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About Subjective Probability

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The main purpose of this note is to study Anscombe and Aumann's [1963] definition of subjective probability and their contributions to decision theory. We emphasize their main ideas, and we revise the framework that they proposed and which became one of the most used in axiomatizations in decision theory. We also develop a new framework to derive subjective probabilities based on the Cartesian product of two sets. We show how two simple axioms allow us to define a subjective probability on a state space of primary interest, given an auxiliary set endowed with an objective probability.

À PROPOS DE LA PROBABILITÉ SUBJECTIVE

L'objectif principal de cette note est d'étudier la définition de la probabilité subjective d'Anscombe et Aumann [1963] et leurs contributions à la théorie de la décision. Nous mettons l'accent sur leurs idées principales et revisitons le cadre de travail qu'ils ont proposé et qui est devenu l'un des plus utilisés dans les axiomatiques de théorie de la décision. Nous proposons également un nouveau cadre pour la dérivation de la probabilité subjective basé sur le produit cartésien de deux ensembles. Nous montrons que deux axiomes simples permettent de définir une probabilité subjective sur un ensemble d'états de la nature d'intérêt principal à partir d'un ensemble auxiliaire équipé d'une probabilité objective.

Keywords: subjective probability, objective probability, Anscombe and Aumann framework, independence.

Mots clés: probabilité subjective, probabilité objective, cadre d'Anscombe et Aumann, indépendance.

JEL Codes: D81.

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INTRODUCTION

The two key ingredients of expected utility (EU) theory are, as the name itself suggests, probabilities and utilities. A utility function $u: X \to \mathbb{R}$ measures the happiness $u(x)$ of a decision maker (DM) when she is given an outcome x in an outcome space X. A probability P is used to evaluate the likelihood of receiving the outcome x. Denoting $\mathbb{E}_p$ the expectation operation, then EU suggests evaluating "acts" using the criterion

$$\mathbb{E}_p[\mathrm{u}(\cdot)]. \tag{1}$$

While this framework looks natural, there are two questions that arise. How do we construct the probability measure P in case of subjective uncertainty? And the second one is of course: What is an act?

In their classical paper "A Definition of Subjective Probability," Anscombe and Aumann [1963] (AA henceforth) proposed a powerful methodological framework to analyze decisions under uncertainty that answers both questions. This framework became one of the classical ones in axiomatizations in decision theory. In particular, it contributed to developing a rich literature on decisions under uncertainty starting with the classical contributions of Schmeidler [1989] and Gilboa and Schmeidler [1989].

The first section of this paper describes how AA answered the two questions above. In our opinion, these two answers form the main contributions of AA's paper to decision theory. The first main idea of AA was to construct subjective probabilities using objective probabilities as a yardstick. Consider an event to which one cannot assign a probability using empirical frequencies, e.g., a horse race. AA proposed to calibrate the subjective probability attached to this event using an auxiliary event for which empirical frequencies can be calculated, e.g., a roulette lottery. The second contribution of AA's paper consists of proposing a framework that easily allows the comparison of objective and subjective probabilities. First, they defined an act as a function from a subjective state space (to which we want to associate a subjective probability) to a space of objective lotteries. Next, they postulated that agents have EU preferences over objective lotteries that give as prizes precisely those acts.

The second section of our paper proposes a construction of subjective probabilities using a different framework. Inspired by the recent paper of Grabisch, Monet and Vergopoulos [2022], we consider a state space defined as the Cartesian product of two sets. The first set represents a subjective state space. Our aim is to associate a subjective probability to this space. The second one is used to model objective uncertainty and it is equipped with an objective probability measure. As in AA's work, we use the objective probability measure to build the subjective one. This is achieved through two axioms. The first one is a standard dominance axiom. The second one is an independence axiom that exploits the notion of stochastic independence of events evaluated through the objective probability measure.

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ANSCOMBE AND AUMANN'S CONTRIBUTION TO DECISION THEORY

This section reviews Anscombe and Aumann [1963] and describes their main contributions.

