Chapter 1
Symmetric Equilibria in Double Auctions with Markdown Buyers and Markup Sellers

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Abstract Zhan and Friedman (2007) study double auctions where buyers and sellers are constrained to using simple markdown and markup rules. In spite of the alleged symmetry in roles and assumptions, buyers are shown to have the upper hand both in the call market and in the continuous double auction. We replicate the study and show that their formulation of the sellers' markup strategies, while seemingly natural, exhibits a hidden asymmetry. We introduce a symmetric set of markup strategies for the sellers and show how it explains away the paradox of buyers' advantage in three different double-sided market protocols.

1.1 Introduction

In a recent paper, Zhan and Friedman (2007) study the continuous double auction protocol for a standard exchange market in an environment populated by simulated traders that follow a simple markup (and markdown) rule. As stated by the authors themselves, the goal of the paper is not to approximate human behavior, but rather to gain insight into how traders' profit motives influence the performance of the protocol.

For simplicity, traders' strategies are reduced to a single dimension, called markup for the sellers and markdown for the buyers. (We occasionally encompass the two terms under the single heading of *markup* strategy.) Each seller uses a

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markup m_u over his cost c to post an ask price a; analogously, each buyer adopts a markdown m_d from his value v to issue a bid price b.

The leading case postulated in Zhan and Friedman (from now on, ZF) is the *standard markup* formulation according to which a seller i and a buyer k decide their offers using the rules

$$a_i = c_i(1+m_u) \qquad \text{and} \qquad b_k = v_k(1-m_d) \tag{1.1}$$

where $m_u, m_d \geq 0$. ZF assume $m_u = m_d$ and describe the effects of this standard markup rule on allocative efficiency and traders' surpluses in a simple call market and (in much greater detail) in a continuous double auction.

Our curiosity was picked upon reading that their results run against the symmetry inherent in the $m_u = m_d$ assumption of their model, prompting them to "consider two alternative markup specifications" (p. 2990). The *exponential markup* posits $a_i = c_i e^{m_u}$ and $b_k = v_k e^{-m_d}$. The *shift markup* prescribes

$$a_i = \min\{c_i + m_u, 1\} \qquad \text{and} \qquad b_k = \max\{v_k - m_d, 0\}. \tag{1.2}$$

Note that ZF omit the truncation in the formulation of the seller's ask for the shift markup rule, but this is irrelevant for their and our results. Kirchkamp and Reiß (2008) rename "absolute" the shift markdown in their study of markdown bidders in first-price auctions.

We present the following results. We replicate ZF's study for call markets and continuous double auction and extend it to the bilateral trading model by Chatterjee and Samuelson (1983). We explain the source of the bias in the standard formulation[1] and propose a fourth markup rule that is linear and symmetric but, differently from the three ZF's formulations, also satisfies obvious constraints of incentive compatibility and weak dominance. Finally, we examine in detail ZF's methodology for finding the equilibria of the continuous double auction and show that refining the search space may expand the set of equilibria, affecting some of their results.

The structure of the paper is the following. Section 1.2 describes ZF's model. Section 1.3 studies the call market. Section 1.4 analyzes the bilateral trading model. Section 1.5 examines the continuous double auction. Section 1.6 draws our conclusions.

1.2 The Model

We use the same setup as ZF (2007). Following Smith (1982), we identify three distinct components for our exchange markets. The environment in Section 2.1 describes the general characteristics of the economy, including agents' preferences and endowments. The protocols provide the institutional details that regulate the functioning of an exchange. We study three protocols associated respectively with the

[1] The asymmetry applies also to the exponential case; we disregard it for lack of space.

call market, the bilateral trading model, and the continuous double auction: each is described at the beginning of its dedicated Section.

Finally, the behavioral assumptions specify how agents make decisions and take actions. We assume that, whenever requested to do so, sellers (respectively, buyers) utter their asks (bids) deterministically according to one of the markup (markdown) strategies. Contrary to other behavioral assumptions such as zero-intelligence, a trader shouts always the same offer. Following ZF, we assume that all traders in the same market obey the same family of strategies; however, we do not impose $m_u = m_d$.

