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Three essays on speculation and welfare in dynamic economies

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Contents

Introduction

This thesis investigates the impact of belief heterogeneity on individual and aggregate investment choices, focusing on welfare and policy implications. The common thread entwining the three chapters is the assumption that agents "agree to disagree" about an exogenous stochastic event and make consumption and saving choices consistently with the system of subjective beliefs they are endowed with. Disagreement is taken as given and persistently affects the dynamics of the outlined economies.

For a long time, the widespread rational expectation assumption has been justified by the idea that people eliminate systematic forecast errors over time, learning the true data generating process. However, disagreement unambiguously affects many socio-economics fields. Theoretically, there are two reasons why differences in opinions may persist in the long-run. First, agents observe the same information starting with heterogeneous prior beliefs. Convergence to the truth, or one single model, is not guaranteed in such a case. Rational Bayesian agents will learn the truth, or at least the best approximation among the available models, provided that their prior beliefs are absolutely continuous with respect to the data generating model, or what comes closest to it. In other words, all trades must attribute a positive weight on the true parameter since the beginning. Second, agents observe different signals but have a common prior. The Aumann's agreement theorem states that, even observing private information, Bayesian learners with a common prior must necessarily end up with the same subjective probability, if their posterior beliefs are common knowledge. However, this requires an extensive communication among market participants and rules out the possibility of making mistakes in processing the available information. Given that transfers of information usually take place in complex networks and there is evidence that behavioral biases emerge in interpreting a common signal, rational learning is quite difficult to achieve in practice.

Although the belief updating process is not explicitly modeled in this work, both the explanations can justify the assumption of persistent disagreement used throughout the thesis.

Another point investigated all along the thesis is the validity of the *Market Selection Hypothesis* (MSH) in financial markets. As originally introduced by Friedman (1953), this assumption states the market ability to select rational over irrational agents and, for a long time, it has been used to justify the use of the rational expectation paradigm in economics. Due to the evolutionary forces of financial markets, agents endowed with less accurate predictions experience larger losses on average and, as a consequence, they vanish in the long-run. Using different frameworks, I verify whether disagreement is thus a transient or a persistent feature of the economy checking the validity of the market selection argument.

Despite being closely connected, each chapter focuses on different aspects.

Chapter 1 is an assessment of the impact of disagreement on the dynamic of a one-sector growth model characterized by a complete financial market structure. Expectation biases alter the real economy even in a representative agent framework, however, disagreement about the probability governing a technological shock, enhances the volatility experienced by individual and aggregate consumption patterns. The paper also confirms the validity of the MSH in production economies, given that most of the existent literature focuses on endowment economies. Disagreement has thus a transient nature since inaccurate agents are drained out of the market. Despite guaranteeing accuracy of the long-run allocation, there are realized welfare losses due to the progressive impoverishment incurred by less accurate agents. In this vein, welfare gains arises when the set of the available contingent claims is restricted and, provided that the truth lies somewhere in the between of the agents' beliefs, benefits extend to the real sector as well.

Chapter 2 studies the optimal *Financial Transaction Tax* (FTT) and its implications on the long-run equilibrium, in a dynamic exchange economy where traders hold different beliefs about the states of Nature and trading exclusively arises for speculative reasons. Without any policy intervention the MSH holds and less accurate agents are driven out of the market. This result is not guaranteed when trading activity is limited by means of a tax set on the agents' security exchanges. Inaccurate agents may dominate, leading to a severe miss-pricing in the long-run. Depending on the policy purpose, the optimal tax rate may either mitigate or completely eliminate speculative trades. Overall, the paper provides a better understanding of the trade-off arising between speculation and accuracy reductions, due to the distortion induces by a trading cost as the FTT.

Chapter 3 studies the effect of a FTT in a speculative production economy, similar to the one developed in Chapter 1. The paper sheds light on the negative spillovers from financial speculation to the real sector, drawing the attention to potential regulatory measure aimed at mitigating the distortions of macroeconomics outcomes. Consistently with Chapter 2, the paper confirms that a FTT may undermine the validity of the MSH, however it focus more on the impact that this tax produces on the aggregate and individual consumption choices. The overall effect depends on the size of the taxation and the position of the truth with respect to the agents' subjective probabilities. In particular, when the truth lies somewhere in the between of the agents' beliefs, a FTT partially corrects individual and aggregate choices toward the ones implied by the truth. By contrast, when the truth lies elsewhere, the tax further distorts real outcomes. Therefore, provided that the Government does not observe the true probability, the effects of this measure on the real sector are impossible to anticipate.

Chapter 1

Speculation and welfare in a stochastic growth model with heterogeneous beliefs[1]

Abstract

This paper investigates the role of beliefs heterogeneity about the capital marginal productivity in a stochastic one-sector growth model. Agents are endowed with heterogeneous subjective beliefs about the probability of a technology shock and are allowed to trade in complete sequential markets, speculating on such opinion divergences. Using an analytical solvable structure, we first study the implication of biased beliefs on the economy dynamics of a representative agent framework. Then, we consider the same setting under the assumption that heterogeneous and biased opinions coexist in the same economy. We analyze the effect of disagreement on output, individual and aggregate consumption, and define the conditions under which heterogeneity is persistent or transient. We show that, everything else equal, the Market Selection Hypothesis (MSH) holds in production economies: in the long-run state prices and the resulting production decisions are determined only by the agent with the most accurate beliefs. Finally, we show that there are welfare gains when speculation is not allowed and agent's beliefs are biased in different directions.

Keywords: Heterogeneous Beliefs, Real Investments, Speculative Markets, Market Selection Hypothesis, Growth.
JEL Classification: E21, E22, G11

1.1 Introduction

For a long time, the *Rational Expectation Hypothesis* (REH) has been the standard approach used to model expectations in macroeconomics. The REH posits full rationality of the expectation formation process: it requires economic agents to collect and process all relevant information needed for the decision making process. Rational expectations relies on a strong assumption regarding both the amount of available information and the capacity of agents to make the best use of it. In particular, it is given for granted that agents probabilistic models of the economy shocks are well specified. A strong argument in favor of this assumption is the *Market Selection Hypothesis* (MSH) of Friedman (1953), that is, the capability of competitive economies to transfer resources to the agents who use the most correct probabilistic model.

[1] This chapter is based on joint work with Pietro Dindo.

Using a standard production framework, the aim of this paper is to evaluate the effect of expectation biases on the macroeconomic dynamics, when production is characterized by high and low productivity states. Agents are endowed with an exogenous static prior about the probability of states of the Nature and they solve a consumption-saving problem accordingly.

In a representative agent framework, expectation biases alter the aggregate production level affecting the capital accumulation path. Deviations from REH depends on both the size and the direction of the agent's belief distortion. As one would expect, over(under)-estimation of the high productivity state increases (decreases) the agent's propensity to invest leading to over(under)-investment compared to the rational expectation case.

In a economy with disagreement, expectation biases affect the real sector through agents' speculative trades. We consider an economy characterized by two types of agents: optimists and pessimists. We call optimist the group that systematically over-estimates the probability of high productivity states and pessimist the one with the opposite vision. Given the existence of a complete contingent-claim market, belief heterogeneity makes investors willing to bet on the probability of future states. In endowment economies, the most accurate wins most of the time, driving the others out of the market. This mechanism allows the market to select the most accurate trader, thus supporting the MSH.

We devote part of the analysis to investigate whether the MSH holds even in production economies, where disagreement does affect not only individual but also aggregate consumption. Aggregate consumption depends on the agents' investment strategies that, in turn, are functions of the individual expected returns.

We find that speculative trading creates and additional source of macroeconomic volatility grounded on the endogenous distribution of agents' wealth. During high productivity states, the optimist gains most of the wealth, leading to an increase in the economy investment rate. The opposite occurs during low productivity states. In other words, the aggregate investment is more pro-cyclical than it would have been in a homogeneous framework. As a result, differences in opinion amplify the business cycle as long as less accurate agents remain in the market. Relatedly, the size of aggregate fluctuations reduces as soon as belief heterogeneity disappears. In this regard, we find that disagreement has a transient nature since, consistently with endowment economies, the market asymptotically selects the most accurate type.

Finally, we study the welfare implications under the ex-post perspective. After the financial crisis, a recent literature (Brunnermeier et al. (2014), Gilboa et al. (2014) and Blume et al. (2018)) claims that Pareto optimality is not a suitable criterion to evaluate welfare in heterogeneous belief frameworks. The argument is that, although increasing the agents' ex-ante utilities, speculative trades are harmful from an ex-post perspective, due to the progressive impoverishment faced by most inaccurate agents. Consistently with these studies, we find that preventing (or mitigating) speculative trades has a positive effect on the realized social welfare. Everything else equal, welfare gains are possible when the set of available assets is reduced, compared to the corresponding complete market framework. Moreover, provided that the true probability is somewhere in the middle of the agents' beliefs (i.e. agents are biased in opposite directions), benefits extends to the real sector as well. This analysis provides a reason for regulators to intervene.

Outline In Section 1.2 we provide an overview of the most related literature contributions. We introduce the representative agent economy, proposing three examples of belief-bias types in Section 1.3. Thereafter, we characterize the heterogeneous belief production economy in Section 1.4. In this section we also present welfare analysis, outlining the conditions under which market incompleteness leads to a welfare improvement in these frameworks. Section 1.5 concludes.

1.2 Related literature

By studying the implications of belief heterogeneity in an economy where agents "agree to disagree", this paper naturally belongs to the MSH literature. This literature originated from the Friedman (1953) idea, according to which, competitive markets select agents with correct beliefs, draining the others out of the economy. Controlling for the agents' discount factor, Sandroni (2000) shows that the MSH holds when the economy is populated by subjective expected utility maximizers and financial markets are complete. Blume and Easley (2006) finds the same conclusions when agents are involved in a Bayesian learning process. These works analyze endowment economies: agents exogenously receive consumption good for all dates and histories and opinion heterogeneity has no impact on the amount of aggregate consumption (although it has an impact on how consumption is distributed among agents).

We contribute to this literature by investigating the MSH in an environment where the aggregate output, and so does consumption, is endogenously determined by the agents' investment strategies.

A few works related with the MSH have studied production economies. Blume and Easley (2002) considers a production economy but with a pretty different economic question. Using a deterministic framework, they investigate whether only profit-maximizer firms survive in the long-run. Close to our purposes, Baker et al. (2016) studies the effect of disagreement in the real and financial sector using a continuous-time production framework. In contrast to our paper, the effect on saving depends on the relative strength of income and substitution effect induced by the size of the risk aversion characterizing the economy. We extend this analysis to log-economies studying the effect of individual and aggregate saving when income and substitution effects perfectly offset.

Beyond the MSH literature, our paper shares some common purposes with Walden and Heyerdahl-Larsen (2015), that likewise evaluate the effect of belief biases and belief heterogeneity in a framework where productive resources need to be allocated. However, in contrast to our work, the paper is set in a static environment so that it cannot study the long-run equilibrium features.

This work endorses some analytical features with Koulovatianos et al. (2009). By introducing learning in the Mirman-Zilcha class of growth models, they study the optimal consumption and investment rules under the assumption that the social planner is either a Bayesian or an adaptive learner. Compared to the rational expectation hypothesis, they derive the conditions for which learning increases or decreases the optimal consumption level.

More broadly, we contribute to the entire literature questioning the REH as one of the cornerstone of the traditional macroeconomics. As originally proposed by Muth (1961), this assumption is based on the idea that information is scarce and, for this reason, it should not be wasted by rational economic actors. Since expectations are informed predictions of future events, there would not be any reason to think the economy outcomes be different from the agents' previsions.

The REH has been one of the foundations of the Neoclassical revolution during the 70s (see Lucas (1972) and Sargent and Wallace (1976)). However, the standard business cycle model fails to match some empirical fact and several extensions have been introduced in this regards. Among the papers investigating the psychological dimension, the REH has been challenged from different perspectives. A branch of the existing literature relaxes the hypothesis of rational expectations in representative agent frameworks. For instance, Eusepi and Preston (2011), Milani (2007) and Milani (2011) claim that exogenous shocks are amplified when the representative agent is involved in a belief updating process[2]. Further, Adam and Marcet (2011) studies a financial economy where homogeneous investors are internal, but not external, rational. Internal rationality entails

[2]Eusepi and Preston (2011) introduces a self-referential system of beliefs that replicates the pattern of forecast errors observed by Survey of Professional Forecasters.

economic choices be consistent with the system of subjective beliefs with which agents are endowed. However, agents are not externally rational when the true probability governing the pay-off process is unknown. Given that, Adam et al. (2017) shows that internally rational investors' expectation explains most of the stock prices fluctuations observed in the post-war US data. An alternative way to challenge the REH is to assume the existence of cognitive biases affecting the agent's decision making process. In this spirit, the news driven business cycle literature (Beaudry and Portier (2004) and Jaimovich and Rebelo (2009)) supports the idea that optimistic and pessimistic waves are grounded on the noisy information that homogeneous agents receive about future fundamentals. In contrast to these works, we depart from the representative agent assumption by introducing differences in opinion among investors. Supporting our choice, there is a consensus about both the non-existence of the representative agent in economic modelling (see Jackson and Yariv (2018)) and the limitations of the common prior assumption. On this last point, Morris (1995) provides a comprehensive overview making a strong rationale for studying belief heterogeneity in economics.

