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Three Essays on Cultural Evolution

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Tesi di dottorato di Fabrizio Panebianco
matricola 955133

Relatore della tesi
Prof. Sergio Currarini

Coordinatore del dottorato
Prof. Agar Brugiavini

a Laura

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The undersigned Fabrizio Panebianco, in his quality of doctoral candidate for a Ph.D. degree in Economics and Organization granted by the Universita' Ca' Foscari di Venezia and Scuola Superiore di Economia-Venezia attests that the research exposed in this dissertation is original and that it has not been and it will not be used to pursue or attain any other academic degree of any level at any other academic institution, be it foreign or Italian.

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Introduction

Cultural evolution, in economic literature, studies how preferences, beliefs, social norms or generic cultural traits are transmitted among agents and how they evolve during time focusing on two key aspects: the coevolution between the social environment and the cultural traits analysed and the influences of social interactions on these processes. With social interaction influences it is usually meant how parents, family, school, friends, peers and other social elements have an impact on the cultural trait analysed. Economists developed a particular interest in these issues since they study the priors behind the choice mechanisms and the endogenization of a preference dynamics.
In this research I analyse three different issues related to cultural evolution: the intergenerational transmission of a cultural trait (interethnic attitude) with a focus on the role of social influences structure in these processes, the role of the evolution of interethnic preferences in the school ethnic segregation problem and, at last, the evolution of a generic social norm in an evolutionary game theoretic framework focusing on the payoff redistributive role that cultural evolution may have in these processes. Apart from the specific contributions of each chapter, the main objective of this study is to better understand how agents' attitudes, preferences and types can be considered endogenous in an economy, which rules they follow in their dynamics and which forces play a crucial role in shaping these rules.

Thus, in the first chapter (*'Driving While Black'*: A theory for Interethnic Integration and Evolution of Prejudice) I analyse how interethnic attitudes are transmitted among generations, and how the structure of social influences affects their long run dynamics. This paper, together with Pichler (2010), is one of the first studies on the intergenerational transmission of a continuous cultural trait analysing the change of the intensity of the trait along time. Previous studies (as Bisin and Verdier (2000, 2001) and all the related literature) analyse the evolution of the distribution of discrete cultural traits with fixed intensities: this literature directly derives from genetists' and anthropologists' studies about cultural evolution (Cavalli Sforza and Feldman (1973, 1981) and Boyd and Richerson (1985, 2005)). By means of this framework, I find that the way in which *Oblique Socialization Schemes* (the way children react to out-of-family stimuli when forming their cultural values) are defined and modelled becomes crucial for the structure of the derived long run equilibria. In particular, I find that steady states implying an ethnic-based social ranking or full integration of ethnicities may be reached depending on whether agents use reciprocity and/or ethnocentrism in their interethnic attitudes formation schemes or not. Then, I study the conditions under which a group puts more effort in the socialization process, it changes more in values and shows more frustration than others. At last, I prtovide three extensions to the model: I first analyse what happens under asymmetric oblique socialization structure, what happens if these structures are time dependent and then I move the first steps towards an endogenization of oblique socialization structures.

The second chapter (Residential and School Segregation and the Evolution of Homophily) contributes to the existing debate about ethnic based school segregation by introducing a simple model of school choice based on spatial residential segregation, analysing when partial ethnic segregation can happen due to the change in interethnic preferences even if the demographic structure is unchanged during time. One of the main driving forces of these choices are interethnic preferences, and the way in which they evolve and change due to social interactions: this paper thus moves away from intergenerational transmission of preferences and starts analysing a more general social influence effect on preferences. I build a theory in which only ethnic preferences concerns affect choices without socio economic biases: while socio-economic motivations has been widely sudied in the literature (for a review see Nechyba (2006)), interethnic preferences motivations have not been given the proper attention. In particular I focus on the case in which groups show a certain degree of *homophily*, meaning a preference for staying with people of their own group (see Currarini et. al. (2009) or Moody (2001)). I introduce a dynamic model in order to analyse how homophily preferences are affected by the social interactions and past homophily preferences: in this way I have a coevolution of preferences and segregation in schools. I then find this cross dependency to be necessary in order to obtain partial segregation results as observed in empirical data. I then analyse deeper these influences on preferences by indentifying the role of two effects on homophily: the School Effect, capturing the effect on preferences of having a higher share of own people atteding the same school, and the Residential Effect, capturing the effect of having more people of own type in own neighborhood that are in contact with other groups.

Chapter 3 (Cultural Evolution as Payoffs Distribution: An Evolutionary Game Approach) analyses the evolution of types in a society with an evolutionary game framework in which types evolve not only depending on how much they are fit but also on how much they are able to persuade that they are fit. Cultural evolution, in fact, directly analyses how fitnesses are formed: in this way I try to answer to what Bowles (1998) had pointed out: *'Evolutionary game theoretic models typically abstract entirely from the process of cultural transmission, representing payoffs associated with particular traits as if they were the only influences on the replication of traits. By contrast, models of cultural evolution typically address what is known about the particulars of the process by which traits are acquired..'*. Again in this framework I move away from intergenerational transmission of preferences and I focus on generic social influences mechanisms. In particular, by allowing agents to have an impact over others' evaluation of own fitness, a kind of payoffs redistribution is created with respect to standard models, so that it is possible to reach equilibria impossible under standard dynamics. I first provide necessary conditions of a generic game matrix and generic class of cultural evolution mechanisms in order to observe polimorphic equilibria, proposing that in these cases standard game classifications are not a good method since, given the specific parametrization, two games of the same class may have different equilibria characterization. I then propose two specific games (one with prisoner's dilemma incentives) and two persuasion rules derived from cultural transmission literature and find when it is possible to observe different classes of equilibria: in particular under a Prisoner's Dilemma type matrix stable coexistence of types and the pareto efficient outcome are possible steady states under some conditions over the parameters. Finally I compare this framework with a standard one without persuasion rules decribing when different types are advantaged by these schemes and comparing the different rules proposed.

Finally conclusions present future research development based on these three contributions.

Chapter 1

'Driving While Black': A theory for Interethnic Integration and Evolution of Prejudice

1.0.1 Abstract

This paper studies the evolution of interethnic attitudes, the integration or segregation dynamics of ethnic minorities and the conditions for the rising of ethnic-based social hierarchies. By means of a cultural transmission framework, a dynamics of interethnic attitudes is provided and conditions for their convergence derived. We find that the way in which *Oblique Socialization Schemes* (the way children react to out-of-family stimuli when forming their cultural values) are defined and modelled becomes crucial for the structure of the derived long run equilibria. In particular, we find that Steady States implying an Ethnic-based social ranking or full integration of ethnicities may be reached depending on whether or not agents use Reciprocity and/or Ethnocentrism in their interethnic attitudes formation schemes. We study the conditions under which one group puts more effort in the socialization process, it changes more in values and shows more frustration than others. At last, we provide three extensions to the model: we first analyse what happens under asymmetric oblique socialization structure, what happens if these structures are time dependent and then we move the first steps towards an endogeneization of oblique socialization structures.

1.1 Introduction

Interactions among different ethnicities in modern societies have always been a great concern for many academics and politicians. The United States has been the first country to experience problems with interracial relationships, since the US society has always been composed by people of different ethnicities. Now, Europe is also starting to face problems and opportunities deriving from a multicultural and multiethnic society. Moreover, given the actual rates of immigration, we can reasonably think that these issues will become increasingly important for the Western societies.
Thus, this paper studies the evolution of interethnic attitudes, the integration or segregation dynamics of ethnic minorities and the conditions for the rising of ethnic-based social hierarchies by means of a cultural transmission framework. As economist, it is important to answer to these questions since they are at the basis of some works on, for example, marriage markets analysis, job hiring process studies and spatial segregation analysis. With respect to

the first case Bisin et al. (2004b, 2006) proposed models for intergroup marriages in which the intergroup preferences are one of the key elements of the analysis. In this respect, taking care of how interethnic attitudes change along time may be an important element in explaining the dynamics of interethnic marriages. Interethnic attitudes have also been shown to be crucial in job hiring processes, as found by Bertrand and Mullainathan (2004), Carlsson and Rooth (2007) and Rooth (2009). In spatial segregation theories, as, for example, the basic Schelling (1971, 1978) model, the preference or tolerance towards other groups is a crucial element: having a theory that analyses how these elements change endogenously with the composition of the neighbourhood can bring to interesting results.

Some sociological studies find the existence of ethnic hierarchies in the society, meaning that the society converges to an agreement over attitudes towards the ethnic groups. These works rise important questions about the long run role of racism: is the ethnic social ranking we observe stable enough for ethnicity to always play a role in people' choices? Or under which conditions ethnic groups may agree on common attitudes towards anyother such that an 'end of racism' may be observed? From an empirical point of view, this intergroup consensus over ethnic hierarchies has been studied for US (Bobo and Zubrinsky (1996); Duckitt (1992)), Canada (Berry and Kalin (1979, 1996)), Sweden (Snellman and Ekehammar (2005)), the Netherlands (Verkuyten and Kinket (2000)), as pointed out by Listhaug and Strabac (2008) that provide the same evidence for Muslim minorities. From a theoretical point of view Hagendoorn et al. (1998) explain why we observe ethnic hierarchies: this literature identifies the causes of this evolution in the process of prejudice formation, in a form of cultural distance among groups and in the socio-economic status of the group. In this paper we study the first one and, in the last section, we will give some hints on the second point.

Examples of attitudes measures are the ones by Golebiowska (2007, 2009) in which measures of reciprocal tolerance are derived by opinion surveys focusing on interpersonal trust and other social indicators. Other examples are derived by ethnic hierarchies studies in the social psychology field, as Hagendoorn et al.(1998), Listhaug and Strabac (2008), Berry (2006) and Schalksoekar et al. (2004), in which indexes that indicate the attitudes among groups are estimated so that an overview on how ethnic hierarchies may arise is given. Such studies become now easier to be performed by means of surveys as the World Value Survey or the International Study of Attitudes Towards Immigration and Settlement (ISATIS), that make the objectivation of these measures possible. A theoretical paper studying how these prejudices exist and are transmitted is given by Bar-Tal (1997), in which the roles of context, socialization and individual variables are examined[1].

Even though some theoretical economic literature uses network theory in order to understand integration and segregation and its determinants (Jackson (2006) and Currarini et al. (2009)), here we focus on a second line of research: cultural evolution. Cultural evolution theories have their roots in the seminal works of Cavalli-Sforza and Feldman (1973, 1981) and Boyd and Richerson (1985, 2005). These theories are based on the intergenerational transmission and modification of some characters having as peculiar element the coevolution of biological and cultural traits; their focus is on the socialization process (i.e. the process by which children acquire cultural traits), and they distinguish between Vertical Socialization (parents' influence) and Oblique and Horizontal socialization (society influence). We use these theories since they provide instruments to analyse how values are formed, and how they may spread in the society, taking care of the interaction between this process and the environment in which the agents live: we think that these are key elements in the study of a social phenomenon as the evolution of interethnic attitudes. Moreover, given the fact that

[1]Another set of studies that uses these indexes are derived from the Bogardus Social Distance Scale (Bogardus (1926, 1959)). These studies, mainly referred to social psychology and psychometric techniques and developed in Hraba et al. (1999), Randall and Delbridge (2005), Lee et al. (1996) and Parillo and Donoghue (2005), estimate, by means of scaling systems, social dystance measures and indicators of how much groups reciprocally like.

attitudes may be transmitted from one person to the other through a kind of imitation, and, thus, they are susceptible of modification, they can be considered as cultural traits without problems. Therefore in the rest of the paper we sometimes refer to the attitude as 'cultural trait' and to the set of the attitudes of a given ethnic group as 'culture' since the framework we proposed can be extended to the study of other cultural traits than interethnic attitudes. The first works trying to introduce these concepts in the economic debate has been Bisin and Verdier (2000, 2001) in which the transmission of a cultural trait is modelled, and the dynamics of groups population is analyzed. The most interesting contributions are Bisin et al. (2004b, 2006) in which models for religious intermarriages in the US and interethnic preferences in UK are set up. In particular, in Bisin et al. (2004b) a cultural transmission model was used in order to estimate the intensity of ethnic identities depending on the social context children are rised.
In these last contributions, however, the cultural traits that are transmitted from one generation to the other are fixed so that only an analysis of the demographic trends is possible. Brueckner and Smirnov (2007, 2008) start to introduce the possibility of a change in the intensity of the cultural traits providing some sufficient conditions for convergence to a *Melting Pot* equilibrium. An innovating contribution is given by Pichler (2010) in which, in a reinterpretation of the Bisin and Verdier framework, parents can choose which kind of cultural trait transmit to their children and in which cultural values also evolve in intensity during time too, thus introducing complexity in the modelization of the vertical socialization.

This paper goes a little bit further in the analysis, focusing on the role of Oblique Socialization. We define as *'integrated'* two ethnic groups that share the same attitudes towards any ethnic group, *'segregated'* when this does not happen, and differences in attitudes are observed, while *integration* is defined as the process that brings to integrated groups: in our case integration does not mean that two groups have good attitude towards each other, but just that their attitude vector is equal so that their cultural traits are identical. Moreover, we consider fixed ethnicities, so that we are not interested in how and if a melting pot society or mixed identities arise, but only under which conditions different cultural groups converge in attitudes still remaining distingushed. We use, as starting point, the Bisin-Verdier framework in which agents choose how much to socialize children. However, differently from these previous studies, we consider two cultural traits that are contemporarily involved in the dynamics: *'ethnicity'* and *'attitudes'* towards other ethnic groups. Since cultural evolution regards interaction between biological and cultural traits, in our study ethnicity is biologically determined and thus fixed but transmittable, while attitudes are culturally derived and thus are transmitted and changed in the socialization process, so that they are no more fixed[2]. Given this framewok we then consider what is said by Boyd and Richerson (1985) in the first pages of their first contribution to cultural evolution theories in which they argue that a theory for cultural evolution '*should predict the effect of different structures of cultural transmission on the evolutionary process*'. In particular, starting from different schemes of cultural transmission, we derive conditions under which ethnic social rankings, as previously defined, arise in the long run, and when attitudes converge to the same value, providing theoretical answers to these sociological questions. In order to understand these dynamics, we study deeper a key element of cultural evolution theories: the socialization mechanism. In particular, depending on how children react to out of family stimuli (Oblique Socialization), the integration/segregation result may change. We then analyse when a group changes in values faster than others, when it pays much more attention to socialization and when its members are much more frustrated than the other groups' members, finding some condition for the policymaker in order to have faster integration and lower frustration in each group.

[2]In this sense, both Pichler (2010) and this work introduce, in different frameworks, the changing in the intensity of cultural traits as a problematic issue in cultural evolution models in economics.

We then extend the model in three directions: we first provide an analysis of what happens to the convergence process if groups differ in the use of oblique socialization schemes and derive conditions over the interethnic relational structures in order to get again ethnic hierarchies or a deeper integration; we then provide time-dependent socialization schemes, focusing on some conditions for convergence to long-run equilibria; at last we set up the first steps for the endogeneization of socialization mechanisms. In this last case we analyse agents using cultural distance in order to form their network, and we study when this homophily rule brings to integration and when it does not.
Given the nature of this analysis, strong relationship with the theory on the spread of opinion in a network (DeGroot (1974), De Marzo et al. (2003) and Golub and Jackson (2007)) can be found. From a formal point of view we will point out, time by time, the differences in the mathematical structure between this work and De Marzo et al. (2003) which is the closest one from a mathematical point of view and, for section 7, with Golub and Jackson (2009). This relationship between the two theories makes clear that the interethnic attitude problem may be only one aspect of the analysis, and that this framework may be frutifully extended in the direction of a theory of opinion formation in a network. The rest of the paper has the following structure: in section 2 we describe the model, in section 3 we provide a general dynamics for interethnic attitudes as cultural traits studying conditions for convergence. Section 4 introduces the oblique socialization structures, while section 5 studies what happens to optimal socialization values and comparative statics is performed. Sections 6, 7 and 8 present the different extensions to the model. Section 9 ends the paper.

1.2 The Model

Consider a society composed of a continuum of agents. Suppose the population set to be partitioned in n subsets belonging to $E \equiv \{i, j, k, ..., w\}$ each identifying an ethnic group. Suppose that each agent only belongs to one ethnic group. All agents in each ethnic group are supposed to be equal. Agents of a given ethnicity i are also characterized, at each point in time, by a vector $V_t^i \in [0,1]^n$, that we call '*type*', such that every entry is a coefficient associated to an ethnic group. Below an example for a 4 ethnicities world.

$$V_t^i := \begin{bmatrix} V_t^{ii} \\ V_t^{ij} \\ V_t^{ik} \\ V_t^{iw} \end{bmatrix} \quad V_t^j := \begin{bmatrix} V_t^{ji} \\ V_t^{jj} \\ V_t^{jk} \\ V_t^{jw} \end{bmatrix} \quad V_t^k := \begin{bmatrix} V_t^{ki} \\ V_t^{kj} \\ V_t^{kk} \\ V_t^{kw} \end{bmatrix} \quad V_t^w := \begin{bmatrix} V_t^{wi} \\ V_t^{wj} \\ V_t^{wk} \\ V_t^{ww} \end{bmatrix}$$

All agents belonging to the same ethnic group have the same type. This vector is supposed to be observable and common knowledge and represents the vector of attitudes of each group's agents towards all other groups' agents. Take a generic type V_t^i, then the element V_t^{ij} represents the attitude i agents have towards j agents. An attitude value equal to 0 corresponds to the worst attitude possible and an attitude value equal to 1 to the best attitude possible. Normalization on the [0,1] interval is arbitrary and using different normalizations does not change the results of the paper.
Given these priors, the structure of the model is the following: at each point in time each agent reproduces asexually[3], so that a child is born from every agent. Once born, each child

[3] The model can be extended to the case of sexual reproduction following the marriage matching mechanism as in Bisin and Verdier (2004, 2006). In this paper the model is kept as simple as possible in order to analyse only the effect of oblique socialization structures.

has the same ethnicity of the parent, but has no 'type' formed yet[4]. Thus each parent of a generic ethnic group i produces a socialization effort $\tau_t^i \in [0,1]$ in order to influence own child's type. In particular, each parent would like to perfectly transmit own type to own child, otherwise she experience a loss. However, socialization has a cost $c(\tau_t^i)$ for the effort produced. The child type then is formed considering the effect of the parental (or Vertical) socialization, and the societal (or Oblique) socialization. Oblique Socialization, in particular, is how other adults with well formed types influence the children' socialization process. After this socialization process has taken place, children become adults with defined types, can reproduce and start again the socialization of their children. Thus a dynamics of cultural traits is endogenously derived.
Following the standard cultural transmission literature, the socialization rule is defined in the following way:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1-\tau_t^i)\bar{V}^{ij}, \forall i,j \in E, \forall t \tag{1.1}$$

in which $\tau_t^i V_t^{ij}$ is the vertical socialization part of the process and $(1-\tau_t^i)\bar{V}^{ij}$ is the oblique socialization part. We suppose $\bar{V}^{ij} \in [0,1]$ and $\bar{V}^{ij}$ to be independent from τ_t^i. For the time being we do not characterize the Oblique Socialization term $\bar{V}^{ij}$. We call this process a *Cavalli-Sforza Feldman Socialization Dynamics* (from now on CSF Dynamics)[5].
Call W_t^i the utility an agent of a generic group i derives from having a child. Since parents are happier the more effectively they can trasmit their type to own children, we thus have the following:

$$W_t^i = V^* - \sum_k (V_t^{ik} - V_{t+1}^{ik})^2 \tag{1.2}$$

Thus, if the child has the same values in all the type vector entries as the parent, then the parent has the highest possible utility from the child, V^*. Otherwise, she additionally experiences a loss depending on the difference of the entries of the types vectors.
Consequently each parent faces the following problem:

$$\underset{\tau_t^i}{Max}\; W_t^i - c(\tau_t^i) \tag{1.3}$$

Define $\Delta \bar{V}_t^i \equiv \sum_k (V_t^{ik} - \bar{V}_t^{ik})^2$ representing the difference between the effect of oblique socialization over all type entries and parents type: this is a general measure of the parent's loss. Substituting (1) into (2) and we get that the parent wants to maximize

$$V^* - (1-\tau_t^i)^2 \Delta \bar{V}_t^i - c(\tau_t^i) \tag{1.4}$$

Thus the higher the effort, the lower the general loss of the parent, but the higher the cost associated to this effort. Moreover, unless $\Delta \bar{V}_t^i = 0$, the marginal utility of τ_t^i is positive and

[4]Even though a pure genetic derivation of ethncity may be questionable, we use this simplificative approach in order to observe what happens to groups that do not experince mixed identities. Moreover an analysis of data as IPUMS (Integrated Public Use Microdata Series, for the US) shows that this genetic approximation follows the data: self-assessments of ethnicity generally follows the ethnicity belonging of the parents, when both parents declare to belong to the same ethnic group.

[5]This equation has been introduced, in different forms, by Bisin and Topa (2003) when introducing the possibility for the socialization of a continuous cultural trait. A more extensive use of this has been done in Pichler (2010), called, in that framework, *parental socialization techniques*. The first insights of this formulation can be found in Cavalli Sforza and Feldman (1981) when analysing the cultural transmission for a continuous trait, in chapter 5.

decreasing at a constant rate and is zero at maximum socialization effort.

Assumption 1: Assume that the socialization cost function has the following properties: $c(\tau) : [0,1] \mapsto \Re^{+}$, $c'(\tau_t^i)|_{\tau_t^i=0} = 0$ and $c''(\tau_t^i) \geq 0$

We can now state the following:

Proposition 1. *If Assumption 1 holds , then* $\tau_t^{i*} = Argmax[W_{t,i}^i(\bar{V}_t, \tau_t^i) - c(\tau_t^i)]$ *exists and is unique* $\forall t, i$. *Moreover if* $V_t^{ij} \neq \bar{V}^{ij}$ *for at least one* j, *then* $\tau_t^{i*} \in (0,1)$.

Proof. See Appendix B. □

Assumption 1 states that costs should be flat at zero socialization level and have non-negative slope elsewhere. This weak assumption ensures the formation of an internal optimal socialization effort. This result is supported from the evidence that both society and parents actually enter in the children socialization process and influence his values. Only if $V_t^{ij} = \bar{V}_t^{ij}$ then the family and the society have the same effect on the children' type so that, being the vertical socialization costly, parents choose not to socialize children, since what children can take from the society is the same they can transmit to the offpring so that no incentives for vertical socialization are present.
The society we describe here is very conservative in the sense that no agent has utility derived from diversity, but everyone would like to have children with her own very same preferences. An usual explanation for this is that parents judge their offpring by means of own preferences so that they use what is called *'imperfect empathy'* (Bisin and Verdier (2000)); we will maintain this behavioural assumption along all this work[6].

1.3 Cultural Dynamics

In equation (1.1) the dynamics of the cultural traits crucially depends on how oblique socialization is defined since, depending on it, parents experience different losses and thus may choose different socialization efforts. In particular $\bar{V}^{ij}$ identifies the generic oblique socialization effect on the element V_{t+1}^{ij}. In this section we introduce a generic specification for this element.
We propose a very general specification in which each generic type entry V_t^{ij} can potentially depends on any other entry V_t^{kw}, so that we do no restrict to any particular social influence. Thus we have:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{ij}^{kw} V_t^{kw}) \tag{1.5}$$

in which w_{ij}^{kw} is an exogenous weight identifying the impact of V_t^{kw} in the V_{t+1}^{ij} dynamics. We consider these weights such that $0 \leq w_{t,ij}^{kw} \leq 1 \ \forall i, j, k, w$ and $\sum_{k,w} w_{t,ij}^{kw} = 1$. Thus the

[6]It has to be noted that these standard assumptions over socialization schemes imply that, since parents know the exact outcome of Oblique Socialization, then they can fully determine their children type and they are sure that their actions maximize their ex-ante and ex-post utilities. Moreover, in this simplified framework, children have only a passive role. Even though simplificative, for the time being we take these assumptions as true, just recalling the limits of this view since, in reality, children actually play an active role in their socialization process and there is also an element of uncertainty in oblique socialization that parents cannot control for, so that oblique socialization is subjected to a form of ambiguity or, at least, of randomness.

matrix of these parameters is row normalized and gives a full characterization of the oblique socialization technology at any time t. These weights could be a measure of similarity of situations, of trust or other factors and could also be a function of the population shares. We call this cultural dynamics a *Generalized Cavalli-Sforza and Feldman Socialization Dynamics* (from now on GCSF Dynamics). This may be a first approximation of reality if population shares do not change during time or if the strucutre of the society (schools, neighborhoods, for example) are almost stable in time.
In section 7 we make these weights time dependent thus having:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{t,ij}^{kw} V_t^{kw}) \tag{1.6}$$

Example 1: *(Population weights) In order to give an example, the simplest way to intend oblique transmission of a cultural trait is taking the social average for that trait. This imply that the child randomly meets agents belonging to the parents' generation in the society and thus take the average value from these encounters (for example teachers or other cultural models in the society). Equation (1.1) will thus become:*

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k} p_t^k V_t^{kj}) \tag{1.7}$$

This can represent the most frictionless society we can imagine. For example the case in which children live in a neighborhood with no biases in group shares, or attend schools with professor of different ethnic groups in quota proportional to the population shares in the overall society or ethnic messages are reported by media respecting the proportions of ethnicities in the society.

This formalization is similar to the one in De Marzo et al. (2003) in which they have each group having a single cultural trait influenced by the neighborhood. Their rule can be written as:

$\bar{x}_{t+1} = T_t \bar{x}_t$
$T_t = (1 - \lambda_t)I + \lambda_t T.$

in which $\bar{x}_t$ is the vector of values, T_t is the recursive rule, λ_t is a friction parameter that can recall our τ_t^i, and T is the time independent matrix indicating the network influences. From a mathematical point of view, our model extends this analysis with four progressive steps: we have that each group have more that one value involved in the dynamics instead of one, and thus a multidimensional dynamics arises. Then we allow the friction parameter to be not only time dependent but also group dependent. This will create a different bias for each group: in this framework this will be interesting since this corresponds to the socialization effort whose implications are analysed in section 4.4. Finally, in section 6, we allow the matrix T to be time independent and, in section 7, to be endogenous with the model.
Given this we can state the following:

Proposition 2. *If Assumption 1 holds, then any* GCSF Dynamics *converges to a steady state.*

Proof. See Appendix B. □

This proposition basically states that if socialization is such that parents always have incentive to socialize their children at least a little bit, then convergence towards a steady state happens. Thus, the role of vertical socialization is to ensure convergence since, if for any reason $\tau = 0$ out of equilibrium, convergence may not happen and cycles may arise. Proposition 2 does not say which kind of steady state is reached and thus leaves the door open to different equilibria implying different levels of integration or segregation: this will be the topic of the next sections.
From a technical point of view proposition 2 also generalizes the contributions of Brueckner and Smirnov (2007, 2008) since here we control for parents' socializing role and for a wider range of possible interaction among ethnic groups, not restricting to the cases in which the matrix of weights is irreducible or block diagonal irreducible, thus providing a more general sufficient condition for convergence.
At steady state it happens that $\tau_t^{i*} = 0$. By the definition of steady state $V_t^{ij} = V_{t+1}^{ij}, \forall i, j, t$ so that parents and sons always have the same type. Consequently there is no incentive to socialize children since the loss the parents experience is zero. This implies that, in the long run, parents would not have any role in the socialization: since they care only about having children similar to them, once that this is an outcome of oblique socialization, they do not care anymore about it.

Remark: In the proof of proposition 2 we also show that even for the case of suboptimal socialization efforts, if $\tau_t^i, \forall i$ is strictly positive, convergence happens, even though steady state values may be different from the case in which the optimal τ_t^{i*} is chosen. Now, as we have argued above, in order to choose an optimal socialization effort the parent should know the whole matrix of all attitudes V, and the vector of weights $\bar{w}$ his son is going to use in the oblique socialization process. While the first assumption may be reasonable, since the matrix of V is common knowledged, the second one may be questioned, since oblique socialization influences may not be perfectly predicted by parents. Still, even if the parent has a wrong guess of the relevance parameters, and thus choose an ex-post suboptimal socialization effort, if the chosen $\tau \in (0, 1]$, then convergence happens[7]. Then, far from being useless, different levels of vertical socialization have effect on the levels of the steady state. Morevoer, the introduction of the optimal socialization effort makes the model richer, such that it will be useful in policy and welfare analysis that will be run in section 5.

1.4 Oblique Socialization Schemes and Evolution Analysis

1.4.1 Socialization Schemes

The previous section provided convergence conditions over a wide class of cultural dynamics. A crucial element of the analysis is the specific form of these dynamics since the convergence properties strongly depend on these structures. As Boyd and Richerson (1985) explain, '*the theory should predict the effect of different structures of cultural transmission on the evolutionary process*': in our case a structure of cultural transmission is fully characterized by the structure of the oblique socialization weights matrix and any vector of these weights identifies an *Oblique Socialization Scheme*. In this section, starting from simple socialization schemes we derive long run equilibria that can be considered sensitive in the study of integration and

[7] From a mathematical point of view, $\tau_t^i > 0, \forall i$ make the diagonal entries of the transmission matrix A strictly positive. Thus, the matrix A, or its diagonal blocks, if irreducible, are also acyclic. However, it is not necessary to have $\tau_t^i > 0$ in order to have acyclic matrix, since an acyclic matrix may also derive from some particular structures of oblique socialization. However, since we have not put constraints on the oblique socialization scheme, $\tau_t^i > 0$ ensures acyclic matrix. This will become clear with the next sections.

segregation of groups, so that an analysis of how different socialization structures influence the process is the key element of the rest of the paper.
We use, as a basic rule, an emulation technology such that while forming the V_{t+1}^{ij} attitude the child takes into account the set of all $V_t^{kj}, \forall k$, so that she 'emulates' the attitudes towards j observed in the society.
Starting from this basic emulation rule we suppose that agents use two different additional schemes: Reciprocity and Ethnocentrism. With the first one we mean that people tend to form bad (good) attitudes towards people that have a bad (good) attitude towards them. This means that $w_{ij}^{ji} > 0$, so that V_t^{ji} enters in the formation of V_{t+1}^{ij}. With Ethnocentrism we mean the possibility that agents of a group never question the attitude towards own group people, so that $w_{ii}^{ii} = 1$ and thus $V_t^{ii} = V_{t+1}^{ii}$.[8]
We thus build 4 socialization schemes combining the previous 3 rules. With respect to equation (1.6) the following schemes are restrictions of the most general case since we impose particular structures on the oblique socialization weights matrix.

