with elements $\varepsilon_{R_jk}^{(p)}, \varepsilon_{R_jk}^{(s)}, j \in J, k \in K$, respectively;

constant $n \times r$ matrix γ_R with elements $\gamma_{x_iR_j}, i \in I, j \in J$.

Moreover, let us define

$$A(t) = \begin{pmatrix} C(t) \\ R(t) \end{pmatrix}, t \in [t_0, t_2], \ X(t) = \begin{pmatrix} x(t) \\ A(t) \end{pmatrix}, t \in [t_1, t_2],$$

$$A^0 = \begin{pmatrix} C^0 \\ R^0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} C^2 \\ R^2 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} \delta_C & 0_{nr} \\ 0_{rn} & \delta_R \end{pmatrix}, \quad E^{(p)} = \begin{pmatrix} \varepsilon_C^{(p)} \\ \varepsilon_R^{(p)} \end{pmatrix},$$

$$Q = \begin{pmatrix} \alpha & -\gamma_C & -\gamma_R \\ -\beta & \delta_C & 0_{nr} \\ 0_{rn} & 0_{rn} & \delta_R \end{pmatrix}, E^{(s)} = \begin{pmatrix} \varepsilon_x^{(s)} \\ \varepsilon_C^{(s)} \\ \varepsilon_R^{(s)} \end{pmatrix}, L = \begin{pmatrix} l_x \\ l_C \\ l_R \end{pmatrix},$$

where, as usual, 0_{nr} and 0_{rn} are $(n \times r)$ and, respectively, $(r \times n)$ zero matrices.

Moreover, let e_n and e_d be n-dimensional and, respectively, d-dimensional unit vectors while 0_n and 0_d be n-dimensional and, respectively, -dimensional zero vectors. Finally, let, as usual, "T" means transpose.

This way the linear marketing model with n segments, r retailers and d kinds of communications is

Problem P ***:*** *maximize*

$$p^T x(t_2) - \int_{t_0}^{t_1} \left(c^T m(t) + e_n^T u(t) + q\right) dt - \int_{t_0}^{t_2} e_d^T a(t) dt,$$

subject to

$$\dot{m}(t) = \mu u(t),\ \dot{A}(t) = -\Delta A(t) + E^{(p)} a(t), t \in [t_{0,} t_1],$$

$$m(t_0) = 0_n, \quad A(t_0) = A^0,$$

$$\dot{X}(t) = -QX(t) + E^{(s)} a(t) + q_1 L, \quad t \in [t_1, t_2],$$

$$x(t_1) = 0_n, \quad x(t_2) \le m(t_1), \quad A(t_2) \ge A^2,$$

$$0_n \le u(t) \le \bar{u}, \quad 0_d \le a(t) \le \bar{a}, \quad q \in [0, \bar{q}].$$

Remark that it is possible to replace the vector inequality $x(t_2) \le m(t_1)$ with an equality. In fact, if by contradiction in an optimal solution of P $x(t_2) < m(t_1)$, it would be possible to decrease production and, at the same time, increase the value of the objective function, due to the special form of the motion equations.

The model proposed is rather general and allows different specifications. One of them is the following. Assume the aim of the firm is to determine the optimal expenditure rate for each segment and each retailer. Then $d = n + r$. Moreover, let each kind of communication act only on a single retailer or on a single segment of the market. It means that for every $i \in I$ and $k \in K \setminus \{i\}$ one has $\varepsilon^{(p)}_{C_i k} = \varepsilon^{(s)}_{x_i k} = \varepsilon^{(s)}_{C_i k} = 0$ and that for every $j \in J$ and $k \in K \setminus \{n + j\}$ one has $\varepsilon^{(p)}_{R_j k} = \varepsilon^{(s)}_{R_j k} = 0$.

Another possible interpretation of the model will be given in Conclusions.

3. Decomposition into parametric subproblems

To discuss an approach for solving problem P, let us define

n-dimensional vector $\tilde{m}$, whose elements $\tilde{m}_i$, $i \in I$, are parameters;

$(n + r)$-dimensional vectors $\tilde{A}, \bar{A}$, whose elements $\tilde{A}_i, \bar{A}_i, i \in I, \tilde{A}_{n+j}, \bar{A}_{n+j}, j \in J$, respectively, are parameters too .