1.2.1 The Environment

There is an exchange economy with an equal number n of buyers and sellers, who can each exchange a single unit of a generic good. The market is *thick* for $n = 100$, *medium* for $n = 10$ and *thin* for $n = 4$. (Following ZF, we adopt the thick market as baseline.) Valuations and costs are drawn from stochastically independent uniform distributions on the same interval, which we normalize to $[0,1]$. (ZF use the interval $[0,200]$.) An obvious constraint of individual rationality requires that each seller i must sell his unit at a price $p \geq c_i$ and each buyer k must buy one unit at a price $p \leq v_k$. Hence, it is assumed throughout the paper that $m_u, m_d \geq 0$.

1.3 Call Market

In a call market, each trader simultaneously issues a price offer for a single unit. The protocol collects bids and asks from traders, derives supply and demand functions, chooses a market-clearing price p^* that maximizes trade, and executes all feasible trades at p^*.

ZF assume $m_u = m_d = m$ and obtain the following results. Overall efficiency is decreasing in m: a larger markup implies a reduction in the effective demand and supply. Sellers' surplus is also decreasing in m but, surprisingly, buyers' surplus is initially increasing in m. The intuition provided by ZF for this "buyer bias" asymmetry is the observation that for a high m most bids are near zero while asks spread over the interval $[0,2]$. (ZF fail to remark that asks above 1 never trade.)

We find this explanation wanting. ZF do not offer an explicit argument for the choice of the standard markup rule, except for its intuitive appeal and the plausible requirement of individual rationality associated with $m \geq 0$. The theory of mechanism design offers more specific suggestions; see f.i. part II in Nisan et al (2007). Namely, by incentive compatibility and weak dominance, a trading rule should satisfy three constraints: 1) it should be strictly increasing in the cost (or valuation) of the trader; 2) a buyer with $v = 0$ should bid $b = 0$; 3) and a seller with $c = 1$ should ask $a = 1$. It is immediate to check that the sellers' standard markup strategy does not satisfy the latter constraint unless $m_u = 0$. Hence, the "buyer bias" asymmetry originates in an implausible choice of the markup strategies. A similar argument disqualifies the exponential markup, while the shift markup fails the requirement of strict monotonicity.

We formalize the requirement of symmetry with respect to traders' role as follows. Define the strength of a buyer with valuation v as the distance $|v-0|$ from the valuation of the weakest buyer (who has $v = 0$) and the strength of a seller with cost c as the distance $|1-c|$ from the valuation of the weakest seller (who has $c = 1$). Analogously, define the strength of a bid b as the distance $|b-0|$ from the weakest bid (that is $b = 0$) and the strength of an ask a as the distance $|1-a|$ from the weakest ask (that is $a = 1$). Symmetry holds when the strengths of the bid and the ask issued by traders of equal strength x are the same; formally, we require $b(x) = 1-a(1-x)$.

Among the many rules that satisfy the three constraints, there is only one that is both linear and symmetric with respect to the traders' role for $m_u = m_d$. This unique choice is described by the formulas

$$a_i = c_i + m_u(1-c_i) \qquad \text{and} \qquad b_k = v_k(1-m_d) \tag{1.3}$$

and we call it *convex markup*; see Galavotti (2008). Rewriting them as $a_i = (1-m_u)c_i + m_u \cdot 1$ and $b_k = (1-m_d)v_k + m_d \cdot 0$ makes the role-based symmetry and the origin of the name transparent. Note that the original standard markdown rule is unchanged.

Differently from the standard markup rule, in a call market the convex (or the shift) markup formulations with $m_u = m_d = m$ imply that both allocative efficiency and the two traders' surpluses are decreasing in m; moreover, they yield a ratio of buyers' surplus to sellers' surplus constant in m (and equal to 1). The proof follows as a corollary of the analysis below for general coefficients $m_u, m_d \geq 0$.