Definition 1: Call

- *Emulation Rule* an oblique socialization rule in which
  $(w_{ij}^{kw} = 0, \forall w \neq j, w_{ij}^{kj} > 0, \forall i, j, k, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_k w_{ij}^{kj} V_t^{kj}) \forall i, j, t;$
- *Ethnocentrism Rule* an oblique socialization rule in which
  $(w_{ij}^{kw} = 0, \forall w \neq j, w_{ij}^{kj} > 0, \forall i, j, k, w_{ii}^{ii} = 1, \forall i, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k \neq j} w_{ij}^{kj} V_t^{kj}) \forall i, j, t;$
- *Reciprocity Rule* an oblique socialization rule in which
  $(w_{ij}^{kj} > 0, \forall i, k, j, w_{ij}^{ji} > 0, \forall i, j, \text{ else } w_{ij}^{kw} = 0, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_k w_{ij}^{kj} V_t^{kj} + w_{ij}^{ji} V_t^{ji}) \forall j \neq i, \forall t;$
- *Reciprocity and Ethnocentrism Rule* an oblique socialization rule in which
  $(w_{ij}^{kj} > 0, \forall i, k, j, w_{ij}^{ji} > 0, \forall i, j, w_{ii}^{ii} = 1, \forall i \text{ else } w_{ij}^{kw} = 0, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k \neq j} w_{ij}^{kj} V_t^{kj} + w_{ij}^{ji} V_t^{ji}) \forall j \neq i, \forall t.$

The first scheme has neither Reciprocity nor Ethnocentrism so that the attitude V_{t+1}^{ij} depends on all the attitudes of everyone towards j. In the second case we introduce ethnocentrism so that $V_t^{ii} = V_{t+1}^{ii}$. Since under Assumption 1 $\tau_t^{i^*} \in (0, 1)$, in order to have this we impose that $w_{ii}^{ii} = 1, \forall i$. In this case all the other $V_t^{ij}, \forall i \neq j$ follow the previous emulation rule. In the third case the reciprocity introduces the possibility of having $w_{ij}^{ji} > 0$ so that V_{t+1}^{ij} depends also on the attitude of j towards i. The fourth case just combines the previous two situations. It should be underlined that these four rules are all symmetric in the sense that all agents

[8] Looking at studies in sociology and social psychology it can also be found (Berry (2006) and Berry and Kalin (1979, 1996), for example) that agents actually use *Reciprocity* and *Ethnocentrism* in their attitude formation schemes. In particular, the correlation between inter-group attitudes has been computed and has been found positive so that Reciprocity seems to be an actual way of attitude formation; on the other side ethnocentrism has been proved to exists in all cases even though with different intensities depending on the ethnic group.

of every group use the same rule: everyone use reciprocity towards anyother or no one uses reciprocity. An extension to asymmetric socialization rules will be done in section 6.

1.4.2 Steady State Characterization

In this subsection we focus on steady states, identifying 4 classes of them that may be considered benchmark outcomes of cultural dynamics in relation to their integration or segregation properties.
As previously argued, in some literature there has been found evidence of social hierarchies based on ethnicity: in particular agents seem to agree on a ranking of different ethnicities, so that common prejudices arise. In terms of our model, if a common hierarchy is shown, we have that $\lim_{t\to\infty} V_t^{ij} = \lim_{t\to\infty} V_t^{kj}\ \forall i,k$. We call these kind of steady states *Hierarchy Equilibria (HE)*. This situation is represented by the top left matrix below in which every row is a type vector so that the ij entry is the V_t^{ij}.

HE	i	j	...	k
i	a	b	c	d
j	a	b	c	d
...	a	b	c	d
k	a	b	c	d

IE	i	j	...	k
i	a	a	a	a
j	a	a	a	a
...	a	a	a	a
k	a	a	a	a

HEE	i	j	...	k
i	E	b	c	d
j	a	E	c	d
...	a	b	E	d
k	a	b	c	E

IEE	i	j	...	k
i	E	a	a	a
j	a	E	a	a
...	a	a	E	a
k	a	a	a	E

Suppose, for example, that $a > b > c > d$: then there is an intergroup consensus to assign the *best* attitude to i agents and the *worst* to k agents, noting that k agents has a bad attitude towards themselves too.
A second kind of steady state is the one that predicts the 'end of racism' since $\lim_{t\to\infty} V_t^{ij} = \lim_{t\to\infty} V_t^{kw}\ \forall i,j,k,w$. If a steady state like this is reached, then a process by which all agents will end up with the same attitude towards every ethnic group has taken place. We call these outcomes *Integration Equilibria (IE)*: this equilibrium can be seen as the objective of integrationist policies. In this case, however, it does not happen that all groups merge in one single culture, but only that they do not discriminate among any culture. Thus, given our framework, this cannot be defined as a Melting Pot equilibirum. The top right matrix represents this case.
It must be noticed that Hierarchy and Integration Equilibria are not good states or bad states a-priori. With integration, in fact, we simply mean that all attitudes converge to the same value: it may happen that an IE is reached with very low (high) final values, meaning that everyone has a bad (good) attitude towards anyother. It can be that, in the first case, a ranking is shown but all the values are high (low), and thus represent very good (bad) attitudes. Thus, by the terms 'hierarchy' and 'integration', it is not meant any phenomenon with a specific positive or negative moral meaning.
In some situations it may not be the case that the groups which is assigned the worst attitude, also self-considers bad. In order to control for this problem we add two more cases to the matrices above, implying that ethnocentrism holds. We can thus define Hierarchy Equilibria with Ethnocentrism (HEE) and Integration Equilibria with Ethnocentrism (IEE) (suppose $V_t^{ii} = E\ \forall t, i$) represented, respectively, by the bottom left and bottom right matrices above.

If we suppose that $E = 1$, then the attitude every ethnic group member has towards own group members is maximum. In the second case, in which convergence of all attitudes towards the same value may happen, every agent can only discriminate with his attitudes between members of his own groups and others out of own group.

1.4.3 Long Run Effects of Oblique Socialization Schemes

We now study the relations between socialization rules and steady states presented above. We state the following:

Proposition 3. *A sufficient condition for an HE to be a steady state is that Emulation Rule holds, for an IE to be a steady state is that Reciprocity Rule holds, for an HEE to be a steady state is that Ethnocentrism Rule holds and for an IEE to be a steady state is that Reciprocity and Ethnocentrism Rule holds*

Proof. See Appendix B. □

Thus, starting from socialization rules significant in the attitude formation schemes we are able to prove convergence to four categories of steady states significant for their social properties. In particular all the steady states may be obtained given some conditions over the oblique socialization schemes so that we can also state that different racism levels may be results of factors internal in the children socialization process, indirectly implying that policies that modify these schemes may have important results in term of racism outcomes. In particular a change in socialization structures such as reciprocity, widely changes the final outcome dramatically. We thus give reason of the intuition of Boyd-Richerson about the importance of the analysis of the cultural transmission schemes for the long run equilibrium of the society.[9]

Example 2: *(Numerical Simulations)*
Figure 1.1 provides examples for the dynamics for the case of 3 ethnic groups in order to better understand what goes on. We consider here 3 ethnic groups and $c(\tau_t^i) = \tau_t^{i^2}$ *as a simplest cost function satisfying the requirements of assumption 1. We then set up weights proportional to the population shares, using for simplicity* $p^i = p^j = p^k = \frac{1}{3}$*, but since population shares do not change, this is just a way to give a rule for the socialization weights. The matrix of the initial attitudes is:*

$\bar{V}_0$	i	j	k
i	1	0.1	0.2
j	0.03	1	0.5
k	0.7	0.9	1

Thus a socialization dynamics as in equation (1.7) holds. Weight matrices are reported in Appendix A for all the four cases. Moreover we set up $E = 1$ *in order to get the idea that groups may consider themselves as the best ones in the case in which ethnocentrism holds. However since E refers to type entries that do not experience any dynamics, this value can be changed without any restriction. In the graphs the lines represent the attitudes and the dots the derived optimal socialization efforts for each ethnic group.*

[9] Additionally, proposition 3 makes clear that, if oblique socialization rules are such that $w_{ij}^{ij} > 0, \forall i, j$ parents do not play any role in the determination of the class of the long run equilibrium and convergence may happen without their contribution. However different socialization efforts will have an influence on the final levels.

The top-left simulation regards a HEE equilibrium: after a very short adjustment we have at the top all the $V_t^{ii}, \forall i$, while each horizontal line then represents the common attitude towards a specific ethnic group, so that an ethnic hierarchy arises. The graph below shows the same socialization rule, but without ethnocentrism, so that also reflexive attitudes converge to the common ranking values. The graphs on the right represent, on the top a IEE and on the bottom a simple IE. In the first one we see that all attitudes, but reflexive ones, converge to the same value, while in the second one also reflexive attitudes converge to the common value.

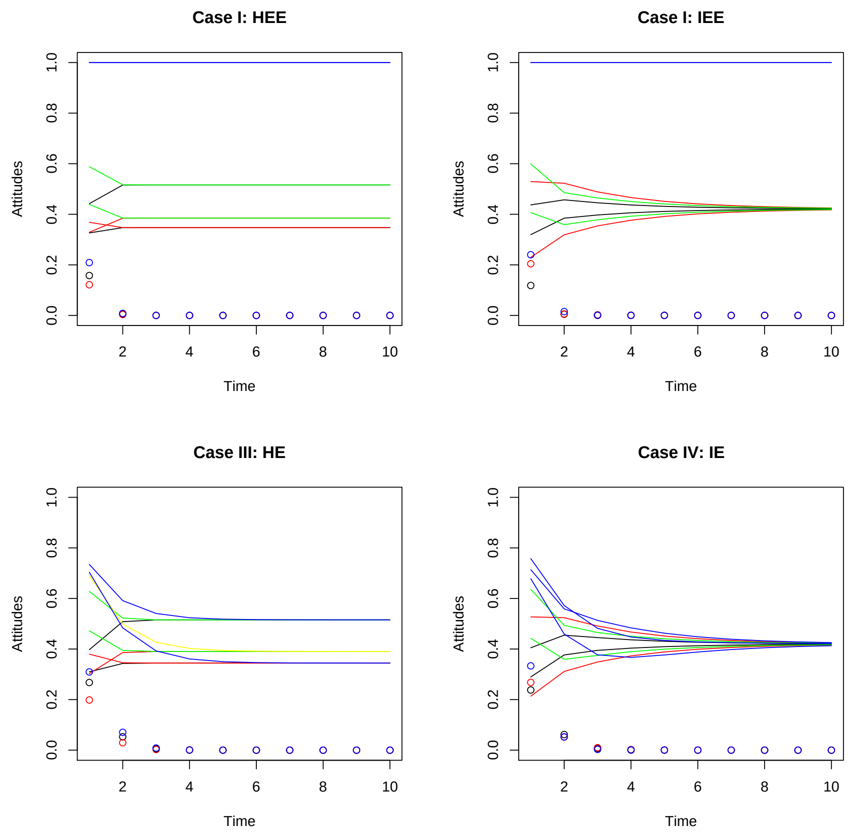

Figure 1.1: Simulations with fixed weights

Remark: *Assume that a steady state has been reached using the proper socialization scheme and that only the 4 socialization schemes previously defined may be used by agents.*

Definition 2: *We say that an IE (resp. IEE, HE, HEE) is invariant under a change in the socialization scheme if the new equilibrium under the new scheme is an IE (resp. IEE, HE, HEE).*

We can thus state the following:

Corollary 1: *An IE is invariant under any change of socialization scheme, while a HEE is not invariant under any change of socialization scheme. An HE is invariant if ethnocentrism is added, while is not invariant if reciprocity is added. An IEE is invariant if reciprocity is removed and not invariant if ethnocentrism is removed.*

Corollary 1 states that in this framework, if only the 4 socialization schemes previously defined hold, once that an IE happened, no changes in the socialization process may alter the equilibrium class. On the contrary, if integration has not happened, then there is room for it to be reached if socialization schemes change appropriately. In particular, starting from a HEE, then any of HE, IEE, IE may be reached adding respectively reciprocity, removing ethnocentrism or doing both actions. If the starting situation is an HE, then by adding reciprocity an IE may be reached, while from IEE, by removing ethnocentrism an IE may be obtained.

Until now we have shown sufficient conditions for convergence. With the next corollary we provide necessary conditions for convergence to the HE-HEE class of equilibria, if every agent of each ethnic group uses the same socialization scheme (*'symmetric'* case) :

Corollary 2: *A Necessary condition for convergence to a HEE or HE with symmetric socialization schemes is that reciprocity does not enter in the socialization schemes.*

Corollary 2 states that an ethnic hierarchy may be sustained in the long run only if no reciprocity holds. This necessary condition may be of some relavance since, in some political talkings on immigration, reciprocity is viewed as a way to introduce incentive for the building of a good attitude world. Sometimes, the subtle justification for these action calls, is in the willingness of maintaining the present ethnic social ranking. With this framework we show that both these reasonings are wrong since reciprocity is the principal scheme for allowing cross-dependence of cultural values and thus for integration, as defined here. On the other side reciprocity, if applied in this symmetric socialization scheme, is incompatible with the preservation of any ethnic social hierarchy.

1.5 Optimal Socialization Effort and Comparative Statics

In this section we analyse the properties of the parents' socialization effort in each point in time. In particual we focus on the situation in which there is an ethnic majority and an ethnic minority, thus building a framework with close relationships with the actual situation in countries experiencing huge migration inflows. Moreover, we study the relationship between the optimal effort and the change in values, meaning the way in which the two groups integrate. Then we see how to measure parents' frustration and if there is some relationship between the effort, the change in values and the frustration levels.
Suppose to have 2 ethnic groups, i and j. Suppose that they act with reciprocity and ethnocentrism so that the only attitudes involved in the dynamics are V_t^{ij} and V_t^{ji}. We restrict the analysis to this case since in the long run both groups show the same reciprocal attitude, that seems a reasonable long run situation for a two groups society. Suppose that oblique socialization weights are given by population shares, as in equation (1.7). Suppose that population shares do not change in time so that $p_t^i = p_{t+1}^i, \forall t$: the case for changes in the population shares will be addressed in section 7 in which the analysis of time dependent weights is run. Suppose that the cost function is $c(\tau_t^k) = (\tau_t^k)^2$ for $k = i, j$. Moreover, define $\Delta_t^i = (V_t^{ij} - V_{t+1}^{ij})^2$, $\Delta_t^j = (V_t^{ji} - V_{t+1}^{ji})^2$ and, from the utility function specification, define the frustration $F_t^i = \Delta_t^i + c(\tau_t^i)$, and the same definition for F_t^k. Δ_t^i indicates the loss of the parents and, contemporarily, is a measure of the change in values between the two generations, while F_t^i indicates the total frustration frustration of the parents being the negative part of the utility function (the sum of the loss and the socialization costs).
We now start studying some conditions under which these two groups differ in the levels of

these measures. We can thus state the following:

Proposition 4. *If there are two groups using ethnocentrism and reciprocity and attitudes dynamics is as in equation (1.7), then a necessary and sufficient condition for* $\tau_t^{i*} > \tau_t^{j*}$*,* $\Delta_t^i > \Delta_t^j$ *and for* $F_t^i > F_t^j$ *is that* $p_t^i < \frac{1}{2}$*.*

Proof. See Appendix B □

The first part of the proposition simply states that minority groups tend always to socialize more, and this is in line with what found by Bisin and Verdier (2000, 2001). In fact in these two groups model, minority values are associated to smaller weight than majority values, and thus minority parents tend to socialize their chidren more in order to reduce the loss. Is this higher effort effective in order to reduce the integration of minority children? Said differently, given $\tau_t^{i*} > \tau_t^{j*}$ are minority children moving slower than majority children towards integration? The second part of the proposition states that $\Delta_t^i > \Delta_t^j$, meaning that even if minority parents put a high effort in the socialization, their chidren have values moving faster than the majority and, in the mean time, minority parents experience a higher loss. As a consequence of the previous two steps, minority parents always experience a higher frustration than majority parents.
We now perform a comparative statics analysis. Call $D_t = (V_t^{ij} - V_t^{ji})^2$, representing the differences in the values of the two groups so that $D_t \in [0, 1]$ with D_t increasing in the distance of the groups' attitudes. We now analyse how τ_t^{i*}, Δ_t^{i*} and F_t^{i*} change, at optimum, with p_t^i and D_t. The values, at optimum, are: $\tau_t^{i*} = \frac{p_t^{j^2} D_t}{1+p_t^{j^2} D_t}$, $\Delta_t^{i*} = \frac{p_t^{j^2} D_t}{(1+p_t^{j^2} D_t)^2}$ and $F_t^{i*} = \frac{p_t^{j^2} D_t}{1+p_t^{j^2} D_t}$.
We first notice that, given the actual cost specification, $\tau_t^{i*} = F_t^{i*}$ so that optimal effort and frustration properties can be analysed contemporarily.
In equilibrium $\frac{\partial \tau_t^{i*}}{\partial p_t^j} > 0$ since the higher the opponent's population share, the higher the weight associated to opponent's attitudes in the oblique socialization process (since (1.7) holds), and thus the higher the parents' effort in order to transmit own attitude value. Since $\tau_t^{i*} = F_t^{i*}$, then the higher the effort the higher the parents' frustration: this happens since in equilibrium $\frac{\partial \Delta_t^{i*}}{\partial p_t^j} > 0$, meaning that, if the opponent's share increases, then i loss increases: thus the greater effort is not effective in preventing higher change in attitudes since it is not able to counterbalance the effect of the shift in oblique socialization weights. Thus i values move faster towards the integration outcomes and consequently F_t^{i*} increases.
After having studied the monotonicity properties of these quantities, we perfom a concavity/convexity analysis.
In equilibrium $\frac{\partial^2 \tau_t^{i*}}{\partial^2 p_t^j} > 0$ if and only if $p_t^j \in [0, \frac{1}{D_t\sqrt{3}}]$: in figure 1.2 this case is represented by the area below the upper curve.

This means that an increase in opponent's population share has a positive effect on τ^*, with decreasing slope only for high p_t^j and high D_t, that means only if i is a strict minority with values very different from the majority. However, since $\dot{D}_t < 0$, permanence in the upper-right part of the graph may be only temporary, and thus, apart from the first periods of the dynamics, it will be more likely to be in the case in which $\frac{\partial^2 \tau_t^{i*}}{\partial^2 p_t^j} > 0$.

If we then look at what happens to Δ_t^{i*} we have that $\frac{\partial^2 \Delta_t^{i*}}{\partial^2 p_t^j} < 0$ if and only if $p_t^j \in [\sqrt{\frac{4-\sqrt{13}}{3D_t}}, 1]$.
In figure 1.2 this is represented by the area above the bottom curve. Notice that this never happens if $p_t^j < \sqrt{\frac{4-\sqrt{13}}{3}}$ (the horizontal line in the graph).
In order to understand the meaning of this analysis, suppose i to be a group under demographic pressure (p^j increases). Suppose to have 3 'worlds' that only differ in the levels of D_t

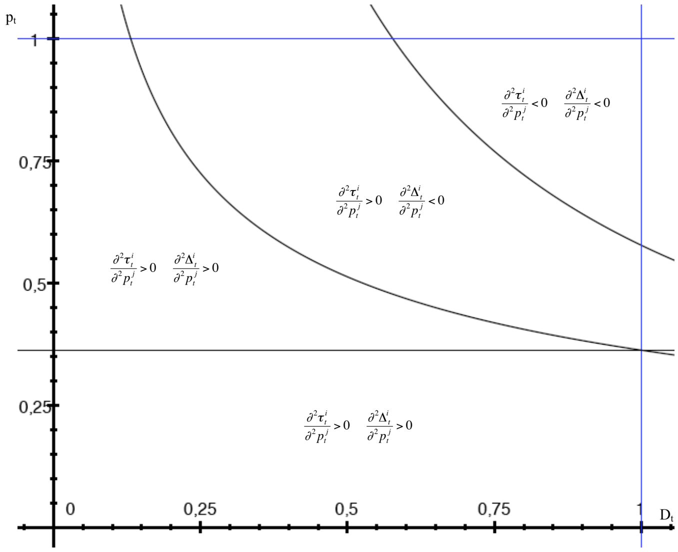

Figure 1.2: Comparative Statics

so that a High-Distance, Middle-Distance and Low-Distance worlds are qualitatively identified.

Consider first the case of a Low-Distance world: the higher p^j the higher the positive reaction of i agents to the demographic shock on all quantities since $\frac{\partial^2\tau_t^{i*}}{\partial^2 p_t^j} > 0$ and $\frac{\partial^2\Delta_t^{i*}}{\partial^2 p_t^j} > 0$ for all p^j levels. This means that if the two groups are 'almost integrated' the more a group is a strict minority, the higher its responce in terms of socialization efforts to demographic pressure, the higher its changes in attitudes and, consequently, the higher its frustration increase.

If we now consider a Middle-Distance world, we still have that $\frac{\partial^2\tau_t^{i*}}{\partial^2 p_t^j} > 0$ for every level of p^j. However $\frac{\partial\Delta_t^{i*}}{\partial p_t^j}$ reaches its maximum for 'medium' levels of p^j. Thus, if two groups have an intermediate level of attitudes distance, then the more one group is a strict minority, the higher its responce in terms of socialization efforts to demographic pressure and the higher its changes in terms of frustration. However middle size groups are the ones that experience the greatest increase in terms of integration of values.

Considering, at last, a High-Distance world, then both τ_t^{i*} and Δ_t^{i*} reach an internal maximum in their increase under a demographic shock: in particular the maximum in the increase of effort and frustration is reached for middle-low size groups, while the maximum for the change in attitude values is reached for middle-high size groups.

This analysis is useful in understanding what happens if in a society with a large majoritarian ethnic group (Natives N) there is an immigration flow of the other ethnic group (Immigrants I). We now analyse the change in the optimal levels of efforts, cultural change and frustration recalling that a minority always has higher levels of these values than the majority (cfr. proposition 4). As far as N agents remain a large majority, then during the immigration process they increase their socialization effort in an always increasing way, so that they always contrast increasingly the cultural change, but their value change is always larger and larger. Thus their frustration level increases at an always faster rate. Now, if the two groups show a low values distance, then the minority group, during the immigration process, always experiences lower and lower socialization efforts, values change and frustration level, thus always contrasting less sharply the cultural change and changing in values always less and less, thus being always less and less frustrated. If there is a middle level values distance between the

two groups, then the minority, while experiencing lower and lower socialization efforts and frustration levels, has its values change decreasing at an increasing rate. If, at last, the two groups attitudes are very different, then the minority experiences an always higher decrease in effort and frustration and an always lower decrease in values change.
The following corollary sums up these results:

Corollary 5: *Suppose there are two groups using ethnocentrism and reciprocity: majority always has values converging at an increasing rate, while the more a minority is culturally closed to the majority the more its integration process speeds up, with very culturally distant minorities having the speed of convergence decreasing. Moreover, the majority has its frustration level increasing at an increasing rate while the more a minority is cuturally closed to the majority the more its frustration level decreases at an increasing rate with very culturally distant minorities having the frustration decreasing at a decreasing speed.*

Until now we have supposed that parents knows perfectly the oblique socialization weights and thus may optimally choose the socialization effort. However this is not always the case. Since it can happen that parents overestimate the impact of opponent groups in the oblique socialization of own children and thus oversocialize children. Suppose $\tilde{p}_t^j$ be the perception i parents have of j population share. Call $\epsilon_t^j = \tilde{p}_t^j - p_t^j$ and $\tilde{\tau}_t^{i*}$ the optimal effort level under $\tilde{p}_t^j$. Then we have that: $\tilde{\tau}_t^{i*} - \tau_t^{i*} = \frac{D_t({\tilde{p}_t^j}^2 - {p_t^j}^2)}{[1+D_t{\tilde{p}_t^j}^2][1+D_t{p_t^j}^2]}$ and $\Delta_t^i = \frac{{p_t^j}^2 D_t}{[1+{\tilde{p}_t^j}^2 D_t]^2}$ and $F_t^i = \frac{D_t({\tilde{p}_t^j}^2 + {p_t^j}^2)}{[1+{\tilde{p}_t^j}^2 D_t]^2}$

We thus have $\frac{\partial \tilde{\Delta}_t^{i*}}{\partial \epsilon_t^j} < 0$ and $\frac{\partial \tilde{F}_t^{i*}}{\partial \epsilon_t^j} > 0$. Thus the overestimation of opponents' share bring to a lower change in children values. However, since the socialization effort is higher, the greater cost offsets the advantage in terms of minor loss and the frustration increases. The opposite hold with underestimation of population shares.
Thus we can state that:

Corollary 6: *Any group that overestimates the opponents' share tend to socialize more, to change less in values and to be more frustrated than in case of perfect population estimation. If a policymaker convince each group that the opponents' share is lower than in reality, then integration of values speed up and agents frustration is lowered.*

1.6 Asymmetric Oblique Socialization Schemes

In this section we propose a first extension to the model. In the previous sections we have analysed the cases in which every agent of each ethnic group follows the same socialization scheme, so that symmetric socialization rules are implied. This assumption, however, limits the analysis since different ethnic groups may show different socialization schemes depending on various social situations: in particular it can be the case that a given group i can consider j's cultural traits as relevant while k's traits as irrelevant during the oblique socialization process. Thus, we now give a proper extension in order to consider these cases.
We introduce a notation borrowed from network theory since, on one side, there is a strong link between the network structure and the transition matrices we use for proving convergence of cultural dynamics and, on the other side, networks give a more intuitive view of the relationships among different ethnic groups in attitudes formation schemes, and thus it is easier to identify oblique socialization structures. Moreover, in order to study steady state classes, there is no need to know the weights intensity, but just their positiveness. In fact,

proposition 3 states that if a given oblique socialization structure exists, then convergence to a particular steady state happens without regards to the the intensity of the single influences. Thus, we are interested in whether the links exist or not, rather than their intensities. The following analysis is also linked with De Marzo et al (2003): in their Appendix 1.C they prove convergence for the case of non-strongly connected matrices. The cases presented here are related to them, with the exception that agents are now allowed to have group dependent τ_t^{i*}.

Suppose each V_t^{ij} is a node, and call U the set of all the nodes.

Then, the directional link $V_t^{ij} \rightarrow V_t^{kw}$ is built if and only if $w_{ij}^{kw} > 0$.

Call P_{ij}^{kw} the set of the possible paths, both direct or non-direct, from V_t^{ij} to V_t^{kw}

Define now a *sink* the set

$S \subset U$: $S \equiv \{V_t^{ij} : P_{ij}^{kw} = \emptyset, P_{ij}^{nx} \neq \emptyset, \forall V_t^{ij} \in S, \forall V_t^{nx} \in S, \forall V_t^{kw} \notin S\}$.

Thus a sink is a set of nodes such that there is no path from any of them to any node outside the sink. The sink may be composed either of only one node or of more than one node. In the first case a node V_t^{ij} is a sink if and only if $w_{ij}^{ij} = 1$ since in this case any other $w_{ij}^{kw} = 0$ and thus no links are formed towards outside. In terms of our model this means that the attitude is not questioned, and thus no dynamics is shown for this trait. As a consequence, if ethnocentrism applies then V_t^{ii} is a sink. In the case in which the sink is composed of multiple nodes, then they are strongly connected. Moreover, given any node not belonging to any sink, there should exist a path that connect it to a sink, otherwise it would belong to a sink itself. This means that if there is only one sink then there should exist a path from any element out of the sink to an element of the sink.

With this framework we have that, depending on the relevance parameters, the structure of the network may differ but, given that they are time independent, the structure of the network does not change with time.

In order to control for asymmetric socialization rules, we start relaxing the Emulation Rule as previously defined. Thus:

Assumption 2: Consider a Socialization rule such that $w_{ij}^{kw} = 0, \forall w \neq j$, and $w_{ij}^{kj} \geq 0, \forall k$.

Note that this is compatible with $w_{ij}^{ij} = 0$, for some i, j, so that the diagonal entries of the weights matrix can be composed of zeros, however the diagonal of the transmission matrix is positive since, at optimum, $\bar{\tau}^* > 0$. Then the following proposition holds:

Proposition 5. *A sufficient condition in order to have an HE is that assumptions 1 and 2 hold, the nodes V_t^{ij} form a single component $\forall i, t$ and each component is strongly connected or has only one sink.*

A sufficient condition in order to have an IE is that assumptions 1-2 and Oblique Socialization Stability holds, the nodes V_t^{ij} form a single component $\forall i, j, t$ and the component is strongly connected or has only one sink.

Proof. See Appendix B. □

The first part of proposition 4 states that HE may be reached under a big variety of socialization structures as far as all the $V_t^{ij}, \forall i$, depend directly or indirectly only from each others (this is the case in which the component is strongly connected) or from a subset of them that formes a sink, so that every other depends on it. Figure 1.3 gives some examples of strongly connected network cases.

The first implication is that is it not needed that every group has contacts with all the other groups or consider them reliable during the socialization process in order to have a hierarchy

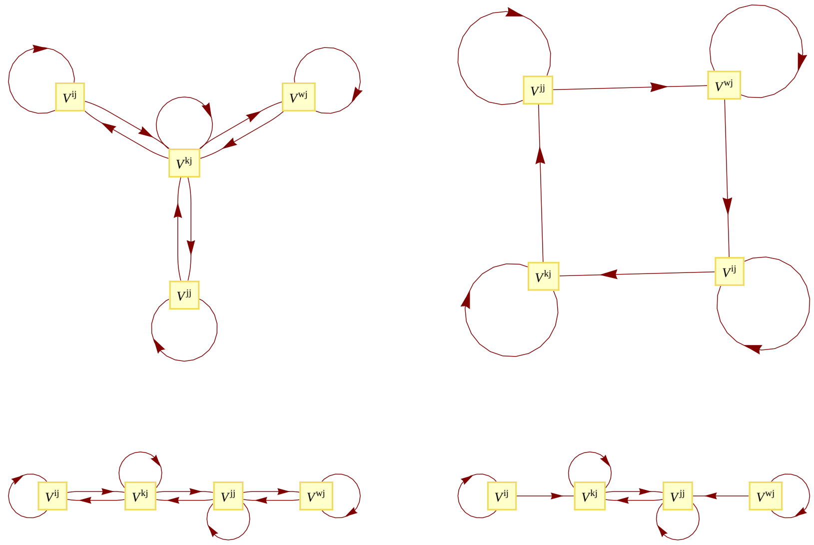

Figure 1.3: Strongly Connected Network

of attitudes. Consequently we can have convergence to a HE even if there are strong frictions in the contacts among groups. Consider, for example, the top-left graph in figure 1.3: in this case only one group enters in the socializaiton process of all the others being thus a *cultural hub*. In this case the 'hub' k is a collector of all others' cultural values, it produces a synthesis and influences the others. In this way everyone gets everyothers' values by means of the cultural hub, so that the weight vector that k uses become crucial for the determination of the steady states values. Considering now the top-right graph we have the case of *cultural circles* in which no group has a predominat role but where each group is directly or indirectly linked to the others. This case also makes clear the role of vertical socialization in ensuring convergence: suppose assumption 1 does not hold and parents do not socialize at all their children so that $\tau_t^i = 0, \forall i, t$. Then convergence does not happen since there is a cyclic matrix and a fluctuation of cultural traits is shown, unless all the values happen to coincide at time 0. In the last graph we have the case in which i and k reciprocally get influenced and j and w do the same. However k and j are also reciprocally linked. These two ethnicities can be considered as *cultural bridges* for ethnic groups that do not have contacts. In order to analyse the case with a sink, consider the bottom right case: V^{kj} and V^{jj} form a sink such that the final attitudes level converge to an average of these two. This happens since i and j respectively consider k and w in attitude formation scheme but the reverse does not happen. The second part of proposition 4 gives similar conditions for an IE to be shown: this differs from the previous one since in this case all cultural values have to be somehow linked each other. In this model the instrument that makes this possible is Reciprocity. However, it is not needed that everyone uses reciprocty towards anyother in order to obtain an IE. For example, suppose, in fact, that every ethnic group uses a Emulation Rule (resp. Ethnocentrism Rule) so that a HE (resp. HEE) is reached. Suppose now that one group starts to use reciprocity towards any other group. Figure 1.4 provides a graphical example for this case (we do not report reflexive arrows).

