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The dynamics of social interaction with agents' heterogeneity

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Abstract. We analyze a class of binary dynamic models inspired by [4] on agents' choices and social interaction. The main feature of our analysis is that agents are heterogeneous, in particular their attitude to interact with the choices of the other agents changes over time endogenously. Although dynamic approaches to the study of models with heterogeneous agents have been already applied in different fields, to our knowledge a complete study of an endogenously varying population of agents has not yet been pursued. As observed in [3], the main problem is given by the fact that with heterogeneous agents the system may be non reversible. We address these problems, we describe the (possible multiple) steady states of the processes involved, we analyze local and global stability and we discuss the similarities and the differences with respect to the literature. Applications are also provided.

Keywords: heterogeneous agent models, intensity-based models, mean field interactions, random utilities, social interactions.

JEL Classification Numbers: D71 - D81 - C62

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1 Introduction

Since the '50s, the general equilibrium theory with complete markets has contributed to propagate the idea that heterogeneity was not an important issue in economic analysis. As a matter of fact, leaving aside distribution matters, in a complete market setting it is enough to consider a one consumer economy to address the main issues such as growth, prices, allocative efficiency, etc. During the last two decades agents' heterogeneity has gained a lot of attention among researchers, e.g. see [6, 1, 12, 19, 13, 14, 17, 8, 11]. This is due to many reasons. First of all, the analysis of an incomplete markets economy with heterogeneous agents may help to explain some puzzles that are not addressed in a one agent framework, e.g. the equity premium puzzle, trading volume, international trade, industrial organization, etc. Moreover there are economic and social phenomena that cannot be analyzed in a market with a price taking setting and require a different framework, e.g., social norms, coordination problems, industrial organization, organization decisions. In other words economists have recognized that it is not possible to reduce social interaction to coordination by prices.

This paper contributes to the literature on social interaction. In the spirit of [4], we analyze a class of binary choice models with interacting agents and random utility function. In this setting a decision maker takes a binary decision (buy or sell an asset, commit or not a crime, technology adoption), there are many agents in the economy and the agent's utility is affected by what the others do, see also [2, 3, 15]. Utility is made up of three components: a private component related to the benefit that the agent attaches to his choice; a *social component* that depends upon what the agents do and in particular on the interaction (distance) between his choice and the average behavior of the economy; a random term that introduces noise and bounded rationality in the model. Each agent forms his belief on the choices of the others and decides taking this datum into account and in particular evaluating his choice with respect to those of the community. The interaction is captured by the social component in the utility function.

This setting allows us to capture cultural and social phenomena and forms of social interaction: (non) conformism, herding, imitation, fashion, strategic complementarity/substitution, herding, externalities in the decision process

of a society. In particular the setting of [4] exhibits strategic complementarities as in [9]: assuming that the agent benefits from the fact that his choice agrees with the one of the majority of the population then we have strategic complementarities, i.e., the marginal utility to one agent of undertaking an action is increasing with the average action taken by the population. This feature generates a sort of *social multiplier*, see [18, 15], i.e., agents' choice change because fundamentals change and/or the behavior of the others does. This type of interaction generates multiple equilibria, a feature that allows us to explain the large variation of social phenomena (think about crime or religion) and the clustering (including segregation) of these phenomena in the time-space dimension.

Compared to the existing literature, the main novelty of our analysis is that we introduce agents' heterogeneity in the social interaction attitude: in [4] the interaction among agents is driven by the positive homogeneous parameter, in our analysis the parameter is stochastic and change over time allowing for conformism (positive parameter), indifference (null parameter), anticonformism (negative parameter). Agents are heterogeneous in their degree of conformism and their attitude changes over time endogenously as a function of the behavior of the agents. We propose two different specifications on the evolution of agents' attitude towards what the others do. In the first specification, we have asymmetric conformism: while in [4] we have a symmetric strategic complementarities towards both possible choices, in our setting we have that conformism reinforces one of the two choices, i.e., (anti) conformism is large when the majority of the population has taken one of the two choices leading to a tendency towards that particular choice. In the second specification instead we assume that the conformity attitude depends on the agreement between the agent's choice and the choices made by the population. Compared to [4], we have that the first specification models a sort of strategic complementarity, instead the second assumes a different behavior.

To our knowledge, dynamically varying classes of agents have not yet been studied in the literature. We show how to introduce this source of complexity maintaining tractability. In particular we describe the (possible multiple) steady states of the processes involved, discussing the similarities

and the differences compared to its homogeneous/static counterparts. As argued in [3, 10], heterogeneity can lead to what is called *non reversibility* of the dynamical system. Reversibility is related to the shape of the generator of the stochastic processes that describes the time evolution of the state variables of the system. It can be shown that when the system is reversible (homogeneous) there are standard techniques useful to describe the stationary distributions and hence the steady states (equilibria) of the system. In the case of heterogeneity it is not always possible to rely on such methodologies; in particular it has not yet been addressed in the literature whether more complex (non reversible) models might exhibit a behavior in line with the findings as in [3, 4].

We fully develop the analysis and we show similarities between our models and [4]. As claimed in [15], strategic complementarity is required to have multiple equilibria. This outcome is confirmed in the first specification but not in the second one. Moreover it is confirmed that to have multiple equilibria we need a small amount of noise, strong complementarities and that the difference between the two choices is not strong.

The paper is organized as follows. In Section 2 we define the binary choice problem. In Section 3 we analyze the dynamics of the dynamical system relying upon results obtained in [10] specifying the two different settings for conformity evolution and we characterize the stationary equilibria of the model. In Section 4 we analyze the system obtained from the first specification. In Section 5 we analyze the system obtained from the second specification. In Section 6 we provide applications of the methodology to crime and scientific revolution.

2 The model

Our setting is a dynamic version of the one proposed in [4]. The economy is made up of I agents facing a discrete binary choice problem $\forall t \geq 0$. There are only two possible choices labeled by -1 and $+1$. We denote by $\omega_i(t) \in \{-1; 1\}$, where $i = 1, \dots, I$ and $t \geq 0$, the choice of the i -th agent at time t and by $\underline{\omega}(t) = (\omega_1(t), \dots, \omega_I(t))$ the vector of the state variables, i.e., agents' decisions at time t .

We develop our analysis in two steps. First, we concentrate our attention on the static decision problem at time t , then we analyze the dynamics of the entire population choices. Presenting the static problem we omit the time indicator t .

Agents' utility function is made up of three components: private utility, social utility and an error term. For agent i we have

$$u_i(\omega_i) = v(\omega_i) + J_i\omega_i\bar{m}_i^e + \epsilon(\omega_i). \quad (1)$$

Let us analyze the three terms.

$v(\omega_i)$ is the private utility associated with the binary choice. Private utility depends upon the choice made by agent. To simplify the analysis we assume that agents have the same private utility function, we can allow for heterogeneity in the private utility function at the cost of computational problems.

$S(\omega_i, \bar{m}_i^e) = J_i\omega_i\bar{m}_i^e$ is the social component of the utility. \bar{m}_i^e is the expectation from the point of view of agent i of the average of the choices of the others: $\bar{m}_i^e = \frac{1}{I-1}E[\sum_{j \neq i} \omega_j]$. J_i is a parameter describing the effect on the agent's utility of the behavior of the others: $J_i > 0$ implies that the agent benefits from the fact that his choice agrees with that of the majority of the population, we refer to this effect as conformism, i.e., the agent tends to follow the behavior of the community as a whole, instead $J_i < 0$ means that the utility of the agent is negatively affected by the fact that his choice is adopted by the majority of the agents, we say that the agent is nonconformist. As noted in [4], $J_i > 0$ captures a social component which is negatively affected by the distance from the average behavior of the population, as a matter of fact $S(\omega_i, \bar{m}_i^e) = -\frac{J_i}{2}(\omega_i - \bar{m}_i^e)^2 = J_i\omega_i\bar{m}_i^e - \frac{J_i}{2}(1 + (\bar{m}_i^e)^2)$, this specification differs from the one employed in (1) only in the level but the factors affecting the choice are already captured in our formulation.