1.3.1 General Markup and Markdown Coefficients

We repeat ZF's analysis of the call market under the more general assumption that m_u and m_d may be different. Normalize the price p and the quantity q to the interval $[0,1]$. Consider first the special case of *truthtelling* or price-taking behavior, when $m_u = m_d = 0$. Assuming away sampling variation, the demand function is $p = 1-q$ and the supply function is $p = q$ so that the competitive equilibrium has $p^* = q^* = 1/2$. Correspondingly, the (realized) traders' surplus is $\text{TS}^* = 1/4 = .25$, equally split between buyers' surplus $\text{BS}^* = .125$ and sellers' surplus $\text{SS}^* = .125$. This allocation is efficient and symmetric, so we adopt it as benchmark.

Assume now that traders adopt the standard markup rules described in (1.1). The demand function is $p = (1-m_d) - q$ and the supply function is $p = (1+m_u)q$ so that the market-clearing price and quantity are $p^s = (1+m_u)(1-m_d)/(2+m_u-m_d)$ and $q^s = (1-m_d)/(2+m_u-m_d)$. The allocative efficiency is $\text{AE}^s = \text{TS}^s/\text{TS}^* = [4(1+m_u)(1-m_d)]/(2+m_u-m_d)^2$ which is respectively split between buyers and sellers as follows:

$$\frac{\mathrm{BS}^s}{\mathrm{TS}^*} = \frac{2(1-m_d)(1+m_d+2m_u m_d)}{(2+m_u-m_d)^2} \quad \text{and} \quad \frac{\mathrm{SS}^s}{\mathrm{TS}^*} = \frac{2(1+2m_u)(1-m_d)^2}{(2+m_u-m_d)^2}$$

The buyer bias asymmetry is apparent because $\mathrm{BS}^s \geq \mathrm{SS}^s$ if and only if $m_d \geq m_u/(1+2m_u)$; in particular, for $m_d = m_u$ the buyers' surplus is always bigger than the sellers'.

Assume now that traders adopt the convex markup rules described in (1.3). The demand function is again $p = (1-m_d) - q$ but the supply function becomes $p = m_u + (1-m_u)q$. To ensure that trade can take place, assume $m_u + m_d \leq 1$. Then the market-clearing price and quantity are $p^c = (1-m_d)/(2-m_u-m_d)$ and $q^c = (1-m_u-m_d)(2-m_u-m_d)$. The allocative efficiency is $\mathrm{AE}^c = \mathrm{TS}^s/\mathrm{TS}^* = [4(1-m_u-m_d)]/(2-m_u-m_d)^2$ which is respectively split between buyers and sellers as follows:

$$\frac{\mathrm{BS}^c}{\mathrm{TS}^*} = \frac{2(1-m_u-m_d)(1-m_u+m_d)}{(2-m_u-m_d)^2} \quad \text{and} \quad \frac{\mathrm{SS}^c}{\mathrm{TS}^*} = \frac{2(1-m_u-m_d)(1+m_u-m_d)}{(2-m_u-m_d)^2}$$

The buyer bias asymmetry disappears because $\mathrm{BS}^c \geq \mathrm{SS}^c$ if and only if $m_d \geq m_u$. In particular, for $m_d = m_u$ the ratio between buyers' and sellers' surplus is constant and equal to 1.

Finally, consider the shift markup rule in (1.2). Under the assumption $m_u + m_d \leq 1$, the market-clearing price and quantity are $p^{sh} = (1+m_u-m_d)/2$ and $q^{sh} = (1-m_u-m_d)/2$. The allocative efficiency is $\mathrm{AE}^{sh} = (1-m_u-m_d)^2$, which is equally split between buyers' and sellers' surplus.

1.3.2 *Ex Ante Equilibria*

Truthtelling assumes that traders are price-takers: there is no reference to a notion of strategic equilibrium. For large n, it is possible to justify the price-taking assumption as an approximation for the Bayesian Nash equilibrium of a large game with private values and incomplete information; see f.i. Rustichini et al (1994) where it is shown that in any equilibrium the allocative inefficiency is $O(1/n^2)$.