In this case group k uses reciprocity towards anyother. As a result, the long run equilibrium will be an IE in which the final attitude of everyone towards anyother is given by an average of the attitudes that everyone had towards k at the beginning, since the set $S = \{V_t^{ij}, \forall i, t\}$ is a sink. In this way, the role of reciprocity is much more clear: if reciprocity is used by an ethnicity that everyone thinks is bad, then a bad attitude of everyone towards anyother will

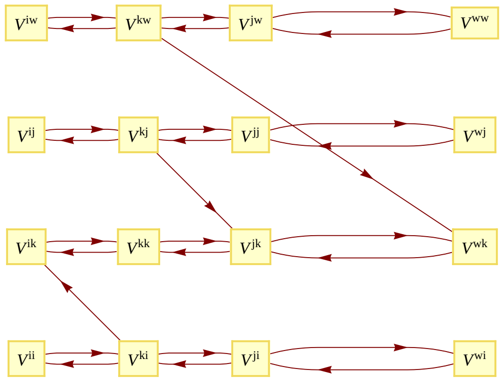

Figure 1.4: Reciprocity

be a result. If, on the reverse, it is used by a well reputed ethnicity, then a long run equilibrium in which everyone have good attitudes towards other may be likely to be observed. Proposition 5 opens the road to different schemes of oblique socialization others that reciprocity such that, linking all nodes together, may be responsible for convergence to an IE.

1.7 Time Dependent Oblique Socialization

Starting from equation (1.5) we have constrained the oblique socialization rule to be fixed along time. However, this may not be the case, since the society composition may change, and consequently weights may change along time following different possible rules. In this section we propose an extension of the model studying what happens if a time dependent specification of the cultural dynamics is taken into account. Considering the formulation in equation (1.6):

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{t,ij}^{kw} V_t^{kw})$$

so that oblique socialization weights may change with time. It is not in the purposes of this paper to produce specific dynamics for these weights: consequently, we analyse sufficient conditions in order to get convergence with time-dependent weights, and sufficient condition for convergence to particular classes of steady states, independently from the specific weights dynamics we consider.

Assumption 3: *(Symmetry)* There is a symmetric oblique socialization rule if $\exists T : w_{t,ij}^{kw} \neq 0 \Leftrightarrow w_{t,kw}^{ij} \neq 0, \forall i, j, k, w, \forall t > 0$.

This assumption is satisfied if after some point in time, if a cultural trait A directly influences cultural trait B, then the trait B influence also the trait A. Speaking with network language this means that all the links that exist in a network, after a period of time T are bidirectional so that the associated matrices are symmetric. To notice that a direct consequence of this assumption is that every component of the derived directed graph is strongly connected.

Assumption 4: *(Temporal Stability)* There is a temporal stable oblique socialziation rule if $\exists T : w^{kw}_{T+t,ij} \neq 0 \Leftrightarrow w^{kw}_{T+t+1,ij} \neq 0$ or $w^{kw}_{T+t,ij} = 0 \Leftrightarrow w^{kw}_{T+t+1,ij} = 0, \forall i, j, k, w, \forall t > 0$.

Under Oblique Socialization Stability, after some periods of time, the way in which ethnicities are influeced each others is stable. Namely, if i agents do not consider j agents, they continue with this scheme forever and if they consider them they continue in this way forever. As a direct consequence of this property $w^{kw}_{T+t,ij} = 1 \Leftrightarrow w^{kw}_{T+t+1,ij} = 1$ and $w^{kw}_{T+t,ij} \in (0,1) \Leftrightarrow w^{kw}_{T+t+1,ij} \in (0,1)$. Moreover, if assumption 4 holds, then after time T the associated network structure is fixed.
Thus, given these assumptions, we can state the following:

Proposition 6. *If Assumptions 1-3-4 hold, then any* GCSF dynamics *converges to a steady state. If Assumptions 1-4 hold and at time T there exists only one sink for each component, then convergence happens.*

Proof. See Appendix B. □

Last proposition states that if we set up any weights dynamics such that symmetry is satisfied and after a time T it is also stable, then convergence happens. For example, if we consider a dynamics as the one represented in equation (1.7), so that the socialization weights are represented by the population weights. As previously argued, this represents the most frictionless society we can imagine. Suppose then that the population dynamics is such that no group ever gets extinguished. Then we can state the following:

Corollary 8: *If cultural dynamics is represented by equation (1.7) and no population ever get extinguished, then convergence to an HE occurs.*

Last corollary put into this context what Brueckner and Smirnov (2007, 2008) found in their works, and shows how this is only one specific case that can be represented in this cultural evolution context.

1.7.1 Optimal Socialization Effort Dynamics

Section 5 introduced the case of change in population shares we used before in order to study the effort properties and its effects on change of values and frustration. In that case only comparative statics analysis was performed: here we complete the analysis by showing what happens to optimal effort levels during the convergence process. Moreover we address here the issue of the monotonicity of τ^*_t during the convergence process.
The setting is similar to the one presented above: suppose to have only 2 groups, Natives (N) and Immigrants (I). Suppose, for simplicity, that they act with ethnocentrism and reciprocity, with weights proportional to the population share. Suppose then that there is a constant inflow of Immigrant in the society such that $p^I_{t+1} > p^I_t$. We can also think at no immigration but at a higher fertility rate of immigrants ($n^I > n^N$) such that the dynamics of population can be derived as $p^N_{t+1} = p^N_t + p^N_t(1 - p^N_t)(n^N - n^I)$. From equation (3) we can thus write the objective function of N parents

$$V^* - (1 - \tau^N_t)^2(1 - p^N_t)D_t - c(\tau^i_t)$$

and thus, in equilibrium we have that $\frac{\partial \tau^{N*}_t}{\partial p^N_t} < 0$ and $\frac{\partial \tau^{N*}_t}{\partial D_t} > 0$. The same happens for the immigrants. Given the definition of the socialization structures, we can state that it always

happens that $D_t \geq D_{t+1}$ since, at each period, values involved in the cultural transmission converge towards a weighted mean.
Now, given $\frac{\partial \tau_t^{i*}}{\partial p_t^i} < 0$ and $\frac{\partial \tau_t^{i*}}{\partial D_t} > 0$ $\forall i \in \{N, I\}$, consider first the case for immigrants: since $D_t \geq D_{t+1}$ and $p_t^I < p_{t+1}^I$, both forces act in order to reduce their socialization effort and thus it happens that $\tau_t^{I*} > \tau_{t+1}^{I*}$. In fact, their increasing population share makes the oblique socialization more biased towards their own values.
Consider now the Native population: since $D_t \geq D_{t+1}$ and $p_t^N > p_{t+1}^N$, the two forces have conflictual effects on the optimal socialization effort, and thus it can happen that $\tau_t^{N*} < \tau_{t+1}^{N*}$, and thus non monotonicity may be observed. In fact if convergence of attitudes reduces the parents' loss, on the other side the reduction in population shares makes the oblique socialization more favourable to the other ethnic group and this has a positive effect on the optimal socialization choice.
Intuitively we can say that if the population change is faster than the convergence in attitudes, then natives will tend to produce more socialization effort. We now provide an esemplificative example.
Suppose to use the standard cost function $c(\tau_t^i) = \tau_t^{i^2}$. Then we have that:

$\tau_t^N = \frac{(1-p_t^N)^2 D_t}{1+(1-p_t^N)^2 D_t}$

$\tau_{t+1}^N = \frac{(1-p_{t+1}^N)^2 D_{t+1}}{1+(1-p_{t+1}^N)^2 D_{t+1}}$

and thus $\tau_{t+1}^N > \tau_t^N$ if and only if $\frac{D_t}{D_{t+1}} < \frac{(1-p_{t+1}^N)}{(1-p_t^N)}$. Consequently:

Corollary 9: *If the rate of growth of I population has been higher than the rate of convergence of attitudes, then an increase in N socialization effort has happened.*

However this can only be an ex-post decription since the rate of convergence depends on optimal efforts, and for the time being we are not able to provide thresholds for the sole fertility rates such that this happens.
This however may have some impact on policy description since, in this simplified framework, immigrants may start with higher socialization efforts given their minority status, but their effort is always declining. On the other side, natives, starting from low levels of socialization, may choose to produce more effort if there is a great imbalance between integration of values and population change.
Moreover, in this framework parents use actual population shares in order to forecast children' oblique socialization. This does not happen in real world, and parents use some expected values that can be biased, especially if fear of immigration holds, tending thus to overvaluate the impact of imigrants on the oblique socialzation. Thus the effect may be more complex and needs a deeper analysis.

Example 3: *(*Numerical Simulations) *We now present a numerical simulation in order to make this result clear. The initial attitudes matrix is the same as in the previous cases, and the initial population vector is* $p_0^i = 0.85$, $p_0^j = 0.05$ *and* $p_0^k = 0.1$ *. Then we set up the following fertility rates (number of children per parent):* $n^i = 2$, $n^j = 3$ *and* $n^k = 2.5$*. In this way the dynamics of i population is given by:*

$p_{t+1}^i = p_t^i + p_t^i\big[(1-p_t^i)(n^i - n^k) - p_t^j(n^j - n^k)\big]$

and the dynamics for the other two groups may be derived in a similar way. We thus want to capture the idea of a society initially composed of a majority i, but in which migration *of two groups happens. Both of them have higher fertility rates than the natives. Given the simplicity of this dynamics, at the end the group with the highest fertility rate will invade the society. In figure (1.5) are represented the simulations for the attitudes in the first graph and for population shares (lines) and socialization efforts (dots) in the second graph.*

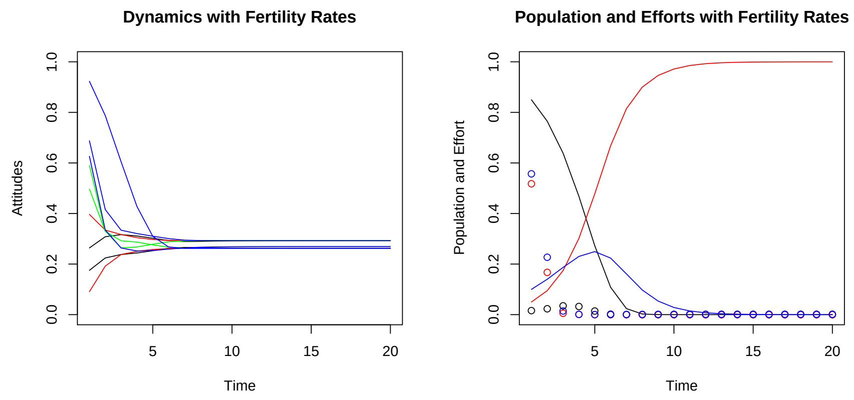

Figure 1.5: Simulations with Fertility Rates

A look at the first graph shows that, even if reciprocity is used, at the end no integration equilibrium happens. However this is simply due to the fact that, at some point, some groups gets extincted and thus they do not enter anymore in the socialization process of others and thus the integration of values cannot happen. In the second graph we can see that, at the end, only the group with the highest fertility rate survives, while the other two are extincted. In particular the second migrant group experience ant the beginning a growth of its population shares, but then, due to the fertility differential with the other immigrant group, it gets extincted as well. If we then look at the socialization efforts we see that one group (the majority i) experience an initial growth in the socialization effort and then a decline. This is due to the fact that at the beginnig the convergence of attitudes is relatively slower than the reduction of i population, and thus i parents, in order to compensate for that, decide to produce an higher effort. When then the rate of population change declines, then the opposite happens and convergence towards the zero level happens.

1.8 Endogenous Oblique Socializations

We present here the last extension to the model that provides a criterion in order to decide which weight structure is more suitable to the analysis in order not to give it exogenously. In equation (1.6) we provided a general dynamic for which any V_{t+1}^{ij} could potentially depend on any V_t^{kw}. Here we consider simpler dynamics, derived from the ones described by the 4 socialization rules previously defined, in which the weights agents assign to other ethnicities' judgments only depend on a measure of similarity of values. In order to study this kind of endogeneity, we introduce the concept of *Cultural Similarity*, $s^{ij} \geq 0$, being a measure of how a given ethnicity i is close to another one j. This similarity will thus have an impact on the weights accordingly to the following basic rules:

- $w_{t,ij}^{kj} > 0 \Leftrightarrow s_t^{ik} > 0$
- $w_{t,ij}^{kj} = 0 \Leftrightarrow s_t^{ik} = 0$
- $w_{t,ij}^{kw} > w_{t+1,ij}^{kw} \Leftrightarrow s_t^{ik} > s_{t+1}^{ik}$

These properties state that the socialization weight is positive if and only if the similarity is positive so that two dissimilar groups do not interact in the attitude formation scheme. Then the weight is increasing in the similarity between the considered ethnicities. Note that if weights only depend on groups similarity then $w^{ji}_{t,ii} = w^{jk}_{t,ik}$ so that ethnocentrism does not hold in this specification.
Define $\Delta^{ij}_t = f(V^i_t, V^j_t) : [0,1]^{n^2} \mapsto [0,1]$ with $\Delta^{ii}_t \leq \Delta^{ij}_t \ \forall i,j$ a measure of cultural values distance among ethnic groups.
Define $s^{ij}_t = s(\Delta^{ij}_t) : [0,1] \mapsto [0,1]$ such that

- $s^{ii}_t > 0$;
- if $\Delta^{ij}_t > \Delta^{ik}_t \Rightarrow s^{ik}_t > s^{ij}_t$
- $\exists \bar{\Delta} : s(\Delta^{ij}_t) > 0 \ \forall \Delta^{ij}_t \leq \bar{\Delta}$, and $s(\Delta^{ij}_t) = 0 \ \forall \Delta^{ij}_t > \bar{\Delta}$, with $\bar{\Delta} \in [0,1]$.

This element is a similarity function such that it is decreasing in the value distance, and self-similarity is always positive. The third condition states that there could exist a threshold $\bar{\Delta}$ under which the similarity is set at 0, and that if $\Delta^{ij}_t = 0$, meaning that the two ethnicities have identical values, then their similarity has to be positive. We call $\bar{\Delta}$ *Openness Propensity*. In fact, for high levels of $\bar{\Delta}$, the agents consider also far ethnicities in their socialization schemes, so that they are open towards big changes in their values. The opposite happens for low levels of $\bar{\Delta}$.
This rule is similar to Golub and Jackson (2009) in which they analyse the case of convergence of opinions in presence of homophily, measured as the willingness to comunicate with closest people using an euclidean metric. In their case, however, since there is no parameter such as $\bar{\Delta}$, convergence to a common value always happens, and the analysis is focused on the speed of convergence. Thus our analysis is mainly focused on the openness parameter and its effect on convergence in presence of parents' socialization effort.

Definition 3: *Call* Basic Cultural Distance *a distance such that* $\Delta^{ij}_t = \sum_k x^k |V^{ik}_t - V^{jk}_t|$ *with* $\sum_k x^k = 1$ *and* $0 \leq x^k \leq 1 \ \forall k$, *and where there is a group independent similarity function* $s(\Delta^{ij}_t)$, *and a parameter* $\bar{\Delta}$.

This is the most simple cultural distance we can think about. In particular the distance between i and j is defined as a weighted mean of the absolute value of the differences of all their entries. If all agents use the same similarity function and have the same openness propensity then $s^{ij}_t = s^{ji}_t$ so that, given two ethnic groups, they agree on the degree of similarity between them. The next analysis is devoted to the study of what happens if $\bar{\Delta}$ is homogenous among groups or if it is groups specific.

1.8.1 Group Independent $\bar{\Delta}$

Before studying the dynamics of all possible attitudes we restrict the analysis to a simplified case in order to understand better the role of the openness propensity parameter.
The model is modified only in the fact that each group i has only two attitudes: $[V^{ii}_t, V^{io}_t]$: a reflexive attitude and an attitude towards the others, so that i agents do not discriminate among the others' ethnic group. Suppose that agents act with ethnocentrism so that only $V^{io}_t, \forall i$ has a dynamics to be analysed and each agent is influenced by the society using higher weights for closer ethnic groups, following the cultural distance rule.
Since now the problem is unidimensional, order all the attitudes in increasing order such that

the following vector arises $V_t = [V_t^1, V_t^2, ..., V_t^{i-1}, V_t^i, V_t^{i+1}, ..., V_t^n]$.
Call $I_t = \{i : V_t^i - V_t^{i-1} > \bar{\Delta}\}$: this is the set of all the V_t^i that does not have a link with the left-neighbour. This means that V_t^{i-1} belongs to a set of attitudes whose dynamics is independent from the dynamics of V_t^i. As a consequence, if we indicate the cardinality of the set as $|I_t|$, we have that the number of the sets of attitudes that are reciprocally independent in their dynamics at each time is $|I_t^i|+1$. Thus call fragmentation of the society at time t $\Phi_t = |I_t^i|+1$.

Proposition 7. *Under Basic Cultural Distance rule, fragmentation can never decrease during time.*

Proof. See Appendix B. □

Last proposition basically states that $\lim_{k\to\infty} |I_{t+k}| \geq |I_t|, \forall t$. This proves that endogeneity based on this kind of homophily rule may be an obstacle towards integration of values. In particular we could be interested in some conditions under which integration may occur in the long run and when, on the contrary, this cannot happen. The following corollaries help us:

Corollary 10: *A necessary condition for integration to happen in the long run is that $\Phi_0 = 1$ or, equivalently, $I_0 = \emptyset$. A necessary condition for $\Phi_t < \Phi_{t+1}$ is that $\exists i : V_t^{i+1} - V_t^i < \bar{V}_t^{i+1} - \bar{V}_t^i$.*

This first necessary condition is very restrictive since it states that, if ethnic groups share the same openness parameters, then if there is positive fragmentation at time zero it is impossible to observe integration. This means that only open societies or societies in which each group is close enough to its neighbour there is hope for integration. It has to be noticed that it is not needed that every group is close to every other. Thus the corollary imposes some conditions on the distribution of initial values. These results are driven by the fact that two 'neighbour' ethnic groups that are too distant in order to be influenced each other can never become closer. The second condition states that if V_t^i and V_t^{i+1} belong to the same influence group, then, in order to have increasing frustration, V_t^i should be more influence by its left neighbors, while V_t^{i+1} must be more influenced by its right-neighbors. We should note that this however may also not end up in the division into two groups, and thus can simply lead to non monotonicity in convergence of attitudes.

We now turn the analysis to the more complex case in which agent can discriminate in their atitudes towards all the groups and not only between themselves and the others.

We can state the following propositions:

Proposition 8. *If $\bar{\Delta} = 1$ or $\bar{\Delta} = 0$, given Basic Cultural Distance then any GCSF Dynamics converges.*
If there are only two ethnic groups, then given Basic Cultural Distance then convergence happens.

Proof. See Appendix B. □

We cannot provide a mathematical proof for convergence of a generic number ($n > 2$) of ethnic groups and a generic $\bar{\Delta}$. However we have run a big number of simulations with different intial values of any parameter findind that if every couple of groups share the judgment over reciprocal similarity, so that they both feel dissimilar or similar each other, whatever the degree of this similarity, then convergence happens. If we better analyse the openness parameter we have that if it is very low we are in front of what we can identify as an exclusive similarity, meaning that agents are very demanding in terms of value similarity in order to consider others in their attitude formation scheme while an inclusive similarity holds if the threshold is high so that agents are not so demanding in terms of similarity in order to question their own values and consider other's attitudes in their socialization process. In the first (second) case, a HE (IE) is likely to be observed.

Example 4: *(Numerical Simulation) In order to better understand the role of this parameter consider the matrices below that report an example. The first one represents the starting values, while the other three represent the equilibrium values for different levels of* $\bar{s} = 1 - \bar{\Delta}$.

$t = 0$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s} = 0.9$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s} = 0.8$	i	j	k
i	0.63	0.63	0.2
j	0.63	0.63	0.2
k	0.8	0.4	1

$\bar{s} = 0.4$	i	j	k
i	0.43	0.43	0.43
j	0.43	0.43	0.43
k	0.43	0.43	0.43

The simulations had been run for $p^i = 0.7$ *and* $p^j = p^k = 0.15$ *so that the difference in the evolution of same size minorities become clearer. Moreover* $s_t^{ij} = [1 - \frac{1}{n}\sum_k |V_t^{ik} - V_t^{jk}|]$, *with* n *the number of groups, and* $w_t^{ij} = \frac{p_t^j s_t^{ij}}{\sum_k p_t^k s_t^{ik}}$[10]. *The original situation is such that there is a bad attitude of* i *and* j *agents towards* k *with initial similarity levels* $s_0^{ij} = 0.8$, $s_0^{ik} = 0.6$ *and* $s_0^{kj} = 0.63$. *In particular, there are two minorities, one of which (k) is considered bad by the other two groups, while both minorities have a good attitude towards the majority. Now, if* $\bar{s}$ *is high, agents are very conservatives meaning that they need a high degree of similarity in order to be influenced by others in their attitudes: this is what we call* exclusive similarity. *As a result no change is shown in the long run. This outcome can be considered similar to what it is usually called* 'closed society'. *In particular we observe that in this case contacts among agents of different groups are not useful in order to get a higher degree of integration. Thus, it is not enough to make two groups in touch in order to achieve at least a higher integration, if they cannot consider the other group's values in their own values formation process. If* $\bar{s}$ *is higher then groups begin to be influenced, and, as a result, some groups will share the same attitudes set (i and j in this case), while others (as k) do not change their attitudes. Only for low levels of* $\bar{s}$ *we have generalized cross influence: this is what we call* inclusive similarity. *In particular an* open society *can be considered a one in which agents are prone towards diversity such that they consider even distant groups in their attitude formation scheme. This open society is the most likely to converge to integration outcomes.*

We should also add that this similarity measure is not entirely endogenous, so that it may take into account some other similarity measures. We can think that some aspect of culture, as religious beliefs, may play a role. Moreover, if we think at some peculiar historical aspects,

[10]In these simulations we weighted also for the population size. However, since population shares re fixed, then they work only as scalars.

as black slavery, this will play for sure a role in the patterns of different ethnic groups. We can thus think that black slavery had an impact on the initial values of the V. Thus, noting that this parameter is such a sensible element in the model, some extensions on how this may change, how it is influenced by institutions and how it can be part of a policy for integration becomes crucially important.

1.8.2 Group Dependent $\bar{\Delta}$

In this subsection we analyse the case with heterogenous openness parameters finding when this can bring to non smooth dynamic processes.If $\bar{\Delta}^i$ is groups specific, it can happen that $s^{ij} \neq s^{ji}$ so that non symmetric socialization rules holds: this happens if the similarity is above the threshold for one group and below for the other[11].
As in the previous analysis, we first restrict the analysis to the case in which agents may only discriminate, in their attitudes, between them and the others. We should re-define the set I_t since now the similarity is no more symmetric and thus the components of the network are not naturally strongly connected. We thus define: $I_t = \{i : V_t^i - V_t^{i-1} > \bar{\Delta}^i \wedge V_t^i - V_t^{i-1} > \bar{\Delta}^{i-1}\}$. In this way the same reasoning as before holds here. Thus we can state that:

Corollary 11: *Under Basic Cultural Distance rule and group dependent openness parameter, fragmentation can never decrease during time. A necessary condition for integration to happen in the long run is that $\Phi_0 = 1$ or, equivalently, $I_0 = \emptyset$. A necessary condition for $\Phi_t < \Phi_{t+1}$ in convergence is that $\exists i : V_t^{i+1} - V_t^i < \bar{V}_t^{i+1} - \bar{V}_t^i$.*

Thus the presence of the homogenous or heterogeneous openness parameters does not influence the fragmentation properties of the model. Still this heterogeneity can capture some phenomena as non smooth convergence processes.

In order to give an idea about these processes we now provide a numerical example for the multidimensioanl case since we are not able to mathematically prove convergence in these cases too.

Example 5: *(Numerical Simulations) Consider again the previous initial situation in which there is a majority i, and two minorities such that one of them, j, is similar to the majority, and the other one, k, that is less similar, while the degree of similarity between the minorities is very low. We analyse now the cases in which one of these groups, in turn, shows a high level of openess ($\bar{s} = 0.4$), while the other two shows a high level of closeness ($\bar{s} = 0.8$). The tables below show the initial and equilibrium values, while the graphs show the cases in which in turns, i, j, and k respectively have a low opennes parameter level.*

$t = 0$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s}^i = 0.4$	i	j	k
i	0.55	0.55	0.55
j	0.55	0.55	0.55
k	0.55	0.55	0.55

$\bar{s}^j = 0.4$	i	j	k
i	0.8	0.8	0.4
j	0.8	0.748	0.4
k	0.8	0.4	1

$\bar{s}^k = 0.4$	i	j	k
i	0.42	0.42	0.42
j	0.42	0.42	0.42
k	0.42	0.42	0.42

[11] This same fact may happen if similarity function is group specific, but we do not consider this case here.

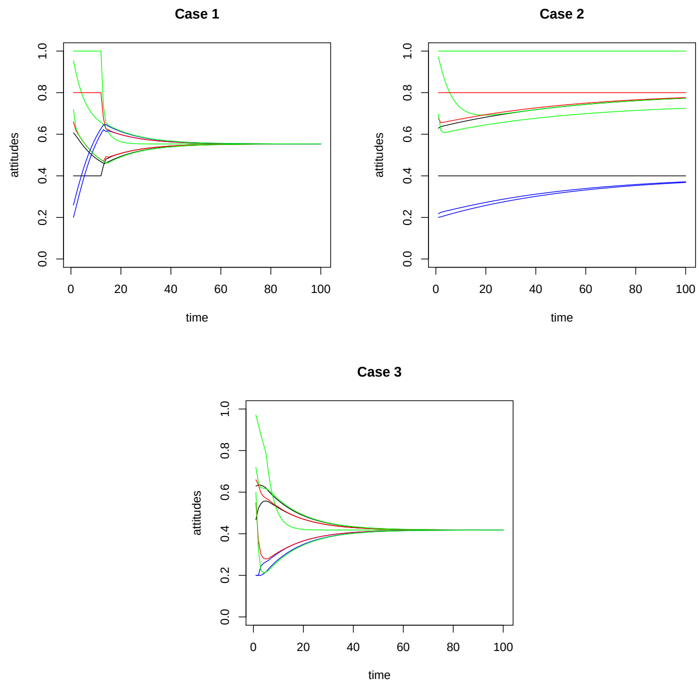

Figure 1.6: Simulations with endogenous weights

The first case represents the one in which the majority is open: we can identify two periods in the convergence. In the first one we have that one ethnic group does not feel similar to anyother and thus it has no contamination nor dynamics: this is the k group for the first 20 generations. After this period of time i agents (and j agents through i's influence) became closer such that now both j and k begin to include the others in the socialization scheme experiencing the convergence of the second period. This irregularity in convergence makes again clearer the role of this parameter in the understanding of short run cross-influences: having in fact a short run view over the dynamics it could be thought that k agents would never wanted to integrate in the society. It was then enough, in this case, to have one group (i) that uses inclusive similarity and is felt similar to a group using exclusive similarity (j) in order to create a bridge for long run integration. In this case, i agents have to be patient and wait for almost 20 generations before hving the first results of their openness: this gives the idea that integration processes may be a matter of decades. The second case represents the situation in which the minority that is closer to the majority is open: then even though some influences from the k groups happen to be observed, in the long run i and j integrate almost perfectly, while the other minority group remains segregated. This also happens for a total open minority: in fact, since j agents are more similar to i than to k, then they will always take more care of i values that of k values, so that their mean will always be biased in favour of the firsts. Consequently it never happens that i and k become sufficiently close to be influenced each other. If, on the other hand, the most dissimilar minority is opened (as in the third case), then integration happens since the ethnicity that was an obstacle for integration removes the closeness prejudice.

With this last part we make clear the effect that heteroegenous propensities towards openness may have on the final outcome. We are consciuos that these measures may be endogenous, but for the time being we consider them as dependent on something out of the model, and dependent on some socio-economic position of the group, as previously argued. However these last results make clear the role that a policy focused on making people more open and tolerant may have on integration policies, since it comes clear that, besides material factors, these are crucial elements of the problem. Moreover, the endogenization of the socialization rule does not help in explaining why cycles in attitudes may be observed, but drives us towards the direction of finding them into the changes of the socio-economic position of the groups, thus providing an exogenous explanation for these phenomena. On the other side, racism per se may be a results of the endogenous dynamics, if agents do no show a sufficiently high level of tolerance and openness. If weights are endogenized in order to depend both on similarity and population shares, and population dynamics shows cycles, then it can be that cycles in racism may be consequently observed.

1.9 Conclusion

Existing economic literature on cultural evolution, referring to Cavalli-Sforza and Boyd-Richerson studies, mainly focuses on what happens if time invariant cultural values are transmitted from one generation to the other (as in the contributions of Bisin and Verdier) and studies the evolution of population shares under this assumption. Only recently, some interest has been devoted to the study of convergence of non-fixed cultural values and to the introduction of complexity in the vertical socialization processes. The more recent contributions studied the conditions under which a *melting pot* equilibrium happens in terms of long run equilibrium, finding that it may happen if there is a general cross influence among cultural values.

Here, starting from the initial intuition of Boyd and Richerson (1985) about the importance of cultural transmission structures, we study what happens if attention is given to different oblique socialization schemes. Using a framework in which there are ethnic groups and parents trying to transmit their attitudes, we are able to understand what happens if different interaction schemes among ethnicities are considered. Using schemes as Reciprocity and Ethnocentrism we prove that, if all agents use the same socialization scheme, then the society may converge both to integration or to a social hierarchy based on ethnicity, thus deriving equilibria consistent with empirical studies. Turning then to the welfare analysis we found that, in a two groups framework, the minority group always put more socialization effort, changes more in values and thus shows more frustration than the majority group. We then analyse what happens if different ethnic groups use different socialization schemes. Using a network-derived framework we underline the role that different groups may have in the convergence process: this framework thus gives an instrument in order to analyse why different minorities may end up with different long run integration equilibria. We then a time dependent socialization structure framework with some sufficient conditions that weights dynamics may satisfy in order to reach integration or segregation equilibria. Finally we provide the first steps for the analysis of convergence with an endogeous homophily rule.

This study opens new roads in which the reasearch may be run: there is space in order to understand what happens if the structure of the interethnic relationships change with time with different mechanism than what we did here, so providing a new endogeneization of socialization schemes. Similarly it would be interesting to study what happens if forms of socialization schemes other than reciprocity and ethnocentrism may be implemented. Again it could be interesting to analyse what happens if horizontal socializiation is taken into account into these schemes. An empirical analysis on some case studies, however, may be necessary.

