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A LINEAR FILTERING APPROACH FOR INCOMPLETE ACCOUNTING INFORMATION MODELS

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Abstract

In this paper, we propose a discrete time model for a firm asset value process, in a context of incomplete accounting information. We model the logarithm Z of the firm asset value process as a Markov chain. The debtholders do not have perfect information about the actual value of the firm: they receive only a discrete noisy stream of reports Y. We study the pair $\{Z, Y\}$ relying on linear filtering techniques. We also characterize the marginal distributions of the filter, discussing some significant properties.

1. Introduction

In this work, we develop a discrete time model under filtering language to provide closed form equation for the conditional densities of the value of the asset of a firm, assuming that debtholders have only partial information. In Duffie and Lando [1], it is studied the effect of *imperfect information* on

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the default probabilities and indeed on the term structure of credit spreads. In that paper, the logarithm of the firm value process Z_t is modeled as a Brownian motion with drift m and volatility σ, assuming that only a stream of discrete noisy reports Y_{t_i} is given to the secondary market. This means that the debtholders only have partial information about the real value of the firm. An important "shadow information", available for both shareholders and debtholders, is given by the fact that they do know in each time if the firm has (or has not) already defaulted. In this model, the default is controlled by the stopping time $\tau := \inf\{t : Z_t \leq \underline{z}\}$.

The problem of computing the density of the unobserved asset price process, conditional on public information, has been analyzed by recent literature by means of (non-linear) filtering schemes (see, among the others, Frey and Runggaldier [2] and Frey and Schmidt [3, 4]). All these models are very general and the filter densities can often be recovered relying on particle filters (see Runggaldier [6]).

What we propose in this paper, is a simple linear filter formulation of this problem on a discretized time scale $(t_1, \ldots, t_n)$, where many computations may be explicitly provided. More precisely, we rely on the following system:

$$\begin{cases} Z_{n+1} = m\Delta t + Z_n + \sigma\sqrt{\Delta t}\, w_n, \\ Y_n = A_n Z_n + B_n v_n, \end{cases}$$

where (v_n) and (w_n) are independent sequences of independent standard normal random variables and $A_n = B_n = \mathbb{I}_{\{t_n \in \mathcal{S}\}}$, being $\mathcal{S}$ the set of times at which the secondary market receives a report.

We study the system $\{Z, Y\}$ via the conditional distributions of Z given Y. It turns out that it is not trivial to link the information given by the discrete observations with the shadow information given by the no default indicator. The final aim is to predict at any time t_n the behavior of Z_{t_n} conditioned on the public (and partial) information. Moreover, we provide an estimate of the filter density in the limiting case where the increment between t_n and t_{n+1} tends to zero (i.e., we are back to a continuous time model).

The remainder of this paper is as follows: In Section 2, we recall the related literature. In Section 3, we propose a discretization of the model, analyzing some important features of the filter density. Section 4 concludes.

2. A Structural Model under Partial Information

In Duffie and Lando [1], it is analyzed the credit risk related to a corporate debt, in the case when debtholders are only partially informed about the real firm asset value of the company. Partial (or incomplete) information leads to the fact that yield spreads on defaultable bonds are non-zero even for short time maturity products (when maturity tends to zero).

Let V represent the asset value of a firm, where

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \tag{1}$$

$\sigma > 0$ and $\mu \in \mathbb{R}$. A solution to (1) is provided by

$$V_t = e^{Z_t}; \quad Z_t = Z_0 + mt + \sigma W_t; \quad m = \mu - \sigma^2/2. \tag{2}$$

The firm is operated by its equity holders. The equity holders' only choice is when to liquidate the firm. Under mild assumptions, it is proved in Leland [5] that the *optimal liquidation policy* τ corresponds to liquidate at the first time in which V is at or below a level $\underline{v}$, so that

$$\tau = \inf\{t : V_t \leq \underline{v}\}. \tag{3}$$

Concerning the bond holders, they only receive imperfect and periodic signals about V_t. More in details, what they observe is the process $(Y_t)_{t \in \{t_1, \dots, t_n\}}$, where $(t_i)_{i \leq n}$ is a stream of imperfect information selected times and where

$$Y_t = Z_t + U_t, \tag{4}$$

being $U_{t_1}, \dots, U_{t_n}$ i.i.d. random variables with distribution $\mathcal{N}(0, 1)$.

