Stochastic Optimisation for Allocation Problem with Shortfall Risk Constraints [1]

Monica Billio [†]
billio@unive.it

Roberto Casarin [‡]
casarin@ceremade.dauphine.fr

ABSTRACT - One of the most important aspects in asset allocation problems is the assumption concerning the probability distribution of asset returns. Financial managers generally suppose normal distribution, even if extreme realizations usually have an higher frequency than in the Gaussian case. We propose a general Monte Carlo simulation approach in order to solve an asset allocation problem with shortfall constraint, and to evaluate the exact portfolio risk-level when managers assume a misspecified tails behaviour. In particular, in a stochastic optimisation problem, we assume that returns are generated by a multivariate Student-t distribution, while in reality returns come from a multivariate distribution where each marginal is a Student-t but with different degrees of freedom. Stochastic optimisation allows us to value the effective risk for managers. In the specific case analysed, it is also interesting to observe that a multivariate density resulting from different Student-t marginal distributions produces a shortfall probability and a shortfall return level that can be approximated adequately by assuming a multivariate Student-t with a common degree of freedom in the optimisation problem. The present simulation based approach could be an important instrument for investors who need a qualitative assessment of the reliability and sensitivity of their investment strategies when their models are potentially misspecified.

KEYWORDS - Misspecification errors, Portfolio optimisation, Stochastic optimisation, Shortfall constraint, Value at Risk.

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[†]GRETA and Dept. of Economics, University of Venice

[‡]GRETA, University of Venice and CEREMADE, University Paris Dauphine

1 INTRODUCTION

In asset allocation problems, the assumption on the probability distribution of future returns is a very important aspect. In many theoretical and empirical works, a normal or log-normal distribution is usually assumed. It is well known that the normal distribution has several attractive properties: it is easy to use and produces tractable results in many analytical exercises; all moments of positive order exist, and it is completely characterized by its first two moments, thus establishing the link with the mean-variance optimization theory. Normal distribution arises as the limiting distribution of a whole class of statistical testing and estimation procedures, and therefore plays a central role in empirical modelling exercises. One of the main characteristics of the normal distribution is that its tails decay exponentially toward zero; thus extreme realizations are very unlikely. However, this seems to contradict empirical findings on asset returns, which evidence that returns' distribution generally exhibits leptokurtic behaviour, i.e. has fatter tails than the normal distribution. This means that extreme returns of either sign occur far more often in practice than predicted by the normal model. For financial managers, who are interested in risk management, these are crucial aspects. It is typically suggested that the use of leptokurtic instead of normal distributions in asset allocation problems leads to more prudent asset portfolios. In other words, it is commonly believed that optimal asset allocations under the assumption of normally distributed returns have a higher Value at Risk (VaR) than the model suggests if returns in reality follow a leptokurtic distribution.

By means of Monte Carlo simulations and following the recent literature on stochastic optimisation in finance (see for example Harvey et al. [8]), we study an asset allocation problem under a given shortfall constraint. The use of simulation methods allows testing the effects of assuming different kinds of returns distribution in modelling asset class returns. To illustrate this point, we consider a simple one-period asset allocation problem with one *shortfall constraint*[1] (see Roy [25], Telser [28] and Kataoka [15]).

As is known from the *safety first principle*, the shortfall constraint reflects the investors typical desire to limit downside risk by putting a (probabilistic) upper bound on the maximum loss. In other words, the investor wants to determine an optimal asset allocation for a given Value at Risk. Results obtained reveal that the degree of shortfall probability plays a crucial role in determining the effects of the choice between a fat tailed and a normal distribution. These effects concern the composition of optimal asset allocations as well as consequences of misspecification of the degree of fat-tailedness for the downside risk measure[2].

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[1]An extensive literature exists since the fifties, known as *Downside Risk Approach*. It tries to explain the risk associated to an investment, exclusively evaluating downside oscillations of returns. The *Downside Risk* is an alternative to the more common *standard deviation* concept, generally used by financial managers in asset allocation problems. Although these approaches were already developed at the beginning of the fifties, they were followed in the mid seventies with the introduction of the *lower partial moment framework* (see Harlow [7]). A special case of lower partial moments is the *safety first principle* (see Roy [25]). This principle allows investors to identify portfolios revealing a minimum probability to fall under a specified return level.

[2]A first analysis of these important aspects was provided by Lucas and Klassen [19] which

If the shortfall is moderately large, say 5%, then the assumption of fat tails results in more aggressive asset allocations. As a consequence, if reality is fat-tailed, optimal asset allocations that are based on the normal distribution may be far too prudent for a given 95% confidence level VaR.

If the shortfall is small, say, 1%, then the use of leptokurtic distribution leads to more prudent asset allocations. Consequently, an optimal asset mix that is based on a normality assumption, will violate a 99% confidence level VaR if reality is leptokurtic. We show that the true VaR may in that case exceed the VaR obtained by simulation by over 30%. After these first results, we analyse the effects, on the portfolio management, when the distribution of each asset returns considered, shows a different behaviour. In fact, it is quite unrealistic to suppose that the probability distribution of each financial class shows the same degrees of leptokurtosis. Through the Monte Carlo method it is possible to study the effects on the true risk when the data come from a mixed distribution while the manager uses a multivariate distribution with identically distributed marginal distributions.

The main result is that a correct estimate of the degrees of freedom for each of them is a necessary condition in order to have no excessive loss of information, an adequate formulation of the optimal strategy and, consequently, a correct perception of the true risk. We have also noted that for the particular combination we used, it is possible to find a multivariate distribution with identically distributed marginal distributions able to approximate the empirical one, with a loss of information that could be minimal. This means that we can simplify the problem with no excessive loss of generalities, which may be very useful when we solve complex mathematical models. The analysis we carry out may be very important to understand which approach a manager could follow to identify the most suitable distribution of probability to use. We concentrate on the interaction between different distribution assumptions made by the manager on the one hand and, on the other, the resulting optimal financial management decisions and downside risk measures. In particular, we pay close attention to the effect of financial policies and VaR if the probability of extreme returns is underestimated.

The structure of the work is as follows. In §2 we present the asset allocation problem and the class of probability distributions considered. In §3 we examine how the problem can be solved through a Monte Carlo approach, and in particular we concentrate on the stochastic optimization aspects. After this analysis, we give some general characterizations of the theoretical effect of fat tails on the problem at hand (see §4) and a numerical illustration of the model, initially using the returns of three financial assets class (cash, stocks and bonds) identically distributed (§5) and then differently distributed (7). Parameters of the probability distribution (mean, variance, correlation matrix), are estimated on three U.S. asset categories. Following the empirical example, in §6 and §7, we study the effect of misspecification of the return distribution on downside risk.

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studied an analogous problem in an analytical way. The main aim of this work is to examine closely these aspects valuing the effects on optimal financial portfolio when each asset studied follows a different marginal probability distribution. Moreover, to make this possible the Monte Carlo optimisation method seems to be the most suitable instrument.

2 THE PORTFOLIO MODEL

We consider a one-period model with n asset categories. At the beginning of the period, the manager can invest the money available in any of the n asset categories and short positions are not allowed.

The objective of the investment manager is to maximize the expected return on the portfolio, subject to a shortfall constraint. This shortfall constraint states that with a sufficiently high probability $1-\alpha$ (with α being a small number), the return on the portfolio will not fall below the *threshold return* r^{low}.

Formally, the asset allocation problem can be written as follows:

$$\underset{x\in\Re^n}{Max} \quad \mathbb{E}\left(\sum_{i=1}^{n} x_i r_i\right) \tag{1}$$

$$\mathbb{P}\left(\sum_{i=1}^{n} x_i r_i < r^{low}\right) \leq \alpha \tag{2}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{3}$$

$$x_i \geq 0 \tag{4}$$

where x_i and r_i , (with $i = 1, 2, \ldots, n$), denote the fraction of capital invested in the asset category i, and the (stochastic) return on asset category i, respectively. The operator $\mathbb{E}(\cdot)$ is the expectations operator with respect to the probability distribution $\mathbb{P}$ of the asset returns. The probabilistic constraint in (2) fixes the permitted VaR for feasible asset allocation strategies. We know that Value at Risk is the maximum amount that can be lost with a certain confidence level in a given period. In the setting of (2) with $r^{low} < 0$, the VaR per Euro invested is $-r_{low}$ with a confidence level of $1-\alpha$.