Objective Lotteries as a Yardstick

"It is widely recognized that the word 'probability' has two very different main senses. In its original meaning . . . is roughly synonymous with plausibility. . . . this kind of probability belongs to logic. In its other meaning . . . belongs to physics. Physical probability can be determined empirically by noting the proportion of successes in some trials" (Anscombe and Aumann [1963], 199).

The very first paragraph of AA's paper recognizes the double nature of the word probability. They note that probabilities can either be subjective or objective. While the definition of objective probability is straightforward, i.e., it is the "proportion of successes in some trials," the one of subjective probability is not. AA proposed a definition of this concept. As they acknowledged, at the time of writing, several authors already gave a definition of subjective probabilities, for instance, Ramsey [1931], Finetti [1937], Savage [1954]. The new idea of AA was to construct subjective probabilities using objective ones as a yardstick. They made this construction by building a mathematical framework in which they could use the EU theorem of von Neumann and Morgenstern [1947]. Von Neumann and Morgenstern [1947] used (objective) lotteries to construct the utility function $u: X \to \mathbb{R}$ in Equation 1. As AA (p. 199) put it: "In this paper we are concerned with the personal or subjective concept of probability, as considered by Ramsey and Savage. Probabilities and utilities are defined in terms of a person's preferences, in so far as these preferences satisfy certain consistency assumptions. The definition is constructive; that is, the probabilities and utilities can be calculated from observed preferences. . . . For such a person his utility can be defined in terms of [objective probabilities] as shown by von Neumann and Morgenstern. The purpose of this note is to define the person's [subjective probabilities] in terms of [objective probabilities]."

In order to construct this subjective probability, AA built a mathematical framework that specifies over which "objects" preferences are defined. Their setting turned out to be so useful in order to build axiomatic models, that in decision theory it is known as *AA framework* and the "objects" are called *AA acts*. We revisit it in the next section.

The AA Framework and Theorem

The intuition of AA framework is the following. There is a state space S with s possible mutually exclusive states of the world. One of the states will realize, however, no probability is defined on S. When the state $j \in S$ realizes, a lottery R_j with objective probabilities is selected. With a famous metaphor, AA called the process of selecting a state $j \in S$ and its associated lottery R_j a "horse lottery." The objective lottery R_j is called by AA a "roulette lottery." Finally, the roulette lottery R_j is played, and a prize in a set A is selected. The function associating states in S to roulette lotteries is known as *AA act*, for the moment we denote it $[R_1, \ldots, R_s]$.

In order to define a subjective probability over the state space S, AA assumed twice EU: (i) over roulette lotteries giving prices in A; (ii) over roulette lotteries whose prices are horse lotteries. Moreover, they assumed two more axioms that connect the two systems of preferences. The first one is a monotonicity axiom, while the second one postulates indifference to the order in which roulette and horse lotteries are played.

Suppose that there is a most desired prize A_1 which is strictly preferred to a least desired prize A_0. Call u^* the von Neumann-Morgenstern utility mentioned in point (ii) of the previous paragraph. Then AA defined the subjective probability of state i as

$$p_i = u^*[A_0, \ldots, A_1, \ldots, A_0],$$

where prize A_1 appears in position i. One can interpret p_i as the unique number making an agent indifferent between the AA act $[A_0, \ldots, A_1, \ldots, A_0]$ and the roulette lotteries giving $[A_1, \ldots, A_1]$ with probability p_i and $[A_0, \ldots, A_0]$ otherwise. This readily follows from a normalization of u^* and using EU linearity properties.

Using p_i as a definition of subjective probability, AA derived the EU representation in Equation 1, in which u is the von Neumann-Morgenstern utility over roulette lotteries and $P = (p_1, \ldots, p_s)$ the subjective probability vector.

We conclude this section by presenting what is the standard AA framework as it appears now in almost all articles in decision theory. This framework is essentially due to Fishburn ([1970], chap. 13), and it has been fruitfully applied to decision theory under uncertainty at least since the seminal papers of Schmeidler [1989] and Gilboa and Schmeidler [1989]. Finally, we conclude this section by stating the AA-Fishburn theorem.