Having introduced the financial model, we now specify the mathematical structure behind the asset value process, the stream of reports and the default indicator. In particular, it is worth to distinguish two different filtrations, the perfect filtration $(\mathcal{F}_t)_t$ generated by V_t and the imperfect bond market filtration $(\mathcal{H}_t)_t$ generated jointly by the noisy accounting report stream and by the indicator of no default.

Definition 2.1 (Imperfect information sigma algebra)**.** Given Y_t as in (4), the imperfect information sigma algebra is $(\mathcal{H}_t)_t$, where

$$\mathcal{H}_t = \sigma(\{Y_{t_1}, \ldots, Y_{t_n}, \mathbb{I}_{\{\tau \le s\}} : 0 \le s \le t\}) \tag{5}$$

for the largest n such that $t_n \le t$, and where τ is as defined in (3).

3. The Discrete Time Model

In this section, we describe the discrete time model. The final aim is to provide a closed formula for the probabilities of default of the firm value process. The state variable Z (corresponding to (2) in the Brownian case) has the following discrete time evolution from time $n\Delta t$ to time $(n+1)\Delta t$:

$$Z_{n+1} = m\Delta t + Z_n + \sigma\sqrt{\Delta t}\, w_n, \tag{6}$$

where the process $(w_n)_{n \in \mathbb{N}}$ is a white noise such that $w_n \sim \mathcal{N}(0, 1)$ and m, σ are defined as in (2). $Z_0 = z_0$ is known.

In the same way, we consider the evolution of the discrete observable variable Y_n as the information we receive at discrete times about the real value of the unobserved process Z_n.

To be consistent with the continuous time model, we assume that the process Y can take place on a subset of the time scale. We call *pricing times* the instants in which the price Z is computed (but not necessarily given to the market) and *report times* the instants when a new signal Y is available to the market. Summarizing, we have:

$$Y_n = A_n Z_n + B_n v_n, \tag{7}$$

where $A_n = B_n = \mathbb{I}_{\{t_n \in S\}}$ and $\mathcal{S} = \{report\ times\} \subset \mathcal{I} = \{pricing\ times\}$. $(v_n)_n$ is a white noise with $v_n \sim \mathcal{N}(0, 1)$ independent of w, and $Y_0 \equiv Z_0 = z_0$ is perfectly known. We obtain the system

$$\begin{cases} Z_{n+1} = m\Delta t + Z_n + \sigma\sqrt{\Delta t}\, w_n, \\ Y_n = A_n Z_n + B_n v_n. \end{cases} \tag{8}$$

In order to provide an explicit expression for the distribution of $\mathbf{Z} | \mathbf{Y}$ (i.e., the distribution of $\mathbf{Z}$ conditioned on the observation $\mathbf{Y}$), we first prove two lemmas concerning the unconditional distribution of the vector $\mathbf{Z}$.

Lemma 3.1. *The vector* $\mathbf{Z} = (Z_1, ..., Z_k)$ *is multivariate normal*, $\mathbf{Z} \sim \mathcal{N}_k(\mathbf{A}, \Sigma)$, *where* $\Sigma = B'B$ *with*

$$\mathbf{A} = z_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + m\Delta \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix}; \quad B = \sigma\sqrt{\Delta} \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & & \cdots & 1 & 0 \\ 1 & & \cdots & & 1 \end{pmatrix}.$$

Proof. By (6), $Z_k = z_0 + km\Delta + \sigma\sqrt{\Delta}(w_1 + \cdots + w_k)$, $\forall k$. Then $\mathbf{Z} = \mathbf{A} + B\mathbf{W}$, being $\mathbf{W} = (w_1, ..., w_k)$ the white noise defined in (6). □

Lemma 3.2. *The joint distribution of* $\mathbf{Z}$, $\mathbf{P}(Z_1 \leq z_1, ..., Z_k \leq z_k)$, *can be written as*

$$\int_{C \subset \mathbb{R}^k} \frac{1}{\sqrt{(2\pi)^k \Delta\sigma^2}} \exp\{\hat{u}_1^2 + (\hat{u}_2 - \hat{u}_1)^2 + \cdots + (\hat{u}_k - \hat{u}_{k-1})^2\} d\mathbf{u},$$

where $C = \{(-\infty, z_1]; ...; (-\infty, z_k]\}$ *and* $\hat{u}_i = u_i - a_i$.

