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Bipartite choices

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Abstract
This piece in the *Milestones* series is dedicated to the paper coauthored by David Gale and Lloyd Shapley and published in 1962 under the title "College admissions and the stability of marriage" on the *American Mathematical Monthly*.

Keywords Two-sided matching · Stable marriage problem · College admission problem · Bipartite graph

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Milestones This series celebrates key contributions that are at least 50 years old. A milestone is widely acknowledged as a seminal paper that has aged into a classic.

"What you choose also chooses you."
Kamand Kojouri

1 Introduction

In 1997 a well-known financial services company launched an advertising campaign for its credit card using the slogan "There are some things money can't buy. For

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everything else there's [the card]." The allusion that what really matters is priceless turned the campaign into a huge success.

Indeed, there are life-defining situations where money is of little use. If you seek a spouse, a job, admission to a university, or organ donation, money cannot buy your choice because "it is not enough to choose. You also have to be chosen" (Persson 2012).

A *two-sided matching* is a pairing that involves the consent of two sides. When the two sides are clearly distinct, as in the archetypal example of heterosexual marriages, the choices are bipartite: both the man and the woman must consent to being matched. If several partners populate the two sides, we have a (two-sided) *matching market* where agents compete for being matched with each other. "The key question [. . .] is: who gets what?" (Persson 2012).

In 1962, David Gale and Lloyd Shapley published on the *American Mathematical Monthly* the first scientific contribution using mathematical reasoning to make sense of matching markets. Their paper, titled "College admissions and the stability of marriage", defines the terms of the question and answers it. This is the milestone we celebrate here. For brevity, we indulge in the contemporary zeal for acronyms—especially by grant-funding institutions—and we call it CASM in the following.

If we are to believe the advice proffered to budding scholars, CASM bears no semblance of what goes for a professionally crafted scientific paper in economics or finance. It is just seven pages long. It has no abstract, no bibliographic references or literature review, no plans (or promises) for future research. It sports no mathematical formulas, and makes sparing use of greek letters (just four).

Notwithstanding such list of no-no's, the WoS Core Collection credits CASM with over 3100 citations. This suggests that the usual paraphernalia expected from a paper published on a scientific journal (including the present one) are not a necessary condition for significant impact.

CASM was not submitted to an economics journal. In fact, it studies matching markets without ever mentioning prices or money—not even the word "market" is ever used. Yet, when Roth (2008) wrote an exhaustive survey on the deferred acceptance algorithm proposed in CASM, he felt compelled to close it with a short unequivocal footnote: "Nobel committee, take note" (p. 564). His advice was heeded in 2012. Following the announcement, freelance journalist Joanna Rose interviewed Per Krusell, Chairman of the Economic Sciences Prize Committee. When she remarked "there is nothing about finance, or money," his retort managed to capture the most important message in CASM: "Economics is not always about money."

CASM's first attempt led to "reject and resubmit". Many of us have faced rejections before publication, including Nobel prize-winners in Economics; see Gans and Shepherd (1994). But CASM is an odd fellow, achieving the rare feat of being published twice (1962, and again in 2013) on the very same journal, the *American Mathematical Monthly*.

The *Monthly* is a well-respected journal founded in 1894. At the start of his tenure as Editor, Halmos explained that "since its beginning, the Monthly has been dedicated to the advancement and promotion of college mathematics. The founders of the Monthly set out to create a publication that would be neither a research journal nor one devoted primarily to educational and pedagogical topics" (1982, p. 3). CASM dwells comfortably in the middle ground between technical research and expository writing. Using far less than college mathematics, CASM attests that “the unreasonable effectiveness of mathematics for the natural sciences” claimed by the physicist Eugene Wigner in 1960 extends to the social sciences.

The republication of CASM is mostly due to the commotion stirred by the Nobel prize. Yet, in 2013 the *Monthly*’s notice to authors—unchanged since 1997—read as follows “the Monthly’s readers expect a high standard of exposition; they look for articles that inform, stimulate, challenge, enlighten, and even entertain.” CASM has not aged: it still fully matches each of these expectations as at the time of its first publication. If you have never read it, pick it up and be ready to be entertained.

CASM has turned 60 years old. Our tribute is arranged around five periods in CASM’s long life. Section 2 covers its origins and structure. Section 3 narrates its first steps, taken in the company of mathematicians and computer scientists. Section 4 recounts how, upon CASM’s coming of age, its charms attracted the attention of economists. Section 5 narrates its maturity, when a serendipitous coincidence led CASM into an unexpected professional course of life. Section 6 dwells on its silver-haired years. Along the way we point the reader to some surveys spanning the rich subject of two-sided matching, but this is no attempt to write one.

2 Origins

2.1 The authors

CASM was brought forth by David Gale (1921–2008) and Lloyd Shapley (1923–2016). This is not the place to showcase their scientific stature, but we share some peripheral tidbits about them.