Similarly, ZF's and our analysis have so far assumed that all traders follow the same rule and use the same markup coefficient. While it may be reasonable to justify the commonality of a specific rule on grounds of bounded rationality, it is far less clear that traders would not act strategically in their choice of the coefficients m_u, m_d. Intuitively, even if each trader has learned to use a given markup rule, it is still up to him to choose the best coefficient.

ZF suggest to account for at least some strategizing by looking at a notion of strategic equilibrium. They restrict all buyers to choose the same m_d and all sellers to the same m_u. In ZF's words, this leads to a two-cartel game for which "a fanciful interpretation is that all buyers belong to a cartel, and all sellers belong to a second cartel, and the members of each cartel agree on a common markup." (p. 2995) We use here the same solution concept, although we prefer to interpret it as an *ex ante* equilibrium where traders must choose a coefficient before learning their types (but knowing which side of the market they will be on). Section 1.5 illustrates a second richer notion of equilibrium and the technical difficulties involved in its calculation.

The unique ex ante equilibrium of the call market under the standard markup rule is $m_u^s = m_d^s = 1/2$. (We do not assume $m_u = m_d$: equality turns out to hold in equilibrium.) The symmetry in coefficients belies an asymmetry in payoffs because the buyers' cartel gets an equilibrium surplus $\mathrm{BS}^s = 1/8$ while the sellers obtain $\mathrm{SS}^s = 1/16$. On the other hand, the unique ex ante equilibrium under the convex markup rule is $m_u^c = m_d^c = 1 - \sqrt{2}/2 \approx .293$. The symmetry in coefficients persists over the equilibrium payoffs: $\mathrm{BS}^c = \mathrm{SS}^c = (\sqrt{2}-1)/4 \approx .104$. Not only the convex markup rule restores the symmetry, but it also improves the allocative efficiency of the *ex ante* equilibrium in the call market. Similar comments apply under the shift markup rule. The unique ex ante equilibrium is in weakly dominant strategies and prescribes truthtelling ($m_u^{sh} = m_d^{sh} = 0$): this maximizes allocative efficiency and preserves symmetry.

1.4 Bilateral Trading

The bilateral trading model by Chatterjee and Samuelson (1983) is a workhorse for the study of how strategic incentives affect the allocative efficiency of a trading protocol. It is not studied in ZF, who concentrate most of their attention on the continuous double auction. However, it is similar to ZF's setup for $n = 1$. The buyer shouts a bid b and simultaneously the seller names an ask a. If $b \geq a$, trade takes place at the price $p = (a+b)/2$. Viewed as a game with incomplete information, this provides a perfect example of an environment where the market power of the sides of the market is exactly balanced. Its large set of equilibria is widely studied under general assumptions, but for consistency we concentrate on the special case where both seller's cost and buyer's valuation are uniformly distributed on $[0,1]$.

The bilateral trading model has several Bayesian Nash equilibria. However, there is only one[2] that is symmetric and based on linear bidding functions similar to the markup rules we are interested in. As shown in Chatterjee and Samuelson (1983), a buyer with valuation v bids $b = (2/3)v + (1/12)$ and a seller with cost c asks $a = (2/3)c + (1/4)$. Consequently, the probability of trading is $9/32 \approx .281$ and the allocative efficiency is $(9/64)/(1/6) = .84375$, which on average is equally shared between buyer's and seller's surplus.

We study what happens when traders play an *ex ante* equilibrium where they are constrained to follow a rule but can choose the markup coefficient. Under the standard markup rule used in ZF, the unique equilibrium is at $m_u^s = m_d^s = 1/3$. Once again, the symmetry is only formal: in equilibrium, any buyer with valuation v in

[2] We leave it understood that uniqueness refers to offers with nonzero probability to be accepted.

$(0,1]$ has a nonzero probability to trade but for a seller this is the case if and only if he has cost $c < 1/2$. Similarly to what happens in a call market, the standard markup rule favors the buyer over the seller. Correspondingly, the probability of trading is $1/4 = .25$ and the expected allocative efficiency is .750, which is split between the buyer's and seller's surplus in the ratio 2:1.