The main ingredients are the ones already there in AA's article. There is a finite state space S, and an outcome space X that is a convex subset of a vector space. For instance, this is the case if X is the set of all finitely valued objective lotteries over an outcome set $\mathcal{A}$. An (AA) act is a function $f: S \to X$. Denote F is the set of all acts. Note that each lottery $x \in X$ can be identified with the constant act $f(s) = x$ for all $s \in S$. F is endowed with a mixture operation performed pointwise. For all $f, g \in F$, for all $\alpha \in [0, 1]$, the mixture act $\alpha f + (1 - \alpha)g \in F$ is given $\forall s \in S$ by

$$(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s).$$

Instead of directly postulating EU, Fishburn [1970] considered a preference relation $\succsim \subseteq F \times F$. Consider now the following axioms.

AA1 (Rationality): $\succsim$ is complete and transitive.

AA2 (Continuity): For all $f, g, h \in F$ s.t. $f \succ g \succ h$ there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

AA3 (Independence): For all $f, g, h \in F$ and $\alpha \in (0, 1]$

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.$$

AA4 (Monotonicity): For every $f, g \in F$,

$$f(s) \succsim g(s) \ \forall s \in S \Rightarrow f \succsim g.$$

AA5 (Non-triviality): There all $f, g \in F$ such that $f \succ g$.

THEOREM 1 (AA – Fishburn). *$\succsim$ satisfies AA1-AA5 if and only if there exist a probability measure P on S and a non-constant function u: $X \to \mathbb{R}$ such that, for every $f, g \in F$*

$$f \succsim g \Leftrightarrow \int_S \mathbb{E}_{f(s)}[u] \mathrm{d}P(s) \geq \int_S \mathbb{E}_{g(s)}[u] \mathrm{d}P(s).$$

Furthermore, in this case, P is unique, and u is unique up to positive affine transformations.

Still to this day, the AA theorem provides one of the simplest approaches to subjective probability and expected utility maximization. But it has also allowed Schmeidler [1989] to obtain an early axiomatization of ambiguity aversion. The original Ellsberg [1961] paradox illustrates ambiguity aversion in terms of a preference for betting on events of known probability rather than ones of unknown probability. But Schmeidler uses the AA framework to explain ambiguity aversion through a preference for randomizing uncertain decisions on objective probabilities and, in this way, smoothing outcomes on uncertain (ambiguous) events.

In greater detail, two acts $f, g \in F$ are said to be comonotonic if there are no $s, s' \in S$ such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$. Intuitively, two comonotonic acts vary in the same direction and hence cannot hedge each other. Schmeidler appeals to the following weak version of AA3:

AA3' (Comonotonic independence): For all $f, g, h \in F$ and $\alpha \in (0, 1]$, if f and h are comonotonic and likewise for g and h,

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.$$

Schmeidler's theorem shows that replacing AA3 with AA3' leads to a version of Theorem 1 where the probability P on S is replaced with a capacity v (that is, a set function v: $2^S \to [0, 1]$ such that $v(S) = 1$, $v(\emptyset) = 0$ and $v(E) \geq v(F)$ for all $E, F \subseteq S$ such that $F \subseteq E$) and the integral with a Choquet integral. The ambiguity that the agent perceives is reflected in the possible nonadditivity of v. Schmeidler's approach allows for various attitudes towards ambiguity. In particular, ambiguity aversion is obtained through the following axiom:

AA3'' (Ambiguity aversion): For all $f, g \in F$ and $\alpha \in (0, 1]$,

$$f \sim g \Rightarrow \alpha f + (1 - \alpha)g \succsim f.$$

The intuition here is that f and g might fail to be comonotonic and hence could hedge each other and reduce the exposure to ambiguity. Then, ambiguity aversion would explain a strict preference for randomizing f and g on some objective probability α over each of f and g. AA3" turns out to characterize convex capacities; that is capacities v on S satisfying

$$v(E \cup F) + v(E \cap F) \geq v(E) + v(F),$$

for all $E, F \subseteq S$. Finally, the Schmeidler approach is only one of the by now many approaches to ambiguity aversion, and they owe a major debt to the AA framework.