David Gale received his Ph.D. in Mathematics from Princeton University in 1949, under the supervision of Albert Tucker. The Mathematics Genealogy Project (`www.mathgenealogy.org`) credits him with 16 students, two of which eventually became economists of great renown. His personality and scientific achievements are commemorated in a special section on *Games and Economic Behavior* (vol. 66, 2009); see especially Sobel (2009).

Gale is credited with 99 publications. We only mention three among the lesser known. Gale (1953) is probably the first formal argument applying backwards induction to games in extensive form, after the method suggested for zero-sum games in von Neumann and Morgenstern (1944). Eisenberg and Gale (1959) develops a market that aggregates individual probability assessment into a consensus using pari-mutuel betting. It has sparked interest among computer scientists (Jain and Vaziarni 2010), and it may be waiting to be rediscovered by the economists.

Gale (2009) reveals a playful personality, and it is a trove of historical anecdotes. We are introduced to four topological games, named Nash, Milnor, Shapley and Gale after their inventors. These games were born in 1948–49 in Princeton, where their namesakes were all students. We learn that Gale built a 14×14 board for the Nash game and donated it to the Fine Hall common room where the game “was an immediate hit” (p. 648). The Nash game, better known as Hex, had already been discovered in 1942 by the Danish polymath and poet Piet Hein. Coincidentally, another game invented by Gale, christened Chomp by Gardner (1973), turned out to be isomorphic to a game of divisors previously proposed by the Dutch mathematician Schuh (1952).

Lloyd S. Shapley (1923–2016) received his Ph.D. in Mathematics from Princeton University in 1953, under the supervision (again) of Albert Tucker. The Mathematics Genealogy Project credits him with 11 students. His personality and scientific achievements are commemorated in a special section on *Games and Economic Behavior* (vol. 108, 2018); see especially Levine (2018).

Shapley is credited with 135 publications, and is a legendary figure in game theory. In his 2005 Nobel Lecture, Robert Aumann called him "the greatest game theorist of all time." Shapley (1953), published on the same year of his dissertation, has garnered over 5100 citations according to Scopus and made the Shapley value an eponym familiar to any game theorist.

Shapley's comment after being awarded the Nobel prize in Economics gives a glint of his forbidding scientific attitude: "I consider myself a mathematician and the award is for economics. I never, never in my life took a course in economics."[1] He served in the United States Army Air Corps, where he broke a code for Soviet weather reports; for this, he received the Bronze Star, a promotion to corporal, and a $4 raise on the monthly pay. According to one of his sons, Shapley mentioned that at the time the raise seemed the most important reward.

2.2 Conception

As the story goes, "it all started with an article in the New Yorker magazine, 10 September 1960, in which a reporter spent several weeks observing the operation of the undergraduate admissions office of Yale University" (Gale 2001, p. 237). Admission officers send out offers ignoring how many students would accept them, and students may attempt to boost their chances faking their first choice. This turns a delicate and important process "into a big guessing game" (p. 238), with a great risk of a suboptimal final outcome.

Gale reduced the problem to a special case where each university has a quota of one. The essential feature is that we are dealing with a (bipartite) matching problem, where partners come from two distinct sets. Soon, this led to a colorful change of scenario, where girls and boys are to be matched for a dance or for marriage. (Heterosexuality is a convention assumed to preserve the bipartite structure.)

Unable to prove the existence of a solution, Gale sent out a letter seeking help to his classmate from Princeton. On 11 October 1960, Shapley "received [the] letter at noon and sent off his reply by 4:00 the same day" (Levine 2018, p. 7).[2] In Shapley's characteristic sharp style, the letter is less than one page long; it opens with "A stable connubiation [sic] always exists. Here is a constructive proof." CASM had been

[1] A few years later, his cowinner Roth mused: "I'm hardly alone among economists of my generation who don't have degrees in economics. A bunch of us rode in on the wave that brought game theory into economics" (Roth 2018, p. 1609).

[2] Notwithstanding his many other achievements, Shapley was awarded the Nobel prize in Economics for CASM. An afternoon's work from a mathematician may go a long way.

conceived. In a second letter dated 4 November 1960, Shapley “gratefully accept[s Gale’s] offer to write a draft of something” because, “without [his] instigation, [he] would do nothing at all.” And Gale was to play midwife for CASM.

2.3 The art of paper maintenance

What lies behind the sempiternal freshness of CASM? A short answer is that it successfully blends four features. First, it has a practical motivation: making the university admission process work better. Moreover, most readers are likely to feel that this problem is compelling, having had a first-hand experience of it. Second, the paper is elegantly written as a narrative, using verbal cues and metaphors to keep the mathematical reasoning buoyant. The clever device of presenting bipartite choices as “marriages” conjures countless examples in the reader’s mind.