Proof. Put $\mathbf{A} = \mathbf{0}$ (the term $\mathbf{A}$ contributes only in a translation of the values z_i, where we compute the joint distribution), so that $\hat{\mathbf{u}} = \mathbf{u}$. Being $\mathbf{Z}$ Gaussian (see Lemma 3.1), it holds

$$\mathbf{P}(Z_1 \le z_1, ..., Z_k \le z_k) = \int_C \sqrt{\frac{|\Sigma^{-1}|}{(2\pi)^k}} \exp\{(\mathbf{z} - \mathbf{A})' \Sigma^{-1} (\mathbf{z} - \mathbf{A})\} d\mathbf{z}.$$

It is easy to see that

$$\Sigma^{-1} = (\sigma^2 \Delta)^{-1} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 \\ \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

In particular, $|\Sigma^{-1}| = 1.$ Moreover,

$$\mathbf{z}'\Sigma^{-1}\mathbf{z} = 2z_1^2 - 2z_1 z_2 + 2z_2^2 + \cdots + z_k^2. \qquad \square$$

We now provide a matrix representation also for the reports' vector $\mathbf{Y}$. Assume that in the interval $[t_0, t_k]$, we see $h \le k$ reports at dates $(t_1, ..., t_h)$. To represent this situation in a reduced form, we define a non-square 0-1 matrix $M \in \{0, 1\}^{h \times k}$ such that

$$M = (M)_{ij}, \text{ where } M_{ij} = 1 \Leftrightarrow t_i = j. \tag{9}$$

Consequently, we can write

$$\mathbf{Y} = M\mathbf{Z} + \mathbf{v}, \tag{10}$$

where $\mathbf{Y} \in \mathbb{R}^h$, $\mathbf{Z} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^h \sim \mathcal{N}_h(\mathbf{0}, I_h)$ is independent from $\mathbf{Z}$.

We can think of M as a "*report policy*" a priori decided by the shareholders.

Given M as in definition (9), we introduce some notations:

$$M'M = \tilde{I}_k \in \mathbb{R}^{k \times k},$$

$$MM' = I_h \in \mathbb{R}^{h \times h},$$

$$V = \Sigma^{-1} + M'M,$$

$$\tilde{V} = (M\Sigma M' + I_h)^{-1}.$$

The square matrix $M'M = \widetilde{I}_k$ is diagonal where entries corresponding to "no report times" are zero (therefore it is generally not invertible). The matrix MM' is simply the identity matrix in $\mathbb{R}^h$.

Concerning $\widetilde{V}$, taking a matrix X, the map $\mathbb{R}^{k\times k} \to \mathbb{R}^{h\times h}$, $X \to MXM'$, operates a "cutting" of the rows and the columns corresponding to times where we do not see a report.

Lemma 3.3 (Matrix Inversion Lemma). *The following equalities hold*:

(i) $\widetilde{V} := (M\Sigma M' + I_h)^{-1} = (MV\Sigma M')^{-1}$;

(ii) $|\widetilde{V}|^{-1} = |V|$;

(iii) $(MV\Sigma M')^{-1} = M(V\Sigma)^{-1}M'$.

Proof. Notice that, according to the definitions of V and M,

$$MV\Sigma M' = M(\Sigma^{-1} + \widetilde{I}_k)\Sigma M' = M(I_k + \widetilde{I}_k\Sigma)M'$$

$$= MM' + M\widetilde{I}_k\Sigma M' = I_h + M\Sigma M',$$

and this proves (i). Thanks to (i), (ii) corresponds to prove that

$$|M(V\Sigma)M'| = |\Sigma^{-1} + M'M|.$$

Since $|\Sigma| = 1$, we have $|\Sigma^{-1} + M'M| = |\Sigma^{-1} + M'M||\Sigma| = |V\Sigma|$. Then we must ensure that

$$|V\Sigma| = |MV\Sigma M'|.$$

Thanks to the particular form of the matrix $(V\Sigma) = (\widetilde{I}_k\Sigma + I_k)$, it can be shown that the determinant of the reduced one (in $\mathbb{R}^{h\times h}$) is the same as the initial one. Indeed, consider row $\bar{k}$ of $V\Sigma$ corresponding to a time of no report (i.e., $(\widetilde{I}_k)_{\bar{k}\bar{k}} = 0$). Then $\widetilde{I}_k\Sigma$ has a zero vector as $\bar{k}$th row. Summing

I_k, we obtain a $\bar{k}$th row of the type $(0, \ldots, 0, 1, 0, \ldots, 0)$. Then the determinant of the original matrix is exactly the same as the determinant of the reduced matrix (in $\mathbb{R}^{k-1\times k-1}$), where we have cut the $\bar{k}$th row and column. This can be done for all rows of no report. Point (iii) is proved arguing in a similar way. □

We are ready to provide a shape for the conditional distribution of $\mathbf{Z}|\mathbf{Y}$.