Appendix A: Weights Matrices for Simulations

We report below the weight matrices we used in the simulations for figure 1.1. With respect to the cases reported in the definition of socialization rules we impose that $w_{t,ij}^{jj} = 0, \forall i, j$ meaning that in forming ij attitude, i agents do not consider the reflexive attitude of j. This does not change the way in which dynamics happens, but just levels. In particular it avoids that in HEE and IEE everything converges to $V = E = 1$, as it is clear from proposition 4-5. Moreover we just write **X** where there is a positive weight. The weights are represented by the population shares so that, for example, $w_{ij}^{kj} = p^k$. Since here population shares are constant, then the weight matrix is fixed.

HE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	**X**	0	0	**X**	0	0
ij	0	**X**	0	0	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	0	0	0
ji	0	0	0	**X**	0	0	**X**	0	0
jj	0	**X**	0	0	**X**	0	0	**X**	0
jk	0	0	**X**	0	0	**X**	0	0	0
ki	0	0	0	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	0	0	**X**	0
kk	0	0	**X**	0	0	**X**	0	0	**X**

IE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	**X**	0	0	**X**	0	0
ij	0	**X**	0	**X**	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	**X**	0	0
ji	0	**X**	0	**X**	0	0	**X**	0	0
jj	0	**X**	0	0	**X**	0	0	**X**	0
jk	0	0	**X**	0	0	**X**	0	**X**	0
ki	0	0	**X**	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	**X**	0	**X**	0
kk	0	0	**X**	0	0	**X**	0	0	**X**

HEE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	0	0	0	0	0	0
ij	0	**X**	0	0	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	0	0	0
ji	0	0	0	**X**	0	0	**X**	0	0
jj	0	0	0	0	**X**	0	0	0	0
jk	0	0	**X**	0	0	**X**	0	0	0
ki	0	0	0	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	0	0	**X**	0
kk	0	0	0	0	0	0	0	0	**X**

IEE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	0	0	0	0	0	0
ij	0	**X**	0	**X**	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	**X**	0	0
ji	0	**X**	0	**X**	0	0	**X**	0	0
jj	0	0	0	0	**X**	0	0	0	0
jk	0	0	**X**	0	0	**X**	0	**X**	0
ki	0	0	**X**	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	**X**	0	**X**	0
kk	0	0	0	0	0	0	0	0	**X**

Appendix B: Proofs of the Propositions

Proof of Proposition 1

Proof. Since $\tau_t^i \in [0,1]$ then $V_{t+1}^{ij}(\tau^i, \bar{p}_t, \bar{V}_t)$ is continous in τ so that W_t^{ii} is continuous in τ_t^i. Since $c(\tau_t^i)$ is also continuous in τ_t^i then $W_{t,i}^i - c(\tau_t^i)$ admits a global maximum in $\tau_t^i \in [0,1]$ so that $\tau_t^{i^*}$ exists. Moreover $W_{t,i}^i$ can be written as $V^* - (1-\tau_t^i)^2 \sum_k (V_t^{ik} - \bar{V}_t^*)^2$ and is strictly concave in τ_t^i. $c(\tau_t^i)$ is strictly convex in τ_t^i then $W_{t,i}^i - c(\tau_t^i)$ is strictly concave in τ_t^i and so $\tau_t^{i^*}$ is unique.
Suppose now that $V_t^{ij} = \bar{V}_t^{ij}, \forall j$, then $\frac{\partial W_t^i}{\partial \tau_t^i} = 0, \forall \tau_t^i$.
In order to be $\tau_t^{i*} = 1$ it should be $c'(1) \leq \frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=1}$. But $c'(1) > 0$, while $\frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=1} = 0$ so that this is impossible. Moreover it cannot be $\tau_t^{i*} \in (0,1)$ since at the optimum it should be $\frac{\partial W_t^i}{\partial \tau_t^i} = c'(\tau_t^i)$, but $\frac{\partial W_t^i}{\partial \tau_t^i} = 0, \forall \tau_t^i$ and $c'(\tau_t^i) > 0, \forall \tau_t^i > 0$.
Thus $\tau_t^{i*} = 0$ since $c'(0) = \frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=0}$ so that $c'(0) \geq \frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=0}$.
Suppose now that $V_t^{ij} \neq \bar{V}_t^{ij}$ for at least one j. In this case $\frac{\partial W_t^i}{\partial \tau_t^i} > 0 \forall \tau_t^i < 1$, and $\frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=1} = 0$.
In order to be $\tau_t^{i*} = 0$ it should be $c'(0) \geq \frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=0}$. But $c'(0) = 0$, while $\frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=0} > 0$ so that it is

impossible.
In order to be $\tau_t^{i*} = 1$ it should be $c'(1) \leq \frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=1}$. But $c'(1) > 0$, while $\frac{\partial W_t^i}{\partial \tau_t^i}|_{\tau_t^i=1} = 0$ so that it is impossible. Thus it must be $\tau_t^{i*} \in (0, 1)$.

□

Proof of Proposition 2

Proof. Equation (1.5) can be written as
$V_{t+1}^{ij} = (\tau_t^i + w_{ij}^{ij} - w_{ij}^{ij}\tau_t^i)V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{ij}^{kw} V_t^{kw})$. Since, by proposition 1, $\tau_t^{i^*}(\bar{p}_t, \bar{V}_t) \in (0, 1)$ and is endogenous, then the dynamics is not linear in $\bar{V}_t$. Consider now any $\tau_t^i \in (0, 1)$ exogenously given at any time and for each group such that $\tau_t^{i^*}(\bar{p}_t, \bar{V}_t) \in (0, 1)$ is only one possible value for τ_t^i. We will prove convergence for any τ_t^i such that convergence for $\tau_t^{i^*}(\bar{p}_t, \bar{V}_t)$ is only a specific case and thus convergence will also happen for every suboptimal $\tau_t^i \in (0, 1)$.
Order the type entries in order to get a $(n^2 X 1)$ vector as the following:

$$\bar{V}_t = [V_t^{ii}, V_t^{ij}, ..., V_t^{in}, V_t^{ji}, V_t^{jj}, ..., V_t^{jn}, ..., ..., V_t^{ni}, ..., V_t^{nn}].$$

We can thus write the dynamics as

$$V_{t+1}^{ij} = [a_{t,ij}^{ii}, a_{t,ij}^{ij}, a_{t,ij}^{ik}, ..., a_{t,ij}^{nn}]\bar{V}_t'$$

in which the non-diagonal terms $a_{t,ij}^{kw} = (1 - \tau_t^i)w_{ij}^{kw}$ if $i \neq k$ and the diagonal term $\forall j \neq w$, and $a_{t,ij}^{ij} = (\tau_t^i + w_{ij}^{ij} - w_{ij}^{ij}\tau_t^i), \forall i, j$.
We thus have the following linear system: $V_{t+1}^{'} = AV_t^{'}$, in which A_t is the $(n^2 X n^2)$ matrix in which the entries are the $a_{t,ij}^{kw}$, $\forall i, j, k, w$, so that A is row-normalized.
Consider now the A matrix. Since $\tau_t^i > 0$, then $a_t^{ii} > 0 \; \forall i$ (*).
Brueckner and Smirnov (2007, 2008) proved that given a linear system $V_{t+1}' = AV_t'$, then if A is irreducible[12] for all t, if at least one diagonal element is positive, then the matrix is also acyclic[13], and thus the dynamics converges to a steady state. Since weights w_{ij}^{kw} are time independent, and $\tau_t^i \in (0, 1)$, the matrix A is always irreducible or always not reducible. Moreover, since (*) holds the matrix, if irreducible, is acyclic. Consequently Brueckner and Smirnov (2007, 2008) immediately applies when the matrix A is irreducible. Moreoever, in this case, they prove that all elements converge to the same value. If A is not irreducible, it can always be rewritten as an upper-triangular-block matrix as B by means of columns and rows transpositions:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & 0 & b_{nn} \end{bmatrix}$$

in which every b_{ii} is a square block, while some non-diagonal blocks may have all zero entries.
Given time independent weights and $\tau_t^i \in (0, 1)$, the strucure of the B matrix is time invariant $\forall t \geq T$.
If B is a block-diagonal matrix, and thus all non-diagonal blocks have all zero entries, then, since every diagonal block is irreducible and, by (*), acyclic, then every element of each block converges and thus overall convergence happens.
If B is not a block diagonal matrix, take the b_{nn} block. Again the structure of the matrix is time

[12] A square matrix A is irreducible if and only if for each i and j there exists some k such that $(a_{ij})^k > 0$, with $(a_{ij})^k$ being the ij entry of the k^{th} power matrix of A. Moreover a matrix is irreducible if and only if the digraph associated to A is strongly connected.

[13] Call d_{ii} the period of the a_{ii} element of the A square matrix. d_{ii} is the greatest common divisors among all k such that $(a_{ij})^k > 0$. A square matrix A is acyclic if and only if $d_{ii} = d_{jj} = 1$, $\forall i, j$.

invariant. b_{nn} elements thus converge since it is irreducible and acyclic.
Consider now the $b_{n-1,n-1}$ block, and analyse its dynamics that depends only on $b_{n-1,n-1}$ and b_{nn} blocks' elements. Consider first the case in which the b_{nn} block is composed of only one element. Then consider the submatrix for the last two blocks:

$$\begin{bmatrix} b_{n-1,n-1} & b_{n-1,n} \\ 0 & b_{nn} \end{bmatrix}$$

Then the corresponding weight matrix is represented as follows:

$$B_t = \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 1-\sum_i \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 1-\sum_i \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 1-\sum_i \gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $X_{t+1} = B_t X_t = B^t X_0$. In terms of Markov processes, this can be identified as a *Non-Homogenous single-unireducible Markov Process.* Given the structure of the process, the limit probability of the markov process represented by the transmission matrix is also the limit of the matrix of weights. Consequently if the limit probability of the markov process exist, then the process converges, and if the limit probability can be identified, then the limit of the matrix of weights can be identified too.
D'amico et al. (2009) proved that, if the probability matrix is non-homogenous sigle-unireducible, then

$$\lim_{t\to\infty} B^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that convergence to the b_{nn} element level happens.
Consider now the case in which the b_{nn} block is composed of more that one element. Since the b_{nn} block is strongly connected and there are no influences by entries not belonging to the block, then all elements of the b_{nn} block converge to the same value I, as proved by Brueckner and Smirnov (2007, 2008). Consequently we have that (for the case of a two-element block, but it can be extended to a n-element case):

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

Thus, we can rewrite the limit of the dynamics of the first $n-1$ entries as:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} + \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} + \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} + \gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \end{bmatrix}$$

This is again the limit of a non-homogenous single-unireducible markov process, so that we have:

$$\lim_{t\to\infty}\begin{bmatrix}\alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it}\\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it\\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it}\\ ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it}\\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it}\end{bmatrix}\begin{bmatrix}x_t\\ y_t\\ z_t\\ ...\\ w_t\\ s_t\end{bmatrix} = \lim_{t\to\infty}\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\...&...&...&...&...&\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}x_t\\ y_t\\ z_t\\ ...\\ I\\ I\end{bmatrix}.$$

If in the whole B matrix the b_{nn} block is the only block that does not depend on any other block, then all other block directly or indirectly depend only on it, so that the same proof can be used to prove convergence of all the entries of the B matrix to the convergence level of the b_{nn} block. Suppose now that this is not the case, and so that $b_{n-1,n}$ block has all zero elements and thus both b_{nn} and $b_{n-1,n-1}$ blocks have dynamics independent each other.
Consider now the $b_{n-2,n-2}$ block. If $b_{n-2,n-1}$ block or $b_{n-2,n}$ block has all zero entries, then the previous result applies for the convergence of $b_{n-2,n-2}$ elements too, since all $b_{n-2,n-2}$ elements have a dynamics that depends on the elements of own block and on a converging block. If this is not the case, then take any V^{ij} belonging to the $b_{n-2,n-2}$ process. Then, we can always find weights α_t, β_t and γ_t such that any element V_t^{ij} of this process can have its dynamic rewritten as

$$V_t^{ij} = \alpha_t V_t^{ij} + \beta_t \bar{b}_{n-2,n-2} + \gamma_t \bar{b}_{n-1,n-1} + \delta \bar{b}_{nn} \tag{1.8}$$

with $\alpha_t \in (0,1]$ (since $\tau_t^i > 0$), $\beta_t \in \Re^+$, $\gamma_t \in \Re^+$ and $\delta_t \in \Re^+$ in which $\bar{b}_{n-2,n-2}$ is the value at which the elements of the $b_{n-2,n-2}$ diagonal block had converged if the dynamics would have been given only by this diagonal block, $\bar{b}_{n-1,n-1}$ is the value at which the elements of the $b_{n-1,n-1}$ diagonal block had converged if the dynamics would have been given only by this diagonal block and $\bar{b}_{nn}$ is the convergence points of the block b_{nn}.[14]
Now, for entries for which $w_{t,ij}^{ij} = 1$ then $\alpha_t = 1, \beta_t = \gamma_t, \delta_t = 0$. Since weights are fixed, then this holds for all periods, so that these entries do not show any dynamics.
Consider now entries with $w_{t,ij}^{ij} \neq 1, \forall i,j$. In this case we can find $\alpha_t \in (0,1)$, $\beta_t \in \Re^+$, $\gamma_t \in \Re^+$ and $\delta_t \in \Re^+$.

Define $\alpha_t! \equiv \prod_{i=0}^{t} \alpha_i$. We can write:

$$\begin{aligned}
V_1^{ij} &= \alpha_1 V_0^{ij} + \beta_1 \bar{b}_{n-2,n-2} + \gamma_1 \bar{b}_{n-1,n-1} + \delta_1 b_{nn}\\
V_2^{ij} &= \alpha_2\alpha_1 V_0^{ij} + \alpha_2\beta_1 \bar{b}_{n-1,n-1} + \alpha_2\gamma_1 \bar{b}_{n-1,n-1} + \alpha_2\delta_1 \bar{b}_{nn} + \beta_2 \bar{b}_{n-2,n-2} + \gamma_2 \bar{b}_{n-1,n-1} + \delta_2 \bar{b}_{nn}\\
V_3^{ij} &= \alpha_3\alpha_2\alpha_1 V_0^{ij} + \alpha_3\alpha_2\beta_1 \bar{b}_{n-1,n-1} + \alpha_3\alpha_2\gamma_1 \bar{b}_{n-1,n-1} + \alpha_3\alpha_2\delta_1 \bar{b}_{nn} + \alpha_3\beta_2 \bar{b}_{n-2,n-2}\\
&\quad + \alpha_3\gamma_2 \bar{b}_{n-1,n-1} + \alpha_3\delta_2 \bar{b}_{nn} + \beta_3 \bar{b}_{n-2,n-2} + \gamma_3 \bar{b}_{n-1,n-1} + \delta_3 \bar{b}_{nn}\\
... &= ...\\
V_t^{ij} &= V_0^{ij} \prod_{i=1}^{t} \alpha_i + \bar{b}_{n-2,n-2}(\beta_t + \beta_{t-1}\alpha_t + \beta_{t-2}\alpha_{t-1}\alpha_t + \beta_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \beta_1\alpha_t\alpha_{t-1}...\alpha_t)\\
\Downarrow &\quad + \bar{b}_{n-1,n-1}(\gamma_t + \gamma_{t-1}\alpha_t + \gamma_{t-2}\alpha_{t-1}\alpha_t + \gamma_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \gamma_1\alpha_t\alpha_{t-1}...\alpha_2)\\
\Downarrow &\quad + \bar{b}_{n,n}(\delta_t + \delta_{t-1}\alpha_t + \delta_{t-2}\alpha_{t-1}\alpha_t + \delta_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \delta_1\alpha_t\alpha_{t-1}...\alpha_2)\\
V_t^{ij} &= V_0^{ij} \prod_{i=1}^{t} \alpha_i + \bar{b}_{n-2,n-2} \sum_{i=1}^{t} \beta_i \frac{\alpha_t!}{\alpha_i!} + \bar{b}_{n-1,n-1} \sum_{i=1}^{t} \gamma_i \frac{\alpha_t!}{\alpha_i!} + \bar{b}_{nn} \sum_{i=1}^{t} \delta_i \frac{\alpha_t!}{\alpha_i!}
\end{aligned}$$

In order to analyse the convergence we should look at $\lim_{t\to\infty} V_t^{ij}$.

Trivially $\lim_{t\to\infty} V_0^{ij} \prod_{i=0}^{t} \alpha_{i+1} = 0$ since $\alpha_t \in (0,1)$.

Consider now the term $\bar{b}_{n-2,n-2} \sum_{i=1}^{t} \beta_i \frac{\alpha_t!}{\alpha_i!}$ and call $z_j = \beta_j \frac{\alpha_t!}{\alpha_j!}$.

$j = t \Rightarrow z_j = \beta_t$
$j = t-1 \Rightarrow z_j = \beta_{t-1}\alpha_t$

[14]Notice that this is not a new dynamic, but is a way in order to find coefficients such that these V_t^{ij} exactly correspond to the values indicated by the original dynamics. In this way the convergence points of the two rules coincide since the second one is built in order to coincide step by step with the original one. Consequently proving convergence for the second one implies proving convergence for the original dynamics.

$j = t-2 \Rightarrow z_j = \beta_{t-2}\alpha_t\alpha_{t-1}$
$j = t-3 \Rightarrow z_j = \beta_{t-3}\alpha_t\alpha_{t-1}\alpha_{t-2}$
...
$j = t-t+1 \Rightarrow z_j = \beta_{t-t+1}\alpha_t\alpha_{t-1}\alpha_{t-2}...\alpha_{t-t+2}$

take $\bar{\beta} = max\{\beta_t\}$ and $\bar{\alpha} = max\{\alpha_t\}$ then
$\sum_{j=0}^{t}\beta_j\frac{\alpha_t!}{\alpha_j!} \leq \sum_{j=0}^{t}\bar{\beta}\frac{\alpha_t!}{\alpha_j!} = \bar{\beta}\sum_{j=0}^{t}\frac{\alpha_t!}{\alpha_j!} = \bar{\beta}(1+\alpha_t+\alpha_t\alpha_{t-1}+\alpha_t\alpha_{t-1}\alpha_{t-2}+...) \leq \bar{\beta}(\sum_{j=0}^{t}\bar{\alpha}^j)$.
Now, $\lim_{t\to\infty}\sum_{j=0}^{t}\bar{\alpha}^j = \frac{1}{1-\bar{\alpha}}$.
Consequently, $\sum_{j=0}^{t}\beta_j\frac{\alpha_t!}{\alpha_j!}$ is an increasing sequence bounded above by a converging sequence, so that it converges.
The same proof holds for $\sum_{j=0}^{t-1}\gamma_{j+1}\frac{\alpha_t!}{\alpha_{j+1}!}$ and for $\sum_{j=0}^{t-1}\delta_{j+1}\frac{\alpha_t!}{\alpha_{j+1}!}$.

Thus $\lim_{t\to\infty} V_t^{ij}$ is a weighted finite sum of converging series so that it converges too. Thus the elements of the $b(n-2, n-2)$ block converge. If we recursively apply this reasoning to all the other blocks until we reach the (b_{11}) block, then convergence to a steady state is proved.

□

Proof of Proposition 3

Proof. A system like $V'_{t+1} = AV'_t$ converges if each diagonal block of the transmission matrix is irreducible, and thus has a strongly connected directed graph, and acyclic. In all cases, socialization rules imply $w_{ij}^{ij} > 0, \forall i, j$, such that all the diagonal blocks of the transmission matrix are acyclic. If Emulation Rule holds, we have that all the entries V^{ij} $\forall i$, with some row and columns transpositions, form a single block which has a strongly connected digraph, since all the links are bidirectional, so that it is also an irreducible block. Thus, by Brueckner and Smirnov (2007, 2008), convergence to a common value for each block, and thus to a HE, happens.
The Reciprocity rule differs from the previous one since each of the previous blocks is connected to the other ones via the double links between V^{ij} and V^{ji} $\forall i, j$, so that all V^{ij} $\forall i, j$ forms a single strongly connected digraph and, for the same reason as before, convergence to a IE happens.
The Ethnocentrism Rule and Reciprocity and Ethnocentrism rule differ from the previous ones in the sense that $V^{ii} = E$ $\forall i$ such that each of these reflexive elements forms a diagonal block per se and do not show any dynamics. The remaining elements have a structure as in the previous two cases thus convergence respectlivey to HEE and IEE happens. □

Proof of Proposition 4

Proof. Given the cost function we have $\tau_t^{i^*} = \frac{(1-p_t^i)^2(V_t^i-V_t^j)^2}{1+(1-p_t^i)^2(V_t^i-V_t^j)^2}$. Thus $\tau_t^{i^*} > \tau_t^{j^*}$ if and only if $p_t^i < p_t^j$ and thus if $\tau_t^{i^*} < \frac{1}{2}$.
Given the optimal efforts we have that $V_t^i - V_{t+1}^i = \frac{(1-p_t^i)^2(V_t^i-V_t^j)^2}{(1+(1-p_t^i)^2(V_t^i-V_t^j)^2)^2}$. Thus $\Delta_t^i = (\frac{(1-p_t^i)^2(V_t^i-V_t^j)^2}{(1+(1-p_t^i)^2(V_t^i-V_t^j)^2)^2})^2$.
So $\Delta_t^i > \Delta_t^j$ if and only if $p_t^i < p_t^j$ so that if and only if $p_t^i < \frac{1}{2}$. Since then $F_t^i = \Delta_t^i + c(\tau_t^{i^*})$, given that cost functions are increasing in the effort, $F_t^i > F_t^j$ if and only if $p_t^i < \frac{1}{2}$. □

Proof of Proposition 5

Proof. Consider the first part of the proposition. Each component forms a dynamics system per se and thus may be considered separately from the others. Consider first the case in which the elements of a group are strongly connected. Then the transmission matrix may be represented as a block diagonal matrix in which each block is irreducible and thus convergence happens. If there is one sink then it can be represented as a upper triangular block matrix in which the sink is the right-bottom block. From proposition 2 in this case convergence holds too. In order to prove that all the elements of the same component converge to the same value, consider first the case in which the sink is composed by only one node. Then the weight matrix is represented as follows:

$$B_t = \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 1-\sum_i \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 1-\sum_i \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 1-\sum_i \gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $X_{t+1} = B_t X_t = B^t X_0$. In terms of Markov processes, this can be identified as a *Non-Homogenous single-unireducible Markov Process*. Given the structure of the process, the limit probability of the markov process represented by the transmission matrix is also the limit of the matrix of weights. Consequently if the limit probability of the markov process exist, then the process converges, and if the limit probability can be identified, then the limit of the matrix of weights can be identified too.

D'amico et al. (2009) proved that, if the probability matrix is non-homogenous sigle-unireducible, then

$$\lim_{t\to\infty} B^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that convergence to the sink level happens.

Consider now the case in which the sink is composed of more that one element. Since the sink is strongly connected and there are no influences by nodes not belonging to the sink, then they converge to the same value I. Consequently we have that (for the case of a two-nodes sink, but it can be extended to a n-nodes sink case):

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 0 & \alpha_{i-1t}+\alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 0 & \beta_{i-1t}+\beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 0 & \gamma_{i-1t}+\gamma_{it} \\ ... & ... & ... & ... & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

Since the last two elements do not show any dynamics and they are fixed on the same values, we can rewrite the limit of the dynamics of the first $n-1$ entries as:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t}+\alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t}+\beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t}+\gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \end{bmatrix}$$

This is again the limit of a non-homogenous single-unireducible markov process, so that we have:

$$\lim_{t\to\infty}\begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix}\begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

Consider now the second part of the proposition. Since all elements forms a single component the proof for convergence is the same as the one for convergence of each component in just exposed.

□

Proof of Proposition 6

Proof. Suppose that ethnocentrism does not hold. If assumptions 3 and 4 hold then the matrix can always be rewritten as a block diagonal matrix in which, at each time, each block is acyclic (since $w^{ij}_{t,ij} > 0, \forall i, j$) and irreducible (because of simmetry). By assumption 4, then, after time T the block structure of the matrix is time invariant. Thus, by Brueckner and Smirnov (2007, 2008), convergence happens. If ethnocentrism hold, then the reflexive traits $V^{ii}, \forall i$ are fixed and each of them form a block with $w^{ii}_{t,ii} = 1$ and thus no dynamics is shown, while the first part of the proof holds for the dynamics of all the other entries.

We prove now the second part of the proposition. If assumption 4 holds, then the matrix structure is stable after time T. Each component of the directed graph forms a dynamics system per se and thus may be considered separately from the others. Consider first the case in which the elements of a component are strongly connected at time T, so that they will always be strongly connected. Then the transmission matrix may be represented as a block diagonal matrix in which each block is irreducible and thus, by Brueckner and Smirnov (2007, 2008) convergence happens. If there is one sink then the matrix can be represented as a upper triangular block matrix in which the sink is the right-bottom block. Consequently the proof follows as the one for proposition 4. □

Proof of Proposition 7

Proof. Suppose that at time t fragmentation $\Phi_t = n$. Thus, there are n disjoined groups of attitudes. In each groups all attitudes move towards their mean. Take now V^i_t and V^{i+1}_t beloging to two different sets. Then $V^i_t > V^i_{t+1}$ and $V^{i+1}_t < V^{i+1}_{t+1}$. Thus the two sets can never be merged together. □

Proof of Proposition 8

Proof. Consider the first part of the proposition. If $\bar{\Delta} = 1$, then $s^{ij}_t > 0, \forall i, j, t$ so that all ethnicities form one single component, the matrix A is thus irreducible and convergence happens.
If $\bar{\Delta} = 0$, then each group forms a component by its own, apart for the case in which two ethnic groups have the very same entries.. In this case, if $V^{ij}_t = V^{ji}_t$ the two groups show no dynamics in type entries. If $V^{ij}_t \neq V^{ji}_t$ then from $t + 1$ they become different in entries so that each of them form a separate component. Thus convergence happens.
Consider now the second part of the proposition. If $\Delta^{ij}_t > \bar{\Delta}$ then they do not influence each others and thus they stay fixed. If $\Delta^{ij}_t < \bar{\Delta}$ they form an irreducible block and they move towards the SS. If they always remain linked, they converge, if they become dissimilar at some point in time the stay fixed for all the rest of the time. Thus convergence happens. □

Chapter 2

Residential and School Segregation and the Evolution of Homophily

2.0.1 Abstract

This paper contributes to the existing debate about school segregation based on ethnicity by introducing a simple model of school choice based on spatial residential segregation, analysing when partial ethnic segregation can happen due to the change in interethnic preferences even if the demographic structure is unchanged during time. We build a theory in which only ethnic concerns affect choices without socio economic biases, and we introduce a dynamic model in order to analyse how homophily preferences are affected by the social interactions and past homophily preferences: in this way we have a coevolution of preferences and segregation in schools. We then find this cross dependency to be necessary in order to obtain partial segregation results as observed in empirical data. We then analyse deeper these influences on preferences by indentifying the role of two effects on homophily: the School Effect, capturing the effect on preferences of having a higher share of own people atteding the same school, and the Residential Effect, capturing the effect of having more people of own type in own neighborhood that are in contact with other groups.

2.1 Introduction

During the last decades, ethnic based school segregation has begun to be a sensible topic in European countries, and has always been a crucial issue in the United Stated debate. The aim of this paper is thus to provide a spatial model for school choice that analyses school ethnic segregation outcomes investigating the role of homophily preferences and the evolution of these preferences due to the social interactions implied by the school choice itselfs. By means of this, it tries to explain when partial ethnic segregation can happen due to the change in interethnic preferences even if the demographic structure is unchanged during time.
A large body of empirical literature investigates this phenomenon analysing the ethnic segregation in spatial terms or introducing the segregation based on public and private school attendance. As far as US are concerned Reardon and Yun (2002), Fairlie (2006), Betts and Fairlie (2003) and Ellen et al. (2002) give an overview of the problematic. For European countries we can find studies for Sweden (Söderström and Urusitalo (2010)), Belgium (Timmerman et al. (2003)), Germany (Kristen (2008)), Denmark (Rangvid (2007)), UK (Burgess et al. (2005, 2009)). In particular Söderström and Urusitalo (2010) and Burgess et al. (2005) study how a school free choice policy may influence the long run segregation in the society

finding that school segregation is always higher than residential segregation. All these studies confirm that higher immigration is generally associated to the *native flight* phenomenon, so that segregation tends to increase. Again Söderström and Urusitalo (2010) and Burgess et al (2005) underline the differences between policies of school free choice and policies making the attendance to a given set of school mandatory. In the first case, in particular, segregation tends to be higher.
This segregation phenomenon can been explained by the difference in socio-economic qualities of natives and immigrants and by the importance of ethnic identity for immigrants. The first set of motivations (socio-economic conditions) has been widely explored by economics literature (look Nechyba (2006) for a review of these works), the second set of motivations (related to homophily reasons) has been largely ignored. Even if ethnic identity has been always cited as a crucial force that may bring to segregation, up to our knowledge there exists only one paper studying the impact of preferences over the school choice (Berniell (2008)) setting a model for public or private school choice.
Homophily literature has not directly applied to the school choice problem but recent insights explore its role in school friendship network formation as in Currarini et al (2009). In particular ethnic based homophily has been proved to exist (Fong and Isajiw (2000), Baerveldt et al (2004) for example), and has social interactions as main influences due to agents' meeting processes and the preferences of individuals (Moody (2001)).
This paper contributes to the existing debate by introducing a simple model of school choice based on spatial residential segregation. We build a theory in which only ethnic concerns affect choices, in order to analyse only homophily effects without socio economic biases. We use then a dynamic model in order to analyse how homophily preferences are affected by the social interactions and past homophily preferences: in this way we have a coevolution of preferences and segregation in schools. In this way we do not impose a specific structure to preferences dynamics (vertical and oblique socialization qith convergence in mean of the values) but we leave it as general as possible finding necessary conditions in order to observe some segregation phenomena. In this sense we try not to restrict the analysis to a specific functional form and we go beyond the intergenerational transmission of preferences as in Bisin and Verdier (2000, 2001) and all the subsequent literature: up to our knowedge this is the first contribution in this sense. We then find this cross dependency to be necessary in order to obtain partial segregation results as observed in empirical data. We then analyse deeper these influence on preferences by indentfying the role of two effects on homophily preferences: the School Effect, capturing the effect on preferences of having a higher share of own people atteding the same school, and the Residential Effect, capturing the effect of having more people of own type in own neighborhood that are in contact with other groups. Along the paper we also try to better understand the role of different structures of residential segregation by analysing two different case: one in which the is a strong residential segregation and the second one in which segregation is weaker.

The paper is structured as follows: in section 2 we present a static model with some implication about free choice policies. Section 3 introduces the dynamic model and section 4 analyses condition over preferences dynamics in order to obtain partial segregation outcome. Section 5 presents the conclusions while in appendix we introduce a possible reinterpretatin of the model for socio-economic groups.

2.2 The Model

Consider a city composed of a space of mass 1 located on the $[0,1]$ line. Suppose the city to be divided in two neighborhoods, N_1 and N_2 with N_1 on the $[0,N]$ segment and with

$N \in (0,1)$, and N_2 on the $[N,1]$ segment. Suppose agents to be located in the space, each at each point in the space, so that there is a continuum of agents of mass 1. Agents are divided in two groups: i agents and j agents. These groups can be ethnic or socio-economic groups. Consider a situation in which a sort of residential segregation holds, such that i agents live in the $[0,p]$ part of the city, with $p \in (0,N)$, and the rest is occupied by j agents. Call $j1$ the j agents living in N_1 and $j2$ the j agents living in N_2: we thus impose a strong residential segregation. Define as I the set of agents of group i, $J1$ the set of agents of group $j1$ and $J2$ the set of agents of group $j2$. We can thus represent the residential composition as in figure 2.1.