Third, the style is pleasantly didascalic. The reader feels reassured that the intent of the paper is not to erect walls before his understanding, but to gently led him towards the solution. Fourth, the paper exercises restraint and trades generality for clarity. The problem is made as simple as possible and the proofs are constructive; some extensions are not mentioned—if one understands the main argument, she can work on them later. Science rhymes with essence.

If we read CASM to be entertained, this answer may suffice. If we read it (as I hope you do) to discover the art of writing a paper as timeless as the mathematics it delivers, we might need to look inside its gears.

One of my favorite instruction manuals explains: “The [. . .] reader is especially encouraged if each article has an introduction [. . .] saying informally what the article is about and how it fits in with more familiar material. Avoid a tight "definition-theorem-proof" format; [. . .] gladly trade brevity for clarity. Avoid too many definitions, especially in the first paragraph or on the first page, and remember, throughout the article, that most people understand words much more quickly and easily than formulas. Keep your sentences short and simple. Do your best to arrange the article so that it doesn’t seem to ramble but hangs together and gives the feeling of an organized structure” (Halmos 1982, p. 3).

CASM, written 20 years earlier, fits all these specifications—moreover, it adds clever cliffhangers before each new section (no spoilers). Section 1 opens with a familiar situation (college admission) and states the problem (admission is fraught with risks for either side). The last paragraph of the section states a bond with the reader—and doubles up as an abstract. “We shall describe a procedure for assigning applicants to colleges which should be satisfactory to both groups, which removes all uncertainties and which, assuming there are enough applicants, assigns to each college precisely its quota” (p. 9).

Section 2 sets up the model (assuming no ties in preferences) and qualifies the question to be investigated: “we wish to determine an assignment of applicants to colleges in accordance with some agreed-upon criterion of fairness” (p. 10). A verbal example shows that swapping applicants across colleges may be Pareto improving, followed by the remark “that the colleges exist for the students rather than the other way around”. These two respectively set the ground for the two definitions that are the gist of this section. The first appears at about 20% of the paper: a matching is *stable* if there are no applicants' swaps that are Pareto improving. The earlier example shows why unstable matchings cannot be satisfactory solutions. The second (and last) definition in the paper formalizes fairness as a stable assignment that is unanimously preferred by applicants to any other stable assignment.

Section 3 shifts the focus from university admission to the marriage problem and inquires about the existence of stable matchings for any profile of preferences. Three examples show that: (a) there may be multiple stable matchings; (b) there may be a unique stable matching where nobody "can get his or her first choice"[3]; (c) bipartite choices (modeled as heterosexual marriages) are a necessary condition for existence. Theorem 1 states that a stable matching always exists, and gives Shapley's constructive proof based on the *deferred acceptance* procedure.[4]

Section 4 dispatches the extension of the existence result to the college admission problem. Section 5 proves by induction that the applicant optimal assignment always exists. Parenthetically, it notes that a similar argument proves the existence of the college optimal assignment. These two sections take about one page: having prepared the ground, the answers flow naturally for the more general case. The progression relies on an additional assumption on colleges' preferences: even though CASM states it, an hurried reader may overlook it; see Roth (1985a).

At this point, the paper has presented two definitions (stability and optimality), two existence theorems, and one algorithmic procedure to find the optimal stable assignment. But there is still one more section of concluding remarks, independent from the rest of the paper and yet motivated by it. If CASM were a fable, this would be its moral.

Section 6 begins with a vindication of applied mathematics, and then moves "to raise the old question once more, [what] is mathematics?" (p. 15). If you could read only one page of CASM, choose this. It is so compelling that The Economist (2012) reprinted almost two paragraphs of it in a piece titled "Priceless".

3 A mathematical upbringing

Most likely, the first scholars to build on CASM were McVitie and Wilson. Their (1971) paper modifies the deferred acceptance procedure to find all stable marriage assignments. Another paper (1970a, 1970b) extends its scope to the case when men and women are not in equal numbers, validating CASM's comment that having "the same number of boys and girls is not essential" (p. 13). More dramatically, this latter paper proves that any person unmatched in a stable marriage solution remains unmatched in any other stable configuration. One can only wonder how many times some corollary

[3] We surmise that this is a common starting point for drama: a (unique) stable matching where everybody feels unfulfilled.

[4] From Shapley's letter: "Let each boy propose to his best girl. Let each girl with several proposals reject all but her favorite, but defer acceptance until she is sure no one better will come her way. The rejected boys then propose to their next-best choices, and so on, until there are no girls with more than one suitor. Marry" (Levine 2018, p. 8).

of this has been used to console an unwilling single, tormented by doubts on whether things might have turned out differently.