$\epsilon(\omega_i)$ is a random term whose distribution is extreme value, i.e.,

$$P(\epsilon(-1) - \epsilon(1) \leq x) = \frac{1}{1 + e^{-\beta x}} \quad (2)$$

where $\beta > 0$ is a measure of the impact of the random component in the decision process. In our formulation of the choice problem, agent i knows $\epsilon(-1)$ and $\epsilon(1)$ at the time of his decision and he chooses $\omega_i = 1$ if $u_i(1) >$

$u_i(-1)$. As shown in [4], a large β means that the deterministic part plays a relevant role in the maximization of the utility; instead when β tends to zero the error term dominates and the choice between $\omega = 1$ or $\omega = -1$ is a coin tossing. The error component can be interpreted as a *bounded rationality* component on the behavior of the agents introducing a noise component.

Differently from [4] and related papers, we assume that conformism (labeled by J_i for $i = 1, \dots, I$) is not the same for all agents and is not constant over time. In particular we analyze a model with two different classes of agents with parameters $J^{(1)}$ and $J^{(2)}$. We make two different assumptions:
 I) $J^{(1)} = 1$ and $J^{(2)} = -1$, there are *conformist* and *nonconformist* agents, the first class of agents follow the average behavior of the population, the second class behaves in the opposite way;
 II) $J^{(1)} = 1$ and $J^{(2)} = 0$, there are conformist and agents indifferent to the social behavior, following [5] we may interpret the first class of agents as *followers* and the others as *leaders*.

Let $h = \frac{1}{2}(v(1) - v(-1))$. When $h > 0$ the choice $\omega = 1$ leads to a higher private value and therefore $\omega = 1$ is risk-dominant in the sense that the agent is more likely to change his choice if $\omega = -1$.

At time t agent i observes his specific noise realization $(\epsilon(1), \epsilon(-1))$ and takes the decision (-1 or 1) comparing $u_i(1)$ and $u_i(-1)$ on the basis of his expectation on the behavior of the population (\bar{m}_i^e) : if $u_i(1) > u_i(-1)$ then $\omega_i = 1$ and -1 otherwise. As the environment is stochastic the choice is not deterministic but satisfies a probability law. Thanks to (2) it can be shown that the agent choice obeys the probability

$$P(\omega_i | \bar{m}_i^e) = \frac{e^{\beta\omega_i(h+J_i\bar{m}_i^e)}}{e^{-\beta\omega_i(h+J_i\bar{m}_i^e)} + e^{\beta\omega_i(h+J_i\bar{m}_i^e)}}. \quad (3)$$

The analysis of the model calls for an assumption on agents' expectations. We can address this issue in a static setting or introducing a dynamics. In the first case the natural candidate is the rational expectations hypothesis: agents possess homogeneous expectations on the average behavior of the population and the expectation is self consistent: the expectations on the average choice coincides with the realization of the economy. Utility maximization and rational expectations allow us to identify equilibria for the average behavior of the population thorough a simple fixed point argument:

with homogeneous J [4] show that the model admits one or three equilibria. [4] address the expectation formation also in a dynamic setting: in a discrete time model the expectation of the behavior of the population at time t is given by the realized behavior at time $t - 1$: agents' expectations are backward looking and myopic. Equilibria of the difference equations system coincide with those of the static problem with rational expectations. Note that their equilibrium analysis cannot be replicated in our setting because J_i are not constant.

A continuous time dynamic version of the model has been proposed in [3]. Agents update their decisions at random Poissonian times: when the Poissonian clock of the i -th agent rings, he takes a decision according to his utility function and to his expectation of the others. Under this specification, the system evolves as a continuous time Markov chain on the state space $\{-1; 1\}^I$. We can study the system dynamics and eventually the *invariant distributions* that characterize the steady states and hence the equilibria of the system.

Inspired by [3, 4], we build a dynamic decision process based on the maximization of the utility (1). Indeed, we transpose the static probability (3) into a dynamic varying (local) probability defined by

$$\lambda_i(t) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} P(\omega_i(t + \tau) \neq \omega_i(t) | \underline{\omega}(t)) = e^{-\beta \omega_i(t)(h + J_i(t)s_I(t))}, \quad i = 1, \dots, I, \quad (4)$$

where now all the state variables are indexed with time and where the expectation by agent i of the behavior of the others (\bar{m}_i^e) has been substituted by the *empirical mean*

$$s_I(t) := \frac{1}{I} \sum_{i=1}^I \omega_i(t). \quad (5)$$

This assumption requires all agents to share the same information at any time, in particular they do know what is the statistical mean of the choices of the entire population. We can also use $\sum_{j \neq i} \omega_j(t)$ in place of $s_I(t)$, i.e., excluding ω_i from the mean. This would only multiply all λ_i 's by a constant term.

In a continuous time model, $\lambda_i(t)$ describes the local rate of probability that agent i changes his choice between time t and t^+ , given the state of the system at time t ($s_I(t)$).

Why should (4) represent a dynamic counterpart of (3)? The idea behind (4) is that the agents change their opinion at random times $\{\tau_n^i\}_{n \in \mathbb{N}}$ such that $\tau_n^i - \tau_{n-1}^i$ are exponentially distributed with mean $1/\lambda_i$. At time τ_n^i , agent i revises his choice according to a process driven by (3).

If we compare (3) and (4), there are basically two main differences: (i) the “minus” in the numerator and (ii) the absence of the denominator. (i) is due to the fact that ω_i has switched its sign between t and t^+ , so that $-\omega_i$ in the numerator of (4) is referred to “the decision of agent i before the jump”. Concerning (ii), the absence of the denominator simply amounts to a re-scaling of the dynamics. In probabilistic terms the presence of the denominator would slightly change the interpretation of the Poissonian clock: from a “mean $1/\lambda_i$ time at which agent i decides to revise his decision” to a “mean 1 time at which the agent is asked whether he wants to change his decision”. It can be proved that the stationary states (the equilibria) of the dynamic systems derived under the two specifications are the same.

We can rewrite (4) in a more compact way:

$$\omega_i \mapsto -\omega_i \quad \text{with intensity} \quad \lambda_i = e^{-\beta\omega_i(h+J_i s_I)}, \quad (6)$$

where $\beta > 0$ and $J_i \in \{-1, +1\}$. The interpretation is as follows: high values of s_I imply a high probability for agent i to choose $\omega_i = 1$ when $J_i = +1$ and to choose $\omega_i = -1$ when $J_i = -1$.

Agents change their choice at random time according to (6), the behavior of agent i is completed by the evolution of J_i . We assume a continuous time Markovian evolution for J_i , $i = 1, \dots, I$, and we consider the case of $J_i = \{-1, 1\}$, the formulation for case $J_i = \{0, 1\}$ is analogous, only the interpretation changes. In particular, we assume the following dynamics:

$$J_i \mapsto -J_i \quad \text{with intensity} \quad \mu_i = e^{\phi(\omega_i, J_i, s_I)}, \quad (7)$$

where $\phi(\cdot)$ is a suitable function and s_I is as defined in (5). We shall in particular concentrate on two cases:

$$\phi^a = -\gamma J_i s_I, \quad (8)$$

$$\phi^b = -\gamma \omega_i J_i s_I, \quad (9)$$

where γ is a constant.

We notice that the i -th agent's decision depends on the system only through the *aggregate statistic* s_I . This variable is indeed an *empirical mean* of the system and incorporates only a partial (averaged) information on the state vector $\underline{\omega}$. This simplifying assumption is called *mean field assumption*: the interaction among different agents only depends on the value of s_I .