PRODUCT SPACE AND SUBJECTIVE PROBABILITY

This section presents a simple framework to axiomatically derive a subjective probability measure over a state space. This space is defined by a product of two sets as in Grabisch, Monet and Vergopoulos [2022]. In this paper, one of the two sets will be of primary interest, as our aim is to construct a subjective probability measure on it. The other one will be equipped with an objective probability measure. We follow AA's lead: we will use the objective probability measure to calibrate the subjective one. Since we are only interested in the construction of a probability measure (and not a utility function), we only need to consider preferences over "bets," i.e., our acts will simply be indicator functions of events. Therefore, our outcome space will consist only of two (non-indifferent) outcomes. We believe that this simple framework can be used fruitfully by experimentalists who want to elicit subjective probabilities.

Formally, consider two state spaces (S_1, Σ_1) and (S_2, Σ_2) where Σ_i are σ-algebras for $i = 1, 2$. The couple (S_1, Σ_1) is the state space to which we want to assign a subjective probability, while (S_2, Σ_2) is an auxiliary source of uncertainty. This latter space comes equipped with an objective probability measure P_2.

We consider an outcome space containing two objects, $X = \{x_0, x_1\}$. We denote by $S = S_1 \times S_2$ and $\Sigma = \Sigma_1 \times \Sigma_2$. Bets on events $E \in \Sigma$ are functions from S to X and can be identified with indicator functions that pay x_1 if E realizes and x_0 otherwise. Consider a decision maker (DM) with a preference relation $\succsim$ over bets. Suppose also that $x_1 \succ x_0$, where we identify x_1 with the bet paying x_1 on S. To simplify notation, we will write 1 instead of x_1, and 0 instead of x_0.

Assumptions on (S_2, Σ_2, P_2).

1. (S_2, Σ_2, P_2) is infinitely rich in the sense that P_2 is nonatomic and that each event E_1 in Σ_1 has an equivalent event E_2 in Σ_2. For example, one can consider $S_2 = [0, 1]$, Σ_2 the Borel σ-algebra and P_2 the Lebesgue measure.

2. Bets depending only on S_2 are evaluated only by their likelihood given by P_2. Formally, for all $E_2, E'_2 \in \Sigma_2$, $S_1 \times E_2 \succsim S_1 \times E'_2$ if and only if $P_2(E_2) \geq P_2(E'_2)$.

Axioms on $\succsim$.

Let us first introduce some notation. Given a set $E \in \Sigma$ and $s_1 \in S_1$ we denote $E(s_1, \cdot)$ the restriction of E on s_1, i.e., $E(s_1, \cdot) = \{(x, y) \in S \mid x = s_1, (x, y) \in E\}$. Note that $E(s_1, \cdot)$ may be empty. Moreover, we simply denote as $E_2 \in S_2$ a bet of the form $S_1 \times E_2$ that only depends on E_2.

Dominance. For all $E, F \in \Sigma$, if $E(s_1, \cdot) \sim F(s_1, \cdot)$ for all $s_1 \in S_1$, then $E \sim F$.

INDEPENDENCE. For all $E_1 \in \Sigma_1$, $E_2, H_2 \in \Sigma_2$ such that E_2 is stochastically independent from H_2,[1] if $E_1 \sim E_2$ then $(E_1 \cap H_2) \cup E \sim (E_2 \cap H_2) \cup E$ for all $E \subseteq S_1 \times H_2^c$.

The two axioms DOMINANCE and INDEPENDENCE express in different ways the common intuition that the two sources of uncertainty S_1 and S_2 are stochastically independent of each other. Indeed, DOMINANCE requires that the agents have well-defined preferences on Σ_2, which are independent from the state obtained in S_1. For instance, it implies the following: For all $E_1 \in \Sigma_1$ and $F_2, F'_2 \in \Sigma_2$,

$$F_2 \sim F'_2 \Rightarrow E_1 \times F_2 \sim E_1 \times F'_2.$$

Hence, observing that some event $E_1 \in \Sigma_1$ obtains, does not affect the initial ranking $F_2 \sim F'_2$ between events in source S_2.