Proposition 3.4. *Given a multivariate vector* $\mathbf{Z}$ *and observations* $\mathbf{Y}$ *as in equation* (10) *and a general report time matrix* M, *we have*

$$\mathbf{Z}|\mathbf{Y} \sim \mathcal{N}_k(\mu, V^{-1}), \tag{11}$$

where $\mu = V^{-1}(\Sigma^{-1}\mathbf{A} + \mathbf{Y}M')$ *and* $V = (\Sigma^{-1} + MM')$.

Proof. Recall that $\mathbf{Z} \sim \mathcal{N}_k(\mathbf{A}, \Sigma)$. From (10), we have

$$\mathbf{Y}|\mathbf{Z} \sim \mathcal{N}_h(M\mathbf{z}, I_h), \tag{12}$$

$$\mathbf{Y} \sim \mathcal{N}_h(M\mathbf{A}, M\Sigma M' + I_h). \tag{13}$$

We need to prove that

$$f(\mathbf{Z}|\mathbf{Y}) = \frac{\sqrt{|V|}}{(\sqrt{2\pi})^k}\exp\left\{-\frac{1}{2}(\mathbf{z}-\mu)'V(\mathbf{z}-\mu)\right\}.$$

Note that

$$f(\mathbf{Z}) = \frac{1}{(\sqrt{2\pi})^k}\exp\left\{-\frac{1}{2}(\mathbf{z}-\mathbf{A})'\Sigma^{-1}(\mathbf{z}-\mathbf{A})\right\},$$

$$f(\mathbf{Y}|\mathbf{Z}) = \frac{1}{(\sqrt{2\pi})^h}\exp\left\{-\frac{1}{2}(\mathbf{Y}-M\mathbf{z})'I_h(\mathbf{Y}-M\mathbf{z})\right\},$$

$$f(\mathbf{Y}) = \frac{\sqrt{|\widetilde{V}|}}{(\sqrt{2\pi})^h}\exp\left\{-\frac{1}{2}(\mathbf{Y}-M\mathbf{A})'\widetilde{V}(\mathbf{Y}-M\mathbf{A})\right\},$$

where $\widetilde{V} = (M\Sigma M' + I_h)^{-1}$ is the variance of $\mathbf{Y}$. Relying on Bayes rule,

$$f(\mathbf{Z}|\mathbf{Y}) = \frac{\sqrt{|\widetilde{V}|^{-1}}}{(\sqrt{2\pi})^k}$$

$$\cdot \frac{\exp\left\{-\frac{1}{2}[\mathbf{A}'\Sigma^{-1}\mathbf{A} + \mathbf{Y}'\mathbf{Y} - 2(\Sigma^{-1}\mathbf{A} + \mathbf{Y}'M)'\mathbf{z} + \mathbf{z}'(\Sigma^{-1} + \widetilde{I}_k)\mathbf{z}]\right\}}{\exp\left\{-\frac{1}{2}(\mathbf{Y} - M\mathbf{A})'\widetilde{V}(\mathbf{Y} - M\mathbf{A})\right\}}.$$

Using the definition of μ and Lemma 3.3, it is not difficult to see that

$$f(\mathbf{Z}|\mathbf{Y}) = \frac{\sqrt{|V|}}{(\sqrt{2\pi})^k} \frac{\exp\left\{-\frac{1}{2}[\mathbf{A}'\Sigma^{-1}\mathbf{A} + \mathbf{Y}'\mathbf{Y} - 2\mu' V\mathbf{z} + \mathbf{z}'V\mathbf{z}]\right\}}{\exp\left\{-\frac{1}{2}(\mathbf{Y} - M\mathbf{A})' M(V\Sigma)^{-1}M'(\mathbf{Y} - M\mathbf{A})\right\}}.$$

Thanks to Lemma 3.3(ii) and (iii) all what we have to prove is

$$\mathbf{A}'\Sigma^{-1}\mathbf{A} + \mathbf{Y}'\mathbf{Y} - (\mathbf{Y} - M\mathbf{A})' M(V\Sigma)^{-1}M'(\mathbf{Y} - M\mathbf{A}) = \mu' V\mu.$$

For simplicity, we suppose $\mathbf{A} = 0$. A more involved calculation provides a generalization for $\mathbf{A} \in \mathbb{R}^k$. We can rewrite the latter equation as

$$\mathbf{Y}'\mathbf{Y} - \mathbf{Y}'M(V\Sigma)^{-1}M'\mathbf{Y} = \mathbf{Y}'MV^{-1}VV^{-1}M'\mathbf{Y}.$$