The two specifications for the dynamics of the agent type with $J_i = \{-1, 1\}$ call for a discussion. The first one, described by (8), says that for $\gamma > 0$ the probability that an agent switch from nonconformism (conformism) to conformism (nonconformism) is high (low) when a large fraction of people adopt choice 1. For $\gamma < 0$ we have the opposite effect: there is a tendency towards conformism when a large fraction of people adopt choice -1 . So this specification allows us to describe the case in which the choices made by the population have an asymmetric effect on the social interaction of the agents: for $\gamma > 0$ we have a reinforcing effect towards $\omega_i = 1$, for $\gamma < 0$ towards $\omega_i = -1$. We refer to this specification as *asymmetric reinforcing conformism*. The second specification, described by (9), says that for $\gamma > 0$ the probability that an agent switch from nonconformism (conformism) to conformism (nonconformism) is high (low) when his choice agrees with the one of the majority of the population. For $\gamma < 0$ we have the opposite effect. So this specification allows us to describe the case in which conformism is reinforced by the fact that the agent's choice is confirmed or not by the community: the fact that the agent's choice is confirmed by the population choice induces the agent to follow the behavior of the population. We refer to this model as *self-confirming conformism*. For model (8) the conformity attitude depends on the choices made by the population, for model (9) the conformity attitude depends on the agreement between the agent's choice and the choices made by the population.

A similar interpretation can be provided in the leader-follower case. In the first case for $\gamma > 0$ (< 0) the tendency to become a follower (leader) increases as the fraction of people adopting $\omega_i = 1$ goes up. In the second case for $\gamma > 0$ (< 0) the tendency to become a follower increases if the agents choices agrees with the behavior of the entire population.

Specifications as (6)-(7) make the state space variables evolve as a continuous-

time Markov chain on $\{-1, 1\}^{2N}$ with the following infinitesimal generator:

$$\begin{aligned} \mathcal{G}_I f(\underline{\omega}, \underline{J}) &= \\ &= \sum_{i=1}^I e^{-\beta \omega_i (h + J_i s_I)} (f(\underline{\omega}^i, \underline{J}) - f(\underline{\omega}, \underline{J})) + \sum_{i=1}^I e^{\phi(\omega_i, J_i, S_I)} (f(\underline{\omega}, \underline{J}^i) - f(\underline{\omega}, \underline{J})) \end{aligned} \quad (10)$$

where $\underline{\omega}^i$ (resp. \underline{J}^i) denotes the vector $\underline{\omega}$ (resp. \underline{J}) where the i -th component has been switched:

$$\omega_j^i = \begin{cases} \omega_j & \text{for } j \neq i \\ -\omega_i & \text{for } j = i. \end{cases}$$

It can be shown that the dynamics induced by (10) are non-reversible hence they do not admit a reversible stationary distribution. Usually, when the dynamics admit a reversible distribution, this distribution can be found explicitly relying on the so called *detailed balance condition*. In the case of non-reversibility it is very difficult to find an explicit formula for the stationary distribution. As already argued, this fact makes the analysis of the dynamics and the relative study of the equilibria more challenging. Nevertheless, we are able to describe the time evolution and the steady states of the system in the *limiting regime* of the system, i.e., when the population size tends to infinity.

3 Dynamic analysis and stationary equilibria

The study of the dynamics of the system induced by (10) is based on a *law of large numbers* on a particular family of probability measures, see Appendix A. We develop our analysis for $J = \{-1; 1\}$ but results do not change in the case $\{0; 1\}$. Before stating the main result of this section we define some aggregate variables that will play an important role in our analysis. Let μ be a probability on $\{-1; 1\}^2$; we define the following expectations:

$$m_\mu^\omega := \sum_{\omega, J=\pm 1} \omega \mu(\omega, J), \quad m_\mu^J := \sum_{\omega, J=\pm 1} J \mu(\omega, J), \quad m_\mu^{\omega J} := \sum_{\omega, J=\pm 1} \omega J \mu(\omega, J). \quad (11)$$

The following Theorem provides us with the dynamical system on the evolution of the fraction of agents choosing $\omega_i = 1$ and on the fraction of

agents with $J = 1$ in a large economy, i.e., when $I \rightarrow \infty$. We characterize the system for the specification ϕ^a (asymmetric conformism) and ϕ^b (experience based conformism).

Theorem 3.1 *Suppose that the distribution at time $t = 0$ of the Markov process $(\underline{\omega}(t), \underline{J}(t))_{t \geq 0}$ with generator (10) is such that the random variables $(\omega_i(0), J_i(0))$, $i = 1, \dots, I$, are independent and identically distributed with law λ .*

Consider the following empirical means

$$s_I^\omega(t) = \frac{1}{I} \sum_i \omega_i(t), \quad s_I^J := \frac{1}{I} \sum_i J_i(t), \quad s_I^{\omega J} := \frac{1}{I} \sum_i \omega_i(t) J_i(t).$$

Then for $I \rightarrow \infty$ the triplet $(s_I^\omega(t), s_I^J(t), s_I^{\omega J}(t))$ converges (in the sense of weak convergence of stochastic processes) to a triplet $(m^\omega(t), m^J(t), m^{\omega J}(t))$ such that

$$m^\omega(0) = m_\lambda^\omega, \quad m^{\omega J}(0) = m_\lambda^{\omega J}, \quad m^J(0) = m_\lambda^J,$$

and

a) in the case of $\phi = \phi^a$

$$\begin{cases} \dot{m}_t^\omega &= -2C(\beta h)C(\beta m_t^\omega)m_t^\omega + 2C(\beta h)S(\beta m_t^\omega)m_t^J \\ &\quad -2S(\beta h)S(\beta m_t^\omega)m_t^{\omega J} + 2S(\beta h)C(\beta m_t^\omega) \\ \dot{m}_t^J &= 2S(\gamma m_t^\omega) - 2C(\gamma m_t^\omega)m_t^J \\ \dot{m}_t^{\omega J} &= 2[S(\gamma m_t^\omega) - S(\beta h)S(\beta m_t^\omega)]m_t^\omega + 2S(\beta h)C(\beta m_t^\omega)m_t^J \\ &\quad -2[C(\gamma m_t^\omega) + C(\beta h)C(\beta m_t^\omega)]m_t^{\omega J} + 2C(\beta h)S(\beta m_t^\omega). \end{cases} \quad (12)$$

where $C(x) = \cosh(x)$ and $S(x) = \sinh(x)$;

b) in the case of $\phi = \phi^b$

$$\begin{cases} \dot{m}_t^\omega &= -2C(\beta h)C(\beta m_t^\omega)m_t^\omega + 2C(\beta h)S(\beta m_t^\omega)m_t^J \\ &\quad -2S(\beta h)S(\beta m_t^\omega)m_t^{\omega J} + 2S(\beta h)C(\beta m_t^\omega) \\ \dot{m}_t^J &= 2S(\gamma m_t^\omega)m_t^\omega - 2C(\gamma m_t^\omega)m_t^J \\ \dot{m}_t^{\omega J} &= -2S(\beta h)S(\beta m_t^\omega)m_t^\omega + 2[C(\beta h)S(\beta m_t^\omega) + S(\gamma m_t^\omega)] \\ &\quad + 2S(\beta h)C(\beta m_t^\omega)m_t^J - 2[C(\beta h)C(\beta m_t^\omega) + C(\gamma m_t^\omega)]m_t^{\omega J}. \end{cases} \quad (13)$$

Proof. See Appendix A.1 ■

This theorem describes the aggregate dynamics of the triplet of sufficient statistics when the number of agents goes to infinity. Relying on this result, we characterize the steady states of the system.

Corollary 3.2 *All the equilibria are solutions of the following fixed point arguments:*

a) *in the case of $\phi = \phi^a$*

$$m^\omega = T(\beta h) + T(\beta m^\omega)T(\gamma m^\omega) - T(\beta h)T(\beta m^\omega)z(m^\omega) \quad (14)$$

where $C(x) = \cosh(x)$, $S(x) = \sinh(x)$, $T(x) = \tanh(x)$ and

$$z(x) = \frac{[S(\gamma x) - S(\beta h)S(\beta x)]x + S(\beta h)C(\beta x)T(\gamma x) + C(\beta h)S(\beta x)}{C(\gamma x) + C(\beta h)C(\beta x)};$$

b) *in the case of $\phi = \phi^b$*

$$m^\omega = \frac{T(\beta h) - T(\beta h)T(\beta m^\omega)z(m^\omega)}{[1 - T(\beta m^\omega)T(\gamma m^\omega)]} \quad (15)$$

where $C(x) = \cosh(x)$, $S(x) = \sinh(x)$, $T(x) = \tanh(x)$ and

$$z(x) = \frac{C(\beta h)S(\beta x) + S(\gamma x) + [S(\beta h)C(\beta x)T(\gamma x) - S(\beta h)S(\beta x)]x}{C(\gamma x) + C(\beta h)C(\beta x)}.$$

Proof. See Appendix A.1.