Our aim is to study the effect of extreme returns on the solution of the asset allocation problem in Equations (1) to (4). We need therefore to introduce a stochastic optimisation technique by simulation and then a class of probability distribution that allows for fat tails. The class of Student-t distributions meets these requirements.

The probability density function of n-dimensional multivariate Student-t distribution is given by:

$$\mathcal{T}_n(r;\mu,\Omega,\nu) = \frac{\Gamma((\nu+n)/2)|\Omega|^{-1/2}}{\Gamma(\nu/2)(\pi\nu)^{n/2}}(1+\frac{(r-\mu)'\Omega^{-1}(r-\mu)}{\nu})^{-(\nu+n)/2} \tag{5}$$

where $\Gamma(\cdot)$ denotes the gamma function, $r = (r_1, r_2, \ldots, r_n)'$ denotes the vector of stochastic asset returns, and μ, Ω^{-1} and ν denote the mean, the precision matrix and the degrees of freedom parameter of the Student-t distribution, respectively.

It's important to note that Ω satisfies the following relation:

$$\Omega = (1-\frac{2}{\nu})V \tag{6}$$

where V denotes the variance-covariance matrix.
The degrees of freedom parameter ν determines the degrees of leptokurtosis. ν has to be strictly positive. The smaller ν, the fatter the tails of the Student-t distribution. The Student-t distribution has the normal distribution as special case: (5) reduces to the normal density with mean μ and covariance matrix Ω if $\nu \to \infty$.

The first two moments of the Student-t distribution play an important role in the subsequent analysis. These moments are given by $\mathbb{E}(r) = \mu$ and $\mathbb{E}((r-\mu)(r-\mu)') = \nu\Omega/(\nu-2)$, and they require $\nu > 1$ and $\nu > 2$, respectively.

Figure (1) displays several univariate Student-t distributions. The distributions are scaled in such a way that they all have zero mean and unit variance. It is clearly seen that for lower values of ν, the tails of the distribution become fatter and the distribution becomes more peaked near the centre $\mu = 0$.

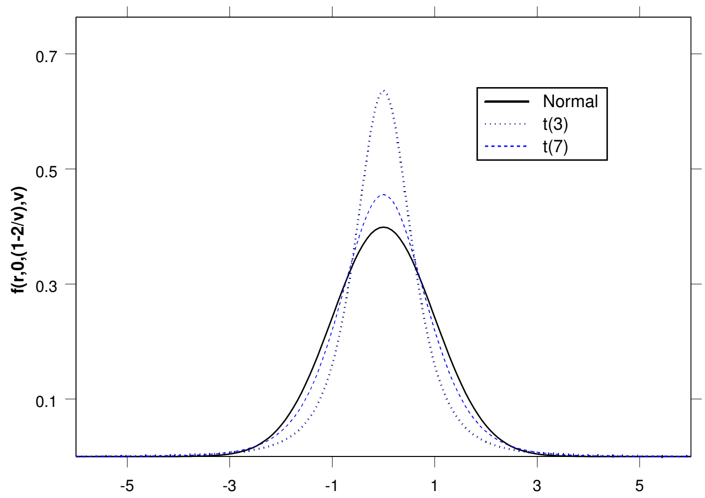

Figure 1: Student-t distribution for various values of degrees of freedom parameter.

We have to note that μ and V are usually unknown and they are therefore estimated (m, V) from historical time series. So we have:

$$\mu \cong m \tag{7}$$
$$\Omega \cong (1 - \frac{2}{\nu})\tilde{V} \tag{8}$$

In our analysis m and $\tilde{V}$ are considered fixed. In the following sections we use the Monte Carlo approach to analyse an asset allocation problem with shortfall constraint, from the manager's point of view.

Our aim is to consider the effects on the portfolio, when the probability distribution of asset returns shows a leptokurtic behaviour, and the concrete risk borne by the financial manager.

3 MONTE CARLO SIMULATION APPROACH TO STOCHASTIC OPTIMISATION

We use a Monte Carlo simulation approach in order to solve the optimization problem, and to compute numerically nontrivial integrals. There are many features that distinguish this method from most of the others generally used. First it can handle problems of far greater complexity and size than most other methods. The robustness and simplicity of the Monte Carlo approach are its strengths. Second, the Monte Carlo method is intuitively based on the law of large numbers and central limit theorem. The probabilistic nature of the Monte Carlo method has important implications. The result of any Monte Carlo procedure is a random variable. Any numerical method has errors, but the probabilistic nature of the Monte Carlo errors puts structure on the errors that we can exploit. In particular, the accuracy of the Monte Carlo method can be controlled by adjusting the sample size. The Monte Carlo method uses pseudo random numbers to solve a given problem; that is, deterministic sequences generated using pseudo random generators, such as *linear congruential generators* (Ripley [22]), that seem to be random [3]. We also know that these generators give an identical sequence of pseudorandom numbers if the same *seed* is set. The random numbers generated are good ones if they are uniformly distributed, statistically independent, and reproducible (Rubinstein [26]). In order to solve our financial problem, we need to simulate assets returns from different probability distributions. Initially we generate random numbers from a normal distribution with mean μ and variance-covariance matrix V and then from a multivariate Student-t distribution with ν degrees of freedom [4].

Let Z have a standard multivariate normal distribution, let Y have a multivariate chi-square distributions with ν degrees of freedom, and let Z and Y be independent, then:

$$X = \frac{Z}{\sqrt{Y/\nu}} \tag{9}$$

has a multivariate Student-t distribution with ν degrees of freedom.

It is known that the Monte Carlo method allows numerical solutions of complex mathematical problems to be obtained, where mathematical procedures seem inadequate. Stochastic optimization problems are an example. We consider a *stochastic optimisation problem* in the following form:

$$x^* = \underset{x \in U}{arg\,max}\, \mathbb{E}\{g(x, Z)\} \tag{10}$$

where Z is a random variable with p.d.f. $h(z)$. The solution of the problem (10) needs the computation of $\mathbb{E}(\cdot)$. A numerical solution can be performed by simulation

[3] In this work we have used the *Mixed Congruential Generator* in the following form: $X_{i+1} = a\,X_i + c\ (mod\, m)$ for $i = 1, 2, \ldots, N$ and with $c = 0$, $m = 2^{31} - 1$ and $a = 397204094$

[4] There exist many algorithms that allow the transformation of random numbers extracted from a uniform distribution into normally distributed random numbers. Many of these techniques are very well explained in Rubinstein [26] and Ripley [22] (see, for example, *Box Müller algorithm*, *Monro algorithm*, etc.).

of the objective function and then by applying standard optimisation techniques. The most simple idea (see Judd [13] for some examples of application in economics, Robert [23], Robert and Casella [24] for an introduction to other stochastic optimisation methods) is to take a sample of size D of the random variable Z, and to approximate $\mathbb{E}\{g(x, Z)\}$ by its sample mean:

$$\frac{1}{D}\sum_{i=1}^{D} g(x, Z_i) \tag{11}$$

Then all standard optimisation techniques can be applied.

We use this approach to solve our portfolio problem. In fact the use of Monte Carlo integration is quite natural for such problems since we are essentially simulating the problem. The solution is denoted by $\tilde{x}^*$ and approximates the true solution x^*, it gives us the fraction of capital has to be invested in each asset to maximise expected return.

The quality of this procedure depends on the size D and how well the integral is approximated by the random sample mean. We are therefore interested in knowing the sample size D and the number of samples N of D draws, necessary to obtain a "good" estimate. To do that, we control the error from the analytical solution [5] and the standard deviation of each estimate. In our context we have seen that for $D = 10,000$ and $N = 100$, we can approximate the underlying distribution adequately and obtain an accurate estimate with a very small standard error[6].