As for INDEPENDENCE, the condition can be decomposed into two requirements. Consider $E_1 \in \Sigma_1$ and $E_2, H_2 \in \Sigma_2$ such that E_2 is stochastically independent from H_2 and, for the moment, let $E = \emptyset$. The first part requires the following implication:

$$E_1 \sim E_2 \Rightarrow E_1 \times H_2 \sim E_2 \cap H_2.$$

In words, observing that some event $H_2 \in \Sigma_2$ from the second source S_2 obtains does not affect the initial ranking $E_1 \sim E_2$ whenever E_2 and H_2 are stochastically independent of each other. Implicit in this logic is the intuition that every event from S_1 is stochastically independent from every event in S_2. The second part of INDEPENDENCE has the flavor of a standard additivity condition that makes the indifference $E_1 \times H_2 \sim E_2 \cap H_2$ result in this other indifference $(E_1 \cap H_2) \cup E \sim (E_2 \cap H_2) \cup E$ for all $E \subseteq S_1 \times H_2^c$.

We now explain INDEPENDENCE with one example in which we use matrices to depict our framework. The same type of matrices will be used in the proof of our theorem. Suppose $S_1 = \{a, b, c\}$ and let $E_1 = \{a, b\}$ and $E_2 \in \Sigma_2$. Columns represent events in S_1, while rows events in S_2. The matrix on the left of the indifference represents a bet on E_1, in fact, this bet is not affected by the realization of a state in S_2. Likewise, the matrix on the right of the indifference relation represents a bet on E_2. Suppose $E_1 \sim E_2$.

E_1		E_1^c	
a	b	c	
1	1	0	E_2
1	1	0	E_2^c

$\sim$

E_1		E_1^c	
a	b	c	
1	1	1	E_2
1	1	0	E_2^c

Before proceeding, let us reformulate the INDEPENDENCE axiom in an explicit way. Suppose that E_2 is stochastically independent from H_2. Then INDEPENDENCE says that $E_1 \times S_2 \sim S_1 \times E_2$ implies $(E_1 \times H_2) \cup (K_1 \times K_2) \sim (S_1 \times (E_2 \cap H_2)) \cup (K_1 \times K_2)$ with $K_1 \subseteq S_1$ and $K_2 \subseteq H_2^c$.

1. Stochastic independence is with respect to the objective probability P_2.

Let us turn back to the example above. Suppose there is $H_2 \in \Sigma_2$ such that E_2 and H_2 are stochastically independent. As proved in Theorem 2 below, we can always suppose that such a set exists. Now take $K_1 = \{b\}$ and $K_2 = H_2^c$. A bet on $K_1 \times K_2$ is represented by the last row of the two matrices below. Suppose that the DM knows that H_2^c realized. Then by Dominance she will be indifferent between the two bets. Suppose now that she knows that H_2 realized. Since H_2 is independent from E_2, knowing H_2 does not change the probability of E_2. Therefore, given H_2 we are back to the bets E_1 vs E_2. Since by hypothesis $E_1 \sim E_2$, she should stay indifferent also knowing H_2.

$$
\begin{array}{ccc|c}
\multicolumn{2}{c}{\overbrace{\hphantom{a \quad b}}^{E_1}} & \overbrace{\hphantom{c}}^{E_1^c} & \\
a & b & c & \\
\hline
1 & 1 & 0 & E_2 \cap H_2 \\
1 & 1 & 0 & E_2^c \cap H_2 \\
0 & 1 & 0 & H_2^c
\end{array}
\quad \sim \quad
\begin{array}{ccc|c}
\multicolumn{2}{c}{\overbrace{\hphantom{a \quad b}}^{E_1}} & \overbrace{\hphantom{c}}^{E_1^c} & \\
a & b & c & \\
\hline
1 & 1 & 1 & E_2 \cap H_2 \\
0 & 0 & 0 & E_2^c \cap H_2 \\
0 & 1 & 0 & H_2^c
\end{array}
$$

Let $\succsim_1$ denote the restriction of $\succsim$ to Σ_1. That is, $E_1 \succsim_1 F_1$ if and only if $E_1 \times S_2 \succ F_1 \times S_2$ for all $E_1, F_1 \in \Sigma_1$. We say that a function $P_1 \colon \Sigma_1 \to [0, 1]$ represents $\succsim_1$ if $E_1 \succsim_1 F_1$ holds if and only if $P_1(E_1) \geq P_1(F_1)$ for all $E_1, F_1 \in \Sigma_1$. We can now state our main result.