1.3 The representative agent economy

We first assess the impact of expectations in the real sector in a representative agent model. Departing from the REH, this analysis serves as a starting point for our study, allowing us to identify the major implications of belief-biases for the dynamics of a stochastic growth model. Having introduced the baseline framework, where the agent's subjective beliefs are correct and coincide with those governing the technological shock, we present three types of belief-biases, outlining the implications that these produces on the optimal capital accumulation path.

1.3.1 The model

In this section, we introduce the baseline framework of the stochastic growth model. Time is discrete and indexed by $t \in \mathbb{N}_+$. A sequence of random variables (θ_t) following a Markov process is the exogenous shock at dates $t = 1, 2, ...,$ while $\theta^t = \{\theta_0, \theta_1, ..., \theta_t\}$ is the vector of the shock realizations up to period t.

We define $\Theta = \{\theta^h, \theta^l\}$ as the possible States of the Nature. Since θ_t is the technology shock affecting the economy capital share, the superscripts on θ stand for the high and the low productivity state, $0 < \theta^l < \theta^h < 1$.

The states' transition probabilities are defined in the following stochastic matrix

$$P = \begin{bmatrix} p_{hh} & p_{hl} \\ p_{lh} & p_{ll} \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

with $p > 1 - p$. Without loss of generality, we assume the two states be persistent to the same extent. Within this assumption, we attempt to capture some features of the actual capital share process, that moves among different levels displaying a certain degree of persistence[3] (see Figure 1.1).

Agent i is endowed with the following subjective probability matrix

$$P^i = \begin{bmatrix} p^i_{hh} & p^i_{hl} \\ p^i_{lh} & p^i_{ll} \end{bmatrix} = \begin{bmatrix} p + b^i & 1 - p - b^i \\ 1 - p + b^i & p - b^i \end{bmatrix}$$

[3]For a long time the division of aggregate income between capital and labour remuneration has been considered stable. However, the apparent decline in labour share observed in the last decades imposed a reconsideration (see Rognlie (2015) for a comprehensive survey about this topic).

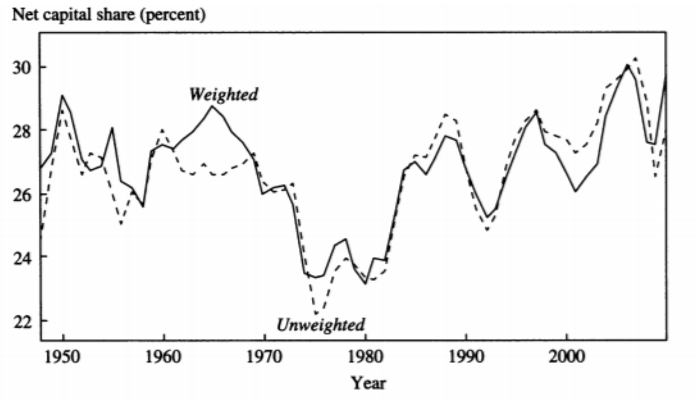


Figure 1.1: Historical series of the average net capital share of the private domestic value-added of the G7 Countries (author's calculations based on national accounts (Rognlie, 2015)).

where b^i is the agent's bias size and $b^i < 1-p$. We restrict the values of p and b to ensure that there exists an invariant distribution of states and name $\pi(\theta^h)$ and $\pi(\theta^l)$ the state θ^h and θ^l invariant probabilities. Depending on the state he believes is more likely with respect to the truth, b can be either a positive or a negative number (i.e. if positive, the agent overestimates the probability related to state θ^h).

Agents do not update their opinions when they observe new shock realizations but keep their belief distortion b^i all along. In a homogeneous framework, the model can be thought of as a proxy of an economy with a miss-specified prior, in the sense that the agent assigns probability 0 to the true state of the world. In a heterogeneous framework, the assumption of persistent disagreement approximates an economy where convergence to one single model is prevented by agents having disjoint probability supports among themselves.

Lastly, to simplify the notation, we drop the reference to the node θ^t when it is not necessary, so that x_t is used in place of $x_t(\theta^t)$.

1.3.2 Firm

The final good is produced by a representative firm whose production between t and $t+1$ depends on the following technology

$$y_t = Ak_t^{\theta_t} \tag{1.1}$$

where A is the *Total Factor Productivity* (TFP) term capturing the long-run level of technology, y_t and k_t are, respectively, the time t level of output and capital, while θ_t is the stochastic technology term affecting the economy capital share. Capital fully depreciates in every period so that k_t also reflect the aggregate investment characterizing the economy.

The firm's technology belongs to the Mirman and Zilcha (1975) class of production functions. Our choice is justified by both analytical tractability and, as detailed in Section 1.3.6, a standing role of the agent's beliefs in the economy dynamics. Finally, consumption, capital and output are the same good, whose price in date $t=0$ is normalized to one.

In every period t, the firm observes θ_t, then demands capital k_{t+1} which - together with θ_{t+1} - determines y_{t+1}. Capital maximized the profit function,

$$\max_{k_{t+1}} P_t = \max_{k_{t+1}} \sum_{\theta_{t+1}|\theta_t} q_{t+1}^t(\theta_{t+1}|\theta_t)\, y_{t+1}(\theta_{t+1}|\theta_t) - k_{t+1} \qquad \text{subject to} \quad y_{t+1} \leq Ak_{t+1}^{\theta_{t+1}} \tag{1.2}$$

that is the sum of next-period contingent productions $y_{t+1}(\theta_{t+1}|\theta_t)$, discounted by the corresponding state-contingent price $q^t_{t+1}(\theta_{t+1}|\theta_t)$, and reduced by the time t cost of capital. The contingent-state price $q^t_{t+1}\left(\theta^h|\theta_t\right)$ is the dated t price of the contract delivering consumption in $t+1$, if the high productivity state will realize after history θ^t. The optimality condition of the above maximization problem is given by

$$\sum_{\theta_{t+1}|\theta_t} q^t_{t+1}(\theta_{t+1}|\theta_t)\theta_{t+1}Ak_{t+1}^{\theta_{t+1}-1} = 1 \tag{1.3}$$

1.3.3 Household

The economy is populated by a unitary mass of infinitely lived identical agents. Therefore, they essentially act as a representative agent facing the following consumption-saving problem

$$\begin{aligned} &\max_{c_t, a_{t+1}(\theta_{t+1}|\theta^t)} \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t \log c_t\right] \\ &\text{subject to} \quad c_t + \sum_{\theta_{t+1}|\theta_t} q^t_{t+1}(\theta_{t+1}|\theta_t)a_{t+1}(\theta_{t+1}|\theta_t) \leq a_t + P_t \end{aligned} \tag{1.4}$$

where $a_0 = k_0$ is the initial capital endowment. We call $a_{t+1}(\theta_{t+1}|\theta_t)$ the Arrow security delivering consumption in state $\theta_{t+1}|\theta_t$, while $\sum_{\theta_{t+1}|\theta_t} q^t_{t+1}(\theta_{t+1}|\theta_t)a_{t+1}(\theta_{t+1}|\theta_t)$ the fraction of disposable wealth destined to the next-period contingent-claim purchases. Trading occurs sequentially and the complete financial structure allows for the existence of as many Arrow securities as states of the Nature. Moreover, since profits are part of the agent's resources, we are implicitly assuming that the representative household is also the firm's owner. The Euler equation is the first-order condition of the above maximization problem

$$q^t_{t+1}(\theta_{t+1}|\theta_t) = \frac{\beta\pi(\theta_{t+1}|\theta_t)c_t}{c_{t+1}} \tag{1.5}$$

1.3.4 Competitive equilibrium

Given the existence of profit-maximizing firms and utility-maximizing consumers, we state the equilibrium conditions so that equilibrium prices are determined in competitive markets.

Definition 1.1. *Given an initial capital k_0, a sequential trading competitive equilibrium is a sequence of prices $\left(q^t_{t+1}(\theta_{t+1}|\theta_t)\right)$, allocations (c_t), Arrow security demands $(a_{t+1}(\theta_{t+1}|\theta_t))$ and output decisions (k_{t+1}, y_t) such that, given the equilibrium prices, the firm maximizes profit as in* (1.2), *the household maximises utility as in* (1.4) *and markets clear as follows*

$$\begin{aligned} c_t + k_{t+1} &= y_t \\ a_{t+1}(\theta_{t+1}|\theta_t) &= y_{t+1}(\theta_{t+1}|\theta_t) \end{aligned} \tag{1.6}$$

The first condition is the feasibility constraint of the economy, for which the total amount of the output produced must be either consumed or invested as capital. The second clearing condition ensures the equality between each state $\theta_{t+1}|\theta_t$ contingent-claim and production.

Solving the problem using dynamic programming techniques, we characterize the agent's optimal decision rules in the next Proposition

Proposition 1.1. *Consumption and investment policy functions are given by*

$$\begin{aligned} c_t &= \left(1 - \beta\mathbb{E}[\theta_{t+1}|\theta_t]\right)Ak_t^{\theta_t} \\ k_{t+1} &= \beta\mathbb{E}[\theta_{t+1}|\theta_t]Ak_t^{\theta_t}. \end{aligned} \tag{1.7}$$

They are both increasing functions in k and non-monotonic in θ. Moreover, they are both continuous in $k \in \mathbb{R}_+$.

Proof. in Appendix A.

The result is due to the structure of the Mirman-Zilcha model, where the time t expected value of the technology shock

$$\mathbb{E}[\theta_{t+1}|\theta_t] = \sum_{j\in\{h,l\}} \theta^j P(\theta_{t+1} = \theta^j|\theta_t = \theta^i) = \sum_{j\in\{h,l\}} \theta^j p_{ij}$$

directly affects both the consumption and the capital investment optimal decisions. The higher is the probability of the high productivity state, the larger is the fraction of the aggregate wealth allocated as capital investment, rather than consumption. In fact, within the Cobb-Douglass class of production functions, capital share also denotes the elasticity of output with respect to capital and, it is not surprising that a larger fraction of wealth will be invested if the next period technology is expected to be more productive. Moreover, the optimal decision rules are affected by the time discount factor as well. The more patient the agent is, the greater would be the propensity for future rather than current consumption[4].

Lastly, replacing the consumption policy rule in (1.7) into the household's optimality condition (1.5), we define the equilibrium state-prices as

$$q^t_{t+1}(\theta_{t+1}|\theta_t) = \frac{\beta\pi(\theta_{t+1}|\theta_t)k_t^{\theta_t}}{k_{t+1}^{\theta_{t+1}}}. \tag{1.8}$$

for any $\theta_{t+1}|\theta_t$.

1.3.5 The asymptotic analysis of a stochastic growth model

The purpose of the asymptotic analysis of a stochastic growth model is to characterize the long-run behavior of the optimal capital stock (see Brock and Mirman (1972)). In fact, in a deterministic environment, the optimal capital path converges to a stable steady state, also known as the modified golden rule. The corresponding definition in a stochastic environment revokes the existence of a unique invariant distribution for capital. Due to the nature of the underlying technology shock process, the sequence of capital stocks $k_{(t,\theta)}$ obtained by iterating the optimal policy rule (1.7) forms a Markov process as well. Let $H(k,\theta) = \beta\mathbb{E}\left[\theta\right] Ak^{\theta}$ be the capital transition function[5] and define

$$H_m(k) = \min_\theta H(k) = \beta\mathbb{E}[\theta]Ak^{\theta^l} \qquad H_M(k) = \max_\theta H(k) = \beta\mathbb{E}[\theta]Ak^{\theta^h} \tag{1.9}$$

as the lower and the upper envelopes of $H\left(k,\theta\right)$. The fixed points of $H_m(k)$ and $H_M(k)$ are given by

$$k_m = \{H_m(k) = k\} \quad k_M = \{H_M(k) = k\}.$$

The following Definition introduces the notion of $\pi-$invariant set, that is essential for the proof of convergence of the capital distribution function

Definition 1.2. *Let S' be a closed interval of S, then S' is said to be $\pi-$invariant if $\pi(\{\theta : H(k,\theta) \in S'\}) = 1$, for each $k \in S'$.*

[4]Although it may seem trivial our result does not hold when the technology shock has a multiplicative nature (i.e. $y_t = \theta_t k_t^\alpha$). See Remark 1 in Appendix A.

[5]$\mathbb{E}[\theta]$ is the unconditional mean of the stochastic term computed using the invariant probabilities of the Markov chain defined in Section 1.3.1.

where S is a closed interval of $\mathbb{R}_+$ and $P(k, S') = \pi(\{\theta : H(k,\theta) \in S'\})$ is a function that measures the probability for capital k to moves in the set S'. Applying the Definition above in our specific setting, we define $F_t(k) = P(k_t \in B)$ the probability that k_t belongs to B, a Borel set on the positive real line. Thus, the next Proposition states the existence of an invariant measure of $\{k_t\}$

Proposition 1.2. *There exists of a unique and invariant distribution $F(k)$ on $\mathbb{R}_+$ and its support is given by $[k_m, k_M]$. Therefore, for any initial capital k_0, $F_t(k) \to F(k)$ as $t \to \infty$.*

Proof. in Appendix A.

Using Figures 1.2 and 1.3, we provide a graphical intuition of some features that derives from the long-run capital convergence process.