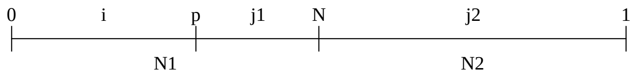

Figure 2.1: City Composition

Consider the case in which in each neighborhood there is a school, indexed with the neighborhood index, so that there is a school S_1 in N_1 and a school S_2 in N_2. Suppose that schools have no capacity costraints.
Consider the case in which, at each point in time, each agent has to decide which school to attend (or has to decide the school her child has to attend).
Suppose that each agent h has a parameter z^h indicating her preference towards opposite group presence in the school. Suppose that each agent inside each group has the same preference parameter[1]. Define τ the cost that an agent has to face if she decides to attend the school out of her neighborhood: this can take the form of transportation costs, bureaucratic costs or other kind of costly frictions and can be due to some structural property of the space or to some element decided by the policy maker. Define β_t^h the share of agents of the same group of agent h staying in the school of own neighborhood. Thus, for ease of notation, let β_t^i represents the share of i agents going to S_1, β_t^{j1} the share of $j1$ agents going to S_1 and β_t^{j2} the share of $j2$ agents going to S_2, at each point in time. Define thus $\bar{\beta}_t$ the vector of all the βs at each point in time. Consequently define $q_{t,i}^{S_n}(\bar{\beta}) = q_t^{S_n}(\bar{\beta})$ with $n \in \{1,2\}$ the share of i agents in school S_n and thus $q_{t,j}^{S_n}(\bar{\beta}) = 1 - q_t^{S_n}(\bar{\beta})$ the share of j agents going to S_n, considering $j1$ and $j2$ agents together as part of the same ethnic or socio-economic group. Call $\bar{q}_t$ the vector of these shares in both schools. Consequently we have that

$$q_t^{S_1} = \frac{\beta_t^i p}{\beta^i p + \beta_t^{j1}(N-p) + (1-\beta_t^{j2})(1-N)}$$

$$q_t^{S_2} = \frac{(1-\beta_t^i)p}{(1-\beta^i)p + (1-\beta_t^{j1})(N-p) + \beta_t^{j2}(1-N)}$$

Agents are endowed with an utility function defined as:

$$U_{t,h}^{S_n} = U(q^{S_n}(\bar{\beta}_t)|p, N, \tau, z^h), \forall h \in \{I, J1, J2\}, \forall n \in \{1,2\}, \forall t \tag{2.1}$$

We suppose that each agent's utility of attending a given school is increasing in the presence of own group in the school, so that $\frac{\partial U_{t,i}^{S_n}}{\partial q_t^{S_n}} \geq 0$ and $\frac{\partial U_{t,j}^{S_n}}{\partial q_t^{S_n}} \leq 0$. Since these shares depend on

[1]Assuming that all agents have the same preference parameter can be a restrictive assumption. This can be justified by processes of prejudice formation when the whole group has an influence on each agent. Cultural transmission theories of continuous traits may give reason of this hypothesis. In particular each agent forms her preferences during a group influence process at comunity level making a synthesis of all the experiences of agents of own group living in own neighborhood

the vector $\bar{\beta}_t$ we can state that the utility of going in own neighborhood's school is increasing in own group β_t. We can thus set up the following set of assumptions over the utility functions:

Assumption 1: *(Homophily Assumption) Assume that* $\frac{\partial U_{t,h}^{S_n}}{\partial q_{t,h}^{S_n}} \geq 0$.

From assumption 1 and from the structure of the vector $\bar{q}_t$ it follows that $\frac{\partial U_{t,h}^{S_n}}{\partial q_{t,-h}^{S_n}} \leq 0$, $\frac{\partial U_{t,h}^{S_h}}{\partial \beta_t^h} \geq 0$ and $\frac{\partial U_{t,h}^{S_{-h}}}{\partial \beta_t^h} \leq 0$.
We now give an example of a possible specification of preferences satisfying Homophily Assumption: we will use this specification only in the examples of the whole paper in order to clarify the different steps of the model.

Example 1: *(Homophilous Preferences I)*
Suppose the utility function to be defined as
$U_{t,h}^{S_h} = \{[q_{t,h}^{S_h} + z^h(1 - q_{t,h}^{S_h})]f(\tau, h, S_h)\}^\alpha$

with $z^h \in [0,1]$ $\forall h \in \{I, J1, J2\}$, $\alpha \in (0,1)$, *with* $f(\tau, h, S_h) = 1$ *and* $f(\tau, h, S_{-h}) = \tau \in (0,1)$, $\forall h \in \{I, J1, J2\}, \forall n \in \{1,2\}, \forall t$. *Consider the* $f(.)$ *function: this means that going to a school in another neighborhood with respect to the residential one imposes a cost. In this case* τ *does not directly represent the cost but is inversely related to it. This specification implies that agents derive positive utility from having both kind of agents in their school, in relation to their share in the school. However they show a certain homophily degree since* $z^h \in [0,1]$. *This means that they always prefer having a higher share of own group agents than higher share of opponent's agents.*

With a little abuse of notation call β_t^h the share of agents of the same group of agent h going to own neighborhood school, and $\bar{\beta}_t^{-h}$ the vector of shares of agents of different groups than h agent going to own neighborhood school, call S_h the school in the same neighborhood of agent h and S_{-h} the school in the opposite neighborhood. Thus, given the monotonicity involved by Assumption 1, for each group there exists a $\beta_t^{h*}(\bar{\beta}_t^{-h}|p, N, \tau, \bar{z}) : U_{t,h}^{S_h} = U_{t,h}^{S_{-h}}, \forall h \in \{I, J1, J2\}$. If $\beta_t^{h*} \in (0,1)$ then any $\beta_t^h \in (0, \beta_t^{h*})$ cannot be sustained in equilibrium and agents act such that $\beta_t^h = 0$ so that this is the group outcome in equilibrium, while if $\beta_t^h \in (\beta_t^{h*}, 1)$ agents' choices is such that $\beta_t^h = 1$ will be the final group outcome for the opposite reason; if $\beta_t^i = \beta_t^{h*}$ then this can be sustained at equilibrium because of the indifference between the two schools, but the social outcome remains unstable since very little deviations from this will make the final outcome being $\beta_t^h = 0$ or $\beta_t^h = 1$. If $\beta_t^{h*} < 0$ then it always happens that $U_{t,h}^{S_h} > U_{t,h}^{S_{-h}}$ so that in equilibrium $\beta_t^h = 1$. If $\beta_t^{h*} > 1$ then it always happens that $U_{t,h}^{S_h} < U_{t,h}^{S_{-h}}$ so that in equilibrium $\beta_t^h = 0$.
We now summarize this by defining as R_t^h the set of possible social outcomes as a function of $\beta_t^{h*}(\beta_t^{-h})$.

$$R_t^h = \begin{cases} \{0, \beta_t^{h*}(\beta_t^{-h}), 1\} & if \beta_t^{h*}(\beta_t^{-h}) \in [0,1] \\ \{0\} & if \beta_t^{h*}(\beta_t^{-h}) > 1 \\ \{1\} & if \beta_t^{h*}(\beta_t^{-h}) < 0 \end{cases}$$

Thus it is possible to define the Nash Equilibrium of the game:

Definition 1: *A Nash Equilibrium of the School Choice Game is defined as a triple* $(\beta_t^i, \beta_t^{j1}, \beta_t^{j2}) \in [0,1]^3 : \beta_t^h \in R_t^h, \forall h \in \{i, j1, j2\}$. *Define an Internal Nash Equilibrium as a Nash Equilibrium of the School Choice game such that* $(\beta_t^i, \beta_t^{j1}, \beta_t^{j2}) \in (0,1)^3$.

Thus, in equilibrium, it is impossible that in each group part of the agents stay in own neighborhood's school and part to opposite neighborhood's school.
Starting from this definition we can state that:

Proposition 1: *The School Choice game does not admit any Internal Nash Equilibrium.*

Proof. In Appendix B. □

Now, since from the data we observe that segregation in school is higher than residential segregation, then we expect the equilibrium to have, in S1, an overrepresentation of i agents with respect to their population share in the neighborhood, and in S2 an underrepresentation of i agents with respect to their population share in the neighborhood. Since i agents are not represented in N2, then there should not be i students in S2 and this happens if and only if all i agents stay in S1. As a consequence $j2$ agents have no incentive to go to S_1 and thus they would stay in S_2. Moreover it is more reasonable to observe $j1$ agents moving to S_2 and $j2$ staying in S_2 than the reverse since $j1$ agents belong to a more diverse neighborhood and they like to go to a school in which more agents are of their own type. In particular it is more likely that agents that are relatively less represented in a neighborhood move away than the same for more represented agents. In order to follow these intuitions we make the following assumption:

Assumption 2: *Assume that* $\beta_t^{j2} = 1, \forall t$, *and* $\beta_t^i = 1, \forall t$.

Given Assumption 2, figure 2.2 represents a possible situation for $j1$ agents. It has to be noticed that, because of Assumption 2, in S2 there are only j type agents, so that the utility of $j1$ agents of going in S2 is independent from the level of β_t^{j1} and thus is a constant.

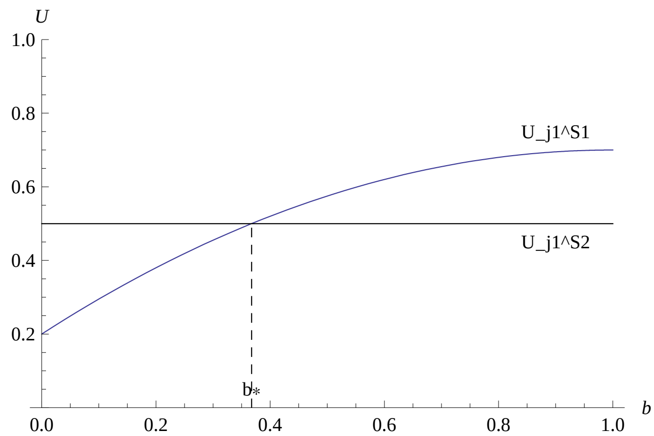

Figure 2.2: Threshold

Thus, depending on the parametrization, we can have that $\beta^{j1*} \in [0,1]$ or not. In figure 2.2 it is represented exactly this case. Depending on the position of β^{j1*}, then different equilibria may be shown. Studying the properties of the threshold we can then state the following:

Proposition 2: *If assumptions 1 and 2 hold, β_t^{j1*} is increasing in p and decreasing in τ. Then, depending on the parametrization, there could exists a p^* such that if $p > p^*$ full segregation always happens.*

Proof. In Appendix B. □

Proposition 2 states that the threshold β_t^{j1*} is increasing in the population of i agents and decreasing in the cost parameter. Thus, if school integration happens with $\beta_t^{j1} = 1$ and segregation with $\beta_t^{j1} = 0$, the basin of attraction of integration is higher for higher cost parameters. Since cost parameters may also be influenced by policy choices, this introduces a first element of impact of the policy on the possibility of having integrated or segregated schools. On the other side the basin of attraction for integration outcomes reduces if i population increases, since β_t^{j1*} increases. For some parametrization levels it could be that, making p higher makes $\beta_t^{j1*} > 1$. Thus it could be that not all levels of i agents population may sustain an integrationist action by $j1$ agents. This captures the idea that demographic conditions may strongly affect the integrationist outcome possibility and in some cases preclude integration possibilities.

Example 2: *(Homophiluos Preferences II)*
We continue the example of homophilous preferences introduced above. Suppose that preferences take the described functional form, assumption 2 holds and, for semplicity, $N = \frac{1}{2}$. Thus we have that $q_i^1 = \frac{p}{p+\beta_t^{j1}(\frac{1}{2}-p)}$ and $q_i^2 = 1$. In order to solve the indifference problem between the two schools we set the two utilities equal and we write $(\frac{p}{p+\beta_t^{j1}(\frac{1}{2}-p)}z^j + \frac{\beta_t^{j1}(\frac{1}{2}-p)}{p+\beta_t^{j1}(\frac{1}{2}-p)}) = \tau$ so that $\beta^{j1*} = \frac{2p(1-\tau)}{(\tau-z^j)(1-2p)}$. Now, since optimal choices depend whether $\beta^{j1*} \in [0,1]$ or not, then we have the following situations:

- *if $\tau \leq z^j$ then $\beta^{j1*} < 0$ and thus $\beta_t^{j1} = 1$ so that all j1 agents go to S1 and full integration happens*
- *if $\tau > z$ and $p > \frac{1-\tau}{2(1-z^j)}$ then $\beta^{j1*} > 1$ and so that $\beta_t^{j1} = 0$ and full segregation happens*
- *if $\tau > z$ and $p \leq \frac{1-\tau}{2(1-z^j)}$ then $\beta^{j1*} \in [0,1]$ and thus we can have $\beta_t^{j1} = \beta^{j1*}$, $\beta_t^{ji} = 0$ or $\beta_t^{j1} = 1$.*

Consider now the threshold $p = \frac{1-\tau}{2(1-z^j)}$, and substitute it into the formula for $q_t^{S_1}$ so that we get $q_t^{S_1} = \frac{1-\tau}{1-z^j}$. This depends only on j interethnic preferences and on the transportation cost. Figure 2.3 helps in understanding this situation.

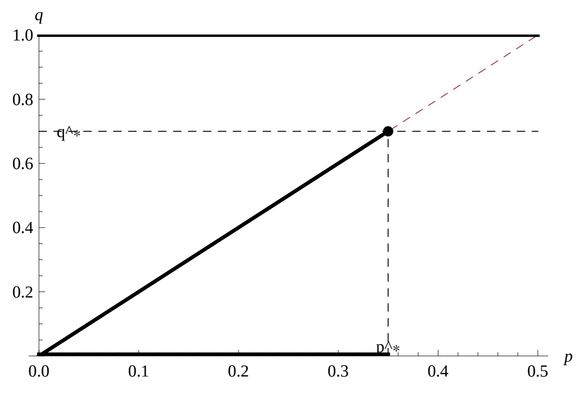

Figure 2.3: Example

On the x axis there is the population share p and on the y axis the induced level of $q_t^{S_1}$. Call $q^* = \frac{1-\tau}{1-z^j}$ and $p^* = min\{\frac{1-\tau}{1-z^j}, \frac{1-\tau}{2(1-z^j)}\} = \frac{1-\tau}{2(1-z^j)} < \frac{1}{2}$. Thus, if $p < p^*$ then it is possible to observe full integration, fulls segregation or partial segregation on β^* even if this last outcome is unstable. If the the share of i agents grows and is such that $p > p^*$ then only full segregation can be observed. Moreover, the maximum level possible of partial segregation is given by q^*. Thus, suppose that p is growing in time. Then, as far as $p < p^*$ we can still observe integration. Then, if p grows too much, we have a tipping point such that when $p \geq p^*$ we start observing segregation. Thus, if p is growing, independently from past integration or segregation outocomes, whenever $p > p^*$ segregation always happens. Considering the threshold $\frac{1-\tau}{2(1-z^j)}$, we have that the higher the cost (so the lower τ) the higher the threshold above which segregation happens. On the contrary the higher the preference parameter the lower the threshold. This simple model thus may help explaining such non smooth dynamics in segregation and can justify the idea that, above some population threshold in a neighborhood, integration is no more possible given the set of parameters.

2.2.1 Integrationist Policies as a Matter of Coordination

In this subsection we propose the first steps for a welfare analysis. A possible interpretation of proposition 2 derives from the fact that if $\beta_t^{j1*} \in (0,1)$ school segregation or integration is also a matter of agents coordination other than pure parmetrization. With coordination we do not mean that agents rationally plan to coordinate on $\beta < \beta^*$ or $\beta > \beta^*$, but that they happen to be casually coordinated on these values. This means that, if $\beta^{j1*} \in [0,1]$, so that if both $\beta_t^{j1} = 1$ and $\beta_t^{j1} = 0$ are possible, then the final equilibrium depends on wether enough $j1$ agents coordinate on one action. If less that β_t^{j1*} agents coordinate on going to S_1, then in equilibrium everyone would go to S_2, and the opposite happens for a coordination level higher than β_t^{j1*}. This may induce some interpretations over the integration policies. In particular the role of free choice policies and, on the other side, the role of obliged school choice can be interpreted. From the time being the model can only say if a no free choice policy is sustainable (meaning that no one would voice against it), while a free choice policy is suposed to be always sustainable. We focus on policies that impose to $j1$ agents to go to S_1, since these are the usual implemented policies. Suppose that $\beta^{j1*} < 0$: in this case all $j1$ agents would prefer going to S_1 and thus a no free choice policy that oblige every agent to go to own neghborhood school is not effective since agents by themselves would go to the prescribed school and thus the policy would not be binding. If, on the contrary, $\beta_t^{ji*} > 1$ then all $j1$ agents would prefer to go to S_2. In this case a policy that obliges to go to the closest school is not sustainable since $j1$ agents would voice against it since they would loose utility. Moreover, i agents would voice against this policy too since they would prefer not having $j1$ agents in their school, while $j2$ agents would be totally indifferent since their utility is the same under both policies. Thus this policy cannot be sustained and, more importantly, it does not help to sustain coordination on an integrationist outcome since agents would change choice as far as the policy is abandoned. If we then consider the case in which $\beta_t^{J1*} \in [0,1]$, then this policy may help $j1$ agents to coordinate on $\beta_t^{j1} = 1$: once this policy is implemented, in fact, no $j1$ agent has incentive to break the rule. Moreover it is also welfare improving for them since $U_{t,j1}^{s_1}(\beta_t^{j1} = 1) > U_{t,j1}^{s_2}(\beta_t^{j1} = 0)$. Still, i agents would not be bettered off with with this policy since they would awlays prefer a segregationist outcome. Thus it is impossible to rank, with respect to the welfare, segregationist and integrationist outcomes, and thus the related policies. Thus we can state that:

Proposition 3: *A no free choice policy may be implemented only if $\beta_t^{j1*} \in [0,1]$. Moreover* *it is impossible to Pareto rank free choice and no free choice policies.*

Proof. Directly derived from previous discussion. □

All this discussion also makes clear that the role of a non free choice policy can also be a coordination role: if for some reason the policy maker thinks that integrated schools are better then it can implement this law without $j1$ protests only if the condition of proposition 3 is satisfied. Under this condition, if the policy is implemented, then agent would continue on this integration road since the policy maker conducted them on a integration path they deive higher utility from.

2.3 The Dynamics

2.3.1 A Standard Dynamics

Integration and segregation processes are, by their nature, dynamic processes since, given some initial conditions, agents choose to imitate some successful choices and, in the long run, this process ends up with full segregation, full integration or some intermediate results. In order to analyse these cases we first propose a simple dynamic model that reproduces, in the long run, the outcome of the static model so that we can also analyse what happens if, during the process, some change occurs. Then, in the next sections, we study the conditions on the dynamic model under which partial segregation outcomes may be observed.
Consider the share β_t^{j1} to be the variable to be described by the dynamic process. We consider the case in which $j1$ agents imitate the school choices of the agents of their own group that performed better in terms of utilities. However, given some frictions, imitation does not happen for everyone immediately. Suppose for the time being that all the rest of the parameters stay fixed. Call $\Delta_t U_t^{j1} = U_{t,j1}^{S_1} - U_{t,j1}^{S_2}$. We thus have that a generic dynamics is represented by equation 2.2:

$$\dot{\beta}_t^{j1} = f(\Delta_t U_t^{j1}) \tag{2.2}$$

This generic dynamics form states that if $U_{t,j1}^{S_1} > U_{t,j1}^{S_2}$ then β_t^{j1} grows since some $j1$ agents will go to S_1, while if $U_{t,j1}^{S_1} < U_{t,j1}^{S_2}$ the reverse happens. Notice that β^{j1*}depends only on fixed parameters, so that it is time independent and represents an equilibrium in the dynamics. If $\beta_t^{j1*} \geq 1$ then it always happens that $U_{t,j1}^{S_1} < U_{t,j1}^{S_2}$ so that, for any initial $\beta_0^{j1} \in (0,1)$ convergence to $\beta^{j1} = 0$ is ensured. On the opposite side, if $\beta_t^{j1*} \leq 0$ then it always happens that $U_{t,j1}^{S_1} > U_{t,j1}^{S_2}$ so that, for any initial $\beta_0^{j1} \in (0,1)$ convergence to $\beta^{j1} = 1$ is ensured. Suppose now that $\beta_t^{j1*} \in (0,1)$. A generic dynamics like this has, thus, the qualitative representation as in figure 2.4. In this case, depending on the inital condition convergence to $\beta^{j1} = 0$ or $\beta^{j1} = 1$ is always possible.

We can thus state that:

Corollary 1: *If a dynamics is expressed as in equation 2.2, then any $\beta \in (0,1)$ can never be a stable steady state.*

This result simply replicates the static one and thus confirm that, for the time being, only full segregation or full integration may be explained as final outcomes of a dynamic process.

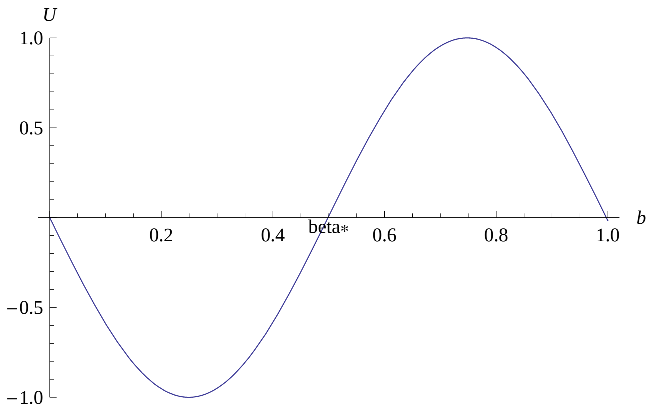

Figure 2.4: Generic Dynamics

Example 3: *(The Replicator Dynamics)*
An usual way in order to model dynamics is to use a replicator dynamics model that can be interpreted as a way in which agents imitate the most sucessfull action performed in the previous period. In particular we write:

$$\dot{\beta}_t^h = \beta_t^h[U_{t,h}^{S_h} - (\beta_t^h U_{t,h}^{S_h} + (1-\beta_t^h)U_{t,h}^{S_{-h}})] = \beta_t^h(1-\beta_t^h)[U_{t,h}^{S_h} - U_{t,h}^{S_{-h}}].$$

Considering thus all the three dynamics (suppose that assumption 2 does not hold), and the consequent endogenous shares of agents in each school we have the following system of differential equations

$$\begin{cases} \dot{\beta}_t^i = \beta_t^i(1-\beta_t^i)[U_{t,i}^1 - U_{t,i}^2] \\ \dot{\beta}_t^{j1} = \beta_t^{j1}(1-\beta_t^{j1})[U_{t,j1}^1 - U_{t,j1}^2] \\ \dot{\beta}_t^{j2} = \beta_t^{j2}(1-\beta_t^{j2})[U_{t,j2}^1 - U_{t,j2}^2] \\ with \\ q_t^{S_1} = \frac{\beta_t^i p}{\beta_t^i p + \beta_t^{j1}(N-p)} \\ q_t^{S_2} = \frac{(1-\beta_t^i)p}{(1-\beta_t^i)p + (1-\beta_t^{j1})(N-p) + (1-\beta_t^{j1})(1-N)} \end{cases}$$

Suppose again for semplicity that $N = \frac{1}{2}$ and that the utility functions are represented as in example 1.
We do not report here all the 17 feasible Steady States of this dynamic system. Given this we can state that it does not exists a Steady State in which the vector $(\beta_t^i, \beta_t^{j1}, \beta_t^{j2}) \in (0,1)^3$, meaning that at least one group has all agents doing the same choice. All the Steady States in which $\beta_t^i \in (0,1)$ or $\beta_t^{j1} \in (0,1)$ or $\beta_t^{j2} \in (0,1)$ are not stable meaning that every stable steady state has all agents in each group performing the same choice. Moreover the set of the NE of the static case is a subset of the set of the SS of the dynamic model (and this is not new from a theoretical point of view). There exists only one SS in which $\beta_t^{j1} \in (0,1)$. This is unstable, and coincides with the equilibrium found in the static model.

2.3.2 A Less Segregated City

One may imagine that these results are peculiar to the case of a very segregated city in which a group i lives only in N1. We now see what happens to the dynamics if we allow agents i

to live in both neighborhoods, in order to have a less segregated city. We can represent the residential segregation as in figure 2.5.

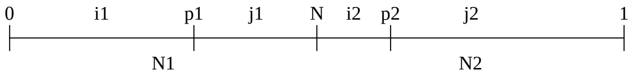

Figure 2.5: Less Residential Segregation

Given this situation all the previous definitions of utilities, βs and all the other parameters are changed in notation just to allow one more group. In this case it still holds the fact that there are not internal nash equilibria for the very same reason as in the more segregated case. In particular it would be incompatible with incentives that $\beta_t^{i1} \in [0,1)$ and $\beta_t^{j1} \in [0,1)$, and in the same way it would be impossible to have in equilibrium $\beta_t^{i2} \in [0,1)$ and $\beta_t^{j2} \in [0,1)$ so that in each neighborhood, agents of at most one group can choose a school in the different neighborhood. Thus we can suppose again that Assumption 1 holds: in this framework this means that there are two groups able to freely choose the school: $j1$ and $i2$ agents. Suppose that i agents are relatively more represented in the first neighborhood while j agents are relatively more represented in the second neighborhood: then assumption 1 means that in each neighborhood moves only the group that is relatively less represented in that neighborhood.

Again in this case the set of possible social outcomes are unchanged with respect ot the previous case and thus the same formulation for R_t^h holds. Thus, given that assumption 1 holds, we can define a Nash Equilibrium for this case as a couple $(\beta_t^{i2}, \beta_t^{j1}) \in [0,1]^2 : \beta_t^h \in R_t^h, \forall h \in \{i2, j1\}$. Again an Internal Nash Equilibrium is defined when there exist a Nash Equilibrium such that $(\beta_t^{i2}, \beta_t^{j1}) \in (0,1)^2$.
Turning now to the dynamic case suppose that $\dot{\beta}_t^{j1} = f(\Delta_t U_t^{j1})$ and $\dot{\beta}_t^{i2} = f(\Delta_t U_t^{i2})$, thus having the same properties of the previously described dynamics. From this it follows that, depending on the parametrization, it may exists a $\bar{q}_t^{S_1}$ and a $\bar{q}_t^{S_2}$ such that $\dot{\beta}_t^{j1} = 0$ with $\Delta_t U_t^{j1} = 0$ and, equivalently, a $\underline{q}_t^{S_1}$ and a $\underline{q}_t^{S_2}$ such that $\dot{\beta}_t^{i2} = 0$ with $\Delta_t U_t^{i2} = 0$. Thus $\bar{q}_t$ and $\underline{q}_t$ represent the shares in each school that make agents indifferent between the two schools. Take first the case of $j1$ agents. This will induce a $\beta_t^{j1*}(\beta_t^{i2})$ such that $\dot{\beta}_t^{j1} = 0$. In particular, given monotonicity properties of the utility functions it must be that $\beta_t^{j1*}(\beta_t^{i2})$ is decreasing in β_t^{i2}. In the same way there exists a $\beta_t^{i2*}(\beta_t^{j1})$ such that $\dot{\beta}_t^{i2} = 0$, with $\beta_t^{i2*}(\beta_t^{j1})$ being decreasing in β_t^{j1}.
In figure 2.6 we give a graphical representation of the space and the involved dynamics depending on the relative slopes of the two lines. On the x axis we represent $\beta_t^{j1*}(\beta_t^{i2})$, and on the y axis $\beta_t^{i2*}(\beta_t^{j1})$

In each case β_t^{j1} is increasing above the $(\beta_t^{j1}(\beta_t^{i2})$ line and decreasing below and, in the same way, β_t^{i2} is increasing above the $(\beta_t^{i2}(\beta_t^{j1})$ line and decreasing below. The intersection of the two lines represents the internal Nash Equilibrium of the static game and we call it $(\beta^{j1*}, \beta^{i2*})$. However, we cannot be sure that $(\beta^{j1*}, \beta^{i2*}) \in (0,1)$ since this strictly depends on the parametrization as in the previous case. However, if an internal equilibrium exists then it is unstable, as shown above. Thus we can state:

Proposition 4: *If residential segregation in each neighborhood is such that both types of agents are present and assumption 1 holds, then a vector* $(\beta^{j1}, \beta^{i2}) \in (0,1)^2$ *can never be a stable steady state. If the static game admits an internal nash equilibrium then this is an unstable steady state.*

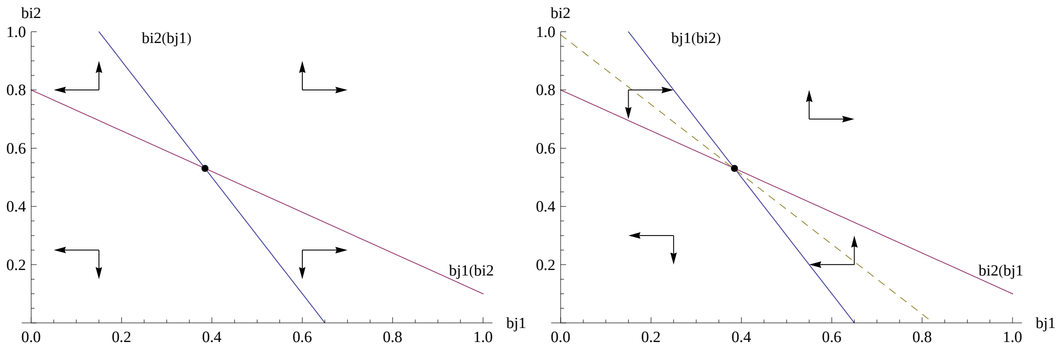

Figure 2.6: Less Residential Segregation Dynamics

Proof. Directly derives from previous discussion. □

Notice that in this case we can have some intermidiate results other than full integration or full segregation. In particular if the vector $(\beta^{j1}, \beta^{i2}) = (0, 1)$ or $(\beta^{j1}, \beta^{i2}) = (1, 0)$ we observed that neither full integration nor full segregation happens. In particular in the first case we observe S_1 to be fully segregated and S_2 to be fully integrated, and the opposite in the second case. These are however extreme results, so that in the following section we provide some conditions over the dynamic in order to produce partial segregation results, usually observed in the data.

2.4 Partial Segregation Equilibria

2.4.1 Endogenous Preferences

In the previous sections we analysed simple dynamics in which the unique possible results are full segregation or full integration. However, in real data intermediate results with partial segregation are observed. This section analyses this case, with a possible alternative explanation introduced in Appendix A.
In terms of our model partial segregation means having a convergence of the dynamics to a value of $\beta^{j1} \in (0, 1)$. In what follows we try to derive some conditions such that the dynamics converges to such a point. First of all we can summarize the previous findings with the following corollary:

Corollary 2: *Given a residential segregation setting, if Assumption 1 holds, utilities are represented as in equation 2.1, the dynamics as in equation 2.2, then if the vector* $(p, N, \bar{z}, \tau)$ *is time independent partial segregation outcomes are impossible to be observed.*

In the following analysis we consider only the first segregation model in which i agents only live in N_1. In the previous sections we argued why Assumption 1 can be reasonable, so that we do not question it here in order to produce partial segregation. In the same way we do not question demographic structure parameters, so that we do not consider changes in p and N: even if it is of certain relevance the fact that changes in demographic variables change the integration properties of possible long run equilibria, the goal of this paper is to consider when partial segregation can happen even if the demographic structure does not change. In the same way we do not consider changes in cost structure also because, given their monotonic effect on the equilibrium thresholds, their effect is straightforward.