Random utility models provide a suitable setting for multiple equilibria. For a constant $J > 0$, [4] show that a unique equilibrium arises if $\beta J < 1$, i.e., the degree of conformism is low and there is a lot of noise in agents' choices. For $\beta J > 1$ and $h = 0$ there exist three equilibria, if $h \neq 0$ a unique equilibrium arises for an h large enough in absolute value, otherwise three equilibria exist. So multiple equilibria arises when there is a small amount of noise, conformism is high and the deterministic part of the choice h is not strongly in favor of one of the two choices.

4 Asymmetric reinforcing conformism

We consider the case of $\phi = \phi^a$ with $J = \{-1, 1\}$. We start by considering the case $h = 0$ and then we will consider $h \neq 0$. For $h = 0$ the differential system for the aggregate indicators reduces to

$$\begin{cases} \dot{m}_t^\omega &= 2S(\beta m_t^\omega) m_t^J - 2C(\beta m_t^\omega) m_t^\omega \\ \dot{m}_t^J &= 2S(\gamma m_t^\omega) - 2C(\gamma m_t^\omega) m_t^J \\ \dot{m}_t^{\omega J} &= 2S(\gamma m_t^\omega) m_t^\omega + 2S(\beta m_t^\omega) - 2[C(\beta m_t^\omega) + C(\gamma m_t^\omega)] m_t^{\omega J}. \end{cases} \quad (16)$$

It is easy to show that the dynamics of $(\dot{m}_t^\omega, \dot{m}_t^J)$ does not depend on $\dot{m}_t^{\omega J}$, therefore the differential system (16) is essentially two dimensional: we have to solve the two-dimensional system

$$(\dot{m}_t^\omega, \dot{m}_t^J) = V(m_t^\omega, m_t^J), \quad (17)$$

on $[-1, 1]^2$ with $V(x, y) = (2 \sinh(\beta x) y - 2 \cosh(\beta x) x, 2 \sinh(\gamma x) - 2 y \cosh(\gamma x))$, and then the third equation (16), which is linear in $m_t^{\omega J}$.

Solutions of the system $V(x, y) = \underline{0}$ are

$$x = \tanh(\beta x) \tanh(\gamma x) ; \quad y = \tanh(\gamma x). \quad (18)$$

We now characterize the equilibrium for $h = 0$.

Proposition 4.1 *Let $h = 0$ and fix $\beta > 0$.*

1. *For $\beta \leq 1$, $x = 0$ is the only solution to (18) whatever the value of γ is.*
2. *For $\beta > 1$ define $\bar{x}(\beta)$ as the unique solution to*

$$1 = \frac{\ell(x)}{\sinh(\ell(x))} + \frac{2\beta x}{\sinh(2\beta x)}, \quad (19)$$

where $\ell(x) = \ln\left(1 + \frac{x}{\tanh(\beta x)}\right) - \ln\left(1 - \frac{x}{\tanh(\beta x)}\right)$. Define also

$$\bar{\gamma}(\beta) = \frac{\ell(\bar{x}(\beta))}{2\bar{x}(\beta)}. \quad (20)$$

It can be proved that $\bar{x}(\beta) \in (0, 1)$ and $\bar{\gamma}(\beta) \in (1, \infty)$. Moreover

- (2.a) *for $\gamma < \bar{\gamma}(\beta)$, $x = 0$ is the only solution to (18);*

(2.b) for $\gamma = \bar{\gamma}(\beta)$ there are two solutions to (18): 0 and $\bar{x}(\beta) > 0$;

(2.c) for $\gamma > \bar{\gamma}(\beta)$ there are three solutions to (18), namely $(0, x^{(1)}, x^{(2)})$, such that $0 < x^{(1)} < \bar{x}(\beta) < x^{(2)}$.

Proof. See Appendix A.2.

Note that for $\gamma < 0$ we reach exactly the same stationary equilibria with $x^{(2)} < \bar{x}(\beta) < x^{(1)} < 0$ and $\bar{\gamma}(\beta) < 0$.

The equilibrium characterization is qualitatively in line with what is found in [4] for the model with a constant J . More closely, in the static case we have multiple equilibria when $\beta J > 1$, i.e., for a high tendency towards conformism and a small amount of noise. In our model we confirm that a small amount of noise ($\beta > 1$) is necessary to find out multiple equilibria. The role of J is played by γ : for γ high enough we have three equilibria. For a comparison among situations where unique or multiple equilibria are found, see Figure 1 where we plot the graph of $f(x) = \tanh(\beta x) \tanh(\gamma x)$ for different values of γ .

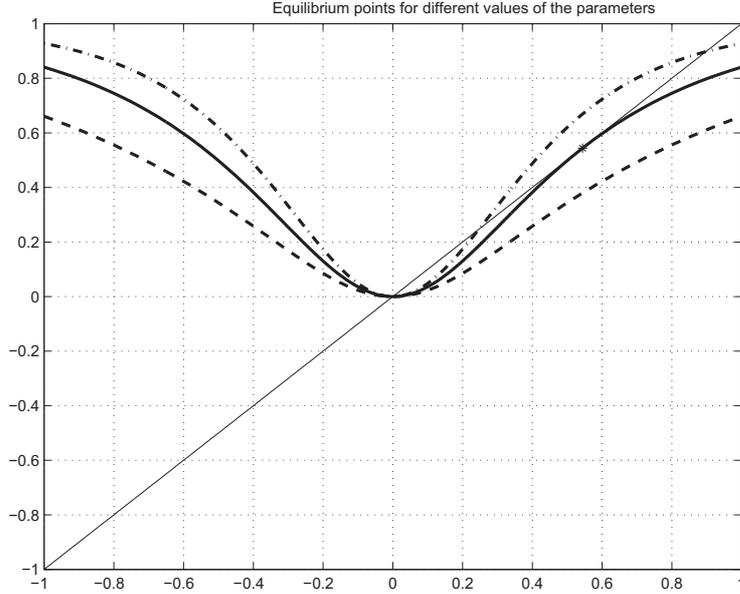


Figure 1: Solutions of (18) for β fixed and different values of γ . Here $\beta = 3$ and $\gamma = 0.8; 1.238; 1.438$ respectively in the dash, solid and dash-dot lines. In this example $\bar{\gamma}(\beta) \sim 1.238$ represents the critical value. The ‘*’marked point corresponds to $\bar{x}(\beta)$ as defined in Proposition 4.1.

β	$\bar{\gamma}(\beta)$	$\bar{x}(\beta)$
2	1.509	0.624
3	1.238	0.544
5	1.103	0.417
10	1.035	0.268

Table 1: Different values of $\bar{\gamma}(\beta)$ and $\bar{x}(\beta)$ when β varies.

Notice also that $\gamma(\beta)$ is a decreasing function of β . In Table 1 we report some values of $\bar{\gamma}$ varying β as found by numerical simulations.

Concerning the case of $h \neq 0$, we can distinguish different situations:

- for $\beta < 1$ or $\beta > 1$ but $\gamma \leq \gamma(\beta)$, there is one unique solution to (14) which equals h in sign;
- for $\beta > 1$ and $\gamma > \bar{\gamma}(\beta)$, there exist two thresholds $h^l(\beta, \gamma)$ and $h^u(\beta, \gamma)$, where $h^-(\beta, \gamma) < 0 < h^+(\beta, \gamma)$ such that
 - for $h < h^l(\beta, \gamma)$ there is only a negative solution $x(h)$ to (14). In particular, $\lim_{h \rightarrow -\infty} x(h) = -1$;
 - for $h^l(\beta, \gamma) \leq h \leq h^u(\beta, \gamma)$ there are three solutions to (14);
 - for $h > h^u(\beta, \gamma)$ there is only a positive solution $x(h)$ to (14)). In particular, $\lim_{h \rightarrow +\infty} x(h) = 1$.