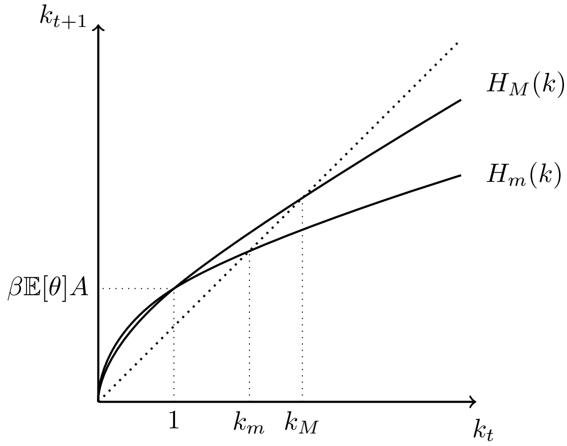


Figure 1.2: Configuration A: the capital stock recurrent set is entirely placed above one.

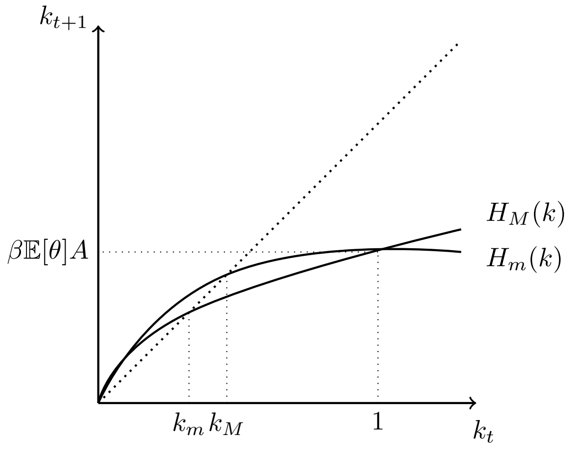


Figure 1.3: Configuration B: the capital recurrent set is entirely placed below one.

Depending on steepness of $H(k, \theta)$, the recurrent set $[k_m, k_M]$ where the capital stock evolves may be located entirely above, below or including one.

Analytically, the steepness of the transition maps depends on the product between three elements: the discount factor β, the time t expected value of the stochastic event $\mathbb{E}[\theta_{t+1}|\theta_t]$ and the

TFP A, that is usually considered a measure of the long-run technological level. Moreover, since both β and $\mathbb{E}[\theta_{t+1}|\theta_t]$ are lower than one, the position of the capital's support is mainly determined by the economy TFP term.

Analysing this feature is essential to understand the impact of the shock realizations on the economy fundamentals. In fact, θ^h and θ^l are the "good" or the "bad" states of the Nature, in the sense that they have a positive or a negative impact on the production level, provided that $k_m > 1$. By contrast, when $k_M < 1$ the meaning of States of the Nature reverts: θ^l and θ^h turns into the "good" and the "bad" states in such a case. Finally, if

$$\beta A \in \left(\left(\mathbb{E}_t[\theta_{t+1}|\theta_t = \theta^h]\right)^{-1}, \left(\mathbb{E}_t[\theta_{t+1}|\theta_t = \theta^l]\right)^{-1} \right)$$

then the support of capital includes one. As a consequence, θ^h and θ^l are the "good" and "bad" states only in the nodes characterized by $k_t > 1$ (see Figure 1.4).

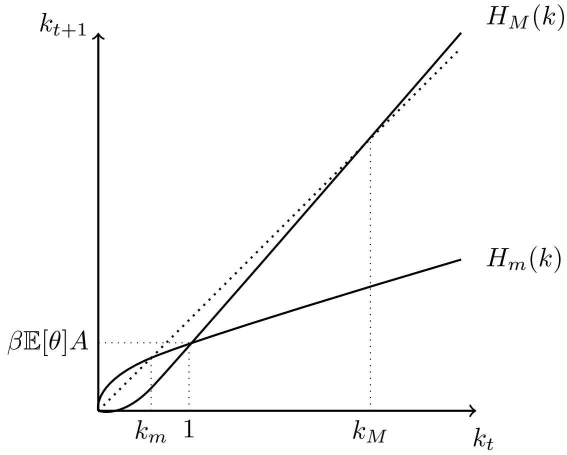


Figure 1.4: Configuration C: the capital recurrent set includes one.

1.3.6 The role of expectations

Having showed the major properties of the baseline framework, we provide some examples to shed light on the role of expectations in the economy dynamics. These examples are intended to capture some behavioral biases that commonly arise when people make investment choices dealing with uncertain returns. We compare economies where people tend to overestimate different states of the Nature.

The optimist As a first case study, we consider the baseline framework under the assumption that the representative agent is an optimist. Essentially, we refer to optimism as a mental attitude leading to a systematic over-estimation of the high productivity state θ^h.

The optimal consumption and investment strategies are set according to the following subjective transition probabilities

$$P^o = \begin{bmatrix} p + b^o & (1-p) - b^o \\ (1-p) + b^o & p - b^o \end{bmatrix} \tag{1.10}$$

where $b^o > 0$ denotes the agent's belief bias. The conditional expected value of the stochastic term is given by

$$\mathbb{E}^o[\theta_{t+1}|\theta_t = \theta^i] = \sum_{j \in \{h,l\}} p^o_{ij}\theta^j \tag{1.11}$$

In the long-run, the invariant probabilities attributes to the random event are such that $\pi^o(\theta^h) > \pi(\theta^h)$ and $\pi^o(\theta^l) < \pi(\theta^l)$. As a result, the unconditional mean is larger than the one computed under the true probability measure

$$\mathbb{E}^o[\theta] > \mathbb{E}[\theta]$$

According to the optimal capital allocation rule (1.7), this bias-type implies over-investment and under-consumption compared to the RE traditional setting. Another element that is worth investigating is the volatility performed by the macroeconomic aggregates. Does a greater capital stock imply a larger investment volatility? We will address the question after presenting the other examples.

The pessimist For thoroughness' sake, we study the economy features in the case the representative agent has an opposite view regarding the technology shock. Therefore, we consider the case where the agent under-estimates the probability of the high productivity state. Thus, the consumption-saving process is grounded on the following transition probability matrix

$$P^p = \begin{bmatrix} p - b^p & (1-p) + b^p \\ (1-p) - b^p & p + b^p \end{bmatrix} \tag{1.12}$$

In contrast to the previous example, the stationary probabilities are such that $\pi^p(\theta^h) < \pi(\theta^h)$ and $\pi^p(\theta^l) > \pi(\theta^l)$ implying

$$\mathbb{E}^p[\theta] < \mathbb{E}[\theta].$$

Further, according to the optimal decision rules, the agent allocates a greater fraction of wealth as consumption, rather than capital investment.

The Trend-follower Finally, we characterize an economy whose representative agent is a trend-follower. This behaviour is often observed in financial markets and it derives from the popular trading strategy of jumping on the trend and ride it, without predicting the market direction. This type of belief distortion is captured by increasing the diagonal probabilities to the extent of a bias $b > 0$. In this way, each state is expected to be more persistent than the truth

$$P^T = \begin{bmatrix} p + b^T & (1-p) - b^T \\ (1-p) - b^T & p + b^T \end{bmatrix}. \tag{1.13}$$

Essentially, the trend-follower behaves as an optimist after having observed θ^h and as pessimist after θ^l. Under the assumption that b is the same in the three outlined examples, the first and the second rows of (1.13) coincides with the first and the second rows of (1.10) and (1.12), respectively. As a consequence, capital evolves in a set $[k_m, k_M]$, sharing the lower k_m and the upper k_M extremes with the pessimistic and optimistic economies.

It should be emphasized that, in contrast to the previous examples, the trend-follower ends up with the correct unconditional expected value of the shock

$$\mathbb{E}^T[\theta] \equiv \mathbb{E}[\theta]$$

as a consequence, it may be considered a less extreme case compared to the optimistic and pessimistic frameworks. However, compared to the REH, there are divergences on the economy dynamics either case.

Using the outlined examples, the rest part of this section is an assessment of the effect of belief-biases on the optimal capital accumulation path. The purpose here is to study the impact of the bias direction, therefore, we consider at this stage economies with the same bias size

$$|b^o| = |b^p| = |b^T|.$$

Results are presented under the assumption that the capital support is entirely placed above one in all the three economies: $\beta\mathbb{E}^i[\theta_{t+1}|\theta^t]A > 1$, $i \in (o, p, T)$. In this way, we can more intuitively refer to θ^h and θ^l as the high and the low productivity states[6].

First, it is worth noting that aggregate investment always moves pro-cyclically with the fluctuations exhibited by the aggregate production due to the shock perturbations. Therefore, having biased beliefs does not affect the sign of the investment best reply to the exogenous shock. However, there are consequences on the mean around which it fluctuates and also on the size of these oscillation, that may amplify or dampen the effect produced by the random events. In this regard, we compute the first and the second moment of k_{t+1} conditional on the same k_t

$$\begin{aligned} \mathbb{E}[k^i_{t+1}|k_t] &= \beta A \left(\pi(\theta^h)\left(\mathbb{E}^i[\theta_{t+1}|\theta^h]k_t^{\theta^h}\right) + \pi(\theta^l)\left(\mathbb{E}^i[\theta_{t+1}|\theta^l]k_t^{\theta^l}\right)\right) \\ Var[k^i_{t+1}|k_t] &= (\beta A)^2 \pi(\theta^h)\pi(\theta^l)\left(\mathbb{E}^i[\theta_{t+1}|\theta^h]k_t^{\theta^h} - \mathbb{E}^i[\theta_{t+1}|\theta^l]k_t^{\theta^l}\right)^2 \end{aligned} \tag{1.14}$$

for $i \in (o, p, T)$.

Conditional to the same time k_t, the investment rate is larger in the optimistic rather than the pessimistic economy, on average. The trend-follower is characterized by a $\mathbb{E}^t[k^i_{t+1}|k_t]$ that is somewhere in the middle of the two extreme cases.

The impact on the investment volatility is not that straightforward. As can be deduced from (1.14), the conditional variance of capital is a positive function of the difference between the conditional individual expected values. The latter is the greatest in the trend-follower economy, followed by the optimistic and the pessimistic frameworks, respectively.

The following Propositions summarizes these results

Proposition 1.3. *Conditional to the same k_t, both the first and the second moments of k_{t+1} are increasing (decreasing) functions of the economy degree of optimism (pessimism). Variance is as well as positively affected by the distance between the conditional individual expected values: $\mathbb{E}^i_t[\theta_{t+1}|\theta^h]k_t^{\theta^h} - \mathbb{E}^i_t[\theta_{t+1}|\theta^l]k_t^{\theta^l}$.*

Proof. in Appendix A.

When capital recurrent sets $[k^i_m, k^i_M]$ are disjoints in the three presented economies, we can extend Proposition 1.3 to the unconditional moments of capital. Specifically, this may be the case of the optimistic and the pessimistic economies. By contrast, as showed in Figure 1.4, the capital recurrent set partially overlaps those of the optimistic and pessimistic cases in the trend-follower economy. The next Lemma provides the parametric condition under which the above Proposition is extended in such a way

Lemma 1.1. *The aggregate capital recurrent sets $[k^i_m, k^i_M]$ are disjoints in the optimistic and pessimistic economy provided that the following inequality holds*

$$(\beta A\mathbb{E}^o[\theta])^{\frac{1}{1-\theta^l}} > (\beta A\mathbb{E}^p[\theta])^{\frac{1}{1-\theta^h}}.$$

Therefore, the unconditional first and second moments of the capital stock are both increasing (decreasing) functions of the level of optimism (pessimism) characterizing the economies.

[6] The results related to the case $\beta\mathbb{E}[\theta_{t+1}|\theta^t]A < 1$ are outlined in the proof of Proposition 1.3.

Proof. in Appendix A. As displayed in Figure 1.5, this happens the larger are both the bias level $|b|$ and the distance between the high and low shock realizations $\theta^h - \theta^l$.

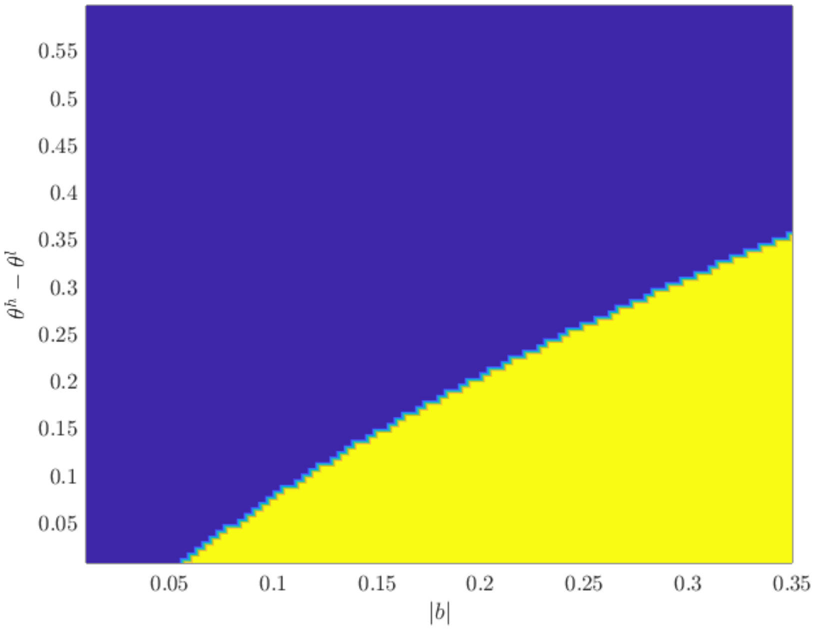


Figure 1.5: The yellow area is the parametric area under which the inequality in Lemma 1.1 holds. Parameters: $|b| \in [0.01, 0.35]$, $\theta^h - \theta^l \in [0.01, 0.6]$, $A = 4$, $\beta = 0.97$, $p = 0.60$.