4 THEORETICAL EFFECTS

We have seen that the parameter ν plays a prominent role in our asset allocation problem, through its presence in the shortfall constraint. Decreasing ν has two effects. First of all, the tails of the distribution become fatter, resulting in a higher probability of extreme events for fixed precision matrix Ω^{-1}. As can be seen in (6), the precision matrix is not independent of ν if the variance of the returns is held fixed. As ν decreases, the eigenvalues of the precision matrix increase. As a result, the distribution becomes more concentrated around the mean μ. The composite effect on the shortfall constraint of altering ν depends critically on the shortfall probability α.

It can be shown that for a sufficiently small value of α, the shortfall constraint becomes less binding if the distribution used tends to normal. The reverse holds if we consider sufficiently large values of α.

It is interesting to present the break-even shortfall probability for the normal distribution obtained by simulation, i.e., the value of α, as a function of ν such that

[5]The *analytical solution* is obtained by calculating integrals "analytically" instead of "numerically". For multivariate distributions (such as normal or Student-t), it can be obtained by using a common calculator.

[6]Although there exist analytical techniques to choose the optimal number of simulation N (see Rubinstein [26]), graphical techniques are often preferred (see Robert [23], Robert and Casella [24]). They permit the choice of the adequate simulations number, necessary to obtain the stabilisation of the solution. The higher N, the better the solution approximation, because the variance of sample mean reduces to zero.

the shortfall constraint for that value of ν is as binding as the shortfall constraint for the corresponding normal distribution (results are similar to those obtained by Lucas and Klaassen [19]). More precisely, given the random variables $u \sim \mathcal{N}(\mu, \sigma)$ and $w \sim \mathcal{T}(\mu, \Omega, \nu)$ and a shortfall probability α, then the quantile associated to α, for the two distributions is

$$\Phi_{\mathcal{N}}(r_{\mathcal{N}}^{low}) = \int_{-\infty}^{r_{\mathcal{N}}^{low}} u\mathcal{N}(u; \mu, \sigma)du = \alpha \quad \Leftrightarrow \quad r_{\mathcal{N}}^{low} = \Phi_{\mathcal{N}}^{-1}(\alpha) \tag{12}$$

$$\Phi_{\mathcal{T}}(r_{\mathcal{T}}^{low}) = \int_{-\infty}^{r_{\mathcal{T}}^{low}} w\mathcal{T}(w; \mu, \Omega, \nu)dw = \alpha \quad \Leftrightarrow \quad r_{\mathcal{T}}^{low} = \Phi_{\mathcal{T}}^{-1}(\alpha) \tag{13}$$

In order to obtain the critical shortfall probability $\alpha(\nu)$, for a fixed value of ν we take $r_{\mathcal{N}}^{low} = r_{\mathcal{T}}^{low}$ and solve w.r.t. α the resulting non linear equation

$$\Phi_{\mathcal{T}}^{-1}(\alpha) = \Phi_{\mathcal{N}}^{-1}(\alpha) \tag{14}$$

Such a value of α, produces the same solution under either normality assumption or the assumption of a Student-t distribution with ν degrees of freedom for the asset returns. The values of $\alpha(\nu)$, are given in Figure 2. This graph indicates the critical shortfall probability ranges from $\alpha = 1.8\%$ for $\nu = 3$ to $\alpha = 3.6\%$ for $\nu = 10$. For values of α below these critical levels, the effect of fat tails on the shortfall constraint dominates the effect caused by increased precision. In these cases, the probability restriction in (2) for the Student-t distribution is more binding for a given asset allocation than in the case of normally distributed asset returns.

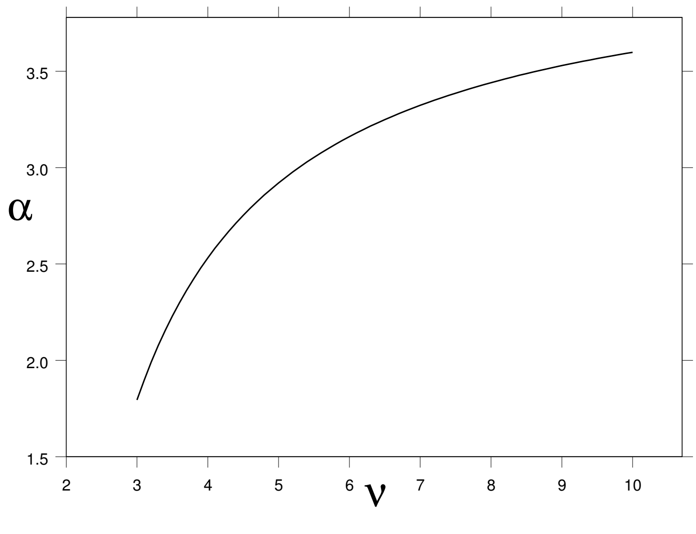

Figure 2: Critical Shortfall Probability $\alpha(\nu)$ for Student-t distribution with ν degrees of freedom (benchmark is the normal distribution).

Again, the reverse holds for values of α above the critical level. In our empirical study, we use $\alpha = 0.5\%$, $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$ in order to illustrate both settings.

5 RESULTS

To illustrate our results, we present and solve through a Monte Carlo approach an asset allocation problem (similar to which studied by Lucas and Klaassen [19]), involving three U.S. asset categories: cash, stocks, and bonds. For cash, we use the return on one month Eurodollar deposits. Stock returns are based on the S&P 500 and include dividends. Bond returns are computed using holding period returns on 10-year Treasury bonds. We consider annual returns over the period 1983-1994. All data are obtained from Datastream. Initially, we need to compute the mean and variance of the returns series. Let x_1, x_2 and x_3 denote the amount invested in cash, stocks, and bonds, respectively, and let the corresponding returns be denoted by r_1, r_2 and r_3. Then $r = (r_1, r_2, r_3)'$ has mean and standard deviation:

$$\begin{array}{rcl} & & \quad Cash \quad Stocks \quad Bond \\ \mu' & = & (6.8\% \quad 17\% \quad 12.3\%) \\ & & \quad Cash \quad Stocks \quad Bond \\ \sigma' & = & (2.3\% \quad 14.7\% \quad 10.5\%) \end{array}$$

and the correlation matrix:

$$\begin{pmatrix} 1 & 0.01 & 0.18 \\ 0.01 & 1 & 0.73 \\ 0.18 & 0.73 & 1 \end{pmatrix} \tag{15}$$

In Figure. 1 we consider results obtained for two values of shortfall probability ($\alpha = 0\%,\ \alpha = 5\%$) and for several values of the shortfall return r^{low} at the levels of 0%, -5%, -10% [7]

Thus, for example, the combination $(\alpha, r^{low}) = (1\%,\, 0\%)$, means that the manager requires an asset mix that results in no loss with a 99% probability. Similarly, the combination $(\alpha, r^{low}) = (5\%, -5\%)$, means that the manager is satisfied with a 5% *Value at Risk* per Euro invested with a confidence level of 95% probability.

Using GAUSS 3.1.4 optimisation library, we compute in simulation the optimal values of x_i satisfying the shortfall constraint in (2) for several values of ν. The main results are presented in Table 1.

Some obvious effects in Table 1 are that the optimal asset mixes become more aggressive if the shortfall constraint is loosened. This can be done by increasing the allowed shortfall probability α or by lowering the required shortfall return r^{low} , i.e. increasing the *Value at Risk* per Euro invested. If we focus on the effects of ν, we note the difference between the $\alpha = 5\%$ and the $\alpha = 1\%$ case.

[7]The studies we made use in particular four values of shortfall probability (0.5%, 1%, 5%, 10%), and five values of shortfall return (0%, -3%, -5%, -7%, -10%). Table 1 only indicates the main results obtained in our work.

Table 1: Optimal asset allocation obtained through Monte Carlo simulation (E* indicates the expected portfolio returns).