Random utility models are static models. Further assumptions are required to introduce a dynamics: in our setting we have introduced a dynamics through Poissonian random clocks, [4] make the assumption of myopic expectations. Stability results are similar: if there exists a unique equilibrium then it is stable, if there three equilibria then the extreme equilibria are stable. We prove this result for the case $h = 0$ in the following proposition.

Proposition 4.2 *Consider the equilibria found in Proposition 4.1, then*

- i) $x = 0$ is always a linearly stable equilibrium of (18).*
- ii) $\bar{x}(\beta)$ is still stable but it has a neutral direction.*
- iii) $x^{(1)}$ is a saddle point and $x^{(2)}$ is linearly stable.*

Proof. See Appendix A.2.

5 Self-confirming conformism

We consider the case of $\phi = \phi^b$ with $J = \{-1, 1\}$. We start by considering the case $h = 0$ and then we will consider $h \neq 0$. For $h = 0$ it is easy to see that (15) reduces to $m^\omega = 0$. Then we have only one (stable equilibrium) whatever the values of β and γ are. Concerning the case $h \neq 0$, looking at the simulations, it seems that the solution of (15) is always unique and its sign coincides with the sign of h .

6 Applications: development of crime rates

The framework described above covers a large set of applications. We analyze in particular crime rates in a society considering a model similar to [17]. Empirical data show that crime is a phenomenon characterized by a high variability that cannot be fully explained by sociological and economic factors. To explain variability, models with multiple equilibria have been invoked: [21, 20] suggest two simple self-reinforcing mechanisms, i.e., as the criminal fraction of the population increases the probability of being arrested decreases and the returns from not being a criminal fall because revenues are stolen by criminals. [17] instead build on a model based on imitation among agents: there are agents with a firm opinion about crime (inclined to crime or not), their choice is not affected by others factors, and there are agents who based their decision on what the others do. A large fraction of agents with no firm choice leads to a high crime variance.

There are I agents, $\omega_i = 1$ means that agent i commits a crime, instead $\omega_i = -1$ means that the agent does not commit a crime. The two specifications on the evolution of J , i.e., (8) and (9), and the choice of possible values for J , i.e., $\{-1, 1\}$ and $\{0, 1\}$, provide us with different modeling of the crime phenomenon.

Assuming (8) with $J = \{-1, 1\}$ and $\gamma > 0$, agent i is more likely to change from anticonformism to conformism, and so to imitate the others, if the majority of the population commits a crime or to change from conformism to anticonformism if the majority of the population doesn't commit a crime. The idea behind this behavioral assumption is that there is a self reinforcing mechanism towards crime. This idea, with a different motivation, agrees

with [21, 20]. Assuming $J = \{0, 1\}$ we have that the fraction of the people affected by the decisions of the others increases as crime goes up and the interpretation is similar to the one described above. With $\gamma < 0$ we have exactly the opposite effect: a self reinforcing mechanism towards a probe behavior.

Assuming (9) with $J = \{-1, 1\}$ and $\gamma > 0$, agent i is more likely to change from anticonformism to conformism if the majority of the population commits a crime and he commits a crime. The idea behind this behavioral assumption is that the agent's degree of conformity is positively influenced by the fact that his choice agrees with the one of the majority of the population.

7 Conclusions

In this paper we have analyzed a class of binary dynamic models inspired by [4] on agents' choices and social interaction. The main feature of our analysis is that agents are heterogeneous, in particular their attitude to interact with the choices of the other agents changes over time endogenously. We have concentrated our attention on two different specifications on the evolution of agents' attitude towards what the others do. In the first specification, we have asymmetric conformism: while in [4] we have a symmetric strategic complementarities towards both possible choices, in our setting we have that conformism reinforces one of the two choices, i.e., (anti) conformism is large when the majority of the population has taken one of the two choices leading to a tendency towards that particular choice. In the second specification instead we assumed that the conformity attitude depends on the agreement between the agent's choice and the choices made by the population.

To our knowledge, dynamically varying classes of agents have not yet been studied in the literature. We have proposed a methodology in order to introduce this source of complexity but maintaining tractability. In particular we have described the (possible multiple) steady states of the processes involved, discussing the similarities and the differences compared to its homogeneous/static counterparts.

References

- [1] Becker Gary and Kevin Murphy (2001) *Social markets: market behavior in a social environment*, Belknap-Harvard University Press.
- [2] Bischi, G., Gallegati, M., Gardini, L., Leombruni, R., Palestrini, A., (2006) Herd behavior and nonfundamental asset price fluctuations in financial markets, *Macroeconomic Dynamics* 10: 502-528.
- [3] Blume, L.E. and Durlauf, S. (2003) Equilibrium concepts for social interaction models, *International Game Theory Review*, 5(3): 193-209.
- [4] Brock, W. and Durlauf, S. (2001) Discrete choice with social interactions, *Review of Economic studies* 68: 235-260.
- [5] Brock, William and Durlauf, Steven (1999) A formal model of theory choice in science, *Economic Theory*, 14: 113-130.
- [6] Brock, W. and Hommes, C. (1998) Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *Journal of Economic Dynamics and Control*, 22: 1235-1274.
- [7] Chang, S. (2007) A simple asset pricing model with social interactions and heterogeneous beliefs, *Journal of Economic Dynamics and Control*, 31: 1300-1325.
- [8] Cont, R. and Löwe M. (2003) Social distance, heterogeneity and social interactions, Centre des Mathématiques Appliquées, Ecole Polytechnique, R.I.No 505.
- [9] Cooper Russel and John A. (1988) Coordinating coordination failures in Keynesian models, *Quarterly Journal of Economics*, 103: 441-464.
- [10] Dai Pra, P., Runggaldier, W.J., Sartori, E., Tolotti, M. (2009) Large portfolio losses, a dynamic contagion model, *The Annals of Applied Probability*, Vol. 19(1): 347-394.
- [11] Ellison Glenn, Fudenberg Drew (1993) Rules of thumb for social learning, *Journal of Political Economy*, 101: 612-643.

- [12] Föllmer, H., Horst, U., Kirman, A. (2005) Equilibria in Financial Markets with Heterogeneous Agents: A Probabilistic Perspective, *Journal of Mathematical Economics* 41(1-2): 123-155.
- [13] Frey, R., Backhaus J. (2008) Pricing and Hedging of Portfolio Credit Derivatives with Interacting Default Intensities, *International Journal of Theoretical and Applied Finance*, 11(6): 611-634.
- [14] Giesecke, K., Weber, S. (2004) Cyclical correlations, credit contagion and portfolio losses, *Journal of Banking and Finance* 28(12): 3009-3036.
- [15] Glaeser, E. and Scheinkman, J. (2002) Non-market interactions, invited lecture Econometric society world congress in Seattle.
- [16] Glaeser, E. and Scheinkman, J. (1999) Measuring social interactions, mimeo.
- [17] Glaeser, E., Sacerdote, B. and Scheinkman, J. (1996) Crime and Social Interactions, *The Quarterly Journal of Economics*, 111: 507-548.
- [18] Glaeser, E., Sacerdote, B. and Scheinkman, J. (2002) The social multiplier, NBER working paper 9153
- [19] Hommes, Cars (2006), Heterogeneous Agent Models in Economics and Finance, in Handbook of Computational Economics, Leigh Tesfatsion & Kenneth L. Judd (ed.): 1109-1186 Elsevier.
- [20] Murphy Kevin, Shleifer Andrei, Vishny Robert (1993) Why is rent-seeking so costly to growth?, *The American Economic Review*, 83: 409-414.
- [21] Sah Raaj (1991) Social osmosis and patterns of crime, *The Journal of Political Economy*, 99: 1272-1295.
- [22] Weisbuch Gérard, Kirman Alan, herreiner Dorothea (2000) market organization and trading relationship, *The Economic Journal*, 110: 411-436.

A Proofs

A.1 Proof of Theorem 3.1

In order to prove Theorem 3.1, we need to develop a suitable *law of large numbers* on a particular family of probability measures.