1.4 The heterogeneous economy

In this section, we consider an environment where different bias types coexist in the same economy. Aiming at preserving the comparability with the results stated above, we maintain the same assumptions for the agent's utility and the firm's technology. Since heterogeneity only concern the households' probabilities, the firm's problem is unaffected by the existence of disagreement and is equivalent to the one described in Section 1.3.2.

Consistently with the previous section, we expect belief heterogeneity affects the real sector, and thus the aggregate consumption process, by means of the capital investment rule. We outline the effect of disagreement in both the individual and the aggregate optimal decision rules. Further, we assess whether the MSH holds in the long-run as it happens in endowment economies. Finally, we study the impact of disagreement in the real economy and evaluate the effect that it produces on the overall social welfare.

Belief heterogeneity is introduced by the following Assumption

Assumption 1.1. *The economy is populated by two equal-sized group of agents: optimist and pessimist. We refer to agent o and p as the representative agent of the group of optimists and pessimists, respectively. Moreover, we assume type i be endowed with the subjective beliefs described in the two examples included in Section 1.3, where $|b^o|$ may be different from $|b^p|$. The two groups share a common discount factor β.*

As in the representative agent framework, each agent $i \in \{o, p\}$ solves a consumption-saving problem by maximizing the stream of future consumptions under his subjective probability measure[7]

$$
\begin{aligned}
&\max_{c_t^i, a_{t+1}^i(\theta_{t+1}|\theta_t)} \mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t \log c_t^i\right] \\
&\text{subject to} \quad c_t^i + \sum_{\theta_{t+1}|\theta_t} q_{t+1}^t\left(\theta_{t+1}|\theta_t\right) a_{t+1}^i\left(\theta_{t+1}|\theta_t\right) = a_t^i + \frac{P_t}{2}
\end{aligned}
\tag{1.15}
$$

1.4.1 Competitive equilibrium

Relying on Definition 1.1 and using dynamic programming techniques[8], we derive the individual optimal consumption and contingent-claim purchases taking place in every $\theta_{t+1}|\theta_t$

$$
\begin{aligned}
c_t^i &= \left(1 - \beta\mathbb{E}^i\left[\theta_{t+1}|\theta_t\right]\right) a_t^i \\
a_{t+1}^i\left(\theta_{t+1}|\theta_t\right) &= \frac{\beta\pi^i\left(\theta_{t+1}|\theta_t\right) a_t^i}{q_{t+1}^t\left(\theta_{t+1}|\theta_t\right)}.
\end{aligned}
\tag{1.16}
$$

Individual optimal choices resemble those characterizing the representative agent problem (see equation 1.7). Each agent consumes a fixed fraction of the current financial wealth a_t^i, that is negatively affected by the expected value of the technology shock. The fraction invested in each contingent claim is a positive function of the state probability assigned and a negative function of its own price.

Using the economy resource constraint included in Definition 1.1, the aggregate capital stock evolves as

$$
k_{t+1} = \beta \sum_{i\in\{o,p\}} \left(\mathbb{E}^i\left[\theta_{t+1}|\theta_t\right]\phi_t^i\right) A k_t^{\theta_t}
\tag{1.17}
$$

where $\phi_t^i = \frac{a_t^i}{Ak_t^{\theta_t}}$ is the node θ^t fraction of the relative wealth owned by agent i, for $i \in \{o,p\}$. Consistently with (1.7), the aggregate investment rule is a convex combination of the agents' expected values of the technology shock. Weights are given by the fraction of relative wealth that each group owns in any node θ^t. As a consequence, the aggregate capital dynamic is mainly driven by the wealthiest type over periods and states.

Consistently with the homogeneous economy, the optimal capital path may evolve in a recurrent set placed above, below or including one. For the rest of the section, we assume that for any θ^t: $\beta\sum_{i\in\{o,p\}}\left(\mathbb{E}^i\left[\theta_{t+1}|\theta_t\right]\phi_t^i\right)A > 1$. In this way θ^h and θ^l are intuitively interpreted as the high and low productivity states.

Lastly, state-prices comes from the equality between Arrow securities' aggregate demand and supply

$$
q_{t+1}^t\left(\theta_{t+1}|\theta_t\right) = \beta\frac{\sum_{i\in\{o,p\}}\pi^i\left(\theta_{t+1}|\theta_t\right)a_t^i}{Ak_{t+1}^{\theta_{t+1}}}
\tag{1.18}
$$

for any $\theta_{t+1}|\theta_t$.

1.4.2 Long-run survival analysis

To study the effect of disagreement in the real economy, we first outline the conditions for which heterogeneity is preserved in the long-run.

Sufficient condition for asymptotic survival is the ownership of a positive fraction of the relative wealth as $t \to \infty$

[7] We assume the firm's profit be uniformly granted among the two groups of agents.
[8] The entire equilibrium proof is showed in Appendix A.

Definition 1.3. *Agent i dominates if*

$$\lim_{t\to\infty} \phi_t^i = 1 \quad a.s.$$

while he vanishes if

$$\lim_{t\to\infty} \phi_t^i = 0 \quad a.s.$$

for $i \in \{o, p\}$.

Agent i vanishes when he is short on cash and, according to (1.16), (1.17) and (1.18), his decisions no longer affects aggregate variables and equilibrium prices. Define

$$z_t = \log\left(\frac{a_t^o}{a_t^p}\right) \tag{1.19}$$

the optimist wealth position relative to the pessimist ($z_t \to \infty \Leftrightarrow \phi^o \to 1$). Replacing the individual wealth dynamic outlined in (1.16), the above may be rewritten as

$$z_t = \xi_t\left(\theta_t|\theta_{t-1}\right) + z_{t-1}$$

where $\xi_t\left(\theta_t|\theta_{t-1}\right) = \log\left(\frac{\pi^o(\theta_t|\theta_{t-1})}{\pi^p(\theta_t|\theta_{t-1})}\right)$ is positive (negative) if the optimist puts relatively more (less) weight on state $\theta_t|\theta_{t-1}$. The conditional drift of the evolution of z_t is given by

$$\mathbb{E}\left[\xi_{t+1}(\theta_{t+1}|\theta_t)\right] = \mathcal{E}_{t+1}^p(\theta_{t+1}|\theta_t) - \mathcal{E}_{t+1}^o(\theta_{t+1}|\theta_t)$$

where $\mathcal{E}_{t+1}^i(\theta_{t+1}|\theta_t) = D_{KL}\left(\pi^i\left(\theta_{t+1}|\theta_t\right)||\pi\left(\theta_{t+1}|\theta_t\right)\right)$ is the node $\theta_{t+1}|\theta_t$ relative entropy of π^i with respect to π (also Kullback-Leibler divergence). Market dominance is stated using the average values of the agents' relative entropies. Let $\mathcal{E}^i = \lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^t \mathcal{E}_\tau^i$, relying on the ergodic theorem: $\mathcal{E}^i = \pi\left(\theta^h\right)\mathcal{E}^i\left(\theta^h\right) + \pi\left(\theta^l\right)\mathcal{E}^i\left(\theta^l\right)$. The next Proposition presents the result related to the market long-run dominance

Proposition 1.4. *The asymptotic value of the drift of the agents log wealth ratio* (1.19) *is given by*

$$\mathcal{E}^p - \mathcal{E}^o.$$

Provided that agent i has the smallest average relative entropy, $\mathcal{E}^i < \mathcal{E}^{-i}$, then $\phi_t^i \to 1$ a.s. and $\phi_t^{-i} \to 0$ a.s.

Proof. in Appendix A.

In a nutshell, the MSH holds even in production economies, supporting rational over irrational investors. Heterogeneity is a transient feature of the market that, sooner or later, converges to an homogeneous economy only populated by the most accurate type. Persistent heterogeneity is possible provided that both the agents are equally distant from the truth and accurate to the same extent: $\mathcal{E}^o = \mathcal{E}^p$.

On top of that, we find an additional aspect that would not emerge in endowment economies. The speed of convergence to the most accurate type is related to the steady state value of production that, in turn, is determined by the dominant type[9].

Specifically, when the optimist is the most inaccurate, the economy convergence to the corresponding homogeneous case faster than he would had been a pessimist. This is true regardless the agents' bias size. Our explanation is that, investing more in the firm, the optimist loose consumption opportunities and waste resources since the firm is less productive then he expects. Controlling

[9] In endowment economies the aggregate production is exogenously given instead.

for the bias level b, the pessimist type would had survived longer. The greater propensity to consume together with the high level of production and investment fuelled by the dominant optimist type, slow down the belief selection process.

To shed light on this asymmetry characterizing the MSH in production economies we define

$$a_t^B \qquad a_t^b$$

the time t financial wealth of the more (B) and the less (b) inaccurate type. Thus,

$$D_{KL}(\pi^B||\pi) > D_{KL}(\pi^b||\pi).$$

Now, we know that in the long-run the MSH holds and the steady state values of the individual financial wealth are given by

$$\begin{cases} a_t^{B*} = 0 \\ a_t^{b*} = y^* \end{cases}$$

where $(\cdot)^*$ denotes the long-run equilibrium values. Since the most inaccurate will be dominated, the negative growth rate of the type B financial wealth is represented by

$$g_{a^B} = \frac{a_{t+1}^B - a_t^B}{a_t^B}$$

take the first-order Taylor approximation of g_{a^B} around the steady state value a_t^{B*}

$$g_{a^B} \approx g_{a^{B*}} + g'_{a^{B*}}(a^B - a^{B*}) \tag{1.20}$$

that is equivalent to

$$g_{a^B} \approx -2\left[\log\left(\frac{\pi^B}{\pi}\right) - \log\left(\frac{\pi^b}{\pi}\right)\right] - \log a^B + \log y^* \tag{1.21}$$

First, the negative sign suggests that the growth rate of a^B is negative indeed. Moreover, as in endowment economies, the speed of convergence to $g_{a^{B*}} = 0$ is positively affected by the difference between the relative distances of π^B and π^b with respect to the truth, π.

However, equation (1.21) implies that the speed increases the highest is the steady state production level y^*. In endowment economies the latter is fixed and it does not depend on the endogenous decision making process. By contrast, it is determined by the dominant type in our model. As a matter of fact, it will be higher if the economy converges to the homogeneous optimistic case and lower otherwise[10]. This is the reason why belief heterogeneity lasts relatively more when the pessimist is the less accurate in the economy.

Proposition 1.5. *The speed with which the market selects for the most accurate beliefs depends on the bias-type. It is faster if the most accurate is a pessimist rather than an optimist.*

Proof. in Appendix A.

Under the same history of shocks θ^t, Figure 1.6 shows the dynamics of ϕ_t^B in two different economies, where the less accurate type B is an optimist (red line) and pessimist (blue line). Clearly, the MSH holds ($\lim_{t\to\infty} \phi_t^B = 0$), however, the convergence is faster when the less accurate type is optimist rather than pessimist.

[10] Remember that the aggregate production is a positive function of capital stock that, in turn, is an increasing function of the degree of optimism (Proposition 1.3).

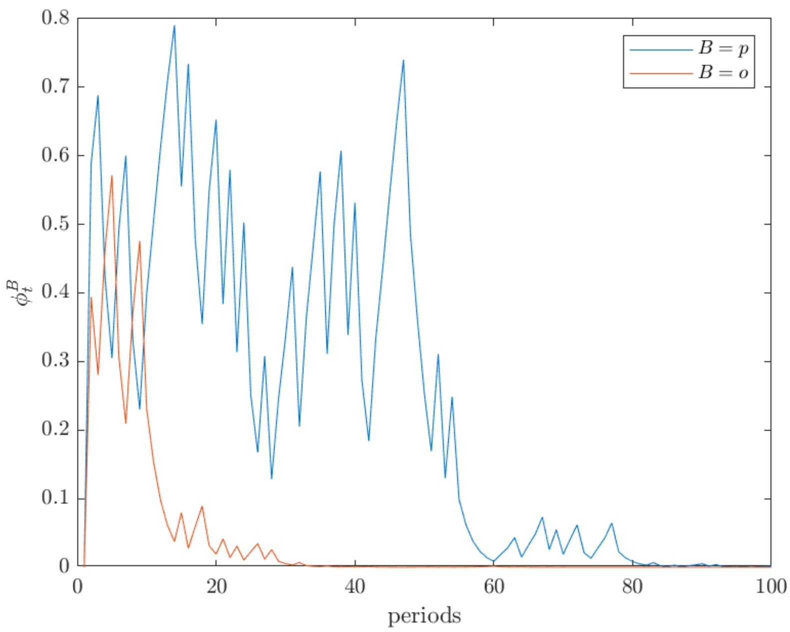


Figure 1.6: Type B relative wealth share in two economies where $B = p$ (blue) and $B = o$ (red). Parameters: $A = 2.5$, $\theta^h = 0.4$, $\theta^l = 0.2$, $\beta = 0.97$, $p = 0.65$.

1.4.3 The real effect of disagreement

We assess the effect of disagreement in the real economy by assuming the two types be biased to the same extent ($|b^o| = |b^p|$ and $\mathcal{E}^o = \mathcal{E}^p$). The explanation lies on the result stated in Proposition 1.4 and the fact that the effect would characterize only the short-term otherwise.