In light of the future developments discussed in Sect. 4, their (1970a) paper is prescient. They set out to discover whether their algorithms could be used in practice to solve the admission problem for British universities. At that time, the admission procedure was run through a clearinghouse called Universities Central Council on Admissions (UCCA), active from 1961. In 1992 UCCA merged with an analogous institution for polytechnics and became the Universities and Colleges Admissions Service (UCAS), that is still currently in service. Using a tenth-scale model on a KDF9 computer,[5] they concluded that porting the admission procedure on "modern computers" was "technically feasible" (p. 430).

Meanwhile, although Shapley never spent much time with CASM, he inaugurated another line of research that may be seen as a distant sibling. CASM is concerned with two-sided matching, where each agent on one side has preferences over the other side. Shapley and Scarf (1974) introduce a one-sided matching problem with indivisible goods. Each agent owns a house (or another indivisible good) and has preferences over the houses in the market. Each agent can own only one house, and there is no money or other medium of exchange. The market is bipartite because there are agents and houses, but the matching is one-sided because only agents have preferences over houses. The market should "redistribute the ownership of the indivisible goods, in accordance with the [...] preferences of the traders" (p. 24).

Shapley and Scarf (1974) prove that this housing market has a nonempty core. Then they give a second proof by producing "a simple constructive method for finding competitive prices" that support a core allocation. This second approach relies on the Top Trading Cycle algorithm suggested to them by Gale, who thus returned Shapley's favor from 1960. In the last section, they point out that the core of the marriage problem is the set of all stable matchings, and go on to describe other models with indivisibilities.

But CASM came to the attention of a wider community through seven expository lectures given by Donald Knuth at the University of Montréal in 1975 and collected in a booklet. A legendary figure in computer science, Knuth initiated the field of analysis of algorithms and gave it a rigorous mathematical foundation. He created and donated the TEX typesetting software to the world community, rebooting the writing routines of countless scholars. And he has always been a brilliantly lucid expositor.

Knuth summarizes his booklet as "a gentle introduction to the analysis of algorithms, using the beautiful theory of stable marriages as a vehicle to explain the basic paradigms of that subject." (www-cs-faculty.stanford.edu/~knuth/ms.html) His keen eye, attracted by CASM's elegance, recognized that "the problem of stable marriage seems ideal [...] since it does not require any prior knowledge of algorithmics and [...] leads naturally to an illustration of the essential techniques of algorithmic analysis" (Knuth 1997, p. xi). With characteristic understatement, Knuth notes that "the level of the discussion [...] is elementary and requires no prior experience either in analysis of algorithms or in marriage."

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5 The KDF9 was built for mathematical and scientific processing. Its logic circuits were entirely solid-state. It weighed 4.7 tonnes. Between 1964 and 1980 only 29 machines were produced.

In fact, CASM is graced with the awareness of computational issues. It makes a passing remark that the deferred acceptance algorithm for a marriage problem with n men and n women requires at most $n^2 - 2n + 2$ stages and provides an example with $n = 4$ where the bound is tight. In other words, the algorithm in CASM has polynomial complexity.

Using the stable marriage problem as a workhorse, Knuth branches out to introduce (and explain) different methods in the analysis of algorithms, drawn from areas as diverse as combinatorics, probability, and computational complexity. Until, in the last lecture, he raises the bar and proposes 12 research questions "far from the level of the famous problems presented by Hilbert in [...] 1900, but [...] worthy of interest and [...] likely to be solved in a finite time" (Knuth 1997, p. 55).

Mathematicians went to their boards: new results began to flow, and CASM branched out in many directions.

Knuth (1997) attributes to Conway the result that the set of stable matchings for the marriage problem (assuming strict preferences) is a lattice with respect to the partial order defined by men's preferences, where the maximum (minimum) element is the men-optimal (women-optimal) matching. This order-theoretic structure surfaces in many later arguments. Its far-reaching consequences include the surprising result that any distributive lattice is isomorphic to the lattice of stable matchings of some marriage problem; see Blair (1984).

Vande Vate (1989) shows that the stable matchings in a marriage problem are the extreme points of a polytope, and thus bring linear programming techniques to the solution of the stable marriage problem. The paper has a lovely title that links linear programming to marital bliss, but its claim to extend "Dantzig's observation by showing that [...] monogamous stable marriage is the best" ignores the original quote's context. Dantzig (1963, p. 322) tells a story where, in an attempt to help a reporter squeeze an interesting story out of his work, he claims that a "by-product [...] is a mathematical proof that monogamy is the best" form of marriage; but the same paragraph ends with the reporter "shaking his head in the negative, 'you've been working with the wrong kind of models'."

Another line of research recasts the stable marriage problem as a problem in graph theory. Abeledo and Isaak (1991), in particular, explain that the bipartite structure is essential to CASM's results: when agents are nodes in a graph where links represent the eligible pairs, the general existence of stable matchings requires the graph to be bipartite.