Simplifying and multiplying the first term by MM', we get

$$\mathbf{Y}'MM'\mathbf{Y} - \mathbf{Y}'M(V\Sigma)^{-1}M'\mathbf{Y} = \mathbf{Y}'MV^{-1}M'\mathbf{Y}.$$

Using the definition of V, the l.h.s. can be rewritten as

$$\begin{aligned}\mathbf{Y}'MM'\mathbf{Y} - \mathbf{Y}'M(V\Sigma)^{-1}M'\mathbf{Y} &= \mathbf{Y}'MVV^{-1}M'\mathbf{Y} - \mathbf{Y}'M(V\Sigma)^{-1}M'\mathbf{Y} \\ &= \mathbf{Y}'M(\Sigma^{-1} + \widetilde{I}_k)V^{-1}M'\mathbf{Y} - \mathbf{Y}'M\Sigma^{-1}V^{-1}M'\mathbf{Y} \\ &= \mathbf{Y}'M\widetilde{I}_kV^{-1}M'\mathbf{Y} = \mathbf{Y}'MV^{-1}M'\mathbf{Y}\end{aligned}$$

which concludes. □

3.1. Adding the information of no default

In this subsection, we compute the distribution of the vector $\mathbf{Z}$ conditioned on both the reports and the new information that default has not yet occurred up to the present time. We have

$$P(\mathbf{Z} \leq \mathbf{z} \mid \mathbf{Y}, \mathbf{Z} > \mathbf{c}) = \frac{P(\mathbf{Z} \leq \mathbf{z}, \mathbf{Z} > \mathbf{c} \mid \mathbf{Y})}{P(\mathbf{Z} > \mathbf{c} \mid \mathbf{Y})} = \mathbb{I}_{\{\mathbf{z}>\mathbf{c}\}} \frac{P(\mathbf{Z} \in (\mathbf{c}, \mathbf{z}] \mid \mathbf{Y})}{P(\mathbf{Z} > \mathbf{c} \mid \mathbf{Y})}.$$

Taking the derivative w.r.t. $\mathbf{Z}$, we can write the joint density $\mathbf{f}_{\mathbf{Z}|\mathcal{H}_n}(\mathbf{z})$ of the process as

$$\mathbf{f}_{\mathbf{Z}|\mathcal{H}_n}(\mathbf{z}) = \mathbb{I}_{\{\mathbf{z}>\mathbf{c}\}} \frac{\varphi(\mathbf{z})}{P(\mathbf{Z} > c \mid \mathbf{Y})}, \tag{14}$$

where $\varphi(\mathbf{z})$ is the density of the multivariate Gaussian distribution found in Proposition 3.4. $\mathbf{f}_{\mathbf{Z}|\mathcal{H}_n}(\mathbf{z})$ can be computed (at least numerically) given the model as in equation (8).

We will now concentrate on the marginal density of the process Z_n given $\mathcal{H}_n = \sigma\{Y_k, \tau > k; k \leq n\}$. We call it $f^n_{Z_n|\mathcal{H}_n}(z)$, where the superscript n indicates that we are looking at the n-dimensional vector. In particular, we show that this density is well defined and has some good properties in the interior of the domain where it is non-zero (i.e., on $\mathcal{C} = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} > \mathbf{c}\}$). Furthermore, we discuss some limit properties of such a density function.

We first marginalize the distribution

$$P(Z_n \leq \zeta \mid \mathcal{H}_n) = \int_{-\infty}^{\zeta} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathbb{I}_{\{\mathbf{z}>\mathbf{c}\}} \frac{\varphi(\mathbf{z})}{P(\mathbf{Z} > \mathbf{c} \mid \mathbf{Y})} d\mathbf{z}, \tag{15}$$

then we can compute $f^n_{Z_n|\mathcal{H}_n}(\zeta)$ as

$$\begin{aligned} f^n_{Z_n|\mathcal{H}_n}(\zeta) &= \frac{d}{d\zeta} P(Z_n \leq \zeta \mid \mathcal{H}_n) \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathbb{I}_{\{\mathbf{z}_{-n}>\mathbf{c}\}} \frac{\varphi(z_1, \ldots, z_{n-1}, \zeta)}{P(\mathbf{Z} > \mathbf{c} \mid \mathbf{Y})} d\mathbf{z}_{-n}, \end{aligned} \tag{16}$$

where $\mathbf{z}_{-k} \in \mathbb{R}^{n-1}$ corresponds to $\mathbf{z}$ without the kth component.