Regarding the capital pattern, we recall from Proposition 1.3 that both the conditional first and second moments of capital stock are increasing (decreasing) functions of the economy degree of optimism (pessimism). Intuitively, we expect disagreement introduces an additional source of volatility due to the coexistence of opposite opinions in the same productive economy. According to (1.17), the investment rate increases (decreases) after high (low) productivity states, not only because of the shock realization, but also because of the increase (decrease) of the aggregate propensity to investment (due to $\phi_t^o \uparrow$ and $\phi_t^p \downarrow$). The volatility induced by the exogenous shock is thus exacerbated by the variability of the agents' wealth distribution over states. This can be seen by looking at the capital-output ratio that, in our context, corresponds to the fraction of wealth allocated as aggregate investment. In homogeneous economies, this is equal to

$$Var\left[\frac{k_{t+1}}{y_t}\right] = \beta^2\pi(\theta^h)\pi(\theta^l)\left((\theta^h - \theta^l)(2p-1)\right)^2 \tag{1.22}$$

and it is neither affected by the agent's bias direction (optimism/pessimism) nor by the bias' magnitude (b). Moreover, it is not state-dependent buy only related to some of the economy parameters (time discount factor, true invariant probabilities and technology). Thus, capital volatility turns out to be an increasing function of optimism because a greater amount of resources are invested and comes to be hit by the high and low shock realizations.

By contrast, the capital-output ratio in an economy with disagreeing agents is given by

$$\begin{aligned} Var\left[\frac{k_{t+1}}{y_t}\right] &= \beta^2\pi(\theta^h)\pi(\theta^l)\left(\sum_{i\in\{o,p\}}\mathbb{E}^i\left[\theta_{t+1}|\theta^h\right]\phi_t^i - \sum_{i\in\{o,p\}}\mathbb{E}^i\left[\theta_{t+1}|\theta^l\right]\phi_t^i\right) \\ &= \beta^2\pi(\theta^h)\pi(\theta^l)\left[(\theta^h-\theta^l)\left(2b[\phi_t^o(\theta^h)-\phi_t^o(\theta^l)]+2p-1\right)\right]^2 \end{aligned} \tag{1.23}$$

and it is always greater than (1.23). Importantly, and in contrast to the homogeneous economy, it is positively affected by the bias size b, implying larger fluctuation the greater is the size of the agents' biases. Variance is also state dependent and related to the agents' distribution of wealth over time and states.

To sum up, disagreement about the stochastic term enhances the capital-output ratio volatility with respect to the corresponding homogeneous economies. Volatility is increased by the size of the agents' biases and the variability of the agents' wealth distribution. Moreover, the average capital-output ratio is always in the middle between the ones performed by the corresponding homogeneous economies.

1.4.4 Complete versus incomplete markets

A complete financial market is one of the condition leading to Pareto optimal results. However, Pareto optimality may be a not suitable definition of the economy social welfare when different opinions coexist in the same environment. A recent literature (see Brunnermeier et al. (2014) and Gilboa et al. (2014)) sheds light on this point, proposing alternative welfare criteria to use in such situations. These works claim that, although increasing the agents' ex ante expected utility, voluntary trades come to be harmful from an ex-post perspective due to the progressive resource depletion faced by the most inaccurate type.

In this section, we aim at characterizing the condition under which an incomplete market structure leads to a welfare improvement compared to the complete market result. In fact, in a contingent-claim production economy, heterogeneous belief agents places bets over future states, beyond financing the aggregate consumption process. In this section trading possibilities are limited to one risky asset paying a stochastic return linked to the marginal productivity of the aggregate capital. The asset purchases are used by agents to transfer resources across periods, however, in contrast to the previous setting, resources cannot be transferred across states of Nature.

In the incomplete market economy, each type $i \in \{o,p\}$ faces the following inter-temporal consumption-saving problem

$$\begin{aligned} &\max_{c_t^i, s_{t+1}^i} \mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t \log c_t^i\right] \\ &\text{subject to}\quad c_t^i + p_t s_{t+1}^i = s_t^i R_t + \frac{P_t}{2} \end{aligned} \tag{1.24}$$

where s_{t+1}^i is amount of the risky asset purchased, yielding a return displayed in the following pay-off vector

$$R_{t+1} = \begin{bmatrix} Ak_{t+1}^{\theta^h} \\ Ak_{t+1}^{\theta^l} \end{bmatrix} \tag{1.25}$$

where k_{t+1} is the node θ^t optimal capital employed.

Definition 1.4. *Given an initial distribution of capital endowments $\{k_0^i\}_{i\in\{o,p\}}$, a competitive equilibrium is a sequence of prices $(p_t)_{\{t,\theta^t\}}$, allocations $(c_t^i)_{\{i,t,\theta^t\}}$, investment strategies*

$\left(s_{t+1}^i\right)_{\{i,t,\theta^t\}}$ *and output decisions* $(k_{t+1}, y_t)_{\{t,\theta^t\}}$ *such that the representative firm maximizes profit as in* (3.3)*, each household maximizes utility as in* (1.24) *and markets clear*

$$\sum_{i\in\{o,p\}} s_{t+1}^i = 1$$
$$\sum_{i\in\{o,p\}} c_t^i + k_{t+1} = Ak_t^{\theta_t} \tag{1.26}$$

for any θ^t*.*

The optimal individual and aggregate decision rules are summarized by the next Proposition

Proposition 1.6. The individual consumption and investment optimal decision rules are given by

$$c_t^i = \left(1 - \beta\mathbb{E}^i\left(\theta_{t+1}|\theta_t\right)\right) s_t^i Ak_t^{\theta_t}$$
$$s_{t+1}^i = \frac{\beta s_t^i Ak_t^{\theta_t}}{p_t} \tag{1.27}$$

with $s_0^i = \frac{k_0^i}{\sum_{i\in\{o,p\}} k_0^i}$. The capital stock evolves as

$$k_{t+1} = \beta\left[\sum_{i\in\{o,p\}} \mathbb{E}^i\left(\theta_{t+1}|\theta_t\right) s_t^i\right] Ak_t^{\theta_t} \tag{1.28}$$

and the equilibrium stock price is given by

$$p_t = \beta Ak_t^{\theta_t} \tag{1.29}$$

Proof. in Appendix A.

By replacing (1.29) into the stock holding optimal rule (1.27), it is easy to see that $s_{t+1}^i = s_t^i$ for any θ^t. Therefore, the latter only depends on the initial distribution of consumption and, provided that neither is endowed with zero initial capital, heterogeneity is a persistent feature of the market. The logarithm of the agents' wealth ratio is constant and given by

$$z_t = z_{t-1} = ... = z_0$$

Figure 3.34 shows the difference of the social welfare achieved in a disagreement complete and incomplete market economy

$$\Delta W = W^{complete} - W^{incomplete} \tag{1.30}$$

where the social welfare function is the sum of the agents' utilities under the true probability measure

$$W = \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t \sum_{i\in\{o,p\}}\left(\log c_t^i\right)\right]. \tag{1.31}$$

Everything else equal, the incomplete market economy achieves a greater social welfare than the corresponding complete market design, for any combination of the agent's belief biases. As one would expect, the complete market economy experiences a severe welfare loss the greater is the distance between the agent' biases. Instead, the difference with the incomplete market reduces along the diagonal, when agents are inaccurate to the same extent and they both survive even in complete market frameworks. In such a case, the welfare level is lower than in the incomplete market setting due to the greater volatility experienced by the aggregate consumption level (see the discussion in Section 1.4.3).

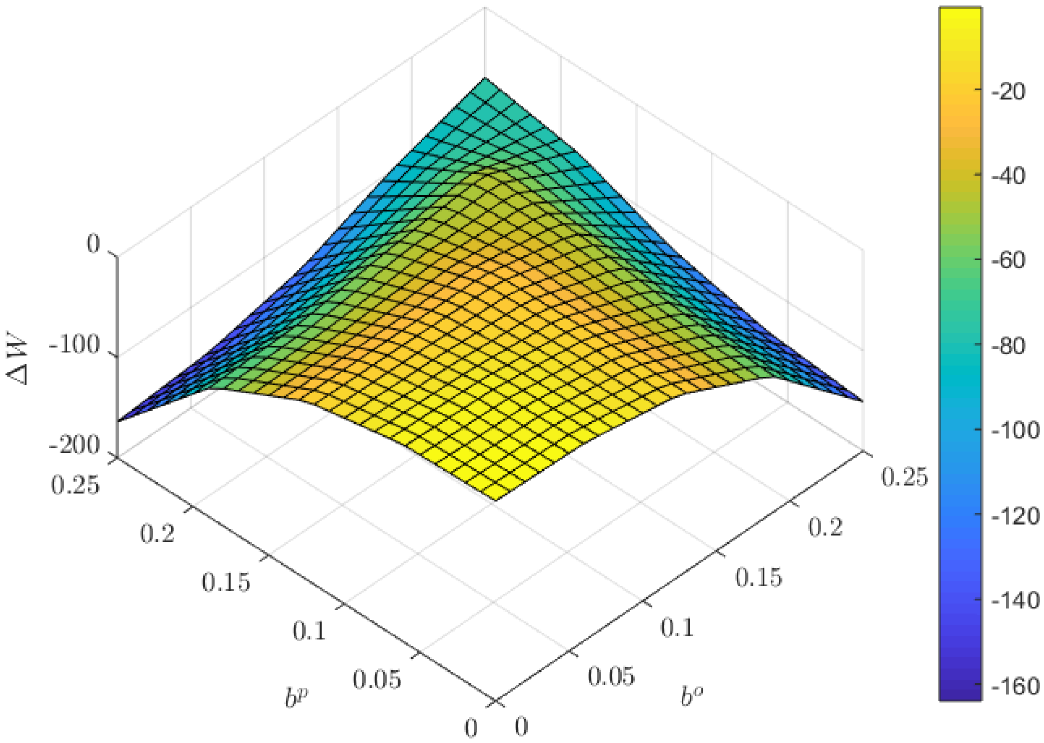


Figure 1.7: Difference of the social welfare between the complete and incomplete market economies, for different combinations of the agents' belief biases, $b^i \in [0, 0.25]$, $i \in \{o, p\}$. When both agents are unbiased, $b^o = b^p = 0$, $\Delta W = 0$. The welfare function is an average value of 10000 simulations of 100 periods economies using the same parameters of Figure 1.6.

Finally, we compare the realized welfare in a heterogeneous incomplete market economy with the one characterizing an homogeneous complete market framework. The result is not straightforward in such a case since the representative agent economy is not exposed to the negative externalities induced by the agents' speculative trades. Figure 1.8 displays ΔW comparing the disagreement incomplete market economy with an homogeneous economy characterized by optimists (left) and pessimists (right). Provided that the truth is in the middle the agents' opinion, belief heterogeneity improves the representative agent result for any combination of belief biases (b^o, b^p). Improvements are more evident the larger is the bias size in the homogeneous belief framework. As can be deduced from the individual consumption (1.27) and the aggregate investment (1.28) rules the incomplete market disagreement economy is equivalent to an homogeneous economy, where the representative agent's belief is the convex combination of the two types $i \in \{o, p\}$ and weights are given by the initial wealth distribution.

Provided that the truth is in the middle of the agents' opinion, welfare gains derives from the reduced over/under-investment that characterize the optimistic/pessimistic homogeneous economies. There are positive implications for the real sector: by convexyfing the market beliefs, there aggregate consumption process is closer to the one implied by the truth. Benefits on the real sector emerges from the comparison with the complete market disagreement economy as well. Preventing speculative trading, disagreement does not enhance the volatility experienced by macro variables due to the endogenous agents' wealth distribution (see discussion in Section 1.4.3). In this case,

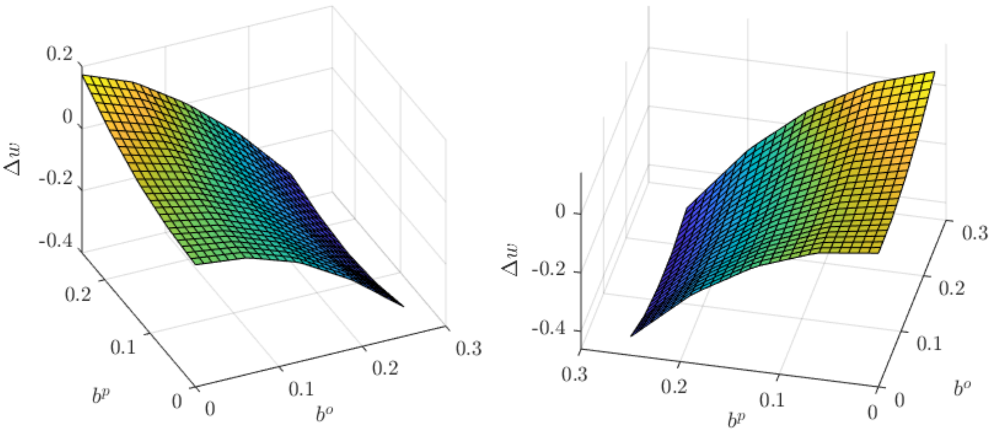


Figure 1.8: Difference in social welfare between a contingent-claim homogeneous economy (optimistic [right] and pessimistic [left]) and an incomplete market heterogeneous economy, where $\{k_0^i = 0.5\}_{i\in\{1,2\}}$. The economies realized welfare are averages of 10000 simulations with $t = 100$. Parameters as in Figure 1.6.

aggregate consumption moves only because of the exogenous shock.