A similar analysis using the convex markup rule gives a unique equilibrium $m_u^c = m_d^c \approx 0.23$. The symmetry is complete: in equilibrium, a seller (respectively, buyer) trades with nonzero probability if and only if he has cost $c < \bar{k} \approx 0.71$ (valuation $v > 1 - \bar{k} \approx 0.29$). Correspondingly, the probability of trading is about .249 and the expected allocative efficiency is about .792, which is equally shared between buyer's and seller's surplus. A direct comparison with the case of standard markup reveals immediately that the probability of trade is almost the same, the allocative efficiency is higher, and the surpluses are equally distributed.

Finally, a shift markup yields symmetric results analogous to the convex rule. The unique equilibrium is $m_u^{sh} = m_d^{sh} = 1/6$. In equilibrium, a seller (respectively, buyer) trades with nonzero probability if and only if he has cost $c < 2/3$ (valuation $v > 1/3$). The probability of trading is $2/9 \approx .222$ and the expected allocative efficiency is $20/27 \approx .741$, which is equally shared between buyer's and seller's surplus. Like convex markups, the shift rule is symmetric but its overall allocative performance is the worst of the three rules.

1.5 Continuous Double Auction

There are many different implementations of the continuous double auction (from now on, CDA). However, the common theme is that traders arrive sequentially and can place limit orders. An order that is marketable is immediately executed; otherwise, it is stored in a book until execution or cancellation. Differently from the double auctions discussed above, the complexity of the CDA makes analytic results remain elusive even under markup trading. Therefore, ZF (2007) suggests to search for the symmetric equilibria by means of simulation techniques.

Using the conventions set up in LiCalzi et al (2008), our simulations assume a market protocol based on the following rules: 1) single unit trading; 2) price-time priority; 3) no retrading; 4) no resampling; 5) uniform sequencing; 6) halting by queue exhaustion (the market closes down when there are no more traders waiting to place an order). The complete set of conventions used by ZF in their implementation is not made explicit, but we found no reason to expect significant differences in practice; see Cervone (2009).

The left-hand side in Figure 1.1 is the analog of Figure 4 in ZF (2007) and is derived from our independent simulations with $n = 100$. Under the assumption that all traders use the standard rule and the same markup coefficient, it shows allocative efficiency as well as buyers' and sellers' surplus. Data are averaged over $t = 2500$ simulations for each $m_u = m_d = m$ over the grid $\{0, 0.1, \ldots, 0.9, 1\}$. There are two sharp conclusions. First, overall efficiency is initially increasing and then decreasing

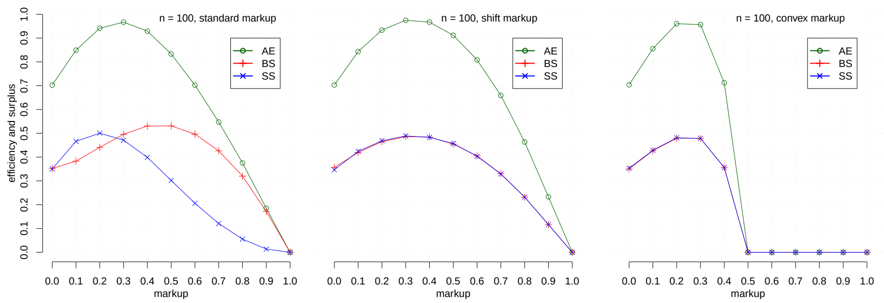

Fig. 1.1 Allocative efficiency and surpluses in the call market under three markup rules.

in m: hence, a larger markup is not necessarily harmful. Second, both sellers' and buyers' surplus are hump-shaped while the ratio between the two favors sellers for low values of m and buyers for high values.

The center and the right-hand side in Figure 1.1 mirror the left-hand side respectively with a shift markup and a convex markup, assuming $m_u = m_d = m$. The general humped shape persists for both the allocative efficiency and the two traders' surpluses, but symmetry is restored because the ratio of buyers' surplus to sellers' surplus is (this being a simulation, virtually) constant in m and equal to 1. The maximum allocative efficiency (computed over the grid $\{0, 0.01, 0.02, \ldots, 0.99, 1\}$) of each rule is .9683 for the standard markup at $m = .29$, .9779 for the shift markup at $m = .34$, and .9778 for the convex markup at $m = .25$. The two symmetric rules (convex and shift) exhibit superior allocative performances.