ν	$r^{low} = 0\%$							
	Cash	Stock	Bond	E^*	Cash	Stock	Bond	E^*
	Shortfall probability 5%				Shortfall probability 1%			
3	28.10%	66.70%	5.10%	13.90%	80.80%	19.20%	0.00%	8.70%
5	46.30%	50.10%	3.50%	12.10%	80.00%	20.00%	0.00%	8.80%
7	49.00%	48.50%	2.50%	11.90%	78.50%	21.50%	0.00%	9.00%
10	50.30%	47.20%	2.50%	11.70%	77.70%	22.30%	0.00%	9.10%
∞	51.60%	45.90%	2.50%	11.60%	75.60%	24.40%	0.00%	9.30%
	$r^{low} = -5\%$							
	Cash	Stock	Bond	E^*	Cash	Stock	Bond	E^*
	Shortfall probability 5%				Shortfall probability 1%			
3	0.00%	100.00%	0.00%	17.00%	58.50%	40.00%	1.50%	11.00%
5	3.80%	88.30%	7.90%	16.30%	58.20%	40.20%	1.60%	11.00%
7	8.00%	84.50%	7.60%	15.90%	56.30%	42.00%	1.80%	11.10%
10	10.20%	82.80%	7.00%	15.60%	54.60%	43.50%	1.90%	11.30%
∞	12.30%	80.60%	7.00%	15.40%	50.70%	46.70%	2.60%	11.70%
	$r^{low} = -10\%$							
	Cash	Stock	Bond	E^*	Cash	Stock	Bond	E^*
	Shortfall probability 5%				Shortfall probability 1%			
3	0.00%	100.00%	0.00%	17.00%	38.80%	57.20%	4.00%	12.90%
5	0.00%	100.00%	0.00%	17.00%	38.70%	57.30%	4.00%	12.90%
7	0.00%	100.00%	0.00%	17.00%	36.20%	59.80%	4.00%	13.10%
10	0.00%	100.00%	0.00%	17.00%	33.40%	62.40%	4.30%	13.40%
∞	0.00%	100.00%	0.00%	17.00%	27.90%	67.00%	5.10%	13.90%

In the 5% case, increasing the fatness of the tails of the asset returns' distribution $\mathbb{P}$ then leads to more aggressive asset allocation. The optimal asset mixes involve less of the relatively safe cash and more of the risky assets, stocks and bonds. The effect is more pronounced if the required shortfall return r^{low} is lower.

Although the results that fat tails lead to more aggressive asset mixes may seem counterintuitive at first sight, it is easily understood, given the results we have seen analysing Figure 2. In fact, decreasing ν while keeping the variance fixed has two opposite effects. First, the probability of extreme (negative) returns increases, leading to more prudent asset allocation strategies. Second, the precision of the distribution increases, leading to more certainty about the spread of the outcome and, thus, to a more aggressive strategy.

For a shortfall probability of 5%, the latter of these two effects dominates. In the case of 1% shortfall probability, the opposite occurs. Decreasing ν now leads to more prudent asset mixes. Again the effect is more pronounced if the required shortfall return r^{low} is lower.

Figure 3 shows the effects on expected portfolio returns, for different distributions, when the shortfall probability vary. [8] Curves obtained by simulations, are

[8] r^{low} has been set equal to 0%.

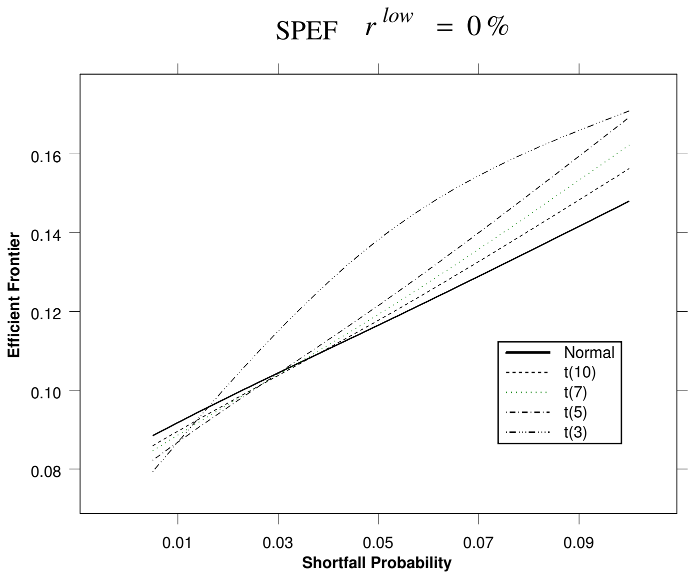

Figure 3: Shortfall Probability Efficient Frontiers obtained in simulation.

labelled "*Shortfall Probability Efficient Frontiers (SPEF)*" [9] (see Rudolf [27]), and represent all the efficient portfolios, given a certain level of risk (expressed by the shortfall probability). On the left side of the graph, we obtain lower expected returns decreasing the degrees of freedom. The reverse holds on the right side. In particular, the SPEF obtained from the normal distribution intercepts all the others in different points for a level of α that is the same decrypted in Figure 2.

Through the Monte Carlo study, it is also possible to determine the effects of a variation of the shortfall probability on the fractions invested in each asset. If we observe Figure 6 (Appendix A), we can note that the higher the level of α, the lower the fraction invested in cash, since to obtain more aggressive portfolios the manager directs capitals in riskier assets. However, while for a sufficiently small value of α (i.e. 0.5%, 1%), a normal distribution shows lower value invested in cash than the other distributions, for higher levels of shortfall probability (i.e. 5%, 10%), the lowest percentages in liquid assets are obtained augmenting the degrees of leptokurtosis. The reverse holds for stocks. In fact for this asset category, the behaviour is exactly reverse to which showed by cash, since there exists a trade-off between liquid and risky asset in order to obtain efficient portfolios. Fractions invested in bonds shows instead a more complex behaviour. In general, we can say that initially the percentage is increasing for smaller value of shortfall probability, and it is decreasing for higher value of shortfall probability.

The intersection between the curve of the normal distribution and the others in this case too, occurs for the values of showed in Figure 2. However, in this particular case each curve intercepts the others in more than one point, so they are more difficult to analyse.

[9] They are an alternative representations of the Efficient Frontier, known from the Portfolio theory.

It is also interesting to observe the expected returns behaviour when the shortfall return varies (given the shortfall probability level). We have choose to graph the expected returns behaviour for $\alpha = 1\%$ and for $\alpha = 5\%$. In Appendix B we can note that the higher the shortfall return value, the lower the expected return, but while for $\alpha = 1\%$ the *fat tail* effect dominates, for an $\alpha = 5\%$ the highest values are obtained when asset returns are leptokurtic. For very high losses levels (ex. 10%), it makes no difference if we use the normal distribution or a Student-t with different degrees of freedom, since the portfolio always contains only risky assets (100% stocks).

We can make the same analysis, observing the effects on the fractions invested in each asset, varying r^{low} for different levels of shortfall probability. The results are indicated in Appendix C.

As we can note, the lower the losses, the higher the fraction invested in cash, since the manager is more "*conservative*"; for the same reason, the fraction invested in stocks is decreasing. For bonds, the behaviour is not regular, and the fraction shows it is be decreasing for $\alpha = 1\%$, while it appears initially increasing and then decreasing for $\alpha = 5\%$. As we know, for $\alpha = 1\%$ the fat tail effect dominates and therefore the fraction invested in risky assets is much higher if data come from the normal distribution; the reverse holds for $\alpha = 5\%$.

6 EFFECTS OF MISSPECIFIED TAIL BEHAVIOUR

The probability distribution $\mathbb{P}$ is taken by the investment manager as a description of the true distribution of the asset returns. We label the distribution used by the manager $\mathbb{P}_m$ and the true distribution $\mathbb{P}_t$. These distributions are characterized by the parameter specifications (μ_m, Ω_m, ν_m) and (μ_t, Ω_t, ν_t) respectively.

Obviously, the manager would do best by matching μ_m, Ω_m and ν_m to μ_t, Ω_t and ν_t, respectively. However, the manager can fail to match all the parameters of the distribution used to solve (1) and (2) to those of the true distribution $\mathbb{P}_t$.

The effects of misspecification of means μ_m and/or covariance Ω_m has been investigated in the literature (see, e.g. Chopra and Ziemba [1]). We concentrate here on the possible mismatch between the true degree of leptokurtosis and the degree of leptokurtosis used by the investment manager, while assuming that means and covariance of the returns are precisely estimated.