On the other side, the idea that preferences stay fixed along the whole integration/segregation process may seem quite unreasonable: preferences may change due to the exposure to one group an thus we analyse what happens if the vector $\bar{z}$ is now time dependent: for this reason we now refer to this vector as $\bar{z}_t$. Suppose for the time being that assumption 1 holds, so that we only care of agents' $j1$ preferences. Notice that if $\bar{z}_t$ is time depedent, now also $\beta^*(\bar{z}_t)$ is time dependent sicne it depends on preferences. From now on we refer to it as β^*_t.
In what follows we do not impose exogenously a functional form for the dynamics, as in Bisin and Verdier framework, but we derive necessary condition that any preference dynamcis must satisfy in order to observe partial segregation outcomes.

In our case preferences are group and neighborhood dependent, meaning that the preference formation process happens at the group level in each neighborhood. Thus the level of preferences in each group is affected by the school choice and school composition of each agent of own group in each neighborhood. We consider two types of effects on preferences: the ***School effect*** and the ***Residential effect***. With School Effect we mean the effect that a change in school composition has on preferences, thus the effect that q_t has on preferences, other things being equal. Suppose that β_t is unchanged but to have an inflow of type i agents. In this way $q^{S_1}_{t,j1}$ is lowered, while $q^{S_2}_{t,j1}$ is unchanged. Then how preferences are affected by this? The School Effect captures the change in preferences due to the change in the share of own agents in a given school. We can in fact suppose that if own group has a higher share in a school, other things being equal, then agents can feel more protected and thus may become less racist, so that z_t may increase. Or the opposite may happen since a higher share of own group agents may make agents more self-confident and make them become more racist so that z_t grows.
The Residential Effect captures the effect on preferences due to the fact that a higher share of own group agents living in own neighborhood are in touch with the other type, being the shares in each school equal. When preferences are formed at neighborhood level, if more agents have been in touch with other type's agents then this can have an effect on intergroup preferences. Suppose for example that there is an i agents immigration inflow and contemporarily a growth in β such that $\bar{q}_t$ is unchanged. What is the effect on preferences due to the fact that β is higher so that more $j1$ people are in touch with i people without any change in exposure intesity given that $\bar{q}_t$ is constant? This Residential Effect has some links with Contact theory that states that the more two groups are in contact each other, the more their cross preferences become more tolerant, so that making two groups in touch is an essential element in promoting integration. Following contact theory ideas the more two groups are in touch the more they should grow tolerant, so that in this case this effect should be positive. On the other side, if contact theory is wrong then making two groups more in contact makes racism grows.
We can clarify the two effects by saying that the school effect captures the intensity of the each interethnic contact, while the residential effect captures the numbers of these contacts. We now analyse how these two effects are operative in the current theoretical framework in order to have an idea of the involved dynamics. Suppose that $\beta_t > \beta^*_t$. If this is the case then β_t grows and, as a consequence, $q^{S_1}_{t,j1}$ grows as well. Thus preferences are affected both by a change in $\bar{q}_t$ and by a change in β_t so that the School Effects and the Residential Effect act together.
Suppose that School Effect and Residential Effect act in the same direction and are both positive, so that the preference dynamics is consistent with contact theory predictions. If this is the case then agents become more tolerant, so that z_t grows. As a consequence even more agents will enter the S_1 school so that β_t dynamics is always increasing and thus the system would end up to perfect integration. If the system starts with $\beta_t < \beta^*$ then the same happens and the system would end up with $\beta = 0$. Thus in this case it would not be possible

to observe partial segregation outcomes. In order to make this result more precise and to go beyond the pure intuition, we make the values of preferences endogenous in the model in particular imposing that:

$$z_t^{j1} = z(\bar{q}_t(\beta_t^{j1}), \beta_t^{j1}) \tag{2.3}$$

with the $\frac{\partial z_t^{j1}}{\partial q_t^{S_1}}$ being the *School Effect* and $\frac{\partial z_t^{j1}}{\partial \beta_t^{j1}}$ being the *Residential Effect.* Given this generic specification we can state the following:

Proposition 5: *Given a residential segregation setting, if Assumption 1 holds, utilities are represented as in equation 2.1, the dynamics as in equation 2.2, then if the vector* (p, N, τ) *is time independent and preferences dynamics is represented as in equation 2.3, a necessary condition in order to observe a partial segregation outcome is* $\frac{\partial z_t^{j1}}{\partial \beta_t^{j1}} < -\frac{(N-p)p}{(\beta_t^{j1}(N-p)+p)^2} \frac{\partial z_t^{j1}}{\partial q_t^j}$.

Proof. In Appendix B. □

Notice that $\frac{(N-p)p}{(\beta_t^{j1}(N-p)+p)^2} > 0$. Thus proposition 5 states that, if previous conditions hold, in order to have convergence to an internal steady state it is impossible that both effects are positive. Suppose first that $\frac{\partial z_t^{j1}}{\partial q_t^j} > 0$ so that School Effect is positive: this can happen since increasing own type agents in a group can make an agent more self confident so that it does not increase her racism levels. If this is the case then, in order to have an internal steady state, it is necessary for the Residential Effect to be negative and 'negative enough'. If on the contrary School Effect is negative so that $\frac{\partial z_t^{j1}}{\partial q_t^j} < 0$ then Residential Effect should be negative or positive (but not too large). We can state this differently by saying that if contact theory is right (positive Residential Effect) then School Effect must be negative so that racism grows with own presence in a school. Proposition 5 does not state anything about conditions for obtaining full segregation or full integration; we can get both these extreme results from having necessary conditions in propostion 5 fullfilled or not, depending on the initial conditions. Under this condition we always have that $\frac{dz_t^{j1}}{d\beta_t} < 0$, so that the necessary condition in proposition 5 is sufficient in order to have an increase in homophily if β_t^{j1} increases.
The main idea behind propostion 5 is that, in order to have convergence to a internal steady state, if $\beta_t > \beta_t^*$ then if β_t grows it should be that β_t^* grows more: the threshold is now time dependent since, depending on preferences z_t^{j1} it now changes in time with $\frac{\partial \beta_t^{j1*}}{\partial z_t^{j1}} < 0$. Figure 2.7 graphically explains this process.

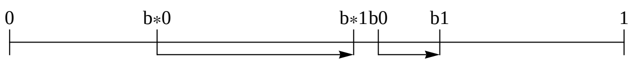

Figure 2.7: Partial Segregation

If the threshold is now time dependent and its dynamcs has the same sign as the dynamics of the β_t then, during the dynamics, it can grow so much until it reaches β_t at some point in time. If this happens then $\beta_t = \beta_t^*$ and this represent a steady state of the process. The very same thing happens for an initial condition such that $\beta_0 < \beta_0^*$ with a negative dynamics. Notice, however, that this is only a necessary condition since, if the process start too close to 1 or 0 then full segregation or integration is reached before the described process may take place.

The process described above may produce partial segregation outcomes even if not all agents in a group satisfy the necessary condition in proposition 5. Suppose in fact that agents are heterogenous in the way in which their preferences react to school composition dynamics, meaning that not all agents have $\frac{dz_t^{j1}}{d\beta_t} < 0$. Suppose that $j1$ agents are divided in two groups, group A such that $\frac{dz_t^A}{d\beta_t} > 0$ and group B such that $\frac{dz_t^B}{d\beta_t} < 0$, so that A agents do not satisfy proposition 5 necessary condition while B agents do. Suppose that $w^A = w$ is the share of A agents in $j1$ group and $w^B = 1 - w$ the share of B agents. Then

$$\beta_t^{j1} = w\beta_t^A + (1-w)\beta_t^B$$

so that

$$\dot{\beta}_t = w\dot{\beta}_t^A + (1-w)\dot{\beta}_t^B.$$

thus in steady state it must be:

$$\dot{\beta}_t^B = -\frac{w}{1-w}\dot{\beta}_t^A$$

so that the forces of the dynamics of the two groups counterbalance. Notice that, for the time being, it is not given that in steady state $\dot{\beta}_t^B = \dot{\beta}_t^A = 0$. Given the structure of the reaction, agents in group A are such that their preference change always go in the direction of fostering the actual dynamics while B agents preferences change such that they always reduce the speed of this dynamics. Suppose that at the first period $\beta_0^* < \beta_0$, then A agents preference dynamics is such that they will reach a steady state in which $\beta^A = 1$ since their preference change do not satisfy proposition 5. However in steady state it should also be that $\dot{\beta}_t^B = 0$, and a necessary condition for this to happen in a $\beta_t^B \in (0,1)$ is the one in proposition 5. The same happens if $\beta_0 > \beta_0^*$. Consequently in both cases the B agents are the ones who are potentially present in both schools. Notice that even if $z_0^A = z_0^B$, the two preference levels become different during the process. Suppose again that $\beta_0^* < \beta_0$. In this case in steady state $z^A > z^B$ because of the different dynamics of preferences: A preferences always grow, while B preferences stop or grow less rapidly. Now if the an internal steady state is reached then in S_1 there will be all A agents and part of B agents, the first one having become more tolerant than the second ones. Thus in S_1 there will be a mixed of tolerant and intolerant agents, while in S_2 only intolerant are present. This makes clear that it may not be true that A agents go to S_1 because of their higher tolerance levels while part B agents go to S_2 because they are intolerant, since this is just the steady state outcome; the process is reversed such that A agents, given their way of reacting, tend to go to S_1 and his makes them more tolerant, while the opposite happens for B agents. Thus, the main driving force of segregation or integration outcome is the way in which preferences react to the dynamic process itself. Suppose now that $\beta_0^* > \beta_0$: in this case we observe the opposite dynamics since, if a polimorphic equilibrium is reached, then $z^A < z^B$. In particular we observe in S_1 only part of B agents, now tolerant, and in S_2 a mixed of tolerant B agents and intolerant A agents. Thus if agent heterogeneity is admitted and a partial segregation equilibrium is observed then, if the process drove towards more segregation we observe tolerant agents in S_1 and a mix of tolerant and intolernat in S_2, while if the process drove towards integration we observe a mix of tolerant an intolerant in S_1 and only intolerant agents in S_2.

Example 4: *(Endogenous Preferences)*
We now provide an example of a dynamics of β driven to stability by endogeneity of prefer-

ences. Suppose, for simplicity, that $N = 1/2$ and that utility functions are specified as in the previous examples. Again suppose that Assumption 1 holds so that only the case of $j1$ agents is analysed.

In what follows we propose a new way of analysis: instead of analysing changes in β^* we analyse changes in τ^*. We define τ_t^* as the level of τ such that the utility of going in the two schools is equal. Thus $U_{t,j1}^{S_1} > U_{t,j1}^{S_2}$ if and only if $\tau < \frac{\beta-2p\beta+2pz}{2p+\beta-2p\beta} \equiv \tau^*$. In a dynamics system we must have the time indeces so that we have $\tau_t^*(\beta_t)$. For the time being consider parameters z fixed in time. Thus we have that

$$\dot{\tau_t^*} = \frac{2p[(1-2p)(1-z)\dot{\beta}_t+(2p+\beta-2p\beta)\dot{z}]}{(2p+\beta_t-2p\beta_t)^2}.$$

Now, assume that $\frac{\partial \tau_t^*}{\partial \beta_t} > 0$: then $\tau < \tau_t^*$ then $U_{t,j1}^{S_1} > U_{t,j1}^{S_2}$ so that $\dot{\beta} > 0$ and thus $\dot{\tau}_t^* > 0$. In this way, at time $t+1$ the unequality $\tau < \tau_{t+1}^*$ is satisfied at a higher degree so that the process self sustains until $\beta = 1$. The same happens if $\tau > \tau_t^*$ and convergence to $\beta = 1$ happens. Thus if $\frac{\partial \tau_t^*}{\partial \beta_t} > 0$ we always observe full integration or full segregation.

Thus a necessary condition for convergence to a $\beta \in (0,1)$ is that $\frac{\partial \tau_t^*}{\partial \beta_t} > 0$. If population share p is fixed this is possible if and only if z is time dependent. Thus the only way to have the condition satisfied is

$$\frac{\partial z_t}{\partial \beta_t} < -\frac{(1-2p)(1-z_t)}{2p+(1-2p)\beta_t} \equiv z_t^*.$$

Now z_t^* is bounded below by $-\frac{1-2p}{2p}$ so that it is enough to have

$$\frac{\partial z_t}{\partial \beta_t} < -\frac{(1-2p)}{2p}.$$

Under this rule since $\frac{\partial z_t^B}{\partial \beta_t} < 0$, the necessary condition of proposition 5 is satisfied.

Since z_t only depends on β_t and $z_t \in (0,1)$ a suitable differential equation that satisfies this condition is $\dot{z}_t = z_t(1-z_t)\gamma_t\dot{\beta}_t$ with $\gamma_t < -\frac{1-2p}{2pz_t(1-z_t)}$. Suppose, in order to satisfy this condition, to have $\gamma_t = -\frac{1-2p}{2pz_t(1-z_t)} - \alpha$ with $\alpha > 0$. Then the dynamics can be written as follows:

$$\dot{z}_t = -\dot{\beta}_t(1+\alpha z_t(1-z_t)) \tag{2.4}$$

We thus use this specific dynamics in order to analyse cases of internal steady states.

In what follows we propose two sets of numerical simulations using equation (2.4), with $\alpha = 0.1$, $N = \frac{1}{2}$, $p = \frac{1}{4}$ and $\tau = 0.7$. Figure 2.8 represents 3 cases in which $z_0 = 0.5$ and in turns $\beta_0 = 0.1$, $\beta_0 = 0.3$ and $\beta_0 = 0.95$. In each figure the thick line represents the dynamics of observed β_t and the dashed line the dynamics of z_t.

Given these first simulations we recognize the opposite directions of the paths of z_t and β_t because of the dynamics built in order to satisfy proposition 5. Moreover, since this is a necessary condition, we see that only in some cases we have convergence. In particular if the initial values are such that they are too close to 1 or 0 then it is not possible to arrive to a partial segregation outcome since the countervailing force of the dynamics of z_t should take too much time to invert the dynamics of β.

In the second set of simulations (figure 2.9) we take β_0 equal in all simulations and make initial z_0 values change. In particular we set $\beta_0 = 0.5$ and set $z_0 = 0.5$, $z_0 = 0.2$ and $z_0 = 0.8$. In this way we notice the effect that initial preference values have on the steady state of the

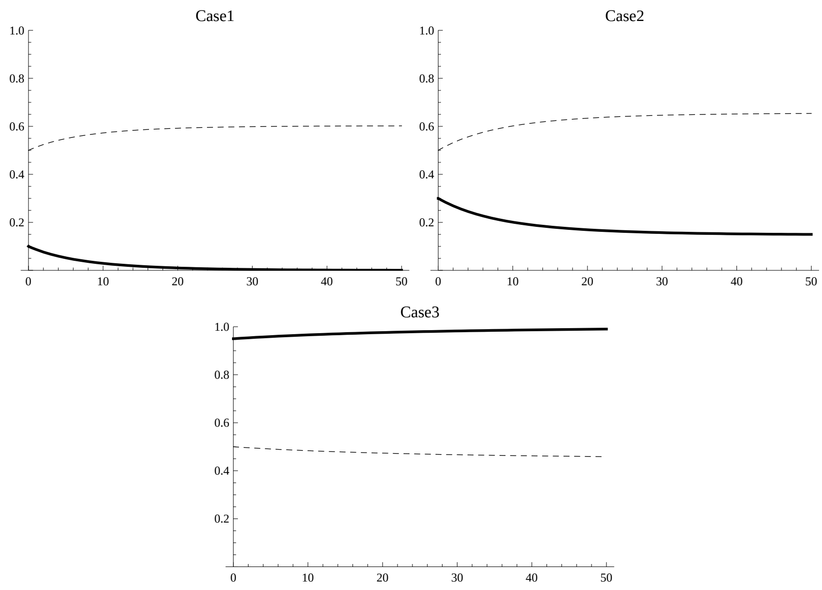

Figure 2.8: Partial Segregation Simulation β_t effect

dynamics. In this case we incidentally have that a partial equilibrium is always observed. Still we have that, given β_0, steady state levels are positively affected by z_0 level. In particual z_0 levels have an impact over the direction of the dynamics. If we look at figure 2.9 we have that in the first two cases ($z = 0.5$ and $z = 0.2$) the dynamics of β_t is declining, while in the third case in which $z = 0.8$ the dynamics is increasing. Thus initial tolerance parameters positively affect the long run integration levels.

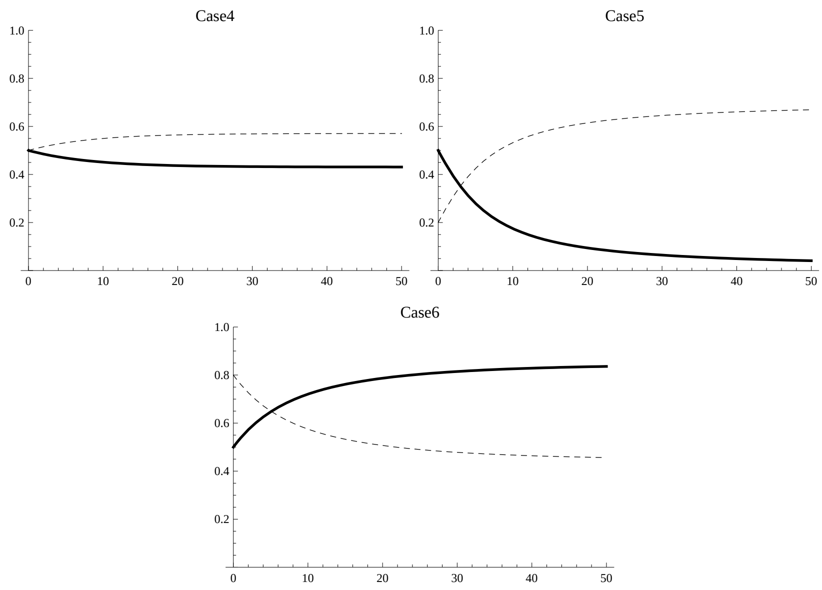

Figure 2.9: Partial Segregation Simulation - z_t effect

2.5 Conclusions

The debate about ethnic segregation in schools rises some issues about the role of homophily based segregation. In this paper we set up a model of spatial segregation in order to better

understand its role. We first analyse a static model in which extreme results are shown but that helps in the analysis and the interpretation of free choice and no free choice policies. We then set up a dynamics model and we find necessary conditions on preferences dynamics that may bring to partial segregation equilibria. In particular we underline the role of School and Residential Effect as two crucial effects on preference changes, and we analyse the conditions under which these effects may bring to observable results.
This study may find interesting extension if some empirical analysis is used in order to better support thiese findings. Moreover some simulations based analysis may be useful in order to understand how different and more complex residential segregation frameworks influence the prefernece dynamics and the long run preferences.

Appendix A: Partial Segregation Outcomes with Socio-Economic Groups

In all the previous analysis we focused on the case in which each group presents a certain degree of homophily so that Assumption 1 holds for each group. This assumption may be considered correct whenever groups identity reasons are such that the willingness of staying with 'own' people prevails over any other factor that may have an adverse impact on utility. Suppose that groups represent ethnicities: if we think that belonging to an ethnic group has some correlations with some socio-economic variables, then this assumption may become incorrect. Suppose, for example, that minority i is also associated with a lower socio-economic status while $j1$ agents are associated with higher status variables. It may be the case that, for example, i agents are immigrants with a lower schooling level or lower income, while j agents are natives with higher socio-economic positions. If agents think that being close to an agents of high socio-economic group has a positive externality on own utility, while the opposite happens for agents of low socio-ecnomic groups, then the net effect over utility may not be represented by assuption 1. Suppose, in particular, that the net effect over utilities is such that assumption 3 holds.

Assumption 3: *(Homophobous Assumption) Assume that* $\frac{\partial U_{t,i}^{S_n}}{\partial q_{t,j}^{S_n}} \geq 0$, $\frac{\partial U_{t,i}^{S_n}}{\partial q_{t,i}^{S_n}} \leq 0$, $\frac{\partial U_{t,i}^{S_1}}{\partial \beta_t^i} \leq 0$ *and* $\frac{\partial U_{t,i}^{S_2}}{\partial \beta_t^i} \geq 0$.

If Assumption 3 holds for i agents while Assumption 1 for j agents then i and j agents look for a school with a high share of j agents. In this case correspondences of possible social outcomes change since the problem is no more symmetric for the agents of different ethnic groups. Thus we have the following:

$$R_t^h = \begin{cases} \begin{cases} \{0, \beta_t^{h*}, 1\} & \text{if } \beta_t^{h*} \in [0,1] \\ \{0\} & \text{if } \beta_t^{h*} > 1 \\ \{1\} & \text{if } \beta_t^{h*} < 0 \end{cases} & \text{if } h = \{J1, J2\} \\ \begin{cases} \{0, \beta_t^{h*}, 1\} & \text{if } \beta_t^{h*} \in [0,1] \\ \{1\} & \text{if } \beta_t^{h*} > 1 \\ \{0\} & \text{if } \beta_t^{h*} < 0 \end{cases} & \text{if } h = \{I\} \end{cases}$$

The definition of the Nash Equilibrium is unchanged and thus Definition 1 holds. Again in equilibrium we cannot have that both $\beta^{j1} \in [0, 1)$ and $\beta^{j2} \in [0, 1)$. However, differently from the previous case, this does not help in the selection of a the Nash Equilibrium sincewe cannot say, ex ante, which of the two situations create higher school segregation with respect to residential segregation. We thus analyse in turn what happens if $\beta^{j1} = 1$ and then if $\beta^{j2} = 1$.

Case 1: $\beta_t^{j1} = 1$
As in the previous analysis, given monotonocity and continuity of the utility functions we can identify a $\beta^{i*}(\beta^{j2})$ such that $\dot{\beta}_t^i = 0$ because of $U_{t,i}^{S_1} = U_{t,i}^{S_2}$. Then this function is decreasing in β^{j2}. In the same way we can identify a function $\beta^{j2*}(\beta^i)$ such that $\dot{\beta}_t^{j2} = 0$ because of $U_{t,j2}^{S_1} = U_{t,j2}^{S_2}$, and again this function is decreasing in β_t^i. Thus, depending on the reltive slopes of the two functions we can have two cases, represented in figure 2.10. The two graphs represent the phase duagrams in the case in which an internal nash equilibrium of the school choice game exists depending on the relative slope of the two functions.

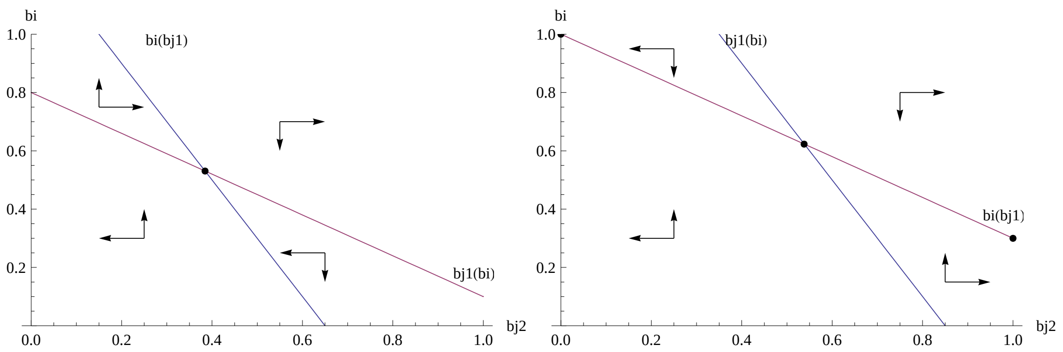

Figure 2.10: Homophoby: Case 1

In the first case we have $\beta^{j2*}(\beta^i)$ to be more elastic than $\beta^{i*}(\beta^{j2})$ while the reverse happens in the second case. We now focus on the analysis of the stability of the internal nash equilibrium. If case 1 holds then depending on the specification of the dynamics we can observe the internal Nash equilibium to be stable or not. In particular we can observe spyrales converging to the Nash Equilibrium, diverging from it or we can also observe cycles around it. In particular, building the Jacobian Matrix and evaluating it in the equilibrium, in order to observe stability it should be that $(\frac{\partial\dot{\beta}^i}{\partial\beta^i} + \frac{\partial\dot{\beta}^{j1}}{\partial\beta^{j1}})|_{(\beta^i_{eq},\beta^{j1}_{eq})} > 0$ and $(\frac{\partial\dot{\beta}^i}{\partial\beta^i}\frac{\partial\dot{\beta}^{j1}}{\partial\beta^{j1}}) - (\frac{\partial\dot{\beta}^i}{\partial\beta^{j1}}\frac{\partial\dot{\beta}^{j1}}{\partial\beta^i})|_{(\beta^i_{eq},\beta^{j1}_{eq})} > 0$. If the second case holds, then it is unstable so that even in this case partial segregation cannot be observed in steady state. However we can have that, depending on the parametrization, two partially internal equilibria may exists. Call the $\bar{\beta}_1 = (\beta_1^i, 0)$ and $\bar{\beta}_2 = (\beta_1^i, 1)$

Case 2: $\beta_t^{j2} = 1$
In this case only agents in $N1$ change neighborhood to attend S_2. We can again identify $\beta^{i*}(\beta^{j1})$ and $\beta^{j1*}(\beta^i)$ as previously defined. In this case, however, $\beta^{j1*}(\beta^i)$ is increasing in β_t^i, and $\beta^{i*}(\beta^{j1})$ is increasing in β_t^{j1}. We thus have the qualitative phase diagram as in figure 2.11

As in the previous case, depending on the specifcations and thus on the slope of the curves, we can have that an internal steady state may exists. Depending on the parametrization we can observe spyrales converging to the Nash Equilibrium, diverging from it or we can also observe cycles around it.

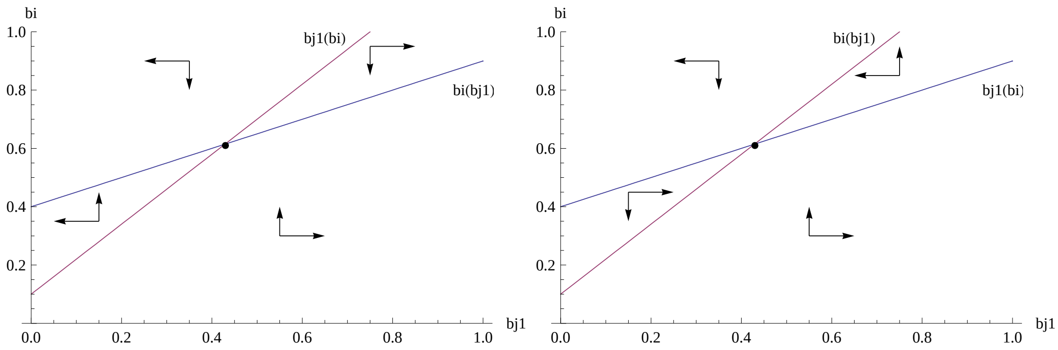

Figure 2.11: Homophoby: Case 2

Appendix B: Proofs of the Propositions

Proof of Proposition 1

Proof. Suppose an internal equilibrium exists. Then $\beta_t^i \in (0,1)$. This happens if and only if $\beta_t^i = \beta_t^{i*}$. Since i agents moving to S_2 fae a cost, then in order to have $U_i^{S_1} = U_i^{S_2}$ it must be that $q_{t,1}^{S_2} > q_{t,i}^{S_1}$. If this is the case then it lways happens that $U_{j1}^{S_1} > U_{j1}^{S_2}$ so that it cannot be that, in equilibrium, $\beta_t^{j1} \in (0,1)$. Thus an internal Nash Equilibrium cannot exists. □

Proof of Proposition 2

Proof. This situation is represented by figure 2. If p increases, then $U_{j1}^{S_2}$ is unchanged, while $U_{j1}^{S_1}$ shifts downwards since for any β, now q si lower and thus the utility is lower. As a consequence β_t^* is decreasing. If τ increases such that cost increases, $U_{j1}^{S_1}$ is unchanged while $U_{j1}^{S_2}$ shifts downwards. Thus β_t^* decreases. Given that β_t^* is increasing in p, then there could be a $p^* \in (0,1)$ such that if $\beta_t^*(p^*) = 1$. Thus if $p > p*$ it must be that $\beta_t^* > 1$ and thus, in equilibrium, $\beta = 0$ and full segregation happens. □

Proof of Proposition 5

Proof. An internal steady state exists if and only if $\beta = \beta^*$. We have defined $\beta_t^*(p, \tau, N, z)$. Given dynamics (2), if $\beta < \beta^*$ then the derived steady state implies $\beta = 0$; if $\beta > \beta^*$ then the derived steady state implies $\beta = 1$. Thus if β^* is time independent then it is impossible to have internal steady state. Suppose thus that $z(\beta_t)$ so that $\beta_t^*(z_t(\beta_t))$. If $\beta_t > \beta_t^*$ then $\dot{\beta}_t > 0$. In order to have an internal steady state it is necessary that $\dot{\beta}_t^* > 0$. Now since $\frac{\partial \beta_t^*}{\partial z_t} < 0$ then, in order to have $\dot{\beta}_t^* > 0$, it is necessary to have $\frac{dz_t}{d\beta_t} < 0$. Now, by (3) we have that $\frac{dz_t}{d\beta_t} = \frac{\partial z_t^{j1}}{\partial \beta_t^{j1}} + \frac{\partial z_t^{j1}}{\partial q_t^j} \frac{\partial q_t^j}{\partial \beta_t^j}$. Given the structure of q^j we have that $\frac{\partial z_t^{j1}}{\partial q_t^j} = \frac{(N-p)p}{(\beta_t^{j1}(N-p)+p)^2}$. Thus substituting this into the $\frac{dz_t}{d\beta_t}$ definition and imposing $\frac{dz_t}{d\beta_t} < 0$ we have that this happens if and only if $\frac{\partial z_t^{j1}}{\partial \beta_t^{j1}} < -\frac{(N-p)p}{(\beta_t^{j1}(N-p)+p)^2} \frac{\partial z_t^{j1}}{\partial q_t^j}$. □

Chapter 3

Cultural Evolution as Payoffs Distribution: An Evolutionary Game Approach

3.0.1 Abstract

This paper analyses the evolution of types in a society with an evolutionary game framework in which types evolve not only depending on how much they are fit but also on how much they are able to persuade that they are fit. This process creates a kind of payoffs redistribution so that it is possible to reach equilibria impossible under standard dynamics. We first provide necessary conditions of a generic matrix and generic class of cultural evolution mechanisms in order to observe polimorphic equilibria, proposing that in these cases standard game classifications are not a good method since, given the specific parametrization, two games of the same class may have different possible equilibria. We then propose two specific games (one with prisoner's dilemma incentives) and two persuasion rules derived from cultural transmission literature and find when it is possible to observe different classes of equilibria: in particular under a Prisoner's Dilemma type matrix stable coexistence of types and the pareto efficient outcome are possible steady states under some conditions over the parameters. Finally we compare this framework with a standard one without persuasion rules decribing when different types are advantaged by these schemes and comparing the different rules proposed.