As in [3] problem P can be solved by first solving the optimal control problems, depending on the parameters $\tilde{m} = m(t_1),\ \tilde{A} = A(t_1),\ \bar{A} \ge A^2$ and then solving a nonlinear programming problem in which the parameters are the decision variables. Denote

$$\tilde{X} = \begin{pmatrix} 0_n \\ \tilde{A} \end{pmatrix}, \quad \bar{X} = \begin{pmatrix} \tilde{m} \\ \bar{A} \end{pmatrix}.$$

The parametric subproblems $P_1 = P(\tilde{m}_1)$, $P_2 = P_2(\tilde{A})$ and $P_3 = P_3(m, q, \tilde{A}, \bar{A})$ can be written this way

Problem P_1 ***:*** *maximize*

$$-\int_{t_0}^{t_1} \left(c^T m(t) + e_n^T u(t)\right) dt, \quad \textit{subject to}$$

$$\dot{m}(t) = \mu u(t),$$

$$m(t_0) = 0_n, \quad m(t_1) = \tilde{m}, \quad 0_n \le u(t) \le \bar{u}.$$

Problem P_2 ***:*** *maximize*

$$-\int_{t_0}^{t_1} e_d^T a(t) dt, \qquad \textit{subject to}$$

$$\dot{A}(t) = -\Delta A(t) + E^{(p)} a(t),$$

$$A(t_0) = A^0, \quad A(t_1) = \tilde{A}, \quad 0_d \le a(t) \le \bar{a}.$$

Problem P_3 ***:*** *maximize*

$$-\int_{t_1}^{t_2} e_d^T a(t) dt, \quad \textit{subject to}$$

$$\dot{X}(t) = -QX(t) + E^{(s)} a(t) + q_1 L,$$

$$X(t_1) = \tilde{X}, \quad X(t_2) = \bar{X}, \quad 0_d \le a(t) \le \bar{a}.$$

Moreover, problem P_1, due to the special form of its motion equations, can be replaced by n simpler problems $P_1^{(i)} = P_1^{(i)}(\tilde{m}_i), i \in I$, this way

Problem $P_1^{(i)}$ ***:*** *maximize*

$$-\int_{t_0}^{t_1} \left(c_i m_i(t) + u_i(t)\right) dt, \quad \textit{subject to}$$

$$\dot{m}_i(t) = \mu_i u_i(t),$$

$$m_i(t_0) = 0, \quad m_i(t_1) = \tilde{m}_i, \quad u_i(t) \in [0, \bar{u}_i].$$

Let $\tilde{F}_1^{(i)} = \tilde{F}_1^{(i)}(\tilde{\mu}_i), i \in I$, $\tilde{F}_2 = \tilde{F}_2(\tilde{A})$ and $\tilde{F}_3 = \tilde{F}_3(m, q, \tilde{A}, \bar{A})$ be the optimal values of problems $P_1^{(i)}, i \in I$, P_2 and P_3 respectively. Then problem P is equivalent to a nonlinear programming problem where the following objective function

$$p^T \tilde{m} - (t_1 - t_0) q + \sum_{i \in I} \tilde{F}_1^{(i)} + \tilde{F}_2 + \tilde{F}_3 \qquad (1)$$

must be maximized. In the following we will assume that the general position condition (GPC) ([7], p.166) holds.

The optimal solutions of problems $P_1^{(i)}, i \in I$, coincide with the ones for the case $n=1$ (see [3]), i.e. for every $i \in I$ the optimal solution of problem $P_1^{(i)}$ is

$$u_i^*(t) = \begin{cases} 0, t \in (t_0, t_{u_i}) \\ \overline{u}_i, t \in (t_{u_i}, t_1) \end{cases},$$

$$m_i^*(t) = \begin{cases} 0, t \in (t_0, t_{u_i}) \\ \mu_i \overline{u}_i (t - t_{u_i}), t \in (t_{u_i}, t_1) \end{cases},$$

where $t_{u_i} = t_1 - \dfrac{\widetilde{m}_i}{\mu_i \overline{u}_i}$. The optimum value is

$$\widetilde{F}_1^{(i)}(\widetilde{m}_i) = \frac{c_i}{2\mu_i \overline{u}_i} (\overline{m}_i)^2 + \frac{\widetilde{m}_i}{\mu_i}, \qquad (2)$$