In what follows we shall denote with $(\omega_i[0, T], J_i[0, T])$ the trajectory on $[0, T]$ of the state indicators of the i -th agent. We also denote with $\mathcal{D}([0, T])$ the Skorohod space of (discontinuous) trajectories on $[0, T]$ endowed with the weak topology. With the notation $\mathcal{M}_1(X)$ we denote the space of probability measures on X .

Let $(\omega_i[0, T], J_i[0, T])_{i=1}^I \in \mathcal{D}([0, T])^{2I}$ denote a path of the system process in the time-interval $[0, T]$ for a generic $T > 0$. We define the so called *empirical measure* of the I -dimensional system as

$$\rho_I = \frac{1}{I} \sum_{i=1}^I \delta_{(\omega_i[0, T], J_i[0, T])}. \quad (21)$$

We may think of ρ_I as a (random) element of $\mathcal{M}_1(\mathcal{D}([0, T]) \times \mathcal{D}([0, T]))$, the space of probability measures on $\mathcal{D}([0, T]) \times \mathcal{D}([0, T])$ endowed with the weak convergence topology.

Let now q be any probability measure on $\{-1; 1\}^2$. Define

$$m_q^\omega := \sum_{\omega, J = \pm 1} \omega q(\omega, J),$$

that can be interpreted as the average choice of any agent under q . The main result of this subsection is the following.

Theorem A.1 *Suppose that the distribution at time $t = 0$ of the Markov process $(\underline{\omega}(t), \underline{J}(t))_{t \geq 0}$ with generator (10) is such that the random variables $(\omega_i(0), J_i(0))$, $i = 1, \dots, I$, are independent and identically distributed with law λ . Then there exists a probability $Q^* \in \mathcal{M}_1(\mathcal{D}([0, T]) \times \mathcal{D}([0, T]))$ such that*

$$\rho_I \rightarrow Q^* \text{ almost surely}$$

in the weak topology. Moreover, if $q_t \in \mathcal{M}_1(\{-1; 1\}^2)$ denotes the marginal distribution of Q^ at time t , then q_t is the unique solution of the nonlinear*

(McKean-Vlasov) equation

$$\begin{cases} \frac{\partial q_t}{\partial t} = \mathcal{L}q_t, t \in [0, T] \\ q_0 = \lambda \end{cases} \quad (22)$$

where

$$\mathcal{L}q(\omega, J) = \nabla^\omega [e^{-\beta\omega(h+Jm_q^\omega)}q(\omega, J)] + \nabla^J [e^{\phi(\omega, J, m_q^\omega)}q(\omega, J)] \quad (23)$$

with $(\omega, J) \in \{-1, ; 1\}^2$ and where $\nabla^x f(x, y) = f(-x, y) - f(x, y)$.

Proof. See [10]. Notice that the theorem can be applied to both the specifications described by (8) and (9) and for $J \in \{-1, 1\}$ and $J \in \{0, 1\}$. ■

Proof of Theorem 3.1

By Theorem A.1, we know that $\rho_I \rightarrow Q^*$ in the weak topology. As a consequence

$$\int f(\omega[0, T], J[0, T])\rho_I(d\omega[0, T], dJ[0, T]) \rightarrow \int f(\omega[0, T], J[0, T])Q^*(d\omega[0, T], dJ[0, T]).$$

We can also consider a projection at time $t \in [0, T]$, choosing a function $f(\cdot)$ of $\omega(t)$ and $J(t)$. Indeed

$$\int f(\omega(t), J(t))\rho_I(d\omega[0, T], dJ[0, T]) \rightarrow \int f(\omega(t), J(t))Q^*(d\omega[0, T], dJ[0, T]).$$

The latter can be rewritten as

$$\frac{1}{I} \sum_{i=1}^I f(\omega_i(t), J_i(t)) \rightarrow \sum_{\omega, J=\pm 1} f(\omega, J)q_t(\omega, J),$$

where q_t solves (22).

If we now choose $f(\omega, J) = \omega$ we have that $\frac{1}{I} \sum_{i=1}^I f(\omega_i(t), J_i(t)) = s_I^\omega(t)$. In the same way, choosing $f(\omega, J) = J$ we obtain $s_I^J(t)$ and for $f(\omega, J) = \omega J$ we have $s_I^{\omega J}(t)$.

The theorem is thus proved if we show that

$$\sum_{\omega, J=\pm 1} \omega q_t(\omega, J) = m_t^\omega, \quad \sum_{\omega, J=\pm 1} J q_t(\omega, J) = m_t^J, \quad \sum_{\omega, J=\pm 1} \omega J q_t(\omega, J) = m_t^{\omega J}; \quad (24)$$

where $(m_t^\omega, m_t^J, m_t^{\omega J})$ solves (12) (or (13)) when $\phi = \phi^a$ ($\phi = \phi^b$).

Case $\phi = \phi^a$:

Consider $m_t^\omega = \sum_{\omega, J} \omega q_t(\omega, J)$, hence $\dot{m}_t^\omega = \sum_{\omega, J} \omega \dot{q}_t(\omega, J)$. Relying on (23) we then have

$$\begin{aligned} \dot{m}_t^\omega &= \sum_{\omega, J} \omega (\nabla^\omega [e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J)] + \nabla^J [e^{-\gamma Jm_t^\omega} q_t(\omega, J)]) = \\ &= \sum_{\omega, J} \omega (e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) - e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J)) + \\ &\quad + \sum_{\omega, J} \omega (e^{\gamma Jm_t^\omega} q_t(\omega, -J) - e^{-\gamma Jm_t^\omega} q_t(\omega, J)). \end{aligned}$$

We now use the following facts

$$\begin{aligned} \sum_{\omega, J} \omega e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) &= - \sum_{\omega, J} \omega e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J), \\ \sum_{\omega, J} \omega e^{\gamma Jm_t^\omega} q_t(\omega, -J) &= \sum_{\omega, J} \omega e^{-\gamma Jm_t^\omega} q_t(\omega, J). \end{aligned}$$

So that

$$\dot{m}_t^\omega = -2 \sum_{\omega, J} \omega e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J) = -2 \sum_{\omega, J} \omega e^{-\beta\omega h} e^{-\beta\omega Jm_t^\omega} q_t(\omega, J).$$

Moreover, it is easy to check that for $\omega, J \in \{-1; 1\}$, it holds

$$\begin{aligned} e^{-\beta\omega h} &= -\omega \frac{e^{\beta h} - e^{-\beta h}}{2} + \frac{e^{\beta h} + e^{-\beta h}}{2}, \\ e^{-\beta\omega Jm_t^\omega} &= -\omega J \frac{e^{\beta m_t^\omega} - e^{-\beta m_t^\omega}}{2} + \frac{e^{\beta m_t^\omega} + e^{-\beta m_t^\omega}}{2}. \end{aligned}$$

Thus, using the definition of $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, we have

$$\begin{aligned} \dot{m}_t^\omega &= -2 \sum_{\omega, J} \omega [-\omega \sinh(\beta h) + \cosh(\beta h)] [-\omega J \sinh(\beta m_t^\omega) + \cosh(\beta m_t^\omega)] q_t(\omega, J) \\ &= -2 \sum_{\omega, J} [-\sinh(\beta h) + \omega \cosh(\beta h)] [-\omega J \sinh(\beta m_t^\omega) + \cosh(\beta m_t^\omega)] q_t(\omega, J); \end{aligned}$$

where we have used the fact that $\omega^2 = 1$. The product into the []-brackets gives

$$\begin{aligned} &\omega J \sinh(\beta h) \sinh(\beta m_t^\omega) - \sinh(\beta h) \cosh(\beta m_t^\omega) \\ &- J \cosh(\beta h) \sinh(\beta m_t^\omega) + \omega \cosh(\beta h) \cosh(\beta m_t^\omega). \end{aligned}$$

Thus

$$\begin{aligned}\dot{m}_t^\omega &= -2 \sinh(\beta h) \sinh(\beta m_t^\omega) m_t^{\omega J} + 2 \sinh(\beta h) \cosh(\beta m_t^\omega) \\ &\quad + 2 \cosh(\beta h) \sinh(\beta m_t^\omega) m_t^J - 2 \cosh(\beta h) \cosh(\beta m_t^\omega) m_t^\omega,\end{aligned}$$

where we have used (11). We use the same approach to compute \dot{m}_t^J and $\dot{m}_t^{\omega J}$.