These results assume that each trader uses the same markup rule and the same coefficient $m_u = m_d = m$. We grant the first assumption for the scope of this paper, but there is no reason to expect that traders on different sides of the market should use the same markup coefficients. This is especially clear for the standard rule, where the asymmetry obviously points buyers and sellers towards different m's. ZF are well aware of this issue and for this reason they suggest taking into account the strategic behavior of traders in the choice of their markup coefficients.

They consider two formulations: the *two-cartel* game leads to equilibria where all the traders on the same side of the market are constrained to use the same markup (or markdown) coefficient; and the *2n-player* game allows for individual deviations from a single trader. We examine the two-cartel game first.

ZF search for the equilibria in pure strategies[3] of the two-cartel game by restricting the choice of m_u and m_d to the 11-point grid $\{0, 0.1, 0.2, \ldots, 0.9, 1\}$. In the baseline case with $n = 100$ (thick market), they compute the average realized surplus for each side of the market over $t = 2500$ simulations. Using these data, they construct a finite bimatrix game between the buyers' coalition and the sellers' coalition with payoffs equal to their average realized gains. Their main findings for the baseline with standard markup can be read on the left-hand side of Table 1.1.

[3] Likewise, we ignore equilibria in mixed strategies throughout this section of the paper.

Table 1.1 Equilibria in the two-cartel and $2n$-player games for the CDA; from ZF (2007)

parameters	two-cartel		$2n$-player	
	standard	shift	standard	shift
	m_d, m_u	m_d, m_u	m_d, m_u	m_d, m_u
$n = 100, t = 2500$	$0.6, 0.5$	$0.6, 0.6$	$0.4, 0.3$	$0.4, 0.4^a$
$n = 10, t = 5000$	$0.5, 0.6$	$0.5, 0.5$	$0.3, 0.3$	$0.4^b, 0.4^c$
$n = 4, t = 25000$	$0.4, 0.5$	$0.4, 0.4$	$0.3, 0.3$	$0.3, 0.3$

^{a} $\varepsilon = .0012$; ^{b} $\varepsilon = .00044$; ^{c} $\varepsilon = .001$

There is a unique equilibrium for each of three markets and for both standard and shift markup formulation. The equilibrium markup coefficients are increasing in n, but they are symmetric only under the shift rule. Moreover, ZF claim that $m_d = 0.6$ is a weakly dominant strategy for the buyers in the baseline under the standard markup rule.

We replicate and improve on ZF's study using the same grid for the choice of the coefficients. Our results are summarized in Table 1.2 that lists also the sample averages for buyers' and sellers' surplus as well as their sample standard deviations. (These additional pieces of information are not provided in ZF.) The average realized surplus is dependent on the sample and on the precision chosen. For instance, using exactly the same parameters as ZF, we find that $m_d = 0.5$ (ZF has 0.6) is a weakly dominant strategy for the buyers only if we truncate the average realized surplus to the third decimal digit. (Otherwise, $m_d = 0.5$ is less profitable than 0.6 against $m_u = 0$.) This accounts also for slight differences in the equilibrium values under standard markup.

Table 1.2 Equilibria, allocative efficiency and surpluses in the CDA for the two-cartel game