Theorem 2. *Suppose that $\succsim$ satisfies Dominance and Independence. Then, there exists a unique probability measure P_1 on (S_1, Σ_1) such that $E_1 \sim E_2$ if and only if $P_1(E_1) = P_2(E_2)$ for all $E_1 \in \Sigma_1$ and $E_2 \in \Sigma_2$. Moreover, P_1 represents $\succsim_1$.*

Proof. By Assumption 1, for all set $E_1 \in \Sigma_1$ there exists a set $E_2 \in \Sigma_2$ such that $E_1 \sim E_2$. Define the function $P_1 \colon \Sigma_1 \to [0, 1]$ for all $E_1 \in \Sigma_1$ as

$$P_1(E_1) = P_2(E_2),$$

where $E_2 \in \Sigma_2$ is such that $E_1 \sim E_2$. The uniqueness claim follows from this definition. We will prove that P_1 is a probability defined on (S_1, Σ_1). First of all, note that P_1 is well defined. In fact suppose there are $E_2, E'_2 \in \Sigma_2$ such that $E_1 \sim E_2$ and also $E_1 \sim E'_2$. By transitivity, $E_2 \sim E'_2$ and by Assumption 2, $P(E_2) = P_2(E'_2)$.

It is easy to see that $P_1(S_1) = 1$. In fact by writing explicitly bets on events, bet S_1 is actually $S_1 \times S_2$, and therefore clearly $S_1 \sim S_2$ and $P_1(E_1) = P_2(E_2) = 1$.

We will prove now that P_1 is additive. Consider two disjoint events $E_1, F_1 \in \Sigma_1$ and let $E_2, F_2 \in \Sigma_2$ be equivalent events. Let also $G_1 = E_1 \cup G_1$ and G_2 be an equivalent event from Σ_2. To show additivity of P_1, it is sufficient to show $P_2(G_2) = P_2(E_2) + P_2(F_2)$. In fact, suppose this is true then $P_1(G_1) = P_2(G_2) = P_2(E_2) + P_2(F_2) = P_1(E_1) + P_1(F_1)$.

We first prove an auxiliary Lemma.

LEMMA 3. *There exists* $H_2 \in \Sigma_2$ *such that* $P_2(H_2) = 1/2$ *and such that* H_2 *is stochastically independent from each of* E_2, F_2 *and* G_2.

Proof. Let Π_2 denote the finite partition of S_2 generated by E_2, F_2 and G_2. For each $K_2 \in \Pi_2$, we can find by nonatomicity $H_2^{K_2} \in \Sigma_2$ such that $H_2^{K_2} \subseteq K_2$ and $P_2(H_2^{K_2}) = P_2(K_2)/2$. Then, define H_2 as the union of all $H_2^{K_2}$ for $K_2 \in \Pi_2$. We have

$$P_2(H_2) = \sum_{K_2 \in \Pi_2} P_2(H_2^{K_2}) = \frac{1}{2} \sum_{K_2 \in \Pi_2} P_2(K_2) = \frac{1}{2}.$$

Furthermore, we have for all $K_2 \in \Pi_2$

$$P_2(H_2 \cap K_2) = P_2(H_2^{K_2}) = \frac{1}{2} P_2(K_2) = P_2(H_2)P_2(K_2),$$

which shows that H_2 and K_2 are independent. In particular, H_2 is stochastically independent from each of E_2, F_2 and G_2. ■

Let us go back to the additivity of P_1. As explained above, bets on events are represented by matrices in which columns are events in Σ_1 and rows are events in Σ_2. In the following matrices, columns will always indicate the same events in Σ_1, therefore we will explicit them only in the first two matrices.