Notice that we are allowed to cut the kth indicator $\mathbb{I}_{\{\zeta > c\}}$ because we know that we are in the interior of the domain $\zeta > c$ by assumption.

In order to study correctly the behavior of the densities of the variables Z_k when increments tend to zero, we refine the discretization of the interval $[0, \bar{t}\,]$ that we are considering.

Definition 3.5. On the interval $[0, \bar{t}\,]$, where $\bar{t} > 0$, define

- $(\varsigma^n)_n$ a sequence of partitions $(0 = t_0 < t_1 < \cdots < t_n = \bar{t}\,)$ on $[0, \bar{t}\,]$;
- $\Delta t^n := |\,\varsigma^n\,| = \bar{t}/n$;
- $t_k^n := k\Delta t^n$, the $(k+1)$th time of the partition ς^n;
- $t_0^n = t_0$ and $t_n^n = \bar{t}$ for all n;
- $n = 2^l$, $l \in \mathbb{N}$.

For a fixed ς^n, define

$$f^n(z, t_k^n) := P(Z(t_k^n) \in dz \,|\, \mathcal{H}_{t_k^n}), \tag{17}$$

as the conditional density of Z_k under the partition ς^n given the available information up to time t_k^n. We denote by

$$f(z, t) := P(Z^*(t) \in dz \,|\, \mathcal{H}_t),$$

the density of a Brownian motion $Z^*(t)$ with drift m and volatility σ, knowing that the trajectory of the Brownian motion up to time t has not fallen under the threshold level c. Notice that, for any ς_n, we have the equality in distribution $Z_{t_k^n} \overset{d}{=} Z^*(t_k^n)$.

Proposition 3.6. *Given the vectors* $\mathbf{Z} \in \mathbb{R}^n$ *and* $\mathbf{Y} \in \mathbb{R}^h$, *where* h *is finite and* $n \geq h$;

(i) *the conditional densities* $f^n(z, t_k^n)$ *of* Z_k *exist and are continuous for each k on the closed domain* $[c, +\infty)$;

(ii) $f^n(z, t_k^n)$ *are analytic functions on the open set* (c, ∞).

Proof. First, we want to show that the probability $P(\mathbf{Z} > c \,|\, \mathbf{Y})$ is strictly positive, independently of the dimension n of the vector $\mathbf{Z}$. Indeed,

$$P(\mathbf{Z} > c) \geq P(Z^*(t) > c, \forall t) =: \widetilde{P}^* > 0, \forall n, \tag{18}$$

where $Z^*(t)$ is a Brownian motion with drift m and variance σ. We can, in fact, see $\mathbf{Z} = (Z_1, ..., Z_n)$ as observations of Z^* made at $(t_1, ..., t_n)$. Indeed, for each n, it is true that

$$\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \overset{d}{=} \begin{pmatrix} Z^*(t_1) \\ \vdots \\ Z^*(t_n) \end{pmatrix}.$$

As a consequence, the event $\{\mathbf{Z} > \mathbf{c}\}$ contains the event $\{Z^*(t) > c, \forall t\}$. $\widetilde{P}^*$ in (18) is defined as the hitting probability of a Brownian motion, which is always strictly positive.

Notice that this idea can be used only in the case when the increment Δt between two pricing times converges to zero when $n \to \infty$.

This implies that, if we fix a final time $\bar{t}$, then the discretization $(0 = t_0, ..., t_n = \bar{t})$ converges to the continuous interval $[0, \bar{t}]$.

We can now substitute the unconditional probabilities in (18) with the corresponding conditional ones, obtaining

$$P(\mathbf{Z} > c \,|\, \mathbf{Y}) \geq P(Z^*(t) > c, \forall t \,|\, \mathbf{Y}) =: P^* > 0, \forall n. \tag{19}$$

To prove (i), let us define $g_{Z_k}^n(\zeta)$ as the marginal density of the variable Z_k conditioned only on $\mathbf{Y}$ (see Proposition 3.4), meaning

$$g_{Z_k}^n(\zeta) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(z_1, ..., \zeta, ..., z_n) d\mathbf{z}_{-k}. \tag{20}$$