	standard	shift	convex		
			1^{st} eq.	2^{nd} eq.	3^{rd} eq.
$n = 100, t = 2500$					
m_d, m_u	$0.5, 0.6$	$0.6, 0.6$	$0.3, 0.4$	$0.4, 0.3$	
BS, SS	.50384,.30223	.40499,.40350	.39215,.48144	.48310,.39289	
std. dev.	.04283,.02625	.03619,.03489	.02559,.02953	.02923,.02541	
$n = 10, t = 5000$					
m_d, m_u	$0.5, 0.4$	$0.5, 0.5$	$0.2, 0.4$	$0.3, 0.3$	$0.4, 0.2$
BS, SS	.49412,.22781	.37903,.37514	.32217,.50071	.41331,.41389	.50281,.32361
std. dev.	.15585,.09228	.11788,.11732	.08947,.11592	.10151,.10284	.11450,.08915
$n = 4, t = 25000$					
m_d, m_u	$0.4, 0.5$	$0.4, 0.4$	$0.2, 0.3$	$0.3, 0.2$	
BS, SS	.38533,.22729	.33786,.33910	.32730,.41307	.41589,.32662	
std. dev.	.25262,.17352	.20701,.20714	.18015,.20905	.20893,.17981	

More interestingly, the equilibrium surplus is split into unequal ratios (favoring the buyers) under the standard rule and symmetrically under the shift rule. But, above all, equilibrium is not unique under the convex rule. However, its allocative efficiency is pretty much the same in each of the equilibria and consistently higher than for the other markup rules. Moreover, for each equilibrium favoring buyers under a convex rule there is a specular equilibrium favoring sellers; hence, symmetry still holds over the set of equilibria. Higher allocative efficiency and set-symmetry of the equilibria suggest that the convex markup rule performs better when traders act strategically.

From a computational viewpoint, the mild discrepancy between ZF's and our results prompted us to refine the grid to a mesh of 0.01 around the equilibrium values found above. This made clear that the weak dominance of the buyers' strategy under the standard markup rule is an artifact due to the limited number of strategies considered. More importantly, we find different sets of equilibria. We illustrate the point with reference to the baseline case of a thick market ($n = 100, t = 2500$). The results for the other two cases are qualitatively similar.[4]

Using a grid with mesh 0.1, the unique equilibrium under standard markup is $(m_d = .5, m_u = .6)$ with allocative efficiency .806. Refining the analysis to a grid with mesh 0.01, we find two equilibria: $(m_d = 0.54, m_u = 0.5)$ and $(m_d = 0.55, m_u = 0.52)$, respectively with average efficiency .801 and .785. The difference in the equilibrium markups is smaller, but both of them still exhibit the usual 2:1 ratio among traders' surpluses in favor of the buyers.

When we repeat the analysis for the convex markup rule, symmetry is restored (up to the inevitable sampling errors and computational approximations). Using a grid with mesh 0.1, we find two equilibria with coefficients lying in the interval $[0.3, 0.4]$. A blow-up based on a grid with mesh 0.01 reveals three equilibria: $(m_d = 0.32, m_u = 0.34)$, $(m_d = 0.34, m_u = 0.33)$ and $(m_d = 0.36, m_u = 0.32)$, respectively with average realized efficiencies .917, .910, and .900. All three equilibria are consistently more efficient than those under standard markup. They slightly favor the side with higher markup attributing a surplus that is respectively about 4%, 2%, and 9% higher than the other side.

For the shift markup rule, using a grid with mesh 0.01 leads to the unique equilibrium $(m_d = 0.58, m_u = 0.59)$ with an overall allocative efficiency of .828. The sellers' surplus is just 1.6% higher than buyers'. Similarly to the case of bilateral trading, both the convex rule and the shift rule are symmetric but the latter one yields a worse allocative efficiency under strategic behavior.

Turning now to the case of $2n$-player game, ZF search for symmetric equilibria where no single trader has individually profitable deviations before learning his type. That is, the markdown coefficient m_d^* for a buyer must be *ex ante* optimal assuming that all other buyers use m_d^* and all sellers use m_u^*; and similarly for the optimal seller's m_u^*. (Clearly, m_d^* may differ from m_u^*.) Imposing that all traders on the same side of the market use the same markup coefficient makes their notion of symmetric Nash equilibrium quite restrictive, but we stick with it for ease of comparison.

[4] For $n = 4$ and $n = 10$, in some of several simulations with the shift markup rule we found also spurious equilibria attributable to sampling errors and computational approximations.