In the (1997) English translation of his booklet (originally published in French in 1981), Knuth acknowledges that over two decades "the theory of stable marriages has advanced greatly [...] and significant new results continue to be discovered" (p. 69). For example, Knuth et al. (1990) use probabilistic analysis to prove that, when preferences are random, with probability 1 the number of spouses available for a person across all stable matchings is between $(1/2)\ln n$ and $\ln n$.

A fair picture of the mathematical upbringing of CASM is in the survey by Gusfield and Irving (1989), who take the viewpoint of discrete mathematics and computer science, including the status of Knuth's original 12 questions, and providing an updated list of 12 open problems. This is nicely complemented by Roth and Sotomayor (1990), whose extensive survey opts for a game-theoretic perspective.

4 Making friends

Meanwhile, CASM had begun to attract the attention of another group of scholars, who developed their insights away from the analysis of algorithms or pure mathematics. Many had a more or less direct connection with David Gale, working at the crossroads of mathematics, economics and operations research.

4.1 Machiavellian courtships

In his 20 October 1960 reply to Shapley's short note, Gale conjectures that the deferred acceptance procedure leading to the men-optimal matching cannot be profitably manipulated by men, but that women might be able to get a better outcome by faking different preferences. In the original document, we find an emphatic "yes!" handwritten by Shapley (Levine 2018, p. 9). However, except for a passing remark about students who can misrepresent their first choices, CASM pretend that preferences are known or that agents will honestly reveal them to the matching algorithm.

There has long been an understanding that interested parties may game a collective decision. Given CASM's topic, an appropriate example is the concern that moved Charles Dodgson (aka Lewis Carroll) to write three pamphlets between 1873 and 1876, after some troubling decisions about student admissions taken by the faculty at the Christ Church college. He worried that the actions of a few skilled people might prevail over the wish of the majority.

In the '70 s, Gibbard (1973) and Satterthwaite (1975) independently gave formal proofs that non-dictatorial voting schemes involving more than two candidates may be profitably manipulated by electors who misrepresent their preferences. This led to a flurry of activity exploring the manipulability of matching procedures.

Dubins and Freedman (1981) bring in Machiavelli (and co-conspirators) and study how much havoc their scheming can bring into the college admission problem. They speak colorfully of fair play and foul play. Gale and Sotomayor (1985) undertake a similar analysis for the stable marriage problems. They incidentally warn that these theoretical results may be a matter of concern for an existing procedure that assigns graduates of medical schools to hospital programs where "they are to fulfill a residency requirement" (p. 262). We return to this important watershed in the next section.

In both cases, the deferred acceptance procedure may lead some agents to fake their preferences and achieve a better matching for themselves. As put by Bergstrom and Manning (1983), "any [...] follower of the soap operas will notice that [...] we have neglected the rich possibilities for duplicity in courtship. A natural question [...] is whether courtship à la Gale and Shapley is cheatproof." They worked on this question, and proved that there exists no mechanism for the stable marriage problem that ensures a stable matching and simultaneously discourages agents from cheating. Their work was never published because, when they sent it to a journal, they learned that Roth (1982) had won the race. Graciously, they acknowledge that "Roth's paper was not only earlier, but deeper and better."[6]

[6] The online introduction to Bergstrom and Manning (1983) incorrectly refers to Roth (1984a) instead of Roth (1982).

4.2 The Monthly retreat

Besides the Machiavellian contributions, over the years the Monthly has published at least two other papers related to CASM. We mention them in passing.

Balinski and Raitier (1998) popularize an alternative approach based on directed graphs, after the observation in Maffray (1992) that a stable matching can be viewed as the kernel of a directed graph. The approach is developed in Balinski and Raitier (1997), who take this opportunity to revisit and survey many known results. They speak modestly of "pictures at an exhibition" (p. 576), but their pictures are elegantly arranged in a coherent structure.

Balinski and Raitier (1998) praise "the light and airy style of [CASM. . .] in stark contrast with the awkwardness of the formalities and the weight of the notation of much of the ensuing work" (1998, p. 461). Curiously, the review of their (1997) paper published on MathSciNet exemplifies this contrast by reformulating CASM's "light and airy" statement of the stable matching problem as follows: "Given a bipartite graph $G = (V, E)$ and, for every vertex $v \in V$, a total order $>_v$ on the edges incident with v, a matching $M \subseteq E$ is stable if for any edge $e = (m, v) \in E \backslash M$ either m is matched with an edge f such that $f|_m e$ or w is matched with an edge g such that $g >_w e$."

The latest entry in the Monthly is Deijfen et al. (2017), whose title put frogs next to the stable marriage problem. The paper makes no mention of a prince (Grimm and Grimm 1812).