Many issues are linked to incomplete markets since they imply a significant risk management limitation. Therefore, preventing market completeness, when this is feasible, is not the best idea to overcome the problem. However, introducing some kind of trading restrictive measures may be desirable from the standpoint of a benevolent policy maker.

1.5 Conclusion

We present a stochastic one-sector growth model where agents have biased beliefs about the true probability governing the technology shock. First, we study the implications that this produces in a representative agent model. Biased expectation affects both the first and the second conditional moments of the aggregate investment process. Second, using the same framework, we characterize a two-agent disagreement economy. Consistently with the MSH literature, the economy selects the most accurate agent even when aggregate consumption is endogenously determined. However, the speed of convergence depends on the steady-state aggregate production level, which in turn depends on the dominant type. The steady-state aggregate production is greater when the latter is an optimist, slowing down the pessimist extinction process. Finally, welfare gains arises when the set of the available assets is restricted. An incomplete market heterogeneous belief economy achieves a greater realized welfare compared to the corresponding complete market framework. This result calls for further research grounded in the policy maker perspectives.

Chapter 2

Long-run effect of a transaction tax in a speculative financial market[1]

Abstract

This paper studies the optimal financial transaction tax, and its implication on long-run asset prices, in a dynamic exchange economy where trading arises exclusively for speculative reasons. Investors hold different beliefs about the occurrence of states of Nature and choose their portfolio composition accordingly. Without any policy intervention the market selection hypothesis holds and prices eventually reveal the beliefs of the most accurate agent. This result is not guaranteed when the trading activity is limited by a tax on the value of agents' securities exchange. The optimal taxation depends on the aim of the Government's intervention. When the Government aims at maximizing the agents' welfare, a no-trade result emerges and state prices are undetermined in the convex combination of the agents beliefs. Conversely, when the purpose of the policy is purely to raise revenues, the optimal tax rate is positive and increasing in agents' disagreement. The tax mitigates, although does not eliminate, the agents' willingness to trade. Dominance of the most accurate traders is not guaranteed any more, and taxation could imply that the most accurate agent vanishes with positive probability, leading to severe miss-pricing in the long-run.

Keywords: Heterogeneous Beliefs, Speculative Markets, Tobin Tax, Market Selection Hypothesis, Asset prices.
JEL Classification: D53, H30

2.1 Introduction

After the financial crisis, the debate about *Financial Transaction Taxes* (FTT or Tobin tax) has flared up again in public economics. Two motivations are commonly offered in favor of this policy instrument. First, the revenue raising capacity. Due to the increase in the debt-levels observed in the past few years, this tax would provide additional revenues to Governments that are pursuing fiscal consolidation programs. Second, the reduction of short-term speculation characterizing financial markets.

In this paper, we investigate the effect of a linear FTT in a dynamic exchange economy where trading arises for speculative purposes. Within this environment, we assess the extent to which, on the one hand, the FTT reduces short-term speculation while, on the other hand, it modifies the market selection landscape and thus affects asset prices in the long-run. To do so, we propose a theoretical framework where trading exclusively originates from differences in investors' beliefs.

[1] This chapter is based on joint work with Pietro Dindo

Agents value terminal consumption and are endowed with the same amount of goods at the end of any possible history of states. However, they are willing to trade because they assign different probabilities to the realization of states of Nature. That happens because the equilibrium state prices, reflecting a convex combination of agents' beliefs, turn out to be under (or over) evaluated in the eyes of some traders, thus providing the incentive to trade. We name this trade speculative in that it would not arise if traders had homogeneous beliefs. In our model, the financial structure is sequentially complete so that in every period agents are not restricted in the amount of speculative trade.

The FTT is set by a Government, who intervenes in the market with both revenue raising and welfare maximizing concerns. Although public expenditure is not explicitly modeled in this work, the amount of fiscal revenues collected may be used as a proxy of welfare benefits that would arise from that. Beliefs' distortion creates room for a normative analysis to study the welfare consequences of this market behavior. Despite Pareto-improving trades are viewed as desirable in traditional frameworks, a growing literature finds this argument not fully compelling when agents have different beliefs (see Brunnermeier et al. (2014) and Gilboa et al. (2014)). The reason is that speculation creates a negative externality that stems from the progressive impoverishment of the more inaccurate agents. In general, there is a controversial argument about the probability measure over which the policy intervention may lean on. Taking a strand of the truth, we study the optimal taxation problem under a set of reasonable belief that the Government may decide to use. Specifically, in the spirit of Brunnermeier et al. (2014), we refer to the set of reasonable beliefs as the convex hull of all the agents' subjective probability measures.

To study the optimal taxation features when the Government's purpose is two-fold, we allow for the following structure. First, we consider the problem when the planner's aim is purely welfare maximizing. In this case, a no-trade result emerges. In other words, speculative trading always reduces social welfare, leading the Government to set a tax rate under which traders are no longer willing to exchange. Second, we characterize the same problem when the policy maker aims at raising fiscal revenues. In this case, we find that the optimal taxation is an increasing function of the difference of the agents' belief and it is not affected by the Government probability measure. Third, we derive the optimal taxation when the Government's aim is a convex combination of two. The optimal tax inherits the features of both the policy problems in such a case.

The dynamic structure of the model allows us to investigate the impact of the FTT on the long-run characteristics of the economy. Without any intervention, it is well known that the market selects the most accurate agent, letting the others vanish sooner or later (Blume and Easley (2006) and Sandroni (2000)). For a set of parameters, the tax challenges this result, removing the link between survival and accuracy. As one would expect, this happens when the optimal tax rate is high enough and neither agent is sufficiently close to the truth. The *Market Selection Hypothesis* (MSH) is no longer guaranteed and, depending on the history of shocks, less accurate agents may eventually dominate in the long-run. The explanation lies on the linearity of the policy instrument: we find the tax impact be decreasing in the agents' level of wealth. Consequently, survival of the less accurate may occur, for a matter of luck, over those path where he accumulates most of the aggregate wealth.

We finally address the trade off between speculation reduction and market accuracy decay. The FTT, as any other friction, affects the allocative efficiency causing both trading flows and price pattern distortion. However, it produces a significant impact on the accuracy of the market as well. When the MSH holds, prices are relatively correct because they eventually reflect the belief of the most accurate agent. This market feature is no longer guaranteed in our environment, where the most inaccurate agent may eventually dominate in the long-run.

2.1.1 Related literature

Taxing financial transactions is not a new idea but it comes from the past. Historically, Tobin (1978) and Keynes (1936) are known as the first proponents. After the collapse of the Bretton Wood system, Tobin's suggestions was to impose a small tax on all the international currency transaction to curb the fluctuations of the exchange rates. Even before, Keynes claimed the introduction of this tax on equity trades. The purpose was to support fundamentalist over short-term investors and facilitate the capital raising function traditionally attributes to the financial sector.

Conversely, Friedman (1953) famously challenged these viewpoints. His argument was that destabilizing speculation cannot be persistent since it is unprofitable and it quickly drains speculators out of the market.

Still, opposing views foster the ongoing debate about the introduction of this tax (for a comprehensive survey of the literature we refer to McCulloch and Pacillo (2010) and Schulmeister (2009)).

This paper naturally belongs to the literature that evaluates the impact of the FTT, under both theoretical and empirical perspectives. See Stiglitz (1989) and Summers and Summers (1989) for the first contributions after the Tobin's proposal. These papers essentially support the introduction of the tax in order to reduce the instability of financial market and enhance the allocation of resources in the real sector.

Perhaps the most closely-related work is Davila (2014), that likewise studies the optimal taxation problem when agents hold different opinions and the policy instrument is a FTT. However, in contrast to our paper, the model is set in a two-period framework and it cannot account for the impact of this policy measure on the long-run features of the economy.

One of the negative points mostly stressed by the traditional literature about this tax is the reduction of market liquidity and the increase of the short-term volatility. For instance, Umlauf (1993) and Jones and Seguin (1997) find an empirical positive correlation between price volatility and reduction of trading transaction costs using both Swedish and American data. This result is consistent with Baltagi et al. (2006), that finds the same evidence in the Chinese stock market, after the Government's decision to increase the tax rate on stocks' trading.

More broadly, this paper also contributes to the growing literature about behavioural public economics. Congdon et al. (2009) is one of the first works raising the need to consider behavioural economics findings into tax policy design. In this vein, Gabaix and Farhi (2017) attempts to find a general theory of optimal taxation when agents misoptimize. Previously, O'Donoghue and Rabin (2006) have shown that, when the population is characterized by limited self-control, a sin tax on unhealthy goods is Pareto improving. Still, Chetty et al. (2009) points out that people tend to underestimate the impact of taxes when they are not explicitly included in posted prices. Regarding the FTT, Westerhoff and Dieci (2006) analyses the ability of this tax to stabilize financial markets. Using a chartist-fundamentalist model, they find that if a transaction tax is imposed in only one market, then speculators migrate enhancing the instability of other trading venues. If it is evenly set in all the available markets, then unjustified speculation is dampened in favor of an increase of the overall financial stability. In contrast to these works, traders are not affected by cognitive biases in our paper. Instead, they are subjective expected utility maximizers (in the sense of Savage (1951)) and they act in their perceived best interest.

Finally, this work also relates to the recent literature focusing on alternative welfare criteria, rather than Pareto optimality, to use when voluntary trades stem from different opinions coexisting in the same economy. Among these, Brunnermeier et al. (2014) defines as belief-neutral efficient, an allocations that is efficient under whatever convex combination of the agents' beliefs. According to Gilboa et al. (2014), there exist a no-betting-Pareto dominant allocation if there is a single belief that, if shared, makes every agent better off. Besides, Blume et al. (2018) investigates the impact of some trading restrictive measures (borrowing constraints and market incompleteness) on the social welfare, when the latter is computed using the true probability measure. In contrast to these works, we do not propose an alternative welfare criterion to use in such situations. Rather we evaluate the effect of a policy measure that limits the trading possibilities when the economy is characterized by belief heterogeneity and financial markets are complete.

Outline Section 2.2 is a brief description of the traders' problem and the competitive equilibrium without FTT. We refer to this structure as our benchmark. Section 2.3 outlines the decentralized equilibrium with a linear FTT. Section 2.4 contains the implications on the long-run equilibrium. We describe the Government's problem and the optimal taxation features in section 2.5. Section 2.6 concludes.

2.2 Baseline framework without FTT

2.2.1 The model

In this section, we introduce the baseline framework of the model. Time is discrete and indexed by $t \in (0, 1, ..., T)$, with $T > 0$. A sequence of random variables (s_t) following an i.i.d. process is the exogenous shocks at dates $t = 1, 2, ..., T$, while $s^t = (s_0, s_1, ..., s_t)$ is the vector of the shock realizations up to t.

At each point of time t there are, without loss of generality, two possible states of the world $s_t \in \{1, 2\}$, while $\pi = (\pi, 1 - \pi)$ are the probabilities[2] of state 1 and 2, respectively.

We assume the economy be populated by two equal-sized group of agents, using a subjective probability measure π^i for $i \in \{1, 2\}$ and $\pi^1 \neq \pi^2$. From now on, we refer to agent 1 and 2 as the representative agent of group 1 and 2. Subjective beliefs are constructed so that agents agree on both the shock realizations and the i.i.d. nature of the stochastic process but each group is endowed with a different $\pi^i = (\pi^i, 1 - \pi^i)$. Agents do not update their opinions when they observe new shock realizations but keep their beliefs all along. The assumption of persistent disagreement can be thought of as a proxy of an economy where, convergence to one single model, is prevented by agents having disjoint probability supports among themselves.

Lastly, we assume that $\pi^1 > \pi^2$ implying that agent 1 assigns more probability to state 1 relative to agent 2. However, we do not make any assumption about the distance between the objective and the subjective probability measures.

Each agent is endowed with the same unit of consumption y^i at time T and regardless the history s^T. However, they have the possibility to sequentially trade claims on terminal consumption in order to change their consumption profiles. Specifically, in every node s^t, each agent may decide the amount of wealth to invest in each asset that delivers consumption at maturity T and conditional on the realization of $s_{t+1} \in \{1, 2\}$. Financial markets are, therefore, dynamically complete.

Figure 2.2 helps to clarify. It displays the stochastic environment in a three-periods economy where $T = 2$. At time $t = 0$, each agent decides the fraction of his wealth to allocate in asset 1 or 2. Both assets have a two-period maturity. Asset 1 (2) is a claim on a unitary amount of consumption good in the final period T if state $s_1 = 1$ ($s_1 = 2$) realizes. At time $t = 1$, when state s_1 is realized, agents may decide to trade in claims on final period consumption conditional on the

[2] With an abuse of notation, we denote with π both the vector of probabilities and its first entry, that is the probability of state 1, $\pi = Pr\,(s = 1)$. Since probabilities sum to one, we refer to $1 - \pi = Pr\,(s = 2)$.