First note that by definition of G_1 and G_2 we have

$$\begin{array}{ccc|c} \multicolumn{2}{c}{\overbrace{}^{G_1}} & & \\ E_1 & F_1 & G_1^c & \\ \hline 1 & 1 & 0 & G_2 \\ 1 & 1 & 0 & G_2^c \end{array} \quad \sim \quad \begin{array}{ccc|c} \multicolumn{2}{c}{\overbrace{}^{G_1}} & & \\ E_1 & F_1 & G_1^c & \\ \hline 1 & 1 & 1 & G_2 \\ 0 & 0 & 0 & G_2^c \end{array}$$

By INDEPENDENCE, we can add a common third row of 0.

$$\begin{array}{ccc|c} 1 & 1 & 0 & H_2 \\ 1 & 1 & 0 & H_2^c \end{array} = \begin{array}{ccc|c} 1 & 1 & 0 & G_2 \cap H_2 \\ 1 & 1 & 1 & G_2^c \cap H_2 \\ 0 & 0 & 0 & H_2^c \end{array} \sim \begin{array}{ccc|c} 1 & 1 & 1 & G_2 \cap H_2 \\ 0 & 0 & 0 & G_2^c \cap H_2 \\ 0 & 0 & 0 & H_2^c \end{array} \quad (2)$$

Since $P_2(H_2) = \frac{1}{2}$, by Assumption 2 and DOMINANCE we have

$$\begin{array}{ccc|c} 1 & 1 & 0 & H_2 \\ 0 & 0 & 0 & H_2^c \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 0 & H_2 \\ 0 & 1 & 0 & H_2^c \end{array} \quad (3)$$

By definition of E_1 and E_2 one has

$$\begin{array}{ccc|c} 1 & 0 & 0 & E_2 \\ 1 & 0 & 0 & E_2^c \end{array} \quad \sim \quad \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \\ 0 & 0 & 0 & E_2^c \end{array}$$

Then thanks to Independence, we can add a common third row with a bet on $F_1 \times H_2^c$

$$\begin{array}{ccc|c} 1 & 0 & 0 & H_2 \\ 0 & 1 & 0 & H_2^c \end{array} = \begin{array}{ccc|c} 1 & 0 & 0 & E_2 \cap H_2 \\ 1 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & H_2^c \end{array} \sim \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & H_2^c \end{array} \quad (4)$$

Similarly, by definition of F_1 and F_2 one has

$$\begin{array}{ccc|c} 0 & 1 & 0 & F_2 \\ 0 & 1 & 0 & F_2^c \end{array} \quad \sim \quad \begin{array}{ccc|c} 1 & 1 & 1 & F_2 \\ 0 & 0 & 0 & F_2^c \end{array}$$

and using once again Independence we can add a bet on $E_2 \cap H_2$

$$\begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & H_2^c \end{array} = \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & F_2 \cap H_2^c \\ 0 & 1 & 0 & F_2^c \cap H_2^c \end{array} \sim \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 1 & 1 & 1 & F_2 \cap H_2^c \\ 0 & 0 & 0 & F_2^c \cap H_2^c \end{array} \quad (5)$$

Summarizing all indifferences in Equations 2, 3, 4 and 5 we obtained

$$\begin{array}{ccc|c} 1 & 1 & 1 & G_2 \cap H_2 \\ 0 & 0 & 0 & G_2^c \cap H_2 \\ 0 & 0 & 0 & H_2^c \end{array} \quad \sim \quad \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 1 & 1 & 1 & F_2 \cap H_2^c \\ 0 & 0 & 0 & F_2^c \cap H_2^c \end{array}$$

The chain of indifferences proves that a bet on $G_2 \cap H_2$ is indifferent to a bet on $(E_2 \cap H_2) \cup (F_2 \cap H_2^c)$. Therefore, using Assumption 2, we have

$$P_2(H_2 \cap G_2) = P_2(H_2 \cap E_2) + P_2(H_2^c \cap F_2).$$

This proves the result as H_2 is stochastically independent from each of E_2, F_2 and G_2 by Lemma 3. Finally, consider E_1, $F_1 \in \Sigma_1$ and let E_2, $F_2 \in \Sigma_2$ be their equivalents in S_2. Then, $E_1 \gtrsim_1 F_1$ is equivalent to $E_2 \gtrsim F_2$ which is finally equivalent to $P_1(E_1) \geq P_1(F_1)$ by Assumption 2 and the definition of P_1. ■

Theorem 2 provides a simple variant of the main idea of AA of using objective probabilities on an auxiliary source of uncertainty for defining subjective probabilities on a source of uncertainty of primary interest. It is also interesting in that it models the auxiliary source explicitly in terms of a state space and, in doing so, uncovers the role of stochastic independence in the construction of subjective probabilities, a role that remains implicit in the AA original argument and, for instance, in AA3.