4.3 Money matters

A posthumous joke by Goethe says that "Mathematicians are like a certain type of Frenchmen: whatever you say to them they translate into their own language and then it is something completely different." Economic theorists oblige this saying in their own way: they interpret CASM as the study of a two-sided market for indivisible goods where money is not allowed. Reading CASM under this perspective, the deferred acceptance procedure is a decentralized (priceless) mechanism that achieves a stable matching.

Shapley and Shubik (1971) bring money back into a two-sided market, allowing for side payments between buyers and sellers who trade indivisible goods. (Think of introducing dowries in the marriage game.) They call their model an *assignment game*, because the core of their market can be computed as the set of solutions for the (linear programming) dual of the optimal assignment[7] problem (Koopmans and Beckmann 1957). They prove that the core is never empty, but "only rarely consists of just a single imputation" (p. 119). Moreover, the core of any assignment games coincides with the set of its competitive price equilibria. Exploring the structure of the core, they find that it contains both a seller-optimal and a buyer-optimal competitive equilibrium.

Calling on CASM, they uncover a deep analogy. The set of stable matchings in the marriage problem coincides with its core. Under the deferred acceptance procedure,

[7] Our paper conflates assignments and matchings, as in the old literature. Nowadays, it is customary to speak of matchings when both sides of the market have preferences, and of assignments otherwise.

letting men propose leads to the men-optimal allocation (and the opposite holds if women propose). Therefore, the choice of the proposing side in CASM decides which sex fares better in the marriage problem. Just as well, in the assignment game, having buyers bid prices up achieves the buyers-optimal allocation; and, switching roles, having sellers bring prices down leads to the seller-optimal allocation. Shapley and Shubik (1971) conclude that indivisibilities create a conflict of interest between the two sides that prices alone cannot solve.

Contrary to the case where the competitive equilibrium is unique, the richness of the core for two-sided problems with indivisibilities "may allow market institutions to influence equilibrium outcomes in a systematic and perhaps unsuspected way" (Crawford and Knoer 1981, p. 439). As they say, the devil may lurk in the details.

The intuition that there is a polarization of interests between the two sides persists for several many-to-one matching analogs of the college admission problem where money is involved. Kelso and Crawford (1982) consider a market where each (salaried) worker must be employed only by one firm and study a wage adjustment process where firms offer workers progressively higher salaries. This implicit auction can be interpreted as a firm-proposing deferred acceptance algorithm. Roth (1985b) restores symmetry and show that the polarization of interests persists even if a worker is allowed to time-share across firms and is paid from each of them. Demange and Gale (1985) describe the limitations to successful agents' manipulation of the equilibrium outcomes.

More recently, Hatfield and Milgrom (2005) provide a unified framework for markets with indivisibilities with and without (monetary) transfers, including auctions. Fleiner (2003) clarifies the mathematical foundation of this generalization, calling on lattice theory and a fixed-point approach. Chiappori (2017) surveys the theory of matching with transfers, and applies its tools to current problems in family economics.

5 Professional life

5.1 A turn of events

Medical internships were introduced at the turn of the 20th century. Postgraduates viewed them as an opportunity for professional advancement and hospitals were happy to access a fresh supply of inexpensive labor.

The assignment of internships at hospitals shares many features with the college admission problem. In practice, different attempts to arrange the system of assignments ran into serious problems, until in 1951 the Association of American Medical Colleges introduced a centralized procedure currently known now as the National Resident Matching Program (NRMP, for short); see Roth (2003).

Unbeknownst to Gale and Shapley, the NRMP was discovered by trial-and-error. And yet it shares a common approach with the deferred acceptance procedure, conceived out of mathematical reasoning: "in 1975, when [Gale] finished a talk about the stable marriage problem, some physician [...] told him that the Gale-Shapley algorithm was very similar to the one [...] used by the National Resident Matching Program." After some inquiries, Gale realized that their procedure "was mathematically equivalent [...], but in the reverse: instead of producing the optimal stable matching for the students it produced the optimal stable matching for the hospitals. This fact was spread orally" (Sotomayor 2008).

Roth (1984b) turned the oral tradition into a case study in game theory. He reviews the market failures preceding NRMP's introduction and shows that the NRMP acts as a centralized clearinghouse that yields the hospital-optimal stable matching. This study initiated CASM to practical applications, and was instrumental in the process that turned CASM's insights into a technical powerhouse.

Later on, Roth compared analogous clearinghouses set up by the British National Health Service across several regions: their local differences helped him to pinpoint the successful features. The overarching conclusion is that "deferred acceptance algorithms [...] capture a "folk idea" of how [orderly] markets operate" (Roth 2008, p. 550).

Meanwhile, the NRMP had begun to face new challenges. For example, the social background had changed. In 1951, there were (predominantly) male doctors seeking single internships: their families usually followed them. Forty years later, there were many (romantically involved) couples of doctors: they looked for complementary arrangements, so as to grow a family together without giving up on their individual careers.