The most obvious example of such a situation is the use of the normal distribution for solving (1), while the asset returns are actually fat-tailed. The mismatch between ν_m and ν_t can have important effects for the feasibility and efficiency of the optimal asset mixes. We assume that for a given value of ν_m , the manager chooses μ_m and Ω_m, in such a way that the mean and variance of $\mathbb{P}_m$ match the corresponding moments of the true distribution $\mathbb{P}_t$. This amounts to setting $\mu_m = \mu_t$ and

$$\Omega_m = (1 - 2\nu_m^{-1})\frac{\nu_t \Omega_t}{\nu_t - 2} \tag{16}$$

We assume, that the true mean μ_t and variance $\nu_t\Omega_t/(\nu_t - 2)$ are observed without error. Of course (**??**) and (16) are only estimates of the underlying true parameters.

We abstract from the associated estimation error for exposition purposes and in order to fully concentrate on the effects of fat tails (the results would be very similar if slightly different values for the means and variances were used, thus allowing for estimation error). Let x_m denote the optimal strategy of the investment manager using the distribution $\mathbb{P}_m$ with ν_m degrees of freedom. The appropriate values of x_m can be found in Table 1.

We want to compute the effect of using x_m when the data follow the distribution $\mathbb{P}_t$ instead of $\mathbb{P}_m$. In particular, we are interested in the effect of a discrepancy between ν_m and ν_t on the shortfall constraint.

We can quantify this effect in at least two different ways. First, we can use the strategy x_m while keeping the required shortfall return r^{low} constant and compute the actual shortfall probability α_* under the true probability measure $\mathbb{P}_t$. Alternatively, we can use the strategy x_m while keeping the required shortfall probability α constant and compute the corresponding shortfall return $r^{*,low}$, i.e., the (negative) Value at Risk per Euro invested.

First consider the case of fixed r^{low}. We then compute

$$\alpha^* = \mathbb{P}_t \left(\sum_{i=1}^{3} x_{m,i}(1 + r_i) \leq 1 + r^{low} \right) \tag{17}$$

where $x_{m,i}$, is the optimal asset allocation to category i for ν_m ; see Table 1.

So $(1 - \alpha^*)$ is the true confidence level of the investment manager's value at risk, given the asset allocation x_m.

In particular, we generate in simulation the asset returns from a distribution with ν_t degrees of freedom, and evaluate the shortfall constraint under the strategy x_m. This allows us to estimate the true risk for the financial manager.

We have seen that different values for r^{low} produce similar results, so we present only the case with $r^{low} = 0\%$. The results are given in Figure 4. The right panel in the figure gives the results if the optimal strategy is computed with $\alpha = 5\%$. The first thing to note is that, as expected, the true shortfall probability α^* is equal to α, if and only if the investment manager uses the correct distribution, i.e., $\nu_m = \nu_t$.

Second, if the investment manager uses a distribution that has thinner tails than those of the true distribution, then the manager is conservative in the sense that the shortfall constraint in (2) is not binding.

This holds even though the manager may believe the constraint to be binding based on the (misspecified) distribution $\mathbb{P}_m$ of the asset returns. As a result, efficiency could be gained by using the correct degree of leptokurtosis. By contrast, if the manager uses a distribution with a fatter tail than reality, the shortfall constraint is violated.

If we consider the case $\alpha = 1\%$, the results are reversed. If a thin-tailed distribution is assumed for the asset returns, e.g., the one based on normality, and if reality is leptokurtic, then the shortfall constraint is violated. Moreover, if $\nu < \nu_t$, the shortfall probability constraint is not binding. These results are directly relevant for risk management, because one minus the true shortfall probability equals the manager's required confidence level for the value at risk $-r^{low} > 0$. For example, for a 99% confidence level VaR, our results imply that the true confidence level of

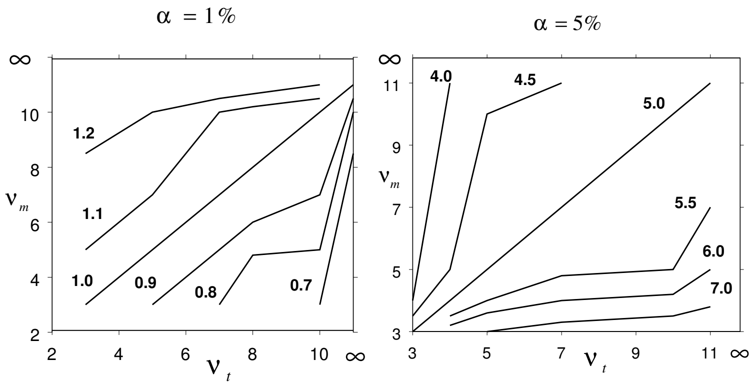

Figure 4: True Shortfall Probability for several combinations of ν_m and ν_t, obtained by simulation.

the manager's computed VaR is smaller than 99% if the manager uses the normal distribution while reality is fat-tailed. Note that, although the absolute difference between α^* and α in Figure 4 is smaller for $\alpha = 1\%$ than for $\alpha = 5\%$, the relative differences are approximately equal for different combinations of (ν_m, ν_t).

To illustrate the effect on the shortfall return r^{low}, for fixed α, we compute the required shortfall return $r^{*,low}$ such that:

$$\alpha = \mathbb{P}_t \left(\sum_{i=1}^{3} x_{m,i}(1 + r_i) \leq 1 + r^{*,low} \right) \tag{18}$$

The differences $(r^{*,low} - r^{low})$, in basis points, obtained in simulation, are presented in Table 2. Remember that r^{low} is the Value at Risk per Euro invested if $r^{low} < 0$. Therefore, $r^{*,low}$ in (18), is the managers true VaR if the investment policy x_m based on $\mathbb{P}_m$ is used.

The qualitative results are similar to those in Figure 4. For high values of α, using a distribution $\mathbb{P}_m$, which has thin tails compared to reality $\mathbb{P}_t$, produces a conservative strategy. Again, the opposite holds for small values of the shortfall probability α.

The impact of using the normal distribution for $\mathbb{P}_m$ if reality is fat-tailed is quite substantial. Assume the postulated required minimum return r^{low} is -5%, i.e., a Value at Risk of 5 cents per Euro invested. That is, with a maximum probability of α, the manager is willing to take losses exceeding 5% of the invested notional principal.

If $\alpha = 5\%$, the true shortfall return can be as much as 352 basis points above the postulated level, implying a shortfall return of about -1.45% instead of -5%, with a probability of 95%. Exploiting the fat tail property in this case can lead to more aggressive asset allocations and, therefore, efficiency gains for a given level of shortfall.

Alternatively, consider the case $\alpha = 1\%$. Using a normal scenario generator ($\nu_m = \infty$), for a reality with $\nu_t = 3$ now leads to a violation of the shortfall constraint. While the manager believes the maximum loss with a 99% probability is 5%

Table 2: Differences ($r^{*,low} - r^{low}$) in basis points.

ν_m	$r^{low} = 0\%$			$r^{low} = -5\%$			$r^{low} = -10\%$		
	ν_t			ν_t			ν_t		
	3	7	∞	3	7	∞	3	7	∞
Shortfall probability 5%									
3	0	-249	-296	202	-153	-218	-	-	-
7	178	0	-33	313	0	-57	-	-	-
∞	201	32	0	352	54	0	-	-	-
Shortfall probability 1%									
3	0	43	88	0	53	164	0	71	243
7	-45	0	51	-56	0	117	-74	0	181
∞	-105	-60	0	-210	-135	0	-301	-205	0

of the invested notional, the actual loss, given that probability, may be about 210 basis points higher, or 7.12%, an increase of over 40%. In this case a correct assessment of the degree of leptokurtosis will lead to a more correct assessment of risk and to the exclusion of infeasible strategies. All these effects are even more pronounced if the manager is, *a priori*, willing to take higher losses, i.e., if r^{low} is lower. These results have obvious important consequences for Value at Risk analyses. Using a distribution with an incorrect tail behaviour may lead to portfolios with a minimum required return $r^{*,low}$ that can be as much as 301 basis points below the minimum return r^{low} imposed by the model. In value at risk calculations, this implies that the true VaR may deviate by more than 30% from what an incorrectly specified Extreme Returns in a Shortfall Risk Framework model suggests. This illustrates the importance of trying to get the tail behaviour of the distribution used to solve (1) and (2) right.