3.1 Introduction

Evolutionary game theory, by analysing the spread of population types in a society, has, at its core, a purely cultural evolution problem, particularly if types are intended as cultural traits or cultural norms. In analysing this process, different ways of modelling the diffusion of cultural traits have been studied. If, for example, we focus on the evolution of cooperation in a Prisoner's Dilemma, the introduction of elements as reputation (Nowak (2006)) or altruistic preferences, preferences against inequality, strong reciprocity, preferences for other agents' outcome and other sort of social preferences (see Fehr and Fischbacher (2002, 2004) and Gintis et al. (2005)) has been proved to modify the way in which types spread in the society with respect to the cases in which type fitnesses are given and payoffs only depend on them. In all these cases, as, in general, in all the literature, payoffs (and fitnesses) are fixed, given the matching, and thus through a replicator dynamics intendend as an imita-

tion process (Weibull (1997)) a population dynamics is derived. However, quoting Bowles (1998):*'Evolutionary game theoretic models typically abstract entirely from the process of cultural transmission, representing payoffs associated with particular traits as if they were the only influences on the replication of traits. By contrast, models of cultural evolution typically address what is known about the particulars of the process by which traits are acquired..'*.
There is a set of papers addressing this issue by introducing the transmission of preferences in a game evolutionary setting: Gintis (2003), Bisin et al. (2004a), Mengel (2008) and Calabuig and Olcina (2009); in all these papers cultural transmission is modelled as an intergenerational process by which preferences for cooperation are transmitted from one generation to the other by an indirect evolutionary process.

In this paper we address this problem not using intergenerational models but studying how cultural traits' fitnesses are formed. We introduce a set up in which agents are culturally programmed to play only one action, and population types spread in the society not only depending on their objective fitness, given by the fixed payoff matrix, but also depending on how much agents are able to convince that their type is fit: in this way we transform the process by which fitnesses are formed, introducing a persuasion step modifying the process of cultural transmission. Thus if maladaptive types are able to 'convince' that their type is fit, they obtain a higher payoff and can spread in the society: this is in the spirit of the evolution of cultural traits. We build a framework in which agents of different types are matched in pairs. Once matched they have to perform a task in which they have different productivities: their final payoff however depends on how much they are able to convince their 'employer' about their fitness level so that they produce a costly effort in order to signal this level and compete with the other agents for a higher final payoff. Then standard replication on these modified payoffs happens. In this way a type spreads in the society depending on how much it is objectively fit and, in the same time, on how much it is able to convince about this fitness level. The employer-employees framework we use here is thus just used as example for a wider possible set of application of this kind of model.
We use persuasion rules derived from cultural transmission theory (Cavalli Sforza and Feldman (1973, 1981) and Bisin and Verdier (2001)). In our framework, the way in which agents compete for payoffs with the costly effort have also some links with both hold-up models (Ellingsen and Robles (2002), or see Olcina and Penarrubia (2002) for a model of hold-up with intergenerational preference transmission) and with conflict theory model (for a review Garfinkel and Skaperdas (2007)).
In this way we first analyse equilibria for generic game matrices and generic class of persuasion mechanisms, focusing on the fact that standard game classification is not helpful in these kinds of cultural processes since payoffs are redistributed by the mechanism itself and thus different equilibria characterization can be found in the same class of games depending on the specific parametrization. We then specify two classes of games (one of the two having a prisoner's dilemma structure) and two different persuasion rules directly derived from cultural transmission literature. In particular, in this way, we find that given this persuasion mechanism in a prisoner's dilemma then polimorphic equilibria (stable and unstable) can also be possible depending on the specific parametrization. We then compare the persuasion rules proposed with the standard framework without the costly effort finding when different types of agents are advantaged by this mechanism. Then we perform a comparison rule analysis in order to study which rule can be better in order to achieve social optimum, so that also in a Prisoner's Dilemma framework the pareto efficient outcome can be reached.
The paper has the following structure: in section 2 we present the model with the first conditions of existence and we present the main rules and matrix structures used. Section 3 analyses what happens under the two specific structures analysed while section 4 provides a comparison of the rules. Section 5 provides conclusions.

3.2 The Model

Suppose to have continuum of agents of mass 1. Agents can be of two types, A and B. Call A the set of all agents belonging to the A group, and B the set of all agents belonging to the B group. The two groups are present in the society with proportions, at time t, $p_t^A = p_t$ and $p_t^B = 1 - p_t$ respectively. Suppose that agents are workers that, in order to complete a task, have to work in pairs. Thus, at each point in time, agents are randomly matched. Assume that, when matched, agents can recognize each others' type. The matrix of productivities for each matching is the following:

	A	B
A	α, α	β, γ
B	γ, β	δ, δ

Thus agents differ only in terms of productivity in each matching. Suppose that $(\alpha, \beta, \gamma, \delta) \in [0,1]^4$. Once matched, agents, in order to have their productivity assessed by the employer, have to produce an effort in order to persuade the employer about their contribution to the joined work (e.g. think for example at a presentation of the work) so that the effort is used by the employer as a signal useful in assessing the productivity level of the agent. Thus each agent i produces an observable effort $\tau_t^{ij} \in [0,1]$ dependent on the ij matching. This effort is costly since the employer knows that part of the working force has been devoted to the persuasion effort and not to the production, so that the agent's salary will be reduced in relation to the amount of effort with a technology of $c(\tau_t^i) \in [0,1]$. Thus the employer looks at the productivities matrix and at the efforts and consequently decides the salaries. Take an ij matching (e.g. AB matching). The employer looks at the efforts, and chooses $q_t^{ij}(\bar{\tau}_t) \in (0,1)$ and $q_t^{ji}(\bar{\tau}_t) \in (0,1)$, the first one being the share of i's productivity (β) that the employer is persuaded to impute to i agent, so that $1 - q_t^{ij}(\bar{\tau}_t)$ is the share of the productivity (β) imputed to j agent. Similarly $q_t^{ji}(\bar{\tau}_t) \in (0,1)$ is the share of j's productivity (γ) imputed to j, with $q_t^{BA}(\bar{\tau}_t)$ being the share imputed to i. Assume that these shares are such that the higher the effort, the higher, ceteris paribus, the productivity share assessed to the agent and, thus, his salary; in the same way, higher opponent's effort reduces the productivity share the agent is assessed. Thus we have the following assumption:

Assumption 1: *Assume that* $\frac{\partial q_t^{ij}}{\partial \tau_t^{ij}} \geq 0$ *and* $\frac{\partial q_t^{ji}}{\partial \tau_t^{ji}} \geq 0$. *Assume also that* $\frac{\partial q_t^{ij}}{\partial \tau_t^{ji}} \leq 0$ *and* $\frac{\partial q_t^{ji}}{\partial \tau_t^{ij}} \leq 0$.

Call π_t^{ij} the salary at time t an i agent gets when matched with a j agent. Thus we have:

$$\begin{cases} \pi_t^{ij} = \alpha q_t^{ij}(\bar{\tau}_t) + \alpha(1 - q_t^{ji}(\bar{\tau}_t)) - c(\tau_t^{ij}) \text{ with } i,j \in \{A\} \\ \pi_t^{ij} = \delta q_t^{ij}(\bar{\tau}_t) + \delta(1 - q_t^{ji}(\bar{\tau}_t)) - c(\tau_t^{ij}) \text{ with } i,j \in \{B\} \\ \pi_t^{ij} = \beta q_t^{ij}(\bar{\tau}_t) + \gamma(1 - q_t^{ji}(\bar{\tau}_t)) - c(\tau_t^{ij}) \text{ with } i \in \{A\} \text{ and } j \in \{B\} \\ \pi_t^{ij} = \gamma q_t^{ij}(\bar{\tau}_t) + \beta(1 - q_t^{ji}(\bar{\tau}_t)) - c(\tau_t^{ij}) \text{ with } i \in \{B\} \text{ and } j \in \{A\}. \end{cases} \tag{3.1}$$

The problem that each agent faces is to maximize his salary given a matching, by choosing the optimal effort level.
Consider first the cases of homogenous matchings. In an AA matching the problem is symmetric since both agents are equal so that, by the previous equations, it follows that $\pi_t^{ij} = \alpha - c(\tau_t^{ij})$ while in a BB matching $\pi_t^{ij} = \gamma - c(\tau_t^{ij})$. Thus if agents are in a homogenous matching then their optimal effort will be 0, so that $\tau_t^{ii*} = \tau_t^{jj*} = 0$. Thus it follows that:
$\pi_t^{ij*} = \alpha$ with $i,j \in \{A\}$

$\pi_t^{ij*} = \delta$ with $i, j \in \{B\}$
Consequently we only analyse the case of heterogenous matching. For simplicity of notation call $\tau_t^{ij} = \tau_t^A$ if $i \in \{A\}$ and $j \in \{B\}$ and $\tau_t^{ij} = \tau_t^B$ if $i \in \{B\}$ and $j \in \{A\}$. The agent objective, depending on his type, is thus to $\underset{\tau^A}{Max}\ \pi_t^{AB}$ and $\underset{\tau^B}{Max}\ \pi_t^{BA}$. We can thus state:

Proposition 1: *If Assumption 1 holds,* $c'(\tau^i)|_{\tau^i=0} = 0$ *and* $c(\tau^i)$ *is 'convex enough', then the optimal persuasion effort* τ^{i*} *exists, is unique and* $\tau^{i*} \in (0,1)$

Proof. Proof in Appendix □

After the salary has been given to the workers, they can observe the salaries each type has taken at current time, and can decide to imitate the others' type if, on average, being of the other type has produced a higher salary than the average salary of own type agents. Recalling that $\tau_t^{AA^*} = 0$ and that $\tau_t^{BB^*} = 0$, we can define average salaries for each type as:

$$\pi_t^A = p_t\pi_t^{AA} + (1-p_t)\pi_t^{AB} = p_t\alpha + (1-p_t)\pi_t^{AB}$$
$$\pi_t^B = p_t\pi_t^{BA} + (1-p_t)\pi_t^{BB} = p_t\pi_t^{BA} + (1-p_t)\delta$$

The described imitation process can be approximated using a standard replicator dynamics (Weibull (1997)) so that we have the following rule for the evolution of population types:

$$\dot{p}_t = p_t(1-p_t)(\pi_t^A - \pi_t^B) \tag{3.2}$$

In this way we have a set up in which population evolves depending on the salaries, being a measure of how productive a type is and how much agents of a given type are able to be convincing about the fitness of their own type so that two major forces of cultural transmission and evolution hold. Next sections are devoted to the study of the implications of this set up.

3.2.1 Polimorphic Equilibria

We define as *Polimorphic Equilibrium* a $p_t^* \in (0,1) : \dot{p}_t|_{p_t^*} = 0$, so that in p_t^* no type dynamics is shown. The analysis of polimorphic equilibria is crucial since it enable us to make comparison with the case of standard dynamics with no efforts involved. We first study a numerosity condition.

Assumption 2: *Assume that the vector of shares* $\bar{q}$ *does not depend on population shares*

Given assumption 2 we can state the following

Proposition 2: *If Assumption 1 and 2 hold, then, for every parametrization and cultural transmission rule, if 3.1 and 3.2 hold, there exists at most one polimorphic equilibrium.*

Proof. Proof in Appendix □

The assumption behind this proposition is quite weak, since it states that during the employer persuasion process the population shares of the different types does not play any role in the pairwise comparison. This means that during the evaluation process the employer does not care about the numerosity of each group so that the salary of a given agent is not influenced by the size of his group but only on productivities and efforts in the specific matching. Thus, if this is the case then we cannot observe multiple polimorphic equilibria. This makes the analysis much easier since it is then enough to study stability of $p = 0$ and $p = 1$ in order to know about the existence of the equilibrium. Thus if these points are both stable or unstable

then the equilibrium exists, while if one is stable and one unstable then the equilibrium does not exist. Morevoer, if they are both stable the derived polimorphic equilibrium is unstable, while if they are both unstable the derived polimorphic equilibrium is stable. Thus it is not needed to compute the potential polimorphic equilibrium, study its feasibility and its stability properties in order to study the set of stable steady states of the system.
Using the previous proposition we now derive some necessary conditions in order to observe in the long run coexistence of the two groups, in a framework with generic $c(\tau_t^{ij})$ and a generic persuasion rule. What follows are conditions over the parameters in order to obtain long run coexistence.

Proposition 3: *If Assumption 1 and 2 hold, under 3.1 and 3.2, necessary conditions in order to observe a polimorphic stable equilibrium in the long run are $\beta + \gamma > \delta$ and $\beta + \gamma > \alpha$.*

Proof. Proof in Appendix □

This proposition gives an idea of the impact of cultural evolution mechanisms over different game structures, and indirectly proposes that standard game characterization may not be useful in cultural evolution frameworks since cultural evolution itself changes the incentives given by the standard distribution of payoffs. If we analyse in turn the two thresholds we have first that $\beta + \gamma > \delta$ is a necessary condition for $p = 0$ to be unstable. If, in fact, $p = 0$ then all B agents get δ. If then an A agent is introduced in the society, he is surely matched with a B agent. If then δ is not high enough to countervail the potential gain of A agents $(\beta + \gamma)$, then the salary of a A agent could be higher than the salary of the average B agent, and thus A agents start imitating B agents and consequently $p = 0$ is not a stable steady state. For the very same reason the second condition $\beta + \gamma > \alpha$ makes $p = 1$ an unstable steady state since B agents could invade a A agents society. This proposition can also be interpreted in the following way: if $\beta + \gamma < \delta$ and $\beta + \gamma < \alpha$ then there exists an unstable polimorphic equilibrium, thus providing sufficient conditions respectively for the stability of $p = 0$ and of $p = 1$. In this way the employer, given any rule satisfying 3.1 and 3.2, can understand if there is space for having homogenous workers population in the long run. If these are satisfied there is no persuasion rule under 3.1 and 3.2 that can invert the result.
We now compare this with the case without cultural transmission, that we call for simplicity *standard case*: necessary and sufficient conditions in order to get a polimorphic stable equilibrium in the long run under replicator dynamics without the efforts $\bar{\tau}_t$ are $\beta > \delta$ and $\gamma > \alpha$, the first one concerning instability of $p = 0$ and the second one instability of $p = 1$. Given these thresholds, the usual characterization of games is possible. For example a Prisoner's Dilemma, with $\beta > \delta > \alpha > \gamma$ satisfies always the first condition and not the second one, so that the only stable equilibrium in the long run is $p = 1$. If we now consider the rules for a generic cultural evolution framework, then we notice that a Prisoner's Dilemma may or may not satisfy both the conditions. Thus, the same incentive structure, as usually classified, may produce totally different outcomes given the fact that cultural transmission modifies the way in which fitnesses and payoffs are evaluated and have an impact over the salaries so that, as usually noticed, in cultural evolution also maladaptive types may survive and win. This happens not because these agents belong to a good type but because they persuade others that their type is good and, if a high enough number of people is persuaded, then they can finally invade the whole population. On the reverse it can happen that cultural evolution may help in catching pareto efficient outcomes in a Prisoner's Dilemma type matrix because of the persuasion power of B types. Thus cultural evolution is involved with persuasion rules that have a *redistributive role* in the mixed matchings. Thus, since $\beta + \gamma$ is the total social outcome under the mixed matching, then the threshold depends on that.
Starting from this, we can find a relationship between the convergence properties in the stan-

dard case and in a cultural evolution framework. In particular:

Corollary 1: *If a game admits a polimorphic stable equilibrium in the standard case, then the parametrization satisfies the necessary conditions for the existence of a polimorphic stable equilibrium in the cultural evolution framework under 3.1 and 3.2 and assumptions 1 and 2.*

This proposition relates the standard results with the cultural evolution framework basically stating that if we analyse a case without the posibility of the effort and we may observe a polimorphic equilibrium, then we observe a polimorphic equilibrium in a cultural evolution framework too, even though this is only a necessary condition. The next analysis goes deeper in the study of this problem by using some sensible specifications.

3.2.2 Persuasion Rules

In order to model the technology of persuasion we use rules taken directly from cultural transmission and cultural evolution literature. These rules are particularly fit since they have been used in contexts in which a cultural trait is transmitted from one agent to another, and thus when an agent has to be convinced about the goodness of a given cultural trait. In this case the employer has to be convinced about the productivity of each type (job culture, way of working, etc), depending on the signal derived from the effort they put in place[1].
We thus analyse two rules: in the first rule the employer is persuaded separately from the two employees and thus decides the salaries on the basis of an absolute judgment over the efforts, while in the second rule he judges on the basis of relative comparison of efforts.

***Rule 1* Absolute Judgment:** The first rule is specified as:

$$\begin{cases} q^{AB} = \tau^A + (1-\tau^A)\frac{1}{2} = \frac{1}{2}(1+\tau^A) \\ q^{BA} = \tau^B + (1-\tau^B)\frac{1}{2} = \frac{1}{2}(1+\tau^B) \end{cases}$$

This rule is directly taken from the probabilistic rule in Bisin and Verdier (2001) derived from Cavalli Sforza and Feldman (1981). Take the first term: $q^{AB} = \tau^A + (1-\tau^A)\frac{1}{2}$. An agent A produces an effort $\tau_t^A \in (0,1)$. The employer has to decide how much of β to assign to the agent. Thus he assigns a share equal to τ_t^A to the agent, and the rest $(1-\tau_t^A)$, that identifies how much he is not truly convinced, to both the agents with equal weights. That is to say that the employer is persuaded linearly with the effort, and for the amount that remains decides to equally distribute among workers. Consequently the share of β that a B agent takes is $1-q^{AB}$ and thus $\frac{1}{2}(1-q^{AB})$. The same happens for B agents in a BA matching. In this rule q^{AB} only depends on A agents' effort, and q^{BA} on B agents' effort, without cross influences. Thus the employer judgment is in absolute terms.

***Rule 2* Relative Judgment:** If we want to analyse a relative terms judgment, we thus modify this rule and obtain the following:

$$\begin{cases} q^{AB} = 1-\tau^B(1-\tau^A) \\ q^{BA} = 1-\tau^A(1-\tau^B) \end{cases}$$

[1] We could have also used rules taken from contest theory, and it can be done in future research, but in this way we build a set up that can be extended to generic cultural transmission problem and that can be immediately used in these contexts.

Take the first term: $q^{AB} = 1 - \tau^B(1 - \tau^A)$. This can have the following interpretation: the share of β the agent A gets is 1 minus how much he is not able to persuade times how much B was able to persuade the employer. If we want to have an interpretation more similar to the previous one we can write $q^{AB} = \tau_t^A + (1 - \tau^A)(1 - \tau^B)$ that is algebrically equivalent to the first one. This means that the share of β an A agent takes is equal to τ plus a share indicating how much he and his opponent were not able to persuade the employer; that is to say: if a worker is not able to persuade and neither his coworker does, then the employer supposes that the worker itself is responsible for that productivity.

We note now that both these rules satisfy the assumption of the independence of the vector $\bar{q}$ from the population share vector so that at most one polimorphic equilibrium exists in both these cases.

3.2.3 Type Productivities

We now introduce two types of productivity matrices that can reproduce sensible situations in workers type productivity, and that we will use in the following sections as particlarly significative cases.

***Case 1:* PD type** In this case A agents are more productive when in a heterogenous matching, while B agents when in a homogenous matching, with the social optimum in terms of productivity is given by a BB matching. We can set up the productivity matrix with coefficients as in a Prisoner's Dilemma with $\beta > \delta > \alpha > \gamma$. We notice that if this specification holds, if agents are just payed the productivity level indicated by the matrix without any persuasion effort, so that the standard case happens, then a suboptimal result is always achieved. This case is thus introduced in order to see if a PD like scheme with this cultural evolution mechanism is able to achieve results impossible under standard dynamic schemes.

***Case 2:* Improductive Workers** In this case we suppose B agents to be less productive than A agents and thus we can set up the matrix such that $\beta > \gamma$ and $\alpha > \delta$. Thus in the heterogenoues matching A agents do better, and the same happens in the heterogenous matching. Notice that however we cannot compare the total potential productivity in heterogenous matching $(\beta + \gamma)$ with the total potential productivities in homogenous matchings. Notice that in this case if agents are payed simply following the matrix, productivity levels without any possibility of effort, then unproductive workers are going to be extincted in the long run. Thus, what happens if agents are given the chance to prove their productivity levels? Is it possible that unproductive workers use this chance in order to gain more than their actual level and thus survive?

Given these two possible situations we would like to know whether it is possible for the employer to use cultural transmission rules in order to obtain an optimal population composition. In particular in the first case we would like to observe only B agents, thus proposing in different terms the problem of cooperative outcomes in a Prisoner's Dilemma. Thus, it is possible that cultural transmission and persuasion rules act in order to make B agents winning. In the second case we have the reversed situation: is it possible that giving unproductive workers the chance to prove their ability gives them an advantage to be exploited and thus to make them winning or at least survive in the long run?
The rest of the analysis tries to answer to these questions with generic matrices and, when necessary, referring to these two particular productivities specifications.

3.3 Dynamic Analysis

In this section we analyse in details how the dynamics of a generic game can be influenced if the two rules of persuasion are introduced.
In the previous section we have described necessary conditions in order to have a mix of strategies in the long run under generic rules. This section is now devoted to give some necessary and sufficient conditions in order to observe long run steady states under specific rules. We thus set these conditions for rule 1 (proposition 4) studying the long run possible outcomes for the different cases of PD like matrix and unproductive workers (proposition 5 and 6). Then we analyse what happens under rule 2 (Corollary 2).

Suppose first that the cost function has a quadratic form, and in particular $c(\tau_t^i) = \frac{\tau_t^{i^2}}{2}$, as generally assumed by cultural transmission models. Analysing first Rule 1 we can state the following:

Proposition 4: *Suppose Rule 1 holds. Then, given any vector $(\beta,\gamma) \in [0,1]^2$, there always exists an $\bar{\alpha}_1(\beta,\gamma) \in (0,1)$ and a $\bar{\delta}_1(\beta,\gamma) \in (0,1)$ such that $p = 0$ is unstable if and only if $\delta < \bar{\delta}_1(\beta,\gamma)$ and $p = 1$ is unstable if and only if $\alpha < \bar{\alpha}_1(\beta,\gamma)$.*

Proof. Proof in Appendix □

This proposition gives necessary and sufficient conditions in order to obtain different long run situations. In particular the two conditions define, in the (α,δ) space, 4 different possible areas depending on $(\bar{\alpha}_1(\beta,\gamma), \bar{\delta}_1(\beta,\gamma))$ as shown in figure 1.

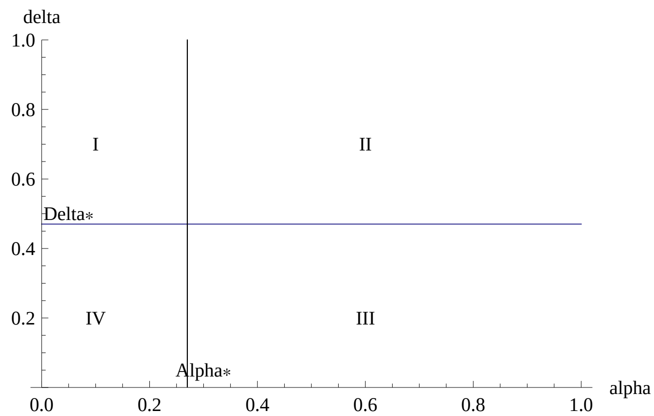

Figure 3.1:

We now analyse what happens in each area in terms of long run equilibria.

- **Area I:** In this area $\delta > \bar{\delta}_1$ and $\alpha < \bar{\alpha}_1$ so that $p = 0$ is a stable steady state and $p = 1$ is an unstable steady state. Thus, since there exists at most one polimorphic equilibrium under rule 1, it follows that no polimorphic equilibrium can exist in this case. This means that in the long run only B agents are present.
- **Area II:** In this area $\delta > \bar{\delta}_1$ and $\alpha > \bar{\alpha}_1$ so that $p = 1$ is a stable steady state and $p = 0$ is a stable steady state. This means that there exists an unstable polimorphic equilibrium and thus, in the long run, a mixed population distribution cannot be observed.
- **Area III:** In this area $\delta < \bar{\delta}_1$ and $\alpha > \bar{\alpha}_1$ so that $p = 1$ is a stable steady state and $p = 0$ is an unstable steady state. Thus, since there exists at most one polimorphic

equilibrium under rule 1, it follows that no polimorphic equilibrium is shown. This means that in the long run only A agents are present.

- **Area IV:** In this area $\delta < \bar{\delta}_1$ and $\alpha < \bar{\alpha}_1$ so that $p = 1$ is an unstable steady state and $p = 0$ is an unstable steady state too. This means that a stable polimorphic equilibrium exists and, in the long run, extreme population distribution cannot be observed.

From this classification we can disentangle the role of α and δ on the stability of extreme points given the rules we have set up. In particular if, given the vector (β, γ), α is augmented then it is more likely that $p = 1$ becomes stable. This however has no effect on the stability of $p = 0$ since it is influenced only by changes of δ. In particular if δ is augmented then it is more likely that $p = 0$ is stable.
We now study how $\bar{\delta}_1$ and $\bar{\alpha_1}$ change with the parameters β and γ

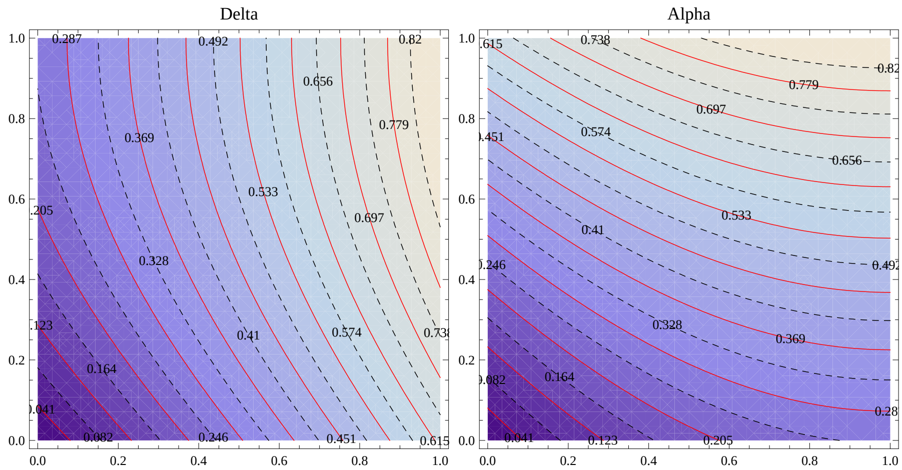

Figure 3.2:

Contour plots in figure 2 represent how the thresholds $\bar{\delta}_1$ and $\bar{\alpha}_1$ change with the parameters β and γ, in which β is on the x axis and γ on the y axis. In this way we can understand what happens if the productivity matrix structure changes. We have the following properties:

- $\frac{\partial \bar{\delta}_1}{\partial \beta} > 0$ and $\frac{\partial \bar{\alpha}_1}{\partial \beta} > 0$: the first property states that an increase in β reduces the areas in which $p = 0$ is a stable steady state so that the space such that A workers can coexist with B workers is enlarged. The second property implies that higher β has the effect of reducing the areas in which $p = 1$ is stable. Thus a higher β has the effect of reducing the set of (β, γ) such that an unstable polimorphic equilbrium exists and to enlarge the areas of existence on a stable polimorphic equilbrium. Effects on areas I and III are unclear.

- $\frac{\partial \bar{\delta}_1}{\partial \gamma} > 0$ and $\frac{\partial \bar{\alpha}_1}{\partial \gamma} > 0$: this case is the same as the previous one so that again higher γ is associated with a larger area for the existence of a stable polimorphic equilibrium and a smaller area for the existence of an unstable polimorphic equilbrium. Again effects on I and III areas are unclear.

It is worth noticing that, since $\bar{\delta}_1$ is independent from α and $\bar{\alpha}_1$ is independent from δ, then the stability of $p = 0$ is independent from the payoff of AA matching, and the stability of $p = 1$ is independent from the payoff of BB matching.
We now use these results in order to see what is the impact on a standard class of game widely studied: Prisoner's Dilemma (PD), and in order to do that we suppose the matrix to be composed as in case 1. Then, standard replicator dynamics make A agents win. If we now

implement a cultural evolution technology, as in Rule 1, we have that $\bar{\delta}_1 > \bar{\alpha}_1$ if and only if $\beta > \gamma$. If $\bar{\delta}_1 > \bar{\alpha}_1$, then in the $\delta > \alpha$ space all 4 cases can happen. But $\delta > \alpha$ and $\beta > \gamma$ are all compatible with a PD payoff matrix, and thus we can state that:

Proposition 5: *If payoffs are represented as in a PD type and Rule 1 holds, then it can always be found a parametrization for each of the following cases to be shown:* $p = 1$ *is the only stable equilibrium,* $p = 0$ *is the only stable equilibrium, there exists a unique unstable polimorphic equilibrium and there exists a unique stable polimorphic equilibrium.*

Proof. Proof in Appendix □

This proposition underlines the fact that cultural transmission can drastically change the population dynamics if compared to the standard case. The Prisoner's Dilemma payoff matrix always produces convergence to the suboptimal steady state if the standard evolutionary approach is used. Introducing the possibility that types' and populations' payoffs are influenced not only by their objective productivity but, as a crucial element in cultural evolution, by their ability to prove how fit they are, the results can be different. The initial matrix, generally classified as PD type matrix, has to be reclassified depending on the incentives that it produces during the cultural evolution and transmission process. This classification depends on the threshold $\bar{\alpha}_1$ and $\bar{\beta}_1$ and is strictly related to the rule used in the persuasion process.
Coming back to the employer-worker framework, if a productivity matrix like this is shown, then there is still a chance for the employer to reach the population compostion of maximum productivity, and it depends on the relative size of the parameters.
Analyse now the *improductive workers* framework: putting together the order of the parameters ($\beta > \gamma$ and $\alpha > \delta$) with the thresholds space we have that Area I can never be feasible so that we can state that:

Proposition 6: *If an improductive workers framework and Rule 1 hold, then it can always be found a parametrization for each of the following cases to be shown:* $p = 1$ *is the only stable equilibrium, there exists a unique unstable polimorphic equilibrium and a unique stable polimorphic equilibrium.*

Proof. Proof in Appendix □

This proposition is based on the same arguments as the previous one. In particular it derives from the fact that $\bar{\delta}_1 > \bar{\alpha}_1$ and, since in an improductive workers framework $\alpha > \delta$, then it is impossible to have a feasible area I. This happens since improductive workers, by means of the effort, may induce the employer to believe that their producitvity is higher than in reality and thus they can get a higher salary. In this way their type can be sustained in the long run. This is a case of how cultural evolution can help maladaptive types to survive. In this case cultural transmission may still favour, during its process, improductive types but these workers can never invade the whole society.
We now analyse what happens if Rule 2 is implemented and if any kind of comparison between the two rules is possible. As in the previous case we can state that:

Corollary 2: *Suppose Rule 2 holds. Then, given any vector* $(\beta, \gamma) \in [0, 1]^2$*, there always exists* $\bar{\alpha}_2(\beta, \gamma) \in (0, 1)$ *and* $\bar{\delta}_2(\beta, \gamma) \in (0, 1)$ *such that* $p = 0$ *is unstable if and only if* $\delta < \bar{\delta}_2(\beta, \gamma)$ *and* $p = 1$ *is unstable if and only if* $\alpha < \bar{\alpha}_2(\beta, \gamma)$*.*

Corollary 2 has the same interpretation of the proposition regarding rule 1. In particual in both cases, depending on the parametrization we can obtain different long run equilibria. The interpretation is qualitatively the same and thus the same 4 areas as in figure 1 can be identified.