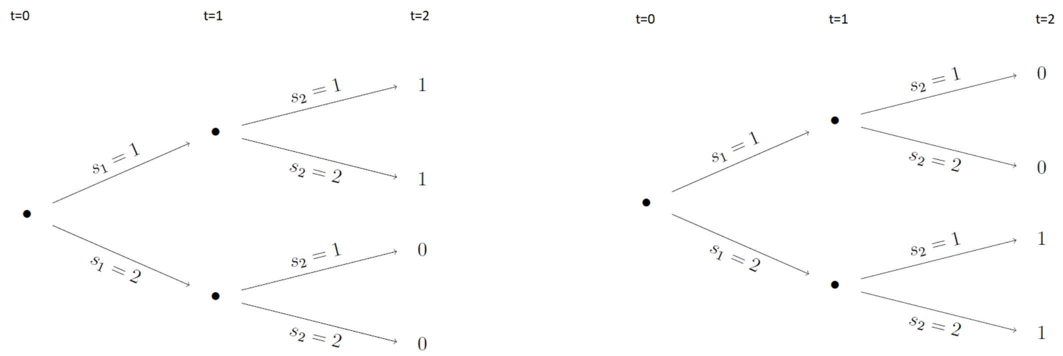


Figure 2.1: Payoffs of the asset 1 [left] and asset 2 [right] traded at time $t = 0$. Asset 1 (2) delivers time $T = 2$ consumption if state $s_1 = 1$ ($s_1 = 2$) realizes.

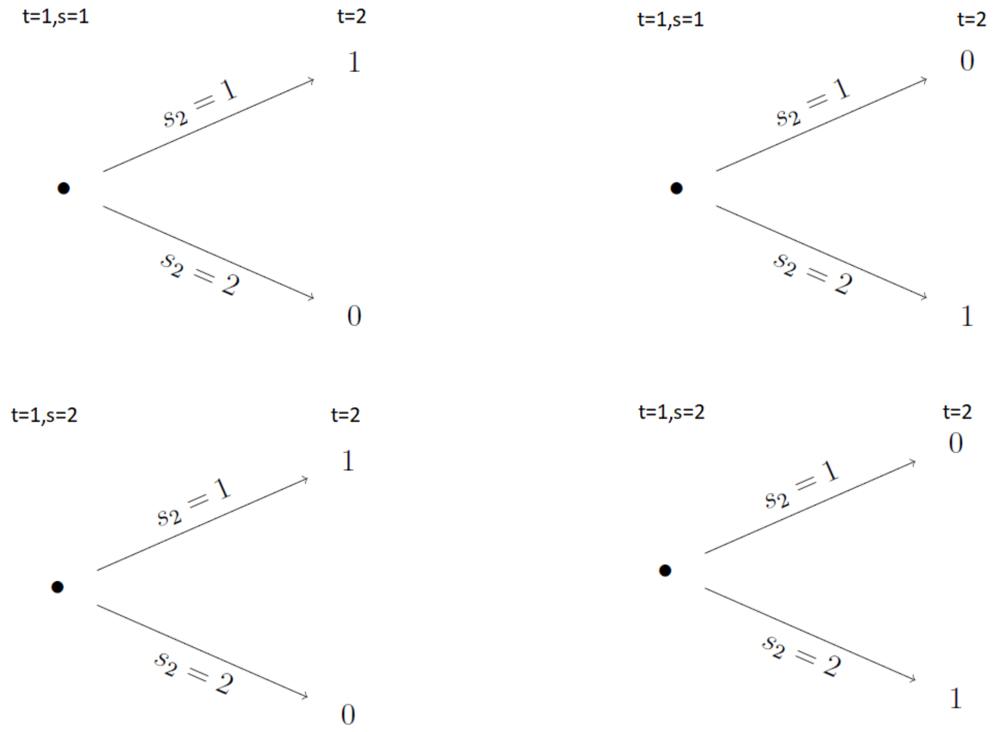


Figure 2.2: Payoffs of the asset 1 [left] and asset 2 [right] traded at time $t = 1$ and conditional on the realization of $s_1 = 1$ [top] and $s_1 = 2$ [bottom]. Notice that, at $t = T - 1$, agents actually exchange Arrow securities.

realization of s_2. Thus, the assets traded in $t=1$ have a one-period maturity. As a consequence, agents exchange assets with different maturities over time (i.e. the assets traded at t have $T-t$ maturity).

To simplify the notation, we drop the reference to the node s^t when it is not necessary, so that x_t is used in place of $x_t(s^t)$. Finally, we denote by $q^t_{t+1,s}$ the dated t price of the asset paying terminal consumption conditional on the realization of $s_{t+1}=s$. The sum of individual endowments and the dated t asset prices is normalized to one

$$\sum_{i\in\{1,2\}} y^i = 1, \qquad \sum_{s\in\{1,2\}} q^t_{t+1,s} = 1.$$

for all s^t. Since the aggregate final consumption has been normalized to 1, this is also the total supply of both the assets traded in each node.

2.2.2 Traders problem without FTT

We first introduce the problem without taxation that we use as a benchmark. The agents' problem is defined as follows

$$\begin{gathered}\max_{c^i_T,\left(a^i_{t+1,s}\right)_{t,s^t,s}} \quad \mathbb{E}^i\left[\log c^i_T\right] \\ \text{subject to} \\ \begin{cases} q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t}\left(q^t_{t+1,1}+q^t_{t+1,2}\right), \quad \forall\,(t,s^t) \\ c^i_T = a^i_{T,s^T} \\ a^i_0 = y^i\left(q^0_{T,1}+q^0_{T,2}\right)\end{cases}\end{gathered} \tag{2.1}$$

where $a^i_{t+1,s}$ is the node s^t agent i purchase of the asset delivering final consumption if $s_{t+1}=s$ realizes. As mentioned in Section 2.2.1, asset prices sum up to one in any s^t, hence $\sum_{s\in\{1,2\}} q^t_{t+1,s} = 1$. Note that each agent evaluates terminal consumption using his belief, hence the notation $\mathbb{E}^i(\cdot)$.

Before the final period T, agents have the possibility to entertain an asset trading activity. Since the market is sequentially complete, in any node s^t they may exchange both the securities in order to transfer their current wealth in the two possible subsequent states. In the final period T, they consume their terminal wealth.

Definition 2.1. *Given a distribution of initial endowments $\{y^i\}_i$, a sequential trading competitive equilibrium is a sequence of prices $\left(q^t_{t+1,s}\right)_{t,s^t,s}$, allocations $\left(c^i_T\right)_{i,s^T}$ and asset demands $\left(a^i_{t+1,s}\right)_{i,t,s^t,s}$ such that, given the equilibrium prices, traders solve the problem in* (2.1) *and the following market clearing holds*

$$\sum_{i\in\{1,2\}} a^i_{t+1,s} = \sum_{i\in\{1,2\}} y^i = 1$$

for all $s\in\{1,2\}$ and in every s^t.

As detailed showed in the Appendix, the agent i optimal portfolio composition rule is given by[3]

$$a^i_{t+1,s} = \frac{\pi^i_s a^i_{t,s^t}}{q^t_{t+1,s}} \tag{2.2}$$

[3] Because of the log utility assumption, the relative amount spent in each asset is constant over time, $q^t_{t+1,s}\frac{a^i_{t+1,s}}{a^i_{t,s^t}} = \pi^i_s$.

while the equilibrium price comes from the aggregation of the individual demands

$$q^t_{t+1,s} = \sum_{i\in\{1,2\}} \pi^i \frac{a^i_{t,s^t}}{\sum_{i\in\{1,2\}} a^i_{t,s^t}} \tag{2.3}$$

for all $s \in \{1,2\}$. Notice that, despite being allowed, short-selling does not occur in equilibrium ($a^i_{t+1,s} > 0$ for all i, s). Agents invest in the two assets to transfer their current wealth in each of the two possible subsequent states. Short-selling one of them would imply ending up with a negative wealth if the state where the latter deliver wealth realized. Therefore, it is always optimal for them holding a positive fraction in both assets. However, depending on the distance between their personal and the market evaluation -that is the equilibrium price-, they decide whether to increase or decrease their position in each asset, with respect to their current wealth a^i_{t,s^t}. If trader i puts relatively more weight on state 1, he will increase and decrease his position in asset 1 and 2, respectively ($a^i_{t+1,1} > a^i_{t,s^t} \quad a^i_{t+1,2} < a^i_{t,s^t}$). This means that he is buying asset 1 and selling asset 2, compared to his current wealth position a^i_{t,s^t}. According to (2.3), prices are a convex combination of the agents' beliefs and both the incentive and the direction of trades originates from the distance between subjective beliefs and equilibrium prices. In a homogeneous economy, equilibrium prices are equal to the representative agent's beliefs and this implies the asset demands be constant over time (i.e. $a^i_{t+1,s} = a^i_{t,s^t}$ for all $s \in \{1,2\}$ and s^t).

2.3 Traders problem with linear FTT

Policy intervention entails a linear[4] transaction tax $\tau \in (0,1)$, paid per unit of dollar traded on each of the asset purchased

$$\tau q^t_{t+1,s} |\underbrace{a^i_{t+1,s} - a^i_{t,s^t}}_{\Delta a^i_{t+1,s}}|$$

$s \in \{1,2\}$. The tax applies on the absolute value of the trading net position because it is a trading cost either direction the agent decide to trade.

Each trader solves the problem in (2.1), where the budget constraint modifies as follows[5]

$$\begin{cases} q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} = \Delta a^i_{t+1,2} = 0 \\ q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} + \tau q^t_{t+1,1}\left[a^i_{t+1,1} - a^i_{t,s^t}\right] + \tau q^t_{t+1,2}\left[a^i_{t,s^t} - a^i_{t+1,2}\right] = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} > 0 \wedge \Delta a^i_{t+1,2} < 0 \\ q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} + \tau q^t_{t+1,1}\left[a^i_{t,s^t} - a^i_{t+1,1}\right] + \tau q^t_{t+1,2}\left[a^i_{t+1,2} - a^i_{t,s^t}\right] = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} < 0 \wedge \Delta a^i_{t+1,2} > 0 \end{cases} \tag{2.4}$$

Let

$$\overline{q}^t_{t+1,s} \div (1+\tau)\, q^t_{t+1,s}, \qquad \underline{q}^t_{t+1,s} \div (1-\tau)\, q^t_{t+1,s} \tag{2.5}$$

be the after-tax normalized prices when the agent buys (left) and sells (right) asset s. These may be used to rewrite the sequence of budget constraints in (2.4) as

$$\begin{cases} q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} = \Delta a^i_{t+1,2} = 0 \\ \overline{q}^t_{t+1,1}a^i_{t+1,1} + \underline{q}^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} > 0 \wedge \Delta a^i_{t+1,2} < 0 \\ \underline{q}^t_{t+1,1}a^i_{t+1,1} + \overline{q}^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t} & \text{if } \Delta a^i_{t+1,1} < 0 \wedge \Delta a^i_{t+1,2} > 0 \end{cases} \tag{2.6}$$

for any node s^t. Given the agent i preferences, the second (third) line of (.20) binds if it is optimal to buy (sell) asset 1 and sell (buy) asset 2. Conversely, the first line binds whenever he chooses

[4]A linear FTT is considered by Davila (2014) as well. By contrast, the FTT is a quadratic function of the agents' security purchases in Blume et al. (2018).

[5]Notice that buying one of the asset implies selling the other in order to ensure the budget constraint clearing condition. For this reason, we rule out the cases where an agent buys (sells) both the assets.

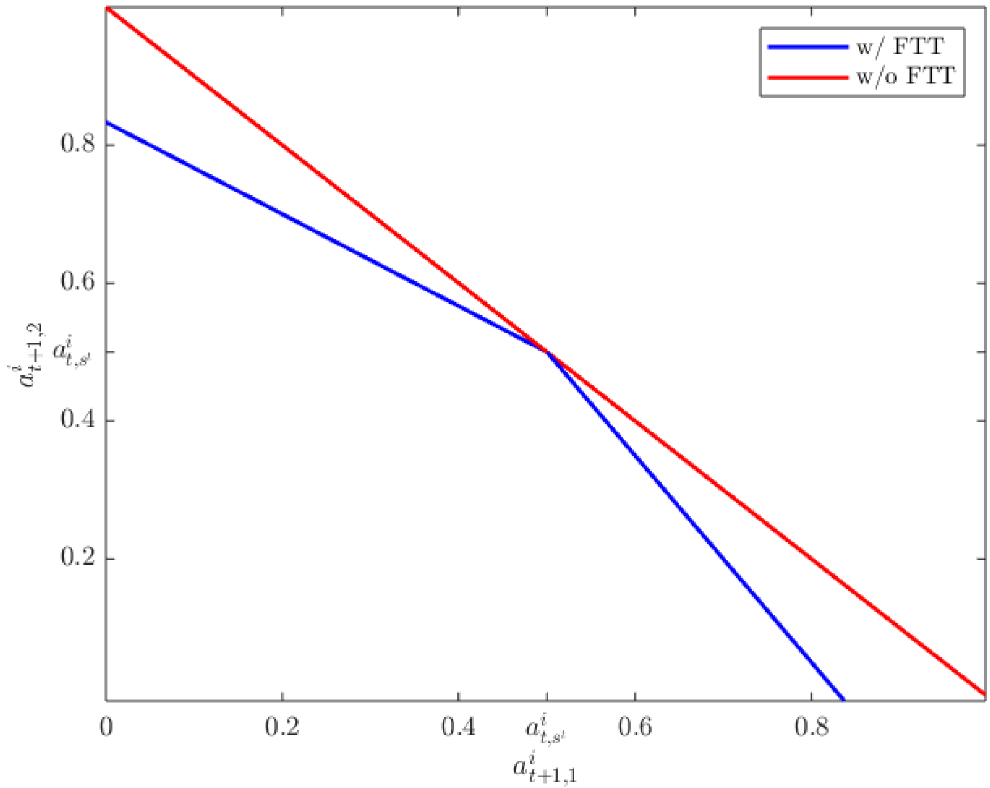


Figure 2.3: Agent i node s^t budget constraint with (blue line) and without (red line) the FTT. The kink point is at a^i_{t,s^t} (here $a^i_{t,s^t} = 0.5$). Above the kink point, the agent buys asset 2 and sells asset 1 ($\Delta a^i_{t+1,1} < 0 \wedge \Delta a^i_{t+1,2} > 0$). The opposite occurs below the kink point ($\Delta a^i_{t+1,1} > 0 \wedge \Delta a^i_{t+1,2} < 0$).