$$\begin{aligned}\dot{m}_t^J &= \sum_{\omega, J} J (e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) - e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J)) + \\ &\quad + \sum_{\omega, J} J (e^{\gamma J m_t^\omega} q_t(\omega, -J) - e^{-\gamma J m_t^\omega} q_t(\omega, J)).\end{aligned}$$

Arguing as before we see that

$$\begin{aligned}\sum_{\omega, J} J e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) &= \sum_{\omega, J} J e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J), \\ \sum_{\omega, J} J e^{\gamma J m_t^\omega} q_t(\omega, -J) &= - \sum_{\omega, J} J e^{-\gamma J m_t^\omega} q_t(\omega, J).\end{aligned}$$

So that

$$\dot{m}_t^J = -2 \sum_{\omega, J} J e^{-\gamma J m_t^\omega} q_t(\omega, J).$$

Moreover, it is easy to check that for $\omega, J \in \{-1; 1\}$, it holds

$$e^{-\gamma J m_t^\omega} = -J \frac{e^{\gamma m_t^\omega} - e^{-\gamma m_t^\omega}}{2} + \frac{e^{\gamma m_t^\omega} + e^{-\gamma m_t^\omega}}{2},$$

Thus

$$\begin{aligned}\dot{m}_t^J &= -2 \sum_{\omega, J} J [-J \sinh(\gamma m_t^\omega) + \cosh(\gamma m_t^\omega)] q_t(\omega, J) \\ &= 2 \sinh(\gamma m_t^\omega) - 2 \cosh(\gamma m_t^\omega) m_t^J.\end{aligned}$$

Concerning $m^{\omega J}$ we have

$$\begin{aligned}\dot{m}_t^{\omega J} &= \sum_{\omega, J} \omega J (e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) - e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J)) + \\ &\quad + \sum_{\omega, J} \omega J (e^{\gamma J m_t^\omega} q_t(\omega, -J) - e^{-\gamma J m_t^\omega} q_t(\omega, J)). \\ &= -2 \sum_{\omega, J} \omega J e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J) - 2 \sum_{\omega, J} \omega J e^{-\gamma J m_t^\omega} q_t(\omega, J).\end{aligned}$$

Thus

$$\begin{aligned}\dot{m}_t^{\omega J} &= -2 \sinh(\beta h) \sinh(\beta m_t^\omega) m_t^\omega + 2 \sinh(\beta h) \cosh(\beta m_t^\omega) m_t^J \\ &\quad + 2 \cosh(\beta h) \sinh(\beta m_t^\omega) - 2 \cosh(\beta h) \cosh(\beta m_t^\omega) m_t^{\omega J} \\ &\quad + 2 \sinh(\gamma m_t^\omega) m_t^\omega - 2 \cosh(\gamma m_t^\omega) m_t^{\omega J}.\end{aligned}$$

Case $\phi = \phi^b$:

Arguing as before we have

$$\begin{aligned}\dot{m}_t^\omega &= \sum_{\omega, J} \omega \left(\nabla^\omega [e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J)] + \nabla^J [e^{-\gamma\omega Jm_t^\omega} q_t(\omega, J)] \right) = \\ &= \sum_{\omega, J} \omega \left(e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) - e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J) \right) + \\ &\quad + \sum_{\omega, J} \omega \left(e^{\gamma\omega Jm_t^\omega} q_t(\omega, -J) - e^{-\gamma\omega Jm_t^\omega} q_t(\omega, J) \right).\end{aligned}$$

We now use the following facts

$$\begin{aligned}\sum_{\omega, J} \omega e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) &= - \sum_{\omega, J} \omega e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J), \\ \sum_{\omega, J} \omega e^{\gamma\omega Jm_t^\omega} q_t(\omega, -J) &= \sum_{\omega, J} \omega e^{-\gamma\omega Jm_t^\omega} q_t(\omega, J).\end{aligned}$$

So that

$$\dot{m}_t^\omega = -2 \sum_{\omega, J} \omega e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J) = -2 \sum_{\omega, J} \omega e^{-\beta\omega h} e^{-\beta\omega Jm_t^\omega} q_t(\omega, J).$$

Thus

$$\begin{aligned}\dot{m}_t^\omega &= -2 \sinh(\beta h) \sinh(\beta m_t^\omega) m_t^{\omega J} + 2 \sinh(\beta h) \cosh(\beta m_t^\omega) \\ &\quad + 2 \cosh(\beta h) \sinh(\beta m_t^\omega) m_t^J - 2 \cosh(\beta h) \cosh(\beta m_t^\omega) m_t^\omega,\end{aligned}$$

We use the same approach to compute \dot{m}_t^J and $\dot{m}_t^{\omega J}$.

$$\begin{aligned}\dot{m}_t^J &= \sum_{\omega, J} J \left(e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) - e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J) \right) + \\ &\quad + \sum_{\omega, J} J \left(e^{\gamma\omega Jm_t^\omega} q_t(\omega, -J) - e^{-\gamma\omega Jm_t^\omega} q_t(\omega, J) \right).\end{aligned}$$

Arguing as before we see that

$$\sum_{\omega, J} J e^{\beta\omega(h+Jm_t^\omega)} q_t(-\omega, J) = \sum_{\omega, J} J e^{-\beta\omega(h+Jm_t^\omega)} q_t(\omega, J),$$

$$\sum_{\omega, J} J e^{\gamma \omega J m_t^\omega} q_t(\omega, -J) = - \sum_{\omega, J} J e^{-\gamma \omega J m_t^\omega} q_t(\omega, J).$$

So that

$$\dot{m}_t^J = -2 \sum_{\omega, J} J e^{-\gamma \omega J m_t^\omega} q_t(\omega, J).$$

Moreover, it is easy to check that for $\omega, J \in \{-1; 1\}$, it holds

$$e^{-\gamma \omega J m_t^\omega} = -\omega J \frac{e^{\gamma m_t^\omega} - e^{-\gamma m_t^\omega}}{2} + \frac{e^{\gamma m_t^\omega} + e^{-\gamma m_t^\omega}}{2},$$

Thus

$$\begin{aligned} \dot{m}_t^J &= -2 \sum_{\omega, J} J [-\omega J \sinh(\gamma m_t^\omega) + \cosh(\gamma m_t^\omega)] q_t(\omega, J) \\ &= 2 \sinh(\gamma m_t^\omega) m_t^\omega - 2 \cosh(\gamma m_t^\omega) m_t^J. \end{aligned}$$

Concerning $m^{\omega J}$ we have

$$\begin{aligned} \dot{m}_t^{\omega J} &= \sum_{\omega, J} \omega J (e^{\beta \omega (h + J m_t^\omega)} q_t(-\omega, J) - e^{-\beta \omega (h + J m_t^\omega)} q_t(\omega, J)) + \\ &\quad + \sum_{\omega, J} \omega J (e^{\gamma \omega J m_t^\omega} q_t(\omega, -J) - e^{-\gamma \omega J m_t^\omega} q_t(\omega, J)). \\ &= -2 \sum_{\omega, J} \omega J e^{-\beta \omega (h + J m_t^\omega)} q_t(\omega, J) - 2 \sum_{\omega, J} \omega J e^{-\gamma \omega J m_t^\omega} q_t(\omega, J). \end{aligned}$$

Thus

$$\begin{aligned} \dot{m}_t^{\omega J} &= -2 \sinh(\beta h) \sinh(\beta m_t^\omega) m_t^\omega - 2 \sinh(\beta h) \cosh(\beta m_t^\omega) m_t^J \\ &\quad + 2 \cosh(\beta h) \sinh(\beta m_t^\omega) - 2 \cosh(\beta h) \cosh(\beta m_t^\omega) m_t^{\omega J} \\ &\quad + 2 \sinh(\gamma m_t^\omega) - 2 \cosh(\gamma m_t^\omega) m_t^{\omega J}. \end{aligned}$$

■

Proof of Corollary 3.2.