ZF's results over the 11-point grid $\{0, 0.1, 0.2, \ldots, 0.9, 1\}$ can be read on the right-hand side of Table 1.1. The general picture is similar to the two-cartel game. There is a unique equilibrium for each of three markets and for both standard and shift markup; the equilibrium coefficients are increasing in n and symmetric (except for the baseline with standard markup). However, this formulation leads to equilibria where traders are less aggressive and make offers close to their values or costs.

Our results are summarized in Table 1.3 using the same conventions as in Table 1.2, in particular for the ε-equilibria. We obtain similar values for the equilibrium coefficients, but the explicit computation of traders' surpluses reveals additional information. Under standard markup, the allocation of surplus is skewed in favor of buyers. Symmetry is restored under convex markup: there is either a unique equilibrium with identical surpluses, or two asymmetric equilibria that are symmetric up to a role reversal between traders. A similar situation occurs using the shift markup. The allocative efficiency is decreasing in n for each markup rule. However, for a given n, the overall realized efficiencies are quite close and hence no markup rule emerges as a clear winner from this point of view.

Table 1.3 Equilibria, allocative efficiency and surpluses in the CDA for the $2n$-player game

	standard	shift		convex	
		1^{st} eq.	2^{nd} eq	1^{st} eq.	2^{nd} eq.
$n = 100, t = 2500$					
m_d, m_u	$0.4, 0.4^a$	$0.4, 0.4$		$0.3, 0.3$	
BS, SS	.53049,.39874	.48408,.48307		.47824,.47848	
std. dev.	.03589,.03160	.04621,.04638		.02950,.02116	
$n = 10, t = 5000$					
m_d, m_u	$0.3, 0.3$	$0.4, 0.4^b$		$0.2, 0.3$	$0.3, 0.2$
BS, SS	.48348,.38273	.42633,.42581		.39485,.49490	.50196,.39136
std. dev.	.13495,.13355	.12262,.12123		.10678,.11171	.11003,.10144
$n = 4, t = 25000$					
m_d, m_u	$0.3, 0.3^c$	$0.3, 0.4$	$0.4, 0.3$	$0.2, 0.3^d$	$0.3, 0.2$
BS, SS	.43503,.29794	.32783,.39542	.39613,.32750	.32730,.41307	.41589,.32662
std. dev.	.23875,.20020	.19969,.22117	.22176,.19985	.18015,.20905	.20893,.17981

^{a} $\varepsilon = .00111$; ^{b} $\varepsilon = .00078$; ^{c} $\varepsilon = .00042$; ^{d} $\varepsilon = .00100$

When we repeat the analysis over a grid with mesh 0.01, the following conclusions emerge. First, while the simulations are affected by sampling errors and computational approximations, the qualitative results are similar. Second, the number of equilibria tends to drop and, in many cases, it goes down to one; this pruning follows because a 10% reduction of the mesh induces a 10-fold increase in the number of strategies tested for a trader. Third, the minor differences in the overall allocative efficiency (notwithstanding the asymmetry in the distribution under standard markup) are further reduced.

1.6 Conclusions

Zhan and Friedman (2007) study three simple families of markup (and markdown) strategies for the continuous double auction. Their main goal is to gain insight into how traders' profit motives influence the performance of the protocol. Our main conclusion is that the standard formulation starring as leading example in their paper is not an appropriate choice, because it fails an elementary test of symmetry in its treatment of buyers and sellers.

We suggest an alternative convex markup rule that is in accordance with the general prescriptions from mechanism design. We test ZF's standard rule, their shift formulation and our convex rule over three different double auction protocols (call market, bilateral trading, and continuous). The standard markup consistently fail the test of symmetry, which is instead passed by the two other formulations. The shift markup rule leads to a higher allocative efficiency only in the call market (and in the baseline for the $2n$-player game), where the strategic interactions are dampened out by the simultaneous aggregation of demand and supply and all nontrivial equilibria are asymptotically efficient under reasonably weak conditions; see Cripps and Swinkles (2006). Therefore, our convex rule seems to offer a more promising route to accommodate strategic behavior while preserving symmetry in a simple behavioral model.

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