In 1995 the Board of Directors of the NRMP commissioned a new algorithm, that was inaugurated in 1998. Roth and Peranson (1999) report on the long and difficult work of "designing, testing, and evaluating the new clearinghouse algorithm" (p. 749). Their design choices were to impact many lives: at the time of their work, the NRMP used to annually fill about 20,000 positions.

Meanwhile, in 1994, the Federal Communications Commission charged Paul Milgrom and Robert Wilson[8] with the design of an auction for the rights to a portion of the electromagnetic spectrum in the USA. Their design was a success, raising about $617 million.

This confluence of applied work on auctions and clearinghouses made clear that "economists [are] called upon not only to analyze markets, but to design them" (Roth 2002, p. 1341). There is an intermediate ground between the normative and the descriptive approach to economics, concerned with the "design and maintenance of markets and other economic institutions." Roth (2002) calls it *design economics*, advocating an engineering mindset. About a decade later, the first handbook on market design was published; see Vulkan et al. (2013).

5.2 More applications

Over 50 years before, CASM had modestly acknowledged that "the practical-minded reader may rightfully ask whether any contribution has been made toward an actual solution of the original problem" (p. 14). To this question, CASM puts forth the opinion that "some of the ideas introduced here might usefully be applied." We mention three of the several domains where CASM's ideas have borne fruit. Each domain has a

[8] Roth (2002) acknowledges Wilson as the Dean of Design. Their thoughts about game theory and market design are lively juxtaposed in Roth and Wilson (2019).

dedicated literature, that we cannot review here; see the surveys by Sönmez and Ünver (2011) or by Abdulkadiroğlu and Sönmez (2013).

Student placement CASM viewed college admissions as the outcome of a decentralized process, as it is typical in the USA Balinski and Sönmez (1999) took inspiration from the case of Turkey, where the assignment of high school graduates to colleges is centralized, to define the *student placement problem*. The main difference is that colleges "have no say in the admissions process" (p. 75): the students submit their preferences and their allocation takes into account priority orderings based on their exam scores. They pinpoint some "serious deficiencies" in the placement mechanism adopted in Turkey and propose an alternative procedure (drawn from CASM), proving that it is the "best mechanism to use in this context" (p. 74).

School choice The assignment of children to public schools has a similar structure to the student placement problem. Children (or, more likely, their parents) have preferences over different programs, while schools are usually passive players.[9] However, there may be many constraints on parents' choices, with different priorities determined by school location (e.g., intra-district or inter-district) and demographics (e.g., having a sibling in the same school). Especially in the USA, there are also concerns about segregation; see *Brown v. Board of Education*. After Abdulkadiroğlu and Sönmez (2003) formulated the *school choice problem*, many U.S. cities reviewed their assignment mechanisms. In particular, in 2005 Boston adopted CASM's student-proposing deferred acceptance procedure.

Kidney exchange Kidney donation increases the chances for transplants. However, when patient A finds a willing donor a, there may still be immunological incompatibilities. In 1986, the medical doctor Rapaport proposed kidney exchanges by which two patients with incompatible donors swap the donors and receive a compatible kidney.[10] Unorganized initiatives sprung out after consensus on the ethical acceptability of this practice materialized, until Roth et al. (2004) observed that paired kidney exchanges are the simplest special cases of the Top Trading Cycle algorithm suggested by Gale in Shapley and Scarf (1974).

The notoriety gained by Roth with the NRMP facilitated the implementation of structured approaches to kidney paired exchanges, expanding their scope both geographically (from regional to national coverage) and numerically (from pairs to longer chains). On July 30, 2008, the Alliance for Paired Donation announced its "first three-state kidney exchange with patients in North Carolina, Colorado, and Alabama receiving a living donor kidney transplant simultaneously."

In a testament to the power of mathematics and to human ingenuity, only three years later, "in August 2011, a complete stranger [. . .] from Riverside, California [. . .] offers his left kidney as an [. . .] altruistic gesture. The recipient's niece is [. . .] asked to donate her kidney to an unknown woman [and so on . . .] The chain does not come to an end until 60 coordinated transplants have taken place across the entire United States" (Persson 2012). One good samaritan and 29 donors have donated kidneys to 30 patients in one single gargantuan feat.

9 There are exceptions: f.i., schools in New York City are active players, whose preferences affect the final allocation.

10 There is no space to go over the many medical and ethical details; see f.i. Chapter 4 in Vulkan et al. (2013).

6 Silver hair

Time has put on CASM's hair a tinge of silver and some laurels. At 60 years' age, CASM is credited by Scopus with over 3600 citations scattered across different fields: each of computer science, mathematics, economics, decisions sciences, and social sciences contribute to the total count with no less than 400.