where

$$\widetilde{m}_i \in [0, \mu_i \overline{u}_i (t_1 - t_0)]. \qquad (3)$$

In problem P_2, due to GPC, the number of switches in the optimal control $a_k^*(t)$ $\forall k \in K$ cannot be more than $n+r$ ([7], p.166). Denote these switching times by $\tau_1^{(k)}, \ldots \tau_{n+r}^{(k)} \forall k \in K$. Let

$$t_0 \le \tau_1^{(k)} \le \ldots \le \tau_{n+r}^{(k)} \le \tau_{n+r+1}^{(k)}, k \in K, \qquad (4)$$

and denote

$$G^{(p)} = e^{t_1 \Delta} \widetilde{A} - e^{t_0 \Delta} A^0, \quad D_k^{(p)}(t) = \overline{a} e^{t_0 \Delta} \Delta^{-1} E_k^{(p)},$$

where $E_k^{(p)}$ is k-th column of matrix $E^{(p)}$. One has

$$G^{(p)} = \begin{cases} \sum_{k \in K} d_k^{(p)}(n+r), & n+r \quad \text{is even} \\ \sum_{k \in K} d_k^{(p)}(n+r+1), & n+r \quad \text{is odd}, \end{cases} \qquad (5)$$

where

$$d_k^{(p)}(l) = \sum_{j=1}^{l} (-1)^j D_k^{(p)}(\tau_j^{(k)}), l \in \{n+r, n+r+1\}.$$

Moreover, $(n+r)$-dimensional vector v exists such that

$$v^T e^{\tau_j^{(k)} \Delta} E_k^{(p)} = 1, k \in K, j \in \{1, \ldots n+r\}, \qquad (6)$$

and the optimum value of the objective function is

$$\widetilde{F}_2 = \begin{cases} \sum_{k \in K} T_k^{(p)}(n+r), & n+r \quad \text{is even}, \\ \sum_{k \in K} T_k^{(p)}(n+r+1), & n+r \quad \text{is odd}, \end{cases} \qquad (7)$$

where

$$T_k^{(p)}(l) = -\overline{a}_k \sum_{j=1}^{l} (-1)^j \tau_j^{(k)}, l \in \{n+r, n+r+1\}.$$

Now consider problem P_3. For sake of simplicity we will consider only the case where all eigenvalues of matrix Q are distinct. If some of them coincide, similar calculations can be made, as in [2]. Let S be the matrix of eigenvectors of matrix Q. Let us denote

$$\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_{2n+r}\} = S^{-1} Q S.$$

Due to GPC, since the eigenvalues of the motion equations matrix are real, then $\forall k \in K$ the number of switches in the optimal control $a_k^*(t)$ cannot be more than $2n+r$. Let us denote the switching times by $\rho_1^{(k)}, \ldots, \rho_{2n+r}^{(k)}$. Let

$$t_1 \le \rho_1^{(k)} \le \ldots \le \rho_{2n+r}^{(k)} \le \rho_{2n+r+1}^{(k)} = t_2, k \in K, \qquad (8)$$

and denote

$$G^{(s)} = e^{t_2 \Lambda} S^{-1} \overline{X} - e^{t_1 \Lambda} S^{-1} \widetilde{X} + (e^{t_1 \Lambda} - e^{t_2 \Lambda}) \Lambda^{-1} S^{-1} L,$$

$$D_k^{(s)}(t) = \overline{a}_k e^{t \Lambda} \Lambda^{-1} S^{-1} E_k^{(s)}, \quad k \in K,$$

where $E_k^{(s)}$ is k-th column of matrix $E^{(s)}$.

Let $M \in I$ be the number of negative eigenvalues of matrix Q.