3.4 Rules Analysis

In this section we analyse the effect of the two persuasion rules in the dynamic process with a particular focus on the ability of each rule to reach the optimal population composition in terms of total productivity. In this way, given the different possible parametrizations, the employer can get some indications on which rule to use in different cases. We expect these two rules to have different impacts in the long run. In order to do this we first compare each rule with the outcome without the costly signalling effort, analysing which type of agents are helped in the dynamics by these processes (Corollary 4 and 5). Then, in proposition 7 we rank the different possible outcomes that can be reached under rule 1, focusing in particular on the polimorphic equilibria optimality. With propostion 8 we see what is the difference in outcome between the two rules and the last two corollaries complete this comparison analysis. Define $\dot{p}_t^S$ the dynamic rule of A type agents for the case in which no effort is permitted, that we called for simplicity standard case, $\dot{p}_t^{R1}$ the dynamic rule for A type's agents if rule 1 holds and $\dot{p}_t^{R2}$ the dynamic rule for A type agents when rule 2 is applied. Take the first two cases: whenever $\dot{p}_t^{R1} > \dot{p}_t^S$ then A agents population share increases always more rapidly or decreases less rapidly under rule 1 than under the case without effort, so that A agents are more present in the society under rule 1 than under the standard case. Since the dynamics has the replicator dynamics form, then it is payoff monotonic so that A agents are always bettered off under rule 1 than under the standard case. The same can be done with rule 2, so that whenever $\dot{p}_t^{R2} > \dot{p}_t^S$ A agents are always bettered off under rule 2 than under the standard case. We start with the analysis of rule 1. Recall that :

$$\dot{p}_t^{R1} = p_t\alpha + (1-p_t)(\tfrac{\beta}{2}(1+\tau_t^{ij*}) + \tfrac{\gamma}{2}(1-\tau_t^{ji*}) - c(\tau_t^{ij*}))$$
$$\dot{p}_t^{S} = p_t\alpha + (1-p_t)\beta$$

so that

$\dot{p}_t^{R1} > \dot{p}_t^S$ if and only if
$$p_t\alpha + (1-p_t)(\tfrac{\beta}{2}(1+\tau_t^{ij*}) + \tfrac{\gamma}{2}(1-\tau_t^{ji*}) - c(\tau_t^{ij*})) > p_t\alpha + (1-p_t)\beta.$$

If we substitute the optimal values of efforts previously determined and we solve the unequality we can state that:

Corollary 4: *If (3.1), (3.2) and Rule 1 hold then A agents are bettered off with respect to the case without persuasion effort if and only if $\beta < \frac{2}{3}\gamma(1-\gamma)$. If matrices are represented as in PD like or unproductive workers cases then A agents are always worsed off.*

The first part of corollary 4 directly derives from the solution of the unequality. The second part is derived by noticing that $\frac{2}{3}\gamma(1-\gamma) < \gamma$, with the left hand side being the threshold for the necessary and sufficient condition of the first part of the corollary. Now, since in the PD like and unproductive Workers parametrization $\beta > \gamma$ then the condition $\beta < \frac{2}{3}\gamma(1-\gamma)$ is

never satisfied and thus it always happens that $\dot{p}_t^{R1} < \dot{p}_t^S$. Thus B agents succeed, during the persuasion process, to be assessed a higher share of productivity by taking part of A agents' productivity and thus to survive longer in the society than in the standard case. From this corollary we have a first intuition: if the employer would like to have always more A agents, then cultural evolution structure involved by rule 1 does not help him. This could be a good news if an unproductive workers structure is presented, while it can be a bad news if the employer faces a PD type matrix structure.
We now analyse what happens under rule 2. In this case, even if it is possible to analytically determine a precise threshold it is more usefull to represent it in a graph. The contour plot in figure 3 represents the difference $\dot{p}_t^{R2} - \dot{p}_t^S$, with β of the x axis and γ on the y axis.

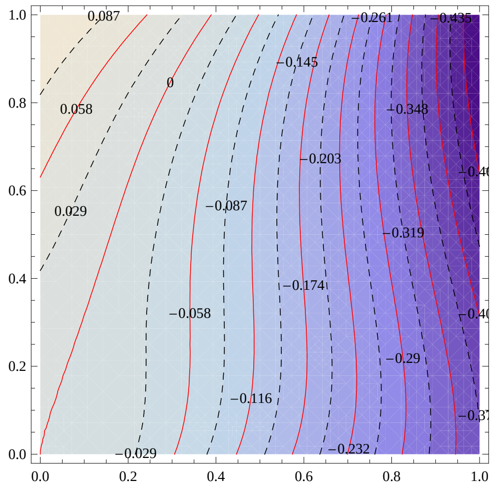

Figure 3.3:

When the difference is negative A agents are worsed off under rule 1 while if the difference is positive they are bettered off. Notice that the 45 degree line (not plotted) represents the case of $\beta = \gamma$ such that PD like and unproductive workers schemes are both represented by parametrizations below the line. Moreover the 45 degree line is always below the 0 level curve. Consequently, since in this case it always happens that $\dot{p}_t^{R2} < \dot{p}_t^S$ we can state the following:

Corollary 5: *If matrix are represented as in PD like or unproductive workers cases then under rule 2 it always happens that A agents are worsed off with respect to the standard case.*

Thus again, even if rule 2 holds we have that B agents are always bettered off. In this way, considering together corollary 4 and 5 we have that both persuasion rules help B agents in the dynamic process if compared to the cases without the costly signaling effort. Thus, this analysis does not help us in discriminating which rule prefer when trying to promote A or B agents, since they seem to drive qualitatively the system towards the same direction. Moreover, since actual efficiency levels in heterogenous matching are changed with respect to the original matrix due the efforts, then we should also study when having more B agents is optimal from the employer point of view and when it is not.
Thus we now start an analysis trying to compare the two rules in terms of the possible outcome they can reach with some hints on the social efficiency levels of these outcomes.
In what follows we particularly focus on the PD like matrix structure. We divide this analysis in two parts: we first analyse when the possible polimorphic equilibrium is reached and is an

improvement with respect to the steady state reached without the efforts ($p^A = 1$). Then we study when $p^B = 1$ is reached. In this way we can compare all the three possible outcomes of the dynamics.
Analysing first the productivity levels of the polimorphic equilibria we can state the following:

Proposition 7: *In a PD like framework, under rule 1 a polimorphic equilibrium is never the optimal outcome, while $p^A = 1$ is the worst outcome if and only if $\alpha + \delta < \frac{\beta+\gamma}{2}$. In both PD like and unproductive workers framework under rule 2 polimorphic equilibrium is always the worst outcome.*

Proof. Proof in Appendix □

Proposition 7 studies the properties of the polimorphic equilibrium under different rules and matrices structures. The first part of the proposition states that if we are in a PD like framework, then even if a polimorphic equilibrium is reached, this is not the best outcome possible. In particular, independently on how large β can be, homogenous population steady states on $p^B = 1$ always brings the system to higher levels of social outcome. In a polimoprhic equilibrium not all the matching are heterogenous, but just a share of $2p(1-p)$. In this matching the total productivity without the effort can be very large due to a large β. Still we have that the final total productivity is higher for $p^B = 1$. This happens since too many resources are destroyed in the persuasion effort. In particular the higher the β and γ, the higher the levels of efforts and thus the level of 'destroyed' productivity. However this is far from saying that a polimorphic equilibrium is the worst possible. In fact, in the second part of the proposition, we derive a condition ($\alpha + \delta < \frac{\beta+\gamma}{2}$) such that a polimorphic equilibrium is always better than $p^A = 1$. This means that if we face a PD and $\alpha + \delta < \frac{\beta+\gamma}{2}$, then applying rule 1 cannot diminish social outcome. In fact if the system ends up with $p^B = 1$ this is by definition superior to $p^A = 1$, while if a polimorphic equilibrium is reached then this outcome is better than the one under no effort in which $p^A = 1$. If, finally, $p^A = 1$ then nothing changes with respect ot the standard case.
If we now turn to the second part of the proposition we have that if rule 2 is applied and the system ends up in a polimorphic equilibrium, then this is for sure the worst outcome possible. In fact, given the technology of the persuasion, in this case even more resources are destroyed in the production of the effort, since the productivity evaluation is in comparative terms, and thus it is never optimal to end up with a polimorphic population in these cases. Now, since in unproductive workers framework $p^B = 1$ is worst than $p^A = 1$, this means that the employer should never use rule 2 in this framework. PD like framework, on the contrary, is different since $p^B = 1$ is associated to an outcome better than $p^A = 1$, so that the employer, by using it may still hope to reach social optimum. Thus the employer under PD like framework would like to now which rule maximises the chances to get $p^B = 1$.
Call $\bar{\gamma} = 2 - \sqrt{2}$ and $\bar{\beta} = 1 - \frac{\sqrt{2-4\gamma+\gamma^2}}{\sqrt{2}}$. We can state the following:

Proposition 8: *If we are in a PD like framework under rule 1 then if $\gamma < \bar{\gamma}$ and $\beta > \bar{\beta}$ a steady state in which $p^B = 1$ is reached. Rule 2 always reduces the space (β, γ) such that $p^B = 1$ is reached.*

Proof. Proof in Appendix □

Proposition 8 underlines the role of persuasion rules in obtaining a better social outcome than $p^A = 1$: both rules are able to reach $p^B = 1$, as previously stated in proposition 4 and

corollary 6. Proposition 8 compare the results under the two rules stating that rule 1 ensures this for a larger region of parameters (β, γ).
Thus, considering the PD like framework, we have found that both rules help B agents if compared to the standard case. However under rule 2 heterogenous matching is always the worst outcome while this is not the case under rule 1, while rule 2 ensures covergence to $p^B = 1$ for a smaller area than rule 1. Thus if PD like framework holds employers should prefer rule 1 over rule 2.
We now perform a more general comparison between the two rules in order to understand how they change the parameter space in order to obtain the different long run outcomes. In order to do this we analyse the impact of each rule on all the thresholds, and thus we compare $\bar{\delta}_1$ with $\bar{\delta}_2$ and $\bar{\alpha}_1$ with $\bar{\alpha}_2$. If $\bar{\delta}_1 > \bar{\delta}_2$ then the space of (α, δ) such that $p = 1$ is unstable is bigger under rule 1 than rule 2, while if $\bar{\delta}_1 < \bar{\delta}_2$ then it is more likely to have $p = 1$ as stable equilbrium under rule 1. On the contrary, if $\bar{\alpha}_1 > \bar{\alpha}_2$ then the space (α, δ) such that $p = 0$ is unstable is higher under rule 1, and the opposite happens for $\bar{\alpha}_1 < \bar{\alpha}_2$. This becomes clear if we look back at figure 1 and see what happens if the thresholds indicated by the lines move along the space and in the same time, see what happens at the extensions of the different 4 areas.
Since the two rules have the same long run effects when the thresholds are the same, we plot the differences of these thresholds in the (β, γ) space in order to analyse the areas in which the long run outcomes of the two rules are different. We thus obtain the figure 4.

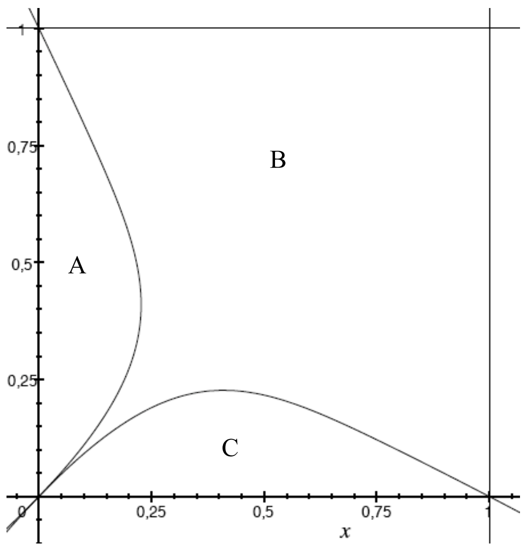

Figure 3.4:

In this figure we have β on the x axis and γ on the y axis. The curves represent the situation in which $\bar{\delta}_1 = \bar{\delta}_2$ and $\bar{\alpha}_1 = \bar{\alpha}_2$. Three cases can be identified:

- *Area A:* In this area $\bar{\delta}_1 > \bar{\delta}_2$ and $\bar{\alpha}_1 < \bar{\alpha}_2$. In this area rule 1, if compared to rule 2, restricts the space of parameters such that $p^B = 1$ is the stable long run outcome of the dynamics while expands the space for the $p^A = 1$ to be the unique stable steady state of the dynamics.

- *Area B:* In this area $\bar{\delta}_1 > \bar{\delta}_2$ and $\bar{\alpha}_1 > \bar{\alpha}_2$. Consequently rule 1 expands the parameter space such that there exists a stable polimorphic equilibrium and reduces the parameter space for the existence of an unstable polimorphic equilibrium.

- *Area C:* In this area $\bar{\delta}_1 < \bar{\delta}_2$ and $\bar{\alpha}_1 > \bar{\alpha}_2$. In this area rule 1, if compared to rule 2, expands the space of parameters such that $p^B = 1$ is the stable long run outcome of the dynamics while restricts the space for the $p^A = 1$ to be the unique stable steady state of the dynamics.

Since it does not exist an area in which both $\bar{\delta}_1 < \bar{\delta}_2$ and $\bar{\alpha}_1 < \bar{\alpha}_2$, thus we can state that:

Corollary 6: *Given the vector (β, γ), Rule 1, if compared with Rule 2, never restricts the space of (α, δ) such that a stable polimorphic equilibrium exists.*

Corollary 6 basically states that given a vector (β, γ) and a random realization of (α, δ), if rule 1 is applied then it is more likely to observe a stable polimorphic equilibrium than if rule 2 holds.
If we now analyse the PD like situation and the unproductive workers framework, we have that in both cases $\beta > \gamma$ so that only B and C areas can be reached given any parametrization. Thus we can state:

Corollary 7 *In both PD like and unproductive workers cases, rule 1 always expands the parameters space in which $p^B = 1$ is stable if compared with rule 2.*

Corollary 7 gives an intuition similar to proposition 8, and implies that in the PD like and unproductive Workers classes, using rule 1 helps, on average, agents B to be present in the long run in the society. The effect of this corollary over employer choice depends on the type of matrix he faces. If a PD matrix is shown, then the employer would like to have $p = 0$ as a stable outcome or a polimorphic steady state and to avoid having $p = 1$ as a stable oucome, so that proposition 7 can be used in order to understand if this is the case.
Thus, suppose that an employer faces a PD like matrix and has to choose between no rule, rule 1 or rule 2. We can state that rule 2 is always dominated by rule 1 since if a polimorphic equilibrium exists under rule 2 is always the worst outcome, and under rule 1 there is a broader region under which $p^B = 1$ is the unique stable steady state. If then the employer compares rule 1 with no rule, if $\alpha + \delta < \frac{\beta+\gamma}{2}$ then rule 1 is at least as good as no rule. If, on the other side $\alpha + \delta > \frac{\beta+\gamma}{2}$ then the choice depends on the specific parametrization since if a polimorphic equilibrium is shown then this can be inferior to $p^A = 1$.

3.5 Conclusion

In this paper, we proposed a model in which agents types spread in the society depending on how the type is fit and on how much agents are able to persuade about this fitness. In this way we first analysed what happens under a generic game with aspecific persuasion rules and we have found conditions for the existence of polimorphic equilibria so that we have proved that also a prisoner's dilemma can be compatible with these conditions. We then have analysed two different persuasion rules derived from cultural transmission literature and two matrix structures, one of which with a prisoner's dilemma structure. We have proved that under a prisoner's dilemma incentive system, if these persuasion mechanisms hold, then it is possible to reach the pareto efficient outcome and a polimoprhic equilibrium. We have then compared these two rues in order to study their total productivity properties. The employer-employee framework is just an exemplificative way of intending this framework that can be extended to a more general case. Moreover a deeper study of other possible classes of persuasion rules can be crucial, maybe introducing some referred also to conflict theory literature. Moreover, a deeper study over the process by which cultural traits are perceived as good and consequently spread in the society partially depending on their objective fitness is needed.

Appendix A: Proofs of the Propositions

Proof of Proposition 1: *Consider a heterogenous matching ij. Then* $\pi_t^{ij} = \beta q_t^{ij}(\bar{\tau}_t) + \gamma(1 - q_t^{ji}(\bar{\tau}_t)) - c(\tau_t^i)$, *with* $\beta q_t^{ij}(\bar{\tau}_t) + \gamma(1 - q_t^{ji}(\bar{\tau}_t))$ *begin the utility of playing a given* τ_t^{ij} *and* $c(\tau_t^i)$ *begin the cost. Call* $MU_{\tau_t^{ij}}$ *the marginal utility and* $MC_{\tau_t^{ij}}$ *the marginal cost of playing* τ_t^{ij}. *Since assumption 1 holds, then* $MU_{\tau_t^{ij}} \geq 0, \forall \tau_t^{ij}$ *and is monotonic in* τ_t^{ij}. *Thus if* $c'(\tau_t^{ij})|_{\tau_t^{ij}=0} = 0$ *and* $c'(\tau_t^{ij})|_{\tau_t^{ij}=1} > (\beta \frac{\partial q_t^{ij}(\bar{\tau}_t)}{\partial \tau_t^{ij}} - \gamma \frac{\partial q_t^{ji}(\bar{\tau}_t)}{\partial \tau_t^{ij}})|_{\tau_t^{ij}=0}$ *is convex enough, then there exist a unique optimal* τ_t^{ij*}, *and* $\tau_t^{ij*} \in (0,1)$.

Proof of Proposition 2:
Suppose that $\bar{q}_t$ *is independent from* p_t. *Given the set of equations 3.1 then also* $\bar{\pi}_t$ *is independent from p. Consider now the dynamics in 3.2. A polimorphic equilibrium exists if and only if there exists a* p^* *such that* $\pi^A = \pi^B$. *Reminding that* π^A *and* π^B *are averages payoffs, then this happen whenever* $p\alpha + (1-p)\pi_t^{AB} - p\pi_t^{BA} - (1-p)\delta = 0$. *Since this is linear in p then there exists at most one* p^* *such that this condition is satisfied.*

Proof of Proposition 3:
Since there exists at most one polimorphic equilibrium, then a necessary and sufficient condition in order to have a stable polimorphic equilibrium is that $p = 0$ *and* $p = 1$ *are unstable. Call* $\Delta_t \equiv \pi_t^{ij} - \pi_t^{ji}$, *then the first condition can be rewritten as* $\Delta_t|_{p=0} > 0$ *and* $\Delta_t|_{p=0} < 0$. *Given 3.2 we have that* $\Delta_{tp=0} = \beta q_t^{ij} + \gamma(1 - q_t^{ji}) - c(_t^{ij]) - \delta}$ *and* $\Delta_t|_{p=1} = \alpha - \beta(1 - q_t^{ij}) - \gamma q_t^{ji} + c(\tau_t^{ji})$.
A necessary condition for $\Delta_t|_{p=0} > 0$ *is that* $c(\tau_t^{ij]) < \beta q_t^{ij} + \gamma - \gamma q_t^{ji} - \delta}$. *Since* $c(\tau_t^{ij]) > 0}$, *in order to have this it is necessary that* $q_t^{ji} < \frac{\beta q_t^{ij} + \gamma - \delta}{\gamma}$. *Since* $q_t^{ji} \in [0,1]$ *it is necessary to have* $\frac{\beta q_t^{ij} + \gamma - \delta}{\gamma} > 0$. *This happens when* $q_t^{ij} > \frac{\delta - \gamma}{\beta}$. *Since now* $q_t^{ij} \in [0,1]$ *it is necessary to have* $\frac{\delta - \gamma}{\beta} < 1$ *and thus* $\beta + \gamma > \delta$
Using the same reasining, take $\Delta_t|_{p=0} < 0$: *this is satisfied when* $c(\tau_t^{ji}) < \gamma q_t^{ji} - \alpha + \beta - \beta q_t^{ij}$. *Since* $c(\tau_t^{ji}) > 0$ *it is necessary to have* $\gamma q_t^{ji} - \alpha + \beta - \beta q_t^{ij} > 0$. *This happens when* $q_t^{ij} < \frac{\gamma q_t^{ji} - \alpha + \beta}{\beta}$. *Since* $q_t^{ij} \in [0,1]$ *this is satisfied only if* $\frac{\gamma q_t^{ji} - \alpha + \beta}{\beta} > 0$, *tht is to say* $q_t^{ji} > \frac{\alpha - \beta}{\gamma}$. *Since* $q_t^{ji} \in [0,1]$ *then this happens only if* $\frac{\alpha - \beta}{\gamma} < 1$ *so that only if* $\beta + \gamma > \alpha$.

Proof of Proposition 4:
Consider agent i. Then, under rule 1 we have $\pi_t^{ij} = \beta\frac{1}{2}(1 + \tau_t^{ij}) + \gamma\frac{1}{2}(1 - \tau_t^{ji}) - \frac{\tau_t^{ij^2}}{2}$. *The first order condition for an internal solution are such that* $\tau_t^{ij*} = \frac{\beta}{2}$.
In the same way $\pi_t^{ji} = \beta\frac{1}{2}(1 - \tau_t^{ij}) + \gamma\frac{1}{2}(1 + \tau_t^{ji}) - \frac{\tau_t^{ji^2}}{2}$ *so that* $\tau_t^{ji*} = \frac{\gamma}{2}$.
We thus get:

$$\pi_t^{ij*} = p_t\alpha + (1 - p_t)[\frac{\beta}{2}(1 + \frac{\beta}{2}) + \frac{\gamma}{2}(1 - \frac{\gamma}{2}) - \frac{\beta^2}{8}]$$
$$\pi_t^{ji*} = (1 - p_t)\delta + p_t[\frac{\beta}{2}(1 - \frac{\beta}{2}) + \frac{\gamma}{2}(1 + \frac{\gamma}{2}) - \frac{\gamma^2}{8}]$$

Substituting into $\delta_t|_{p_t=0}$ *and* $\delta_t|_{p_t=1}$ *we get:*

$$\delta_t|_{p_t=0} = \frac{\beta}{2} + \frac{\beta^2}{4} + \frac{\gamma}{2} + \frac{\gamma^2}{4} + \frac{\beta^2}{8} - \delta$$
$$\delta_t|_{p_t=1} = -\frac{\beta}{2} + \frac{\beta^2}{4} - \frac{\gamma}{2} + \frac{\gamma^2}{4} + \frac{\gamma^2}{8} + \alpha$$

We thus have that $p = 0$ *is unstable if and only if* $\delta_t|_{p_t=0} > 0$, *that means*
$\delta < \frac{1}{8}(\beta(4 + \beta) + 2\gamma(2 - \gamma)) \equiv \bar{\delta}_1(\beta, \gamma)$.

In the same way $p = 1$ is unstable if and only if $\delta_t|_{p_t=1} < 0$, that means
$\alpha < \frac{1}{8}(\gamma(4+\gamma) + 2\beta(2-\beta)) \equiv \bar{\alpha}_1(\beta, \gamma)$.

Proof of Corollary 2:

Following the very same proof as the previous proposition, using Mathematica computations, we get:
$\bar{\delta}_2 = \frac{(2\beta-\beta^2+2\beta^3+2\beta^4-2\beta^2 g-4\beta^3 g-4\beta^4 g-2\beta g^2-\beta^2 g^2+10\beta^3\gamma^2+2\gamma^3+6\beta\gamma^3-6\beta^2\gamma^3-3\gamma^4-2b\beta\gamma^4+2\gamma^5)}{(2+4\beta^2+2\beta^4-8\beta\gamma-8\beta^3\gamma+4\gamma^2+12\beta^2\gamma^2-8\beta\gamma^3+2\gamma^4)}$
and
$\bar{\alpha}_2 = \frac{(2\beta^3-3\beta^4+2\beta^5+2\gamma-2\beta^2\gamma+6\beta^3\gamma-2\beta^4\gamma-\gamma^2-2\beta\gamma^2-\beta^2\gamma^2-6\beta^3\gamma^2+2\gamma^3-4\beta\gamma^3+10\beta^2\gamma^3+2\gamma^4-4\beta\gamma^4)}{(2+4\beta^2+2\beta^4-8\beta\gamma-8\beta^3\gamma+4\gamma^2+12\beta^2\gamma^2-8\beta\gamma^3+2\gamma^4)}$

Proof of Proposition 7:

Given Rule 1 the optimal effort levels are $\tau_t^A = \frac{\gamma}{2}$ and $\frac{\beta}{2}$. Thus the total productivity level in case of heterogeneous matching is $\beta + \gamma - \tau_t^{A*} - \tau_t^{B*} = \frac{\beta+\gamma}{2}$. Thus the total productivity in case of a polimorphic equilbrium is:
$\pi_{pol} = p^2 2\alpha + (1-p)^2 2\delta + 2p(1-p)\frac{\beta+\gamma}{2}$.
Suppose that $\alpha + \delta > \frac{\beta+\gamma}{2}$. Now we compare this outocme with the outcomes in the cases of no polimorphic equilibria. After some algebraic computations we get:
$\pi_{pol} > \pi_{AA}$ *if and only if* $p < \frac{2(\delta-\alpha)}{2\alpha+2\delta-\beta-\gamma} \equiv \bar{p}$
and
$\pi_{pol} > \pi_{BB}$ *if and only if* $p > \frac{4\delta-\beta-\gamma}{2\alpha+2\delta-\beta-\gamma} \equiv \underline{p}$
Now since it always happens that $\underline{p} > \bar{p}$ then it always happens that polimorphic equilibrum is not the optimal outocome.
Suppose now that $\alpha + \delta < \frac{\beta+\gamma}{2}$. Now $\pi_{pol} > \pi_{AA}$ if and only if $p > \frac{2(\alpha-\delta)}{\beta+\gamma-2\alpha-2\delta}$ and $\pi_{pol} > \pi_{BB}$ if and only if $p < \frac{\beta+\gamma-4\delta}{\beta+\gamma-2\alpha-2\delta}$. These conditions are satisfied only if $\gamma < \frac{1}{3}$ and $\delta < \frac{\beta+\gamma}{4}$. However if we substitute in this condition the optimal $p^* = \frac{8\delta-\beta(8-\beta+2\beta\gamma-2\gamma^2)}{8(\alpha+\delta-\beta-\gamma)+\beta^2+\gamma^2}$ and we consider the conditions for the existence of a polimorphic equilbrium ($\alpha < \bar{\alpha}_1$ and $\delta < \bar{\delta}_1$) then these are no more satisfied so that the polimorphic equilbrium is never optimal.
Moreover under the case in which $\alpha + \delta < \frac{\beta+\gamma}{2}$, then it happens that $\pi_{pol} > \pi_{AA}$. Since, by matrix structure, $\pi_{AA} < \pi_{BB}$ it follows that π_{AA} is the worsto outcome.
If we repeat the same for rule two, by substituting now $\tau_t^{A*} = \frac{\beta-\beta\gamma+\gamma^2}{1-(\beta+\gamma)^2}$ and $\tau_t^{B*} = \frac{\beta^2-\beta\gamma+\gamma}{1-(\beta+\gamma)^2}$, it always happens that $\pi_{pol} < \pi_{AA}$ and $\pi_{pol} < \pi_{BB}$ so that under rule 2 a polimorphic equilibrium is always the worst outcome.

Proof of Proposition 8:

The proof of this proposition is entirey based on algebraic computations. Consider the first part of the proposition: we have that, given the PD like matrix ordering, $\delta > \bar{\delta}_1$ and $\alpha < \bar{\alpha}_1$ if and only if $\gamma < \bar{\gamma} = 2 - \sqrt{2}$ and $\beta > \bar{\beta} = 1 - \frac{\sqrt{2-4\gamma+\gamma^2}}{\sqrt{2}}$.
Consider now the second rule. Then we have that $\delta > \bar{\delta}_1$ and $\alpha < \bar{\alpha}_1$ if and only if $\gamma < \bar{\gamma}_2$ and $\beta > \bar{\beta}_2$ with $\bar{\gamma}_2$ being the first root of the following polynomial equation:
$-1 + 4x - 6x^2 + 8x^3 - 6x^4 + 2x^5 = 0$
and $\bar{\beta}_2$ being the first root of the following polynomial equation:
$-\gamma^2(1+2\gamma-2\gamma^2+3\gamma^3)+(6\gamma^2-4\gamma^3+4\gamma^4)x+(-6\gamma-\gamma^2-2\gamma^3)x^2+(2+6\gamma+2\gamma^2)x^3+(-3-4\gamma)x^4+2x^5 = 0$.
Since it always happens that $\bar{\gamma}_2 < \bar{\gamma}$ and $\bar{\beta}_2 > \bar{\beta}$ then under rule 2 the (β, γ) space such that $p^B = 1$ is reduced.

Conclusions

The main message of this thesis is that evolution of cultural traits matters in the study of different phenomena so that a deeper attention should be devoted on the change of attitudes, preferences and social norms: in each chapter one of these elements has been analysed. In all the cases the endogenization of these dynamics seems to be a crucial element to be studied. In the first chapter the endogenization is just the last analysed step and it needs a deeper analysis. In particular future researches can be devoted to the mathematical solution of problems with endogenous socialization weights based on some cultural dystance rules. In this way the problem of children that do not choose their socialization weights can be solved. Moreover there is space in order to understand what happens if the structure of the interethnic relationships change with time with different mechanism than what we did here, so providing different endogeneization of socialization schemes. Similarly it would be interesting to study what happens if forms of socialization schemes other than reciprocity and ethnocentrism may be implemented. Again it could be interesting to analyse what happens if horizontal socializiation is taken into account into these schemes. An empirical analysis on some case studies, however, may be necessary.
The second paper introduced an endogenization in preference dynamics as a necessary condition in order to obtain partial segregation dynamics, identifying two different effects of social interactions on preferences. In this case an effort on the empirical analysis is needed in order to better distinguish between School Effect and Residential Effect. Moreover some simulations based analysis may be useful in order to understand how different and more complex residential segregation frameworks influence the prefernece dynamics and the long run preferences.
As far as the third contribution is concerned, the employer-employee framework is just an exemplificative way of intending this framework that can be extended to a more general case. Moreover a deeper study of other possible classes of persuasion rules can be crucial, maybe introducing some rules linked to conflict theory and hold up theory literature. Moreover, a deeper study over the process by which cultural traits are perceived as good and consequently spread in the society partially depending on their objective fitness is needed.

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