In 2012, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (popularly known as the Nobel Prize in Economics) was jointly awarded to Shapley and Roth "for the theory of stable allocations and the practice of market design". The information to the public released by the Royal Swedish Academy of Sciences (2012) is titled "Stable matching: Theory, evidence, and practical design" and CASM takes centerpiece in the section on Matching theory.

Gale could not partake of the award because he had died in 2008. In his obituary (www.berkeley.edu/news/media/releases/2008/03/18_galeobit.shtml), we read that Roth "had nominated Gale and Shapley to the Nobel Committee for Economics" because "it was past time that [they] share that award." The award ceremony speech acknowledges that Shapley and "Gale are the founders of matching theory, and the deferred-acceptance algorithm [...] is the cornerstone on which theory and applications rest" (Persson 2012).

The prize-winners' contributions are saluted as "one of those unexpected journeys, from basic research motivated by sheer curiosity, to practical use for the benefit of mankind." CASM, born out of sheer curiosity, was the first step in this long journey. Some of the most recent stages are surveyed by Kojima (2017) and Fenoaltea et al. (2021) put forward "an interdisciplinary review" from the viewpoint of physicists' complexity.

In practical applications, an increasingly important approach is to circumvent some theoretical impossibilities by studying approximate market designs: when it is not possible to guarantee a set of desirable properties, one may content to show that they hold with high probability, especially as the market grows large. Moreover, generalizations are being introduced to consider increasingly complex constraints of a regulatory or technical nature. From a more theoretical perspective, instead, the literature has begun to study continuum versions of matching markets, with and without transfers; see Greinecker and Kah (2021).

CASM's insights are being rediscovered under alternative market structures. Problem 11 in Knuth (1997), posed in 1975, asks if CASM's result on the existence of stable matchings could be generalized when three kinds of agents (e.g., men, women, dogs) must form threesomes. Alkan (1988) answers in the negative, for $n = 3$ or more. Twenty years later, however, Ostrovsky (2008) has found that CASM's insights generalize when different classes of agents (or markets) are vertically connected, as in supply chain networks. Rostek and Yoder (2020) study matching environments with complementarities that are inspired by patent licensing or social media examples.

Finally, new technologies are enabling new kinds of matching markets. There now exist portals that match short-term travelers and hosts or passengers with nearby drivers. In this situation, swift changes in demand or supply require procedures that can handle matching dynamically. The literature on this problem is still limited; see f.i. Baccara et al. (2020) or Doval (2022).

6.1 Piecemeal social engineering

CASM's main legacy is probably the emergence of a new discipline called market design. Its achievements and challenges are summarized in Roth (2018); see also Kominers et al. (2017). Van Basshuysen (2022) adds a critical perspective focused on the history of economic ideas.

By now, it should be clear that Alvin Roth played an instrumental role in realizing CASM's potential. He has been running an informative and renowned blog on market design since 2008 (marketdesigner.blogspot.com). Roth (2002) is a manifesto, where suspension bridges and the troubled relationship between surgery and medicine are used to illustrate the potential complications of designing institutions.

Here is how Roth (2018) distills his experience: "Markets and marketplaces are like languages; both are ancient human artifacts. Whole languages are hard to redesign, but smaller parts, e.g., technical vocabularies, are easier. And so it is with marketplaces: a marketplace is a piece of the market, not the whole. Marketplace designers don't have control over the whole strategy space: market participants have lots of options" (p. 1646).

This analogy reminds me of an approach that initially appeared on a paper in 1936, and was later updated and published in a book. I think that this approach exemplifies CASM's spirit, and was passed on as part of its heritage. Read on, and I will reveal you the author at the end.

"Piecemeal social engineering resembles physical engineering in regarding the ends as beyond the province of technology. (All that technology may say about ends is whether or not they are compatible with each other or realizable.) [. . .] Just as the main task of the physical engineer is to design machines and to remodel and service them, the task of the piecemeal social engineer is to design social institutions, and to reconstruct and run those already in existence.

The piecemeal technologist or engineer recognizes that only a minority of social institutions are consciously designed while the vast majority have just 'grown', as the undesigned results of human actions. [. . .] The technologist should study the differences as well as the similarities, expressing his results in the form of hypotheses. And indeed, it is not difficult to formulate hypotheses about institutions in technological form as is shown by the following example: "You cannot construct foolproof institutions, that is to say, institutions whose functioning does not very largely depend upon persons: institutions, at best, can reduce the uncertainty of the personal element, by assisting those who work for the aims for which the institutions are designed, and on whose personal initiative and knowledge success largely depends. (Institutions are like fortresses. They must be well designed and properly manned.)"

This long quotation is from *The Poverty of Historicism* by Popper (1957) where, in a footnote, he feels compelled to admit that "the success of mathematical economics shows that one social science at least has gone through its Newtonian revolution."

Declarations

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