Moreover, the argument just presented in the proof of Theorem 2 can be used to illustrate the relevance of the AA framework to the modelling of ambiguity aversion. Indeed, suppose Independence is weakened into the following:

Comonotonic independence. For all $E_1 \in \Sigma_1$, $E_2, H_2 \in \Sigma_2$ such that E_2 is stochastically independent from H_2, if $E_1 \sim E_2$ then $(E_1 \cap H_2) \cup (K_1 \cap K_2) \sim (E_2 \cap H_2) \cup (K_1 \cap K_2)$ for all $K_1 \subseteq S_1$ such that $E_1 \subseteq K_1$ or $K_1 \subseteq E_1$ and $K_1 \subseteq H_2{}^c$.

To see the connection to AA3', note that two indicator functions 1_{E_1} and 1_{K_1} of subsets $E_1, K_1 \in \Sigma_1$ are comonotonic if and only if $E_1 \subseteq K_1$ or $K_1 \subseteq E_1$. Hence, following the logic of Schmeidler [1989], Comonotonic independence as just defined is indeed a version of Independence that is restricted by comonotonicity. Note also that Comonotonic independence remains strong enough to yield the indifferences obtained in Formulas 2 and 5, but possibly fails to yield Formula 4. In fact, consider the following axiom:

Ambiguity aversion. For all $E_1 \in \Sigma_1$, $E_2, H_2 \in \Sigma_2$ such that E_2 is stochastically independent from H_2, if $E_1 \sim E_2$ then $(E_1 \cap H_2) \cup (K_1 \cap K_2) \succsim (E_2 \cap H_2) \cup (K_1 \cap K_2)$ for all $K_1 \subseteq S_1$, and $K_2 \subseteq H_2{}^c$.

The idea here is that, in those cases where we do not have $E_1 \subseteq K_1$ or $K_1 \subseteq E_1$, betting on $(E_1 \cap H_2) \cup (K_1 \cap K_2)$ allows one to reduce the exposure to ambiguity and can therefore be strictly preferred to betting on $(E_2 \cap H_2) \cup (K_1 \cap K_2)$. To illustrate, consider again the matrices from Formula 4. Applying Ambiguity aversion yields the following preference:

$$\begin{array}{ccc|c} 1 & 0 & 0 & E_2 \cap H_2 \\ 1 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & H_2^c \end{array} \quad \succsim \quad \begin{array}{ccc|c} 1 & 1 & 1 & E_2 \cap H_2 \\ 0 & 0 & 0 & E_2^c \cap H_2 \\ 0 & 1 & 0 & H_2^c \end{array} \qquad (6)$$

There is here a reduced exposure to uncertainty in the left matrix due to the fact that the rows (1, 0, 0) and (0, 1, 0) are not comonotonic. Finally, in the context of the proof of Theorem 2, Ambiguity aversion and Formula 6 deliver the subadditivity of P_1 in the following sense: For all disjoint $E_1, F_1 \in \Sigma_1$,

$$P_1(E_1 \cup F_1) \geq P_1(E_1) + P_1(F_1).$$

Such subadditivity is a property implied by convexity and can be understood as a sign of ambiguity aversion.

CONCLUSION

Anscombe and Aumann [1963] definition of subjective probability is a milestone in the decision theory literature. In our paper, we first revised Anscombe and Aumann's [1963] contribution. We underlined how subjective probabilities are constructed using objective ones as a "thermometer." We also recalled Anscombe and Aumann's framework (as given by Fishburn [1970]), which became the standard mathematical setting in most preference axiomatizations in decision theory under Knightian uncertainty. Second, we presented a new framework, given by the Cartesian product of an objective state space with a subjective one. This framework allowed us to propose a new construction of subjective probability. The probability is derived from preferences over simple bets satisfying the axioms of Dominance and Independence.

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