We set $(\dot{m}_t^\omega, \dot{m}_t^J, \dot{m}_t^{\omega J}) = (0, 0, 0)$. We immediately see that $m_t^J = \tanh(\gamma m_t^\omega)$. The third equation is linear in $m_t^{\omega J}$ so we can write it in the form $m_t^{\omega J} = F(m_t^\omega)$. It is easy to verify that $F(\cdot) = z(\cdot)$ where $z(\cdot)$ is defined in (14). Then we substitute F and $m_t^J = \tanh(\gamma m_t^\omega)$ into the first of (14). Reasoning in the same way we prove also (15). ■

A.2 Proof of Propositions 4.1 and 4.2

Proof of Proposition 4.1.

Notice that $x = \tanh(\beta x) \tanh(\gamma x)$ is a fixed point argument and that $x = 0$ is always solution. Moreover, being $\tanh(\beta x) \tanh(\gamma x) \geq 0$ the possible solutions of (18) can not be negative.

Consider now the case $\beta \leq 1$. Then it is easy to see that $x > \tanh(\beta x)$ for all $x > 0$. Thus, being $\tanh(\gamma x) \leq 1$ for all x , $\tanh(\gamma x) = x / \tanh(\beta x)$ can not admit strictly positive solutions. This proves point 1.

Consider $\beta > 1$. We want to show the existence of $\gamma(\beta)$ (called $\bar{\gamma}$ for brevity) such that (18) has a (unique) positive solution. To this aim, notice that for very small values of γ , the graph of $g(x) = \tanh(\beta x) \tanh(\gamma x)$ lays always below $f(x) = x$ for positive x ; as a consequence $x = 0$ is the only solution to (18). To characterize $\bar{\gamma}$ we use a continuity argument: we conjecture that there exists a unique $\bar{\gamma}$ for which the following system admits a unique solution $\bar{x}(\beta) \in (0, 1)$:

$$\begin{cases} x &= \tanh(\beta x) \tanh(\bar{\gamma} x) \\ 1 &= \frac{2\beta x}{\sinh(2\beta x)} + \frac{2\bar{\gamma} x}{\sinh(2\bar{\gamma} x)} \end{cases} \quad (25)$$

The second equation says that $\frac{d}{dx} \tanh(\beta x) \tanh(\bar{\gamma} x)|_{x=\bar{x}(\beta)} = 1$, hence the curve is tangent to the straight line $f(x) = x$ in the point $\bar{x}(\beta)$. Looking at the concavity of the function $\tanh(\beta x) \tanh(\bar{\gamma} x)$ (convex up to some positive x and then concave) this amounts in showing that $\bar{x}(\beta)$ is the only positive solution to (18) and that $\bar{\gamma}$ is the lowest value for which such a positive solution to (18) does actually exist. We shall now prove this conjecture where moreover $\bar{\gamma}$ and $\bar{x}(\beta)$ are as defined in the statement of the theorem. Then (2.a) and (2.b) immediately follow.

$\beta > 1$ implies that there exists $z \in (0, 1)$ such that $\tanh(\beta z) = z$. Moreover, $\tanh(\beta x) > x$ for all $0 < x < z$. We recall that for small values of γ , $\tanh(\beta x) \tanh(\bar{\gamma} x) < x$ for all $x > 0$. On the other hand, for $\gamma \rightarrow \infty$, $\tanh(\beta x) \tanh(\bar{\gamma} x)$ tends to $\tanh(\beta x)$. As a consequence, for γ large enough there must exist $x > 0$ such that $\tanh(\beta x) \tanh(\bar{\gamma} x) > x$. This implies that there is a threshold value of $\bar{\gamma}$ for which the graph of $\tanh(\beta x) \tanh(\bar{\gamma} x)$ is tangent to $f(x) = x$. The tangent point is exactly \bar{x} . The couple $(\bar{\gamma}, \bar{x})$ is the only solution to (25). Solving the first equation in (25) for γ gives (20), where

$\ell(x) = 2 \tanh^{-1}\left(\frac{x}{\tanh(\beta x)}\right)$. Substituting then in the second, it gives (19). It remains to prove that $\bar{\gamma} > 1$ and $\bar{x} < 1$. If $\bar{\gamma} \leq 1$ then $x/\tanh(\bar{\gamma}x) > 1$ and (18) could not admit positive solutions. Concerning the latter, it is easy to see that $\tanh(\beta x)\tanh(\bar{\gamma}x) < \tanh(\beta x)$. Thus $\bar{x} < z < 1$, where $z = \tanh(\beta z)$ has been defined before. This concludes the proof of (2.a) and (2.b).

To prove (2.c) it is enough to notice that for $\gamma > \bar{\gamma}(\beta)$ there are $x \in (0, 1)$ that lie above the curve $f(x) = x$ and thus necessarily we have two positive solutions $x^{(1)} < x^{(2)} < 1$ to (18). Since for $\gamma > \bar{\gamma}$, $\tanh(\beta x)\tanh(\gamma x) > \tanh(\beta x)\tanh(\bar{\gamma}x)$, we immediately see that $x^{(1)} < \bar{x}(\beta) < x^{(2)}$. ■

Proof of Proposition 4.2.

The matrix of the linearized system is

$$DV(x, y) = 2 \begin{pmatrix} \beta \cosh(\beta x)y - \beta \sinh(\beta x)x - \cosh(\beta x) & \sinh(\beta x) \\ \gamma \cosh(\gamma x) - \gamma \sinh(\gamma x)y & -\cosh(\gamma x) \end{pmatrix},$$

where (x, y) solve (18).

When $(x, y) = (0, 0)$ it is easy to see that the two eigenvalues of $DV(0, 0)$ are negative, thus the equilibrium is linearly stable and this proves point (i).

When (x, y) is non zero, it can be shown that the matrix $DV(x, y)$ can be rewritten as

$$DV(x, y) = 2 \begin{pmatrix} \frac{\beta x}{\sinh(\beta x)} - \cosh(\beta x) & \sinh(\beta x) \\ \frac{\gamma}{\cosh(\gamma x)} & -\cosh(\gamma x) \end{pmatrix},$$

After some algebraic computations one shows that

$$\det(DV(x, y)) = \frac{2}{\cosh(\gamma x)\cosh(\beta x)} \left[1 - \frac{2\beta x}{\sinh(2\beta x)} - \frac{2\gamma x}{\sinh(2\gamma x)} \right]. \quad (26)$$

In the case of $\gamma = \bar{\gamma}(\beta)$, $x = \bar{x}(\beta)$ we have $\det(DV(x, y)) = 0$ by equation (26). Hence in this case $\lambda_1 = 0$ and $\lambda_2 = 2 \cdot \text{Tr}(DV(x, y))$. We have

$$\text{Tr}(DV(x, y)) = \frac{\beta x}{\sinh(\beta x)} - \cosh(\beta x) - \cosh(\gamma x).$$

Notice that $\frac{\beta x}{\sinh(\beta x)} < 1 < \cosh(\beta x)$ then the trace is negative. This implies that there is a critical direction and a stable direction and this proves (ii).

When $\gamma > \bar{\gamma}(\beta)$, we have two positive equilibria, $x^{(1)}$ and $x^{(2)}$. Look at (26). We recall that $\frac{2\beta x}{\sinh(2\beta x)} + \frac{2\gamma x}{\sinh(2\gamma x)} = \frac{d}{dx} \tanh(\beta x)\tanh(\gamma x)$. Thus

$\det(DV(x, y)) < 0$ if and only if $\frac{d}{dx} \tanh(\beta x) \tanh(\bar{\gamma} x) > 1$, and this happens in $x^{(1)}$. As a consequence $x^{(1)}$ is certainly unstable since $\det(DV(x, y)) = \lambda_1 \lambda_2 < 0$ and then one of the eigenvalues is positive. Concerning $x^{(2)}$, we have $\det(DV(x, y)) > 0$. The eigenvalues have thus the same sign, in particular $\text{sign}(\lambda) = \text{sign}(\text{tr}(DV(x, y)))$. Since the sign of the trace is negative, $x^{(2)}$ is stable. ■