7 ASSETS'S RETURNS AND TAILS BEHAVIOUR

The analysis carried out, allowed us to value the true risk for the financial manager, when the degrees of freedom of the empirical distribution are not correctly estimated. Nevertheless, the results we have showed up to now are based on the idea that returns of the three asset classes considered are identically distributed. As we know, this hypothesis is quite unrealistic, given the heterogeneity of the financial indexes studied. We therefore decide to develop our analysis, using a different probability distribution for each index, remaining in the Student-t class. The use of stochastic optimisation techniques allow us to extend the study of the asset allocation problems for particular multivariate distribution for which in general it is not possible to obtain the result in a closed form. This approach makes it necessary to determine which Student-t combination seems able to fit the empirical distribution adequately. To do this, we have analysed, through several tests [10], the behaviour of some distributions

[10]We have used two categories of tests: graphical tests and statistical tests. The first category includes QQ-plots, and the study of the empirical density function and empirical distribution function with respect to theoretical ones. The second category includes Jaque Bera normality

of financial indexes [11], for each asset category (cash, stocks, bonds), valuing for each of them the existence of *fat-tails*, and the parametric distribution that allows us to obtain the best fit for the empirical distribution on the tails. The test results show that the normal distribution is often unable to capture the behaviour of the tails, since financial time series are usually leptokurtic, while the Student-t distribution seems more adequate to capture the fat-tails effect.

In particular, tests indicate that for stocks, the Student-t with 7 degrees of freedom seems a good approximation of the empirical distribution in the extreme returns area. Otherwise, bonds seem to prefer Student-t with 8 degrees of freedom. Finally, cash behaviour does not appear unimodal, and for this reason no Student-t appears suitable to fit the empirical distribution adequately. Yet, some statistical tests seem to accept the null hypothesis that some of the cash indexes studied follow a Student-t distribution with 30 degrees of freedom (well approximated by a normal). For this reason we use this marginal distribution to simulate returns even if we know that it is not the most suitable. In fact our objective is to study the effects on the portfolio model when the empirical distribution is fat-tails, and in particular when each asset class presents a different degree of leptokurtosis. To do this, we generate asset returns simulating from the Student-t class, which present leptokurtic but not plurimodal or asymmetric characteristics.

Making use of tests results, we now calculate the optimal investment strategy and expected portfolio returns, generating asset returns from a multivariate distribution [12], where marginal distributions are $t(30)$, $t(7)$ and $t(8)$. We label this new distribution as "Mixed" with mean μ and variance-covariance matrix V. To impose the desired correlative structure to the simulated series, we use the *calibration method*.

In particular, given two variables x and w, where $x \sim t(\nu_x)$ and $w \sim t(\nu_w)$, we say that the correlation between x and w is ρ^* if there exists a value (τ^*) for the parameter τ such that $\rho^* = f(\tau^*, \nu_x, \nu_w)$, where f is the correlation between the components of the random vector given in (9). It is very difficult to obtain an analytical solution, and therefore we use the Monte Carlo method to calibrate τ step by step, until we obtain the desired value for $\rho(\rho^*)$[13]. In our case, we impose a correlation matrix τ to the multivariate normal in Equation 9 and simulate a sample from the multivariate "Mixed" distribution. Then we vary τ until the correlation matrix ρ estimated on simulated data is equal to the desired correlation matrix ρ^*. This calibration method has been recently used also in Palmitesta and Provasi

tests, Chi-square tests, Anderson Darling tests, Kolmogorov-Smirnov tests and Cramer von Mises tests (see DAgostino and Stephens [2].

[11] We consider annual returns. For stocks, indexes analysed are: S&P 500 over the period (1/1/1970- 1/4/2001), MSCI Europe and MSCI North America over the period (1/1/1986-1/12/1999); for bonds: JP Morgan Great Britain, JP Morgan Japan, JP Morgan France, JP Morgan Belgium over the period (1/1/1986-1/1/2000); for cash: Euro-Mark 3 mth, Euro-Franc 3 mth, Euro-Lire 3 mth, Euro-Yen 3 mth, Euro-£3 mth, and Euro-$ 3 mth over the period (1/1/1985-1/4/2001). All data are obtained from Datastream.

[12] Note that it is not a multivariate Student-t distribution, because each t has different degrees of freedom. Moreover the covariance between marginal random variables is no more proportional to the covariance matrix used in equation 15 to simulate from that multivariate distribution.

[13] ρ^* used is indicated in (15).

[21] in order to fit the parameters of the Koehler-Symanowski distributions on real data. They minimize the distance between the correlation matrix simulate from the multivariate Koehler-Symanowski distribution and correlation matrix estimated on real data.

We now examine the effects on asset allocation strategies. In particular, we can note that results obtained combining Student-t with different degrees of freedom (see Appendix C) are intermediate between those obtained using a multivariate Student-t with 7 degrees of freedom and with 10 degrees of freedom. This aspect is very important in a risk management framework because the use of different marginal distributions gives more information than previous analysis. This means that financial manager can invest with higher precision in the estimates of optimal strategies and of expected returns. Furthermore, it is important to note that for different combinations (for example $t(15)$, $t(3)$ and $t(9)$), where the leptokurtosis degree is very different, the loss of information could be very high if we choose a distribution with identically distributed marginal distributions.

Following the same techniques used in previous sections, we now concentrate our analysis on the study of effective risk associated with the investment if the financial manager uses a distribution $\mathbb{P}_m \neq \mathbb{P}_t$.

We have seen that we can quantify these effects in at least two different ways: first, we can use the strategy x_m while keeping the required shortfall return r^{low} constant and compute the actual shortfall probability α^* a under the true probability measure $\mathbb{P}_t$. Alternatively, we can use the strategy x_m while keeping the required shortfall probability a constant and compute the corresponding shortfall return $r^{*,low}$.

Figure 5 indicates the level of α^* if the manager uses the distribution $t(\nu_m)$ and the data follow the Mixed distribution. This analysis has been made, using $\alpha = 5\%$ and $\alpha = 1\%$ [14]

For $\alpha = 5\%$, if the investment manager uses a distribution that has thinner tails than reality, then the manager is conservative, in the sense that the shortfall constraint in (2) is not binding, while if the investment manager uses a distribution that has fatter tails than reality, the shortfall constraint is violated.

As we know, if we consider the case $\alpha = 1\%$, the results are reversed. If a thin-tailed distribution is assumed for the asset returns, e.g., the one based on normality, and if reality is leptokurtic, then the shortfall constraint is violated. Moreover, if $\nu_m < \nu_t$, the shortfall constraint is not binding.

It is interesting to note the relationship that exists between Mixed distribution, $t(7)$ and $t(10)$.