Then from (15) and (16), we can bound $f^n(\zeta, t_k^n)$ as follows:

$$\begin{aligned} 0 \le f^n(\zeta, t_k^n) &= \frac{\partial}{\partial \zeta} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{\zeta} \cdots \int_{-\infty}^{+\infty} \mathbb{I}_{\{\mathbf{z} > \mathbf{c}\}} \frac{\varphi(\mathbf{z})}{P(\mathbf{Z} > c | \mathbf{Y})} d\mathbf{z} \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathbb{I}_{\{\mathbf{z}_{-k} > \mathbf{c}\}} \frac{\varphi(z_1, ..., \zeta, ..., z_n)}{P(\mathbf{Z} > c | \mathbf{Y})} d\mathbf{z}_{-k} \\ &\le \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\varphi(z_1, ..., \zeta, ..., z_n)}{P(\mathbf{Z} > c | \mathbf{Y})} d\mathbf{z}_{-k} \\ &= \frac{1}{P(\mathbf{Z} > c | \mathbf{Y})} g_{Z_k}^n(\zeta) \le \frac{1}{P^*} g_{Z_k}^n(\zeta) < \infty. \end{aligned}$$

The first inequality follows from the fact that we are integrating a positive function on a bigger domain and the second from (19).

Continuity of the density $f^n(\zeta, t_k^n)$ on the open domain $\{\zeta > c\}$ is ensured by the continuity of $\varphi(\cdot)$. So that point (i) is proved.

Notice also that, being the function $\varphi(\cdot, \zeta)$ uniformly bounded by an integrable function, point (i) can be extended to the closed domain $\{\zeta \ge c\}$. Indeed, consider the limit $\zeta \searrow c$. By the continuous version of Lebesgue's Theorem of Dominated Convergence, we can interchange limit and integral operator.

To prove point (ii), we rely on a recursive argument for $f^n(z, t_k^n)$ in the time variable. Since (n, z) is fixed in this context, to simplify the notation, we will use the notation f^k in place of $f^n(z, t_k^n)$.

We can construct f^n starting from f^0 using a three step recursion: we start from $f^0 = \delta_{z_0}$. The first step is to use equation (8) to pass from t_{k-1} to t_k. Then we add the information given by the variable Y_k and finally the information given by the no default indicator.

We now describe this algorithm, noticing that it preserves the analyticity on the open set (c, ∞).

(1) For any $k = 1, \ldots, n,$ define $\widetilde{f}^n(z, t_n^n)$ the convolution of the two densities: $f^n(\cdot, t_{n-1}^n)$ and $g_n(z),$ where the latter is the density of $\widetilde{w}_n,$ i.e.,

$$g_k(z) := \frac{1}{\sqrt{2\pi}\sigma\Delta t^n} e^{-\frac{1}{2}\left(\frac{z - m\Delta t^n}{\sigma\Delta t^n}\right)^2}$$

so that

$$\widetilde{f}^n(z, t_k^n) = \int_{-\infty}^{+\infty} g_k(x) f^n(z - x, t_{k-1}^n) dx.$$

Note that this is the density function we obtain using the algorithm suggested by the first equation in (8).

After the first run $(k = 1),$ we obtain $\widetilde{f}^1$ which is analytic because it is a convolution of a delta function with an analytic function. For $k > 1,$ $\widetilde{f}^k$ is again always analytic thanks to the convolution with the Gauss kernel.

(2) Conditioning on $Y_k,$ we obtain a density $\hat{f}^k$ which is again Gaussian with different mean and variance (see Proposition 3.4). This step again preserves the analyticity.

(3) In order to obtain $f^k,$ we have to take into account the no default information. We operate a "truncation" at the level c. This implies that all the mass of the interval $(-\infty, c]$ is moved to the complementary set $(c, +\infty)$.

$$f^n(z, t_k^n) = \begin{cases} 0, & \text{if } z \le c, \\ C(k, n)\hat{f}^n(z, t_k^n), & \text{if } z > c \end{cases}$$

for some strictly positive constant $C(k, n)$. On the set $\{z > c\},$ the mass changes but the properties of the function are left unchanged. This proves that $f^n(z, t_k^n)$ is analytic for any k and $z > c.$ □

We now extend the results of Proposition 3.6 also to the limit case where n is letting go to infinity. We prove in particular a weak convergence "in distribution" for the density functions when the increments go to zero.

Proposition 3.7. *For any measurable set* $C \in \mathbb{R}$ *and* $\eta > 0$, *there exists* $\bar{n}$ *big enough such that for any* $n > \bar{n}$,

$$\left| \int_C f^n(\zeta, \bar{t}\,) d\zeta - \int_C f(\zeta, \bar{t}\,) d\zeta \right| \leq \eta \ a.s. \tag{21}$$

Proof. As already noticed, $Z_{t_k^n} \overset{d}{=} Z^*(t_k^n)$, where $Z^*(t)$ is a Brownian motion with drift m and volatility σ. We thus identify $Z_{t_k^n}$ with $Z^*(t_k^n)$.