Let M be even. Then

$$G^{(s)} = \begin{cases} \sum_{k \in K} d_k^{(s)}(2n+r), & r \text{ is even}, \\ \sum_{k \in K} d_k^{(s)}(2n+r+1), & r \text{ is odd}, \end{cases} \qquad (9)$$

where

$$d_k^{(s)}(l) = \sum_{j=1}^{l} (-1)^j D_k^{(s)}(\rho_j^{(k)}), l \in \{2n+r, 2n+r+1\}.$$

Further, there exists a $(2n+r)$-dimensional vector w such that

$$w^T e^{\rho_j^{(k)} \Lambda} S^{-1} E_k^{(s)} = 1, k \in K, j = \{1, \ldots 2n+r\}, \qquad (10)$$

and the optimum value of the objective function is

$$\widetilde{F}_3 = \begin{cases} \sum_{k \in K} T_k^{(s)}(2n+r), & r \text{ is even}, \\ \sum_{k \in K} T_k^{(s)}(2n+r+1), & r \text{ is odd}, \end{cases} \qquad (11)$$

where

$$T_k^{(s)}(l) = -\overline{a}_k \sum_{j=1}^{l} (-1)^j \rho_j^{(k)}, l \in \{2n+r, 2n+r+1\}.$$

Let M be odd. In this case

$$G^{(s)} = \begin{cases} -\sum_{k \in K} d_k^{(s)}(2n+r+1), & r \text{ is even}, \\ -\sum_{k \in K} d_k^{(s)}(2n+r), & r \text{ is odd}, \end{cases} \qquad (12)$$

and there exists a $(2n+r)$-dimensional vector w such that (10) holds, and the optimum value of the objective function is

$$\widetilde{F}_3 = \begin{cases} \sum_{k \in K} T_k^{(s)}(2n+r+1), & r \text{ is even}, \\ \sum_{k \in K} T_k^{(s)}(2n+r), & r \text{ is odd}. \end{cases} \qquad (13)$$

4. The parametric optimization problem

Now we can formulate the parametric optimization problem. Recall that the following conditions hold:

$$\overline{A} \ge A^2, \quad q \in [0, \overline{q}]. \qquad (14)$$

If M is even, then the parametric optimization problem which has to be solved to obtain the solution of problem P is the following

P_+ : *maximize (1) subject to (3),(14),(4)-(6),(8)-(10), where* $\tilde{F}_1^{(i)}$ *is (2)* $\forall i \in I$, $\tilde{F}_2$ *is (7),* $\tilde{F}_3$ *is (11).*

If M is odd, then the parametric optimization problem to be considered becomes

P_- : *maximize (1) subject to (3),(14),(4)-(6),(8),(12),(10), where* $\tilde{F}_1^{(i)} \forall i \in I$ *is (2),* $\tilde{F}_2$ *is (7),* $\tilde{F}_3$ *is (13).*

Both P_+ and P_- are nonlinear programming problems with $3n+2r+1$ parameters $\tilde{m}, q, \tilde{A}, \overline{A}$, with $(3n+2r)\cdot d$ switching time variables $\tau_1^{(k)},\ldots,\tau_{n+r}^{(k)}$, $\rho_1^{(k)},\ldots,\rho_{2n+r}^{(k)}, k \in K,$ with $(n+r)$ -dimensional variable vector v and $(2n+r)$ -dimensional variable vector w, with $(3n+2r)\cdot(d+1)$ nonlinear equality constraints and some box constraints. The objective functions of the parametric problems are rather simple, but the equality constraints are nonlinear and cannot be solved explicitly. Remark that the constraints of the parametric optimization problem are necessary and sufficient conditions for the feasibility of problem P .

Conclusions

We have proposed and investigated an optimal control model with several kinds of communications in order to determine the optimal expenditure rate for each segment and each retailer in a multi-segment market. Remark that if there are only one retailer and one market segment (i.e. $n = r = 1$) and if we consider the whole communication expenditure rate a priori subdivided into two parts, one for the retailer and one for consumers, we obtain the model presented in [1]. Moreover, for this case we have seen in particular that the sign of eigenvalues of the matrix of motion equations determines the type of optimal alternate communication. This implies that if saturation aversion and goodwill decay, which can be considered as negative factors for the firm, are “stronger” than word-of-mouth and goodwill productivity, which are positive for the firm, than it is more convenient to advertise only in the middle of the selling period. Otherwise it is convenient to advertise at the beginning of the selling period and then to refresh the goodwill of the firm at the end of the period. In the model considered here, the type of optimal alternate communication in the selling period depends from the evenness/oddness of the number of negative eigenvalues of the matrix of motion equations.

We hope that a similar kind of interpretation (the relation between positive and negative factors for the firm, the retailers and the market segments) can be found for this general case of the model. At this aim the investigation of the parametric optimization problem P seems to be particularly relevant.

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