If we use a shortfall probability $\alpha = 5\%$ in the model, then the values of true shortfall probability $\alpha^*(\nu_m)$ are not only intermediate to those obtained using the other two distributions, but we can demonstrate that they are very close to what we could have by generating data from a $t(7)$, for every level of ν_m. If we use a shortfall probability $\alpha = 1\%$ in the model, when the data come from "Mixed" distribution, the behaviour of $\alpha^*(\nu_m)$ seems initially intermediate for $\nu_m = 3$, and very close to $t(7)$ curve, for higher degrees of freedom. For example, if the manager uses a distribution

[14] We have set $r^{low} = 0\%$

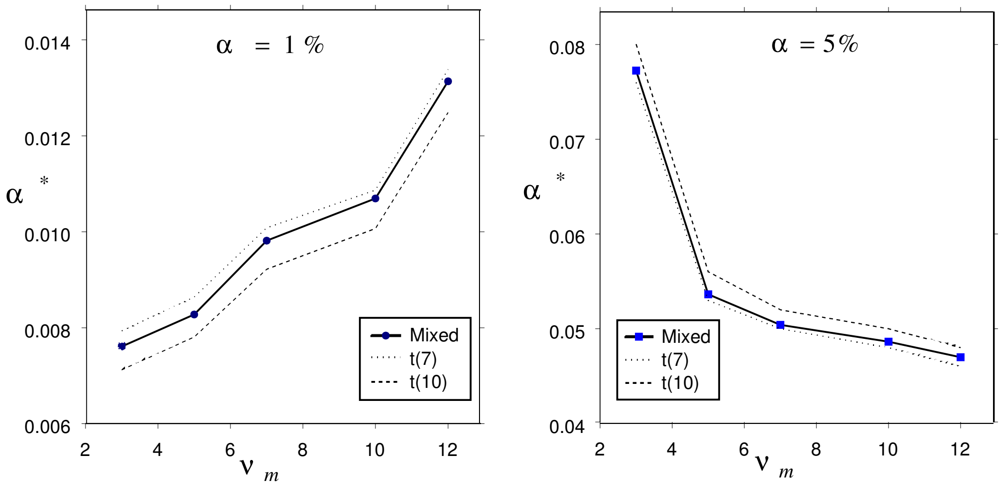

Figure 5: True Shortfall Probability for Mixed distribution, $t(7)$ and $t(10)$; $\alpha = 1\%$ and $\alpha = 5\%$.

$t(3)$, when the data follow a Mixed distribution, we obtain $\alpha^* = 0.75\%$, while if the manager uses a distribution with thinner tails, the true shortfall probability for Mixed distribution tends to the true shortfall probability for $t(7)$ one. It is important to consider that the interval in which the curve oscillates is very small, and for this reason every variation seems imperceptible. The second analysis we can do is to study the differences $r^{low} - r^{*,low}$ in basis points, when data come from Mixed distribution. Results obtained are indicated in Table 3.

For $\alpha = 5\%$, if the investment manager uses a distribution with tails heavier than reality, the loss could exceed r^{low}. For $\alpha = 1\%$, the loss could exceed r^{low} only if the investment manager uses a distribution with tails thinner than reality. Furthermore, for $\alpha = 1\%$ and $r^{low} = -5\%$, the impact of using the normal distribution to explain the empirical distribution behaviour if reality is "Mixed", is quite substantial. In fact the manager is willing to take losses exceeding 5% of 137 basis points, i.e. an increase of over 30%. It is interesting to note that for $\nu_m = 7$, the difference $r^{low} - r^{*,low}$ is very small and for $\alpha = 5\%$ it tends to zero, decreasing the r^{low} level.

For $\alpha = 1\%$, the difference becomes smaller with the reduction of losses. From this study we can conclude that for the case analysed, a multivariate Student-t distribution with 7 degrees of freedom seems a good approximation of the Mixed distribution. For the financial manager it could be simpler to use a multivariate $t(7)$ with identically distributed marginal distributions, without excessive loss of information on the true risk. However, if this result held for the specified combination used $t(30)$, $t(7)$, $t(8)$, in general we can not extend our conclusions for the enormous number of combinations of marginal distributions. This means that before any application, it is wise for the financial manager to analyse whether an approximation of the mixed distribution can cause losses of information and thus undesired effects for risk management.

Table 3: Differences ($r^{*,low} - r^{low}$) in basis points. Data generated from Mixed distribution.

Shortfall probability=5%			
ν_m	$r^{low} = 0\%$	$r^{low} = -5\%$	$r^{low} = -10\%$
3	-253	-157	-
5	-37	-59	-
7	-4	-3	-
10	23	21	-
∞	27	45	-
Shortfall probability= 1%			
3	43	47	71
5	33	39	70
7	3	10	13
10	-16	-39	-67
∞	-62	-137	-198

8 CONCLUSION

In our work, we have investigated the effects of extreme returns on the optimal asset allocation problem with a shortfall constraint, using a Monte Carlo simulation approach. We have seen that financial markets usually show a "non-normal" behaviour, since the tails of returns distributions often appear very heavy. Extreme returns can be modelled by using a statistical distribution with fatter tails than those of a normal distribution. We have used the Student-t distribution to show the salient effects of fat tails in financial decision context. Initially, we have analysed the effects on asset allocation choices when all asset returns are identically distributed. This analysis is not very realistic, but has allowed us to discover the true risk for the financial manager when the behaviour of empirical distribution is incorrectly estimated. We have then tried to simulate data from a multivariate distribution where each marginal distribution has a different degree of leptokurtosis. For each of these two analyses, we may conclude that a correct assessment of the fattailedness of asset returns is important for the determination of optimal asset allocation. If asset allocations are based on the normal distribution, the resulting allocation may be either inefficient or unfeasible if reality is non-normal. Both effects can be quite substantial. Then, it appears that the shortfall probability set by the investment manager plays a crucial role for the nature of the effect of leptokurtic asset returns. If the shortfall probability is set sufficiently high, using normal scenarios for the leptokurtic asset return leads to overly prudent and therefore inefficient asset allocations. If the shortfall probability is sufficiently small, however, the use of normal scenarios leads to unfeasible strategies if reality is fat-tailed. This second result implies that the Value at Risk of a given portfolio may be underestimated if the tail behaviour of asset returns is not captured adequately. Our results show that the actual VaR can be substantially higher than the model suggests in such cases. Extreme Returns in a Shortfall Risk Framework In studying the "Mixed" distribution we have also shown by simulation that another important aspect in financial analysis is the correct specification of the marginal distribution for each asset analysed. This aspect becomes crucial when there is a shortfall constraint in the asset allocation problem and at the same time the asset class return has a different tails behaviour. For our particular case, we have seen that the resulting optimal allocation obtained with the use of Mixed distribution could be adequately approximated by a multivariate Student-t at certain level of shortfall probability and shortfall return. Furthermore we obtain the following result of interest. When asset classes returns have a "Mixed distribution", the assumption of Student-t distribution with misspecified degrees of freedom produces effective shortfall probability and effective shortfall return which differ from the desired ones. However these errors have known upper and lower bounds. Even if this result can be very useful for the financial manager, since he can enormously simplify the mixed problem, in general, given any "mixed distribution", we do not know what its adequate approximation is and so we can not conclude that an adequate approximation always exists for all combinations of marginal distributions. For this reason, the use of stochastic simulation in the study we have performed in this article, is a very effective instrument to choose the optimal strategy to apply in asset allocation problems, and in particular when we use a shortfall constraint. In a more general sense, our findings imply that a good characterisation of the distribution of asset returns is needed in a financial decision context involving downside risk. Such a characterisation may require not only the specification of usual measures like mean and variance, but also a correct specification of additional features of the distribution of asset returns, such as the tail behaviour or degree of leptokurtosis. Specification and estimation of such additional features can proceed along familiar lines, for example using parametric or non-parametric methods. Regardless of the method chosen, our results provide insights into the general effects of different type of leptokurtic distributions on optimal asset allocations and associated risk measures. The results are therefore valuable to investors who require a qualitative assessment of the reliability and sensitivity of their adopted investment strategies in case their models are potentially misspecified.

A.FRACTION INVESTED, VARYING THE SHORTFALL PROBABILITY

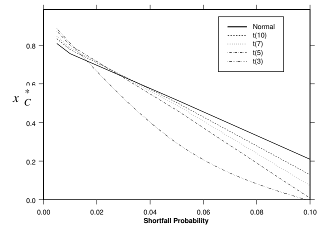

Figure 6: Fraction invested in Cash varying the shortfall probability level.

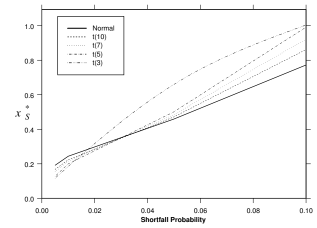

Figure 7: Fraction invested in Stocks varying the shortfall probability level.

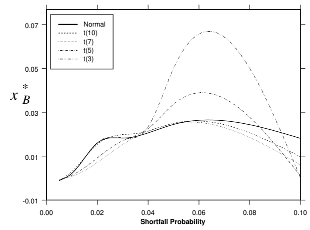

Figure 8: Fraction invested in Bonds varying the shortfall probability level.