Take any measurable set $C \in \mathbb{R}$, by definition of $\mathcal{H}_k$ and since $t_n^n = \bar{t}$ for all n, we can write

$$P(Z_{t_n^n} \in C \,|\, \mathcal{H}_{t_n^n}) = \frac{P(Z_{\bar{t}} \in C \cap \mathbf{Z} > \mathbf{c} \,|\, \mathbf{Y})}{P(\mathbf{Z} > \mathbf{c} \,|\, \mathbf{Y})}$$

$$= \frac{P(\{Z^*(\bar{t}) \in C\} \cap \{Z^*(t_k^n) > c, \forall k \leq n\} \,|\, \mathbf{Y})}{P(\{Z^*(t_k^n) > c, \forall k \leq n\} \,|\, \mathbf{Y})}. \tag{22}$$

Note that the event $\{Z^*(t_k^n) > c, \forall k \leq n\}$ is the union of two disjoint events:

$$[\{Z^*(t_k^n) > c, \forall k \leq n\} \cap \{Z^*(t) > c, \forall t\}]$$

$$\cup [\{Z^*(t_k^n) > c, \forall k \leq n\} \cap \{Z^*(t) < c, \exists t \neq t_k^n\}].$$

It can be rewritten as

$$[\{Z^*(t) > c, \forall t\}] \cup [\{Z^*(t_k^n) > c, \forall k \leq n\} \cap \{Z^*(t) < c, \exists t \neq t_k^n\}].$$

We use this equivalence in (22), obtaining

$$\frac{P(\{Z^*(\bar{t}) \in C\} \cap \{Z^*(t) > c, \forall k\} | \mathbf{Y})}{P(\{Z^*(t_k^n) > c, \forall k \le n\} | \mathbf{Y})}$$

$$+ \frac{P(\{Z^*(\bar{t}) \in C\} \cap [\{Z^*(t_k^n) > c, \forall k \le n\} \cap \{Z^*(t) < c, \exists t \ne t_k^n\}] | \mathbf{Y})}{P(\{Z^*(t_k^n) > c, \forall k \le n\} | \mathbf{Y})}. \quad (23)$$

Now we take the limit for $n \to \infty$ on both sides. Notice that, defining

$$[\{Z^*(t_k^n) > c, \forall k \le n\} \cap \{Z^*(t) < c, \exists t \ne t_k^n\}] = A_n,$$

then

$$P\left(\lim_{n \to \infty} A_n\right) = 0.$$

Thus, the second term in (23) tends to zero (the denominator is always positive). Similarly, we can see that

$$P(\{Z^*(t_k^n) > c, \forall k \le n\} \to \{Z^*(t) > c, \forall t\}) = 1,$$

so that

$$\lim_{n \to \infty} P(Z_{t_n^n} \in C, \mathbf{Z} > \mathbf{c}) = \frac{P(\{Z^*(\bar{t}) \in C\} \cap \{Z(t) > c, \forall t\})}{P(\{Z(t) > c, \forall t\})}.$$

The r.h.s. can be rewritten as $P(Z^*(\bar{t}) \in C, Z^*(\bar{t}) > c, \forall t | \mathcal{H}_{\bar{t}})$ and this is exactly the distribution of the geometric Brownian motion conditioned on $\mathcal{H}_t = \sigma(\{Y_{t_1}, ..., Y_{t_n}, \mathbf{1}_{\{\tau \le s\}} : 0 \le s \le t\})$ defined in the original continuous time model. We obtain

$$\lim_{n \to \infty} P(Z_{t_n^n} \in C | \mathcal{H}_{\bar{t}}) = P(Z^*(\bar{t}) \in C | \mathcal{H}_{\bar{t}}).$$

For any measurable set C, there exist n big enough and η such that

$$\left| \int_C f^n(\zeta, \bar{t}) d\zeta - \int_C f(\zeta, \bar{t}) d\zeta \right| \le \eta \text{ a.s.} \qquad \square$$

4. Conclusions

We showed how to explicitly compute the filtering density of a discretized version of a classical structural firm value model. In particular, assuming that the firm value is not perfectly observable, one has to consider an imperfect filtration generated by a noisy stream of discrete report times. Relying on classical linear filter theory, we provided analytical expressions for the filtering densities and we discussed some interesting properties. In particular, we provided regularity conditions of the density functions and convergence results when the time increments tend to zero.

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