B. THE EFFECTS OF THE SHORTFALL RETURN LEVEL ON THE OPTIMAL ALLOCATION

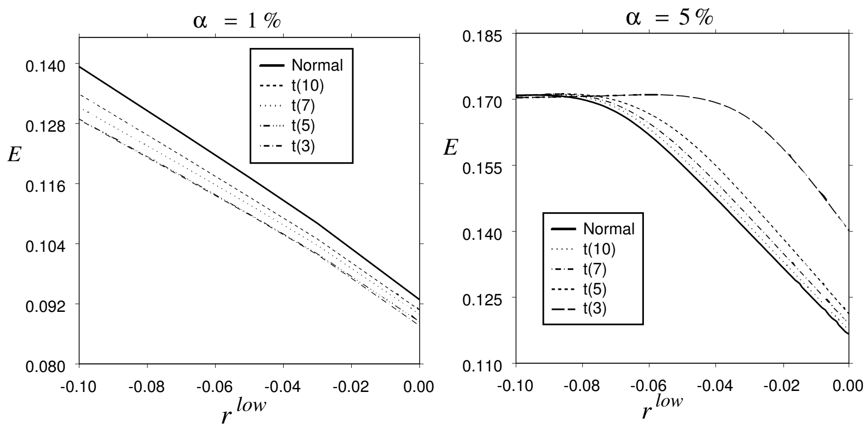

Figure 9: Expected return, varying the shortfall return ($\alpha = 1\%$, $\alpha = 5\%$).

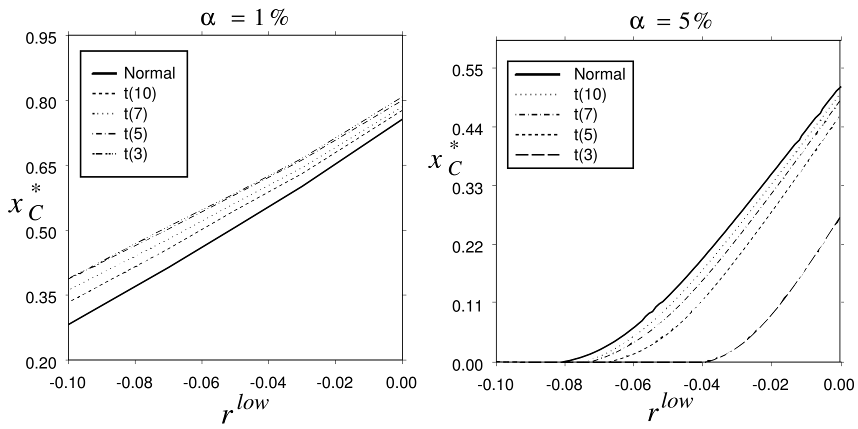

Figure 10: Fraction invested in Cash for different distributions for $\alpha = 1\%$ and $\alpha = 5\%$.

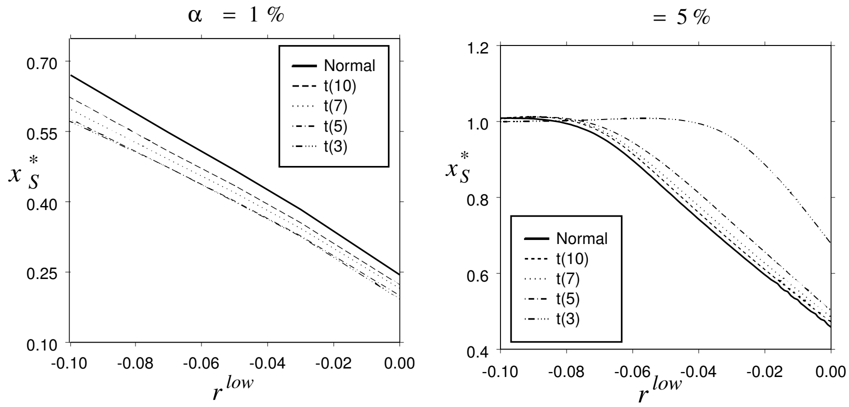

Figure 11: Fraction invested in Stocks for different distributions for $\alpha = 1\%$ and $\alpha = 5\%$

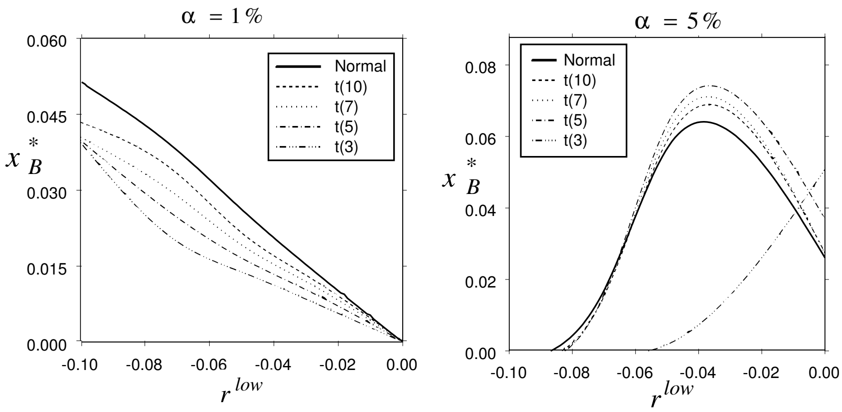

Figure 12: Fraction invested in Bonds for different distributions for $\alpha = 5\%$

C.RESULTS OBTAINED FOR MIXED DISTRIBUTION

Table 4: Expected portfolio return (Mixed distribution).

	α			
r^{low}	0.5%	1%	5%	10%
0%	8.442 (0.012)	8.980 (0.013)	11.825 (0.026)	16.182 (0.043)
-3%	9.612 (0.016)	10.313 (0.016)	14.190 (0.034)	17.000 (0.000)
-5%	10.306 (0.019)	11.127 (0.020)	15.760 (0.043)	17.000 (0.000)
-7%	10.955 (0.021)	11.971 (0.025)	16.982 (0.017)	17.000 (0.0000)
-10%	11.938 (0.026)	13.151 (0.024)	17.000 (0.000)	17.000 (0.000)

Table 5: Fraction invested in each asset (Mixed distribution)

	α				
r^{low}	0.5%	1%	5%	10%	Assets
0%	83.878 (0.111)	78.665 (0.112)	49.254 (0.203)	7.813 (0.343)	CASH
	16.092 (0.110)	21.261 (0.111)	47.643 (0.216)	91.695 (0.361)	STOCK
	0.0285 (0.010)	0.0731 (0.024)	3.101 (0.015)	0.491 (0.026)	BONDS
-3%	72.302 (0.139)	64.982 (0.139)	25.277 (0.232)	0.000 (0.000)	CASH
	27.179 (0.136)	33.776 (0.139)	67.382 (0.232)	1.000 (0.000)	STOCK
	0.518 (0.021)	1.238 (0.006)	7.340 (0.094)	0.000 (0.000)	BONDS
-5%	65.245 (0.164)	56.596 (0.167)	8.516 (0.239)	0.000 (0.000)	CASH
	33.485 (0.165)	41.261 (0.172)	83.476 (0.463)	1.000 (0.000)	STOCK
	1.269 (0.021)	2.142 (0.013)	8.011 (0.224)	0.000 (0.000)	BONDS
-7%	57.928 (0.185)	48.348 (0.202)	0.144 (0.055)	0.000 (0.000)	CASH
	39.164 (0.195)	48.593 (0.214)	99.778 (0.081)	1.000 (0.000)	STOCK
	2.907 (0.011)	3.058 (0.017)	0.077 (0.027)	0.000 (0.000)	BONDS
-10%	47.797 (0.216)	35.883 (0.200)	0.000 (0.000)	0.000 (0.000)	CASH
	48.208 (0.237)	60.088 (0.219)	1.000 (0.000)	1.000 (0.000)	STOCK
	3.994 (0.030)	4.028 (0.023)	0.000 (0.000)	0.000 (0.000)	BONDS

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