to not trade. Figure 2.3 shows the kinked budget constraint faced in any node. Compared to the benchmark (red line), for a given level of prices, the FTT increase (decrease) the budget line slope below (above) the initial asset allocation (that is the kink point implying equal consumption conditional in either state)

$$\frac{\Delta a^i_{t+1,2}}{\Delta a^i_{t+1,1}} = \begin{cases} \frac{1+\tau}{1-\tau}\frac{q^t_{t+1,1}}{q^t_{t+1,2}} & \text{if } \Delta a^i_{t+1,1} > 0 \wedge \Delta a^i_{t+1,2} < 0 \\ \frac{1-\tau}{1+\tau}\frac{q^t_{t+1,1}}{q^t_{t+1,2}} & \text{if } \Delta a^i_{t+1,1} < 0 \wedge \Delta a^i_{t+1,2} > 0. \end{cases}$$

Imposing a cost τ on financial transaction, tilts the budget constraint inducing both an income and a substitution effect. To better understand the problem we describe the two possible cases:

- $\Delta a^i_{t+1,1} > 0 \;\wedge\; \Delta a^i_{t+1,2} < 0$: buy asset 1 and sell asset 2. In this case, the income effect is negative because the agent feels poorer and he would like to cut in expenditure on $a^i_{t+1,1}$. The substitution effect goes in tandem since taxation increases the relative asset 1 price $\left(\frac{\bar{q}^t_{t+1,1}}{\underline{q}^t_{t+1,2}} > \frac{q^t_{t+1,1}}{q^t_{t+1,2}}\right)$. Thus, $\tau \uparrow$ will lead to $a^i_{t+1,1} \downarrow$.
- $\Delta a^i_{t+1,1} < 0 \;\wedge\; \Delta a^i_{t+1,2} > 0$: sell asset 1 and buy asset 2. On the one side, the income effect is still negative because the FTT increases the overall cost of trading. On the other side, the substitution effect pushes in the opposite direction since taxation decreases the relative price of asset 1 $\left(\frac{\underline{q}^t_{t+1,1}}{\bar{q}^t_{t+1,2}} < \frac{q^t_{t+1,1}}{q^t_{t+1,2}}\right)$. The overall effect on $a^i_{t+1,1}$ turns out to be ambiguous and depending on the relative strength of the two effects.

The FTT challenges the trader's optimal portfolio in two different dimensions. First, it affects the decision to trade and, second, it reduces the size of trading. Related to the first point, the agent i reservation prices as a buyer and a seller of asset 1 are given by[6]

$$q^i_{buy} = \frac{\pi^i (1-\tau)}{(1-\tau(2\pi^i - 1))}, \qquad q^i_{sell} = \frac{\pi^i (1+\tau)}{(1+\tau(2\pi^i - 1))}. \tag{2.7}$$

These are the marginal prices under which the trader is willing to buy and sell asset 1. Given an equilibrium price $q^t_{t+1,1}$

$$\begin{cases} \Delta a^i_{t+1,1} > 0, & \forall q^t_{t+1,1} : q^t_{t+1,1} < q^i_{buy} \\ \Delta a^i_{t+1,1} < 0, & \forall q^t_{t+1,1} : q^t_{t+1,1} > q^i_{sell} \\ \Delta a^i_{t+1,1} = 0, & \forall q^t_{t+1,1} : q^t_{t+1,1} \in \left(q^i_{buy}, q^i_{sell}\right) \end{cases}$$

Without any taxes, $q^i_{buy} = q^i_{sell} = \pi^i$ and the investor would buy (sell) asset 1 for any price that is lower (higher) than his personal evaluation π^i. The FTT reduces the set of prices for which the agent is willing to trade in either direction, given that $q^i_{buy} < \pi^i$ and $q^i_{sell} > \pi^i$ when $\tau > 0$. Moreover, by increasing the gap between q^i_{buy} and q^i_{sell}, it creates an inactive region: the investor prefers to not trade if the equilibrium price is in the between of these two thresholds.

Regarding the size of trading, the next Proposition introduces the modified optimal portfolio rules

Proposition 2.1. *The asset demand functions resulting from the optimization problem* (2.1) *subject to the budget constraint* (.20) *are given by*

$$a^i_{t+1,s} = \frac{\tilde{\pi}^i_{t,s} \delta^i_t a^i_{t,s^t}}{q^t_{t+1,s}}, \qquad s \in \{1,2\} \tag{2.8}$$

where $\tilde{\pi}^i_{t,s} = \tilde{\pi}^i \left(q^i_{buy}, q^i_{sell}, q^t_{t+1,s}\right)$ *and* $\delta^i_t = \delta^i \left(q^i_{buy}, q^i_{sell}, q^t_{t+1,s}\right)$ *are the agent i effective beliefs and effective discount factors, respectively.*

Proof. in Appendix B.

As in the benchmark case, the traders' investment rules are still functions of both their current income and the before-tax asset prices. However, the tax introduces a double distortion regarding both the effective beliefs and discount factors. The agent i effective belief consists in his reservation prices

$$\tilde{\pi}^i_{t,1} = \tilde{\pi}^i \left(q^i_{buy}, q^i_{sell}, q^t_{t+1,1}\right) = \begin{cases} q^i_{buy} & \text{if } q^t_{t+1,1} \in \left(0, q^i_{buy}\right) \\ q^i_{sell} & \text{if } q^t_{t+1,1} \in \left(q^i_{sell}, 1\right) \\ q^t_{t+1,1} & \text{if } q^t_{t+1,1} \in \left[q^i_{buy}, q^i_{sell}\right] \end{cases} \tag{2.9}$$

while $\tilde{\pi}^i_{t,2} = 1 - \tilde{\pi}^i_{t,1}$ in every period. The distortion on the state 1 effective belief is showed in Figure 2.4 and it is captured by the distance between the red and the blue line. According to (2.7), reservation prices are equal to the agent's probability π^i when $\tau = 0$ and effective and subjective beliefs coincide $\tilde{\pi}^i_{t,1} = \pi^i$ in any s^t. Conversely, a positive tax makes $q^i_{buy} < \pi^i < q^i_{sell}$, reducing and increasing the agent's effective belief when he buys and sells asset 1.

To explain the source of the distortion on the effective discount factor, we first define T^i_t as the individual tax levy faced in s^t

$$T^i_t = \tau \sum_{s \in \{1,2\}} \left(q^t_{t+1,s} |a^i_{t+1,s} - a^i_{t,s^t}|\right) \tag{2.10}$$

[6] Relying on price normalization, the reservation prices as a buyer and a seller of asset 2 are $1 - q^i_{sell}$ and $1 - q^i_{buy}$, respectively.

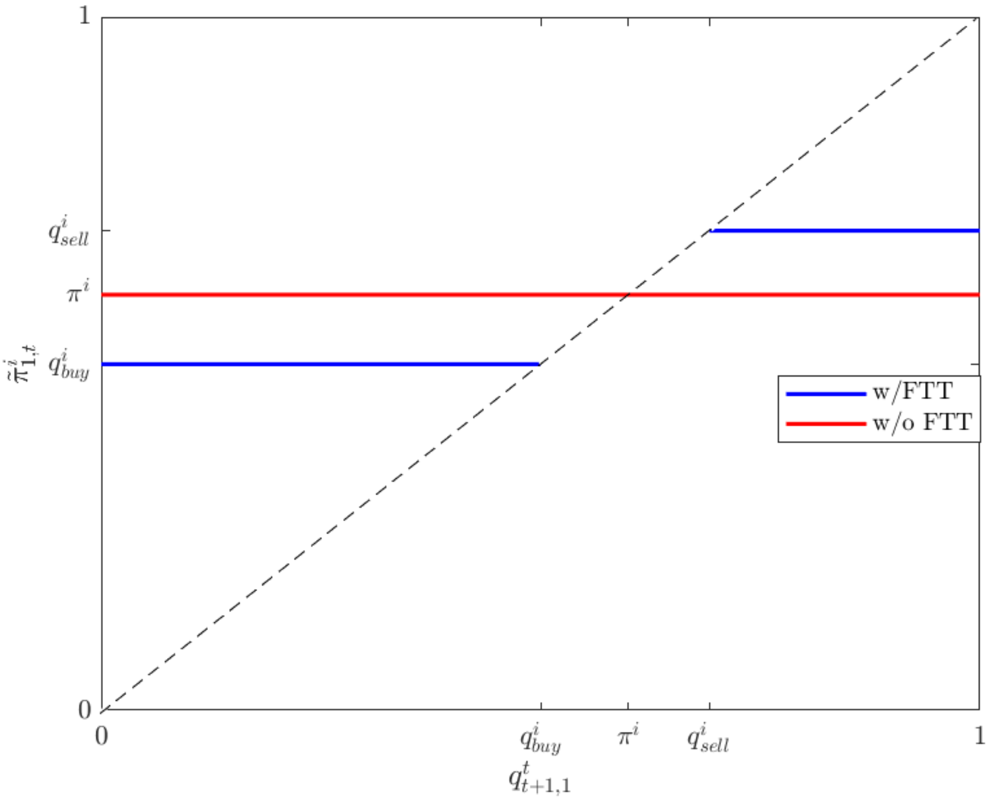


Figure 2.4: Distortion on the state 1 effective belief. Without any tax, the latter is constant and it coincides with the subjective probability π^i. The FTT reduces (increases) the agent's effective belief for the set of prices under which he is willing to buy (sell) the asset 1.

according to which, the budget constraint in (.20) may be equivalently rewritten as

$$q^t_{t+1,1}a^i_{t+1,1} + q^t_{t+1,2}a^i_{t+1,2} = a^i_{t,s^t}\left(1 - \frac{T^i_t}{a^i_{t,s^t}}\right). \tag{2.11}$$

We call effective discount factor the measure of the tax incidence that depends on the agent's wealth

$$\delta^i_t = \delta^i\left(q^i_{buy}, q^i_{sell}, q^t_{t+1,1}\right) = \left(1 - \frac{T^i_t}{a^i_{t,s^t}}\right) = \begin{cases} \frac{1-\tau\left(2\pi^1-1\right)}{1-\tau^2}\left(2\tau q^t_{t+1,1} + 1 - \tau\right) & \text{if } q^t_{t+1,1} \in (0, q^i_{buy}) \\ \frac{1+\tau\left(2\pi^1-1\right)}{1-\tau^2}\left(-2\tau q^t_{t+1,1} + 1 + \tau\right) & \text{if } q^t_{t+1,1} \in (q^i_{sell}, 1) \\ 1 & \text{if } q^t_{t+1,1} \in [q^i_{buy}, q^i_{sell}] \end{cases} \tag{2.12}$$

The lower is a^i_{t,s^t}, and thus the higher is $\frac{T^i_t}{a^i_{t,s^t}}$, the larger is the tax impact on the individual resources. According to (2.8), asset demands reduces with a^i_{t,s^t} not only because the agent has less money to spend, but also because the FTT has a greater relative impact on the poorest agent. The following Corollary formalizes this result

Corollary 2.1. *The limiting values of* $\frac{T^i_t}{a^i_{t,s^t}}$ *are given by*

$$\lim_{\frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}}\to\infty} \frac{T^i_t}{a^i_{t,s^t}} = 0 \qquad \lim_{\frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}}\to 0} \frac{T^i_t}{a^i_{t,s^t}} > 0.$$

The tax impact on agent i becomes less and less relevant as he earns relatively more income than $-i$, therefore

$$\lim_{\frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}} \to \infty} \delta^i_{t,s^t} = 1 \qquad \lim_{\frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}} \to 0} \delta^i_{t,s^t} < 1$$

Proof. in Appendix B.

Figure 2.5 provides an intuition displaying δ^i_t as a function of the price of asset 1. The latter depends on the agents' distribution of wealth: when $\frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}} \to \infty$, agent i has a greater influence on the market and the price formation process. Therefore, the equilibrium price $q^t_{t+1,1}$ tends to approach his reservation prices (q^i_{buy} or q^i_{sell}, depending on whether he is buying or selling the asset 1). The Figure shows that, close to his reservation prices, δ^i_t converges to one, meaning that $\frac{T^i_t}{a^i_{t,s^t}} \to 0$ and hence the tax has a negligible effect on the agent's resources. Conversely, as the equilibrium price moves away from q^i_{buy} or q^i_{sell}, δ^i_t decreases and the tax impact becomes more relevant, $\frac{T^i_t}{a^i_{t,s^t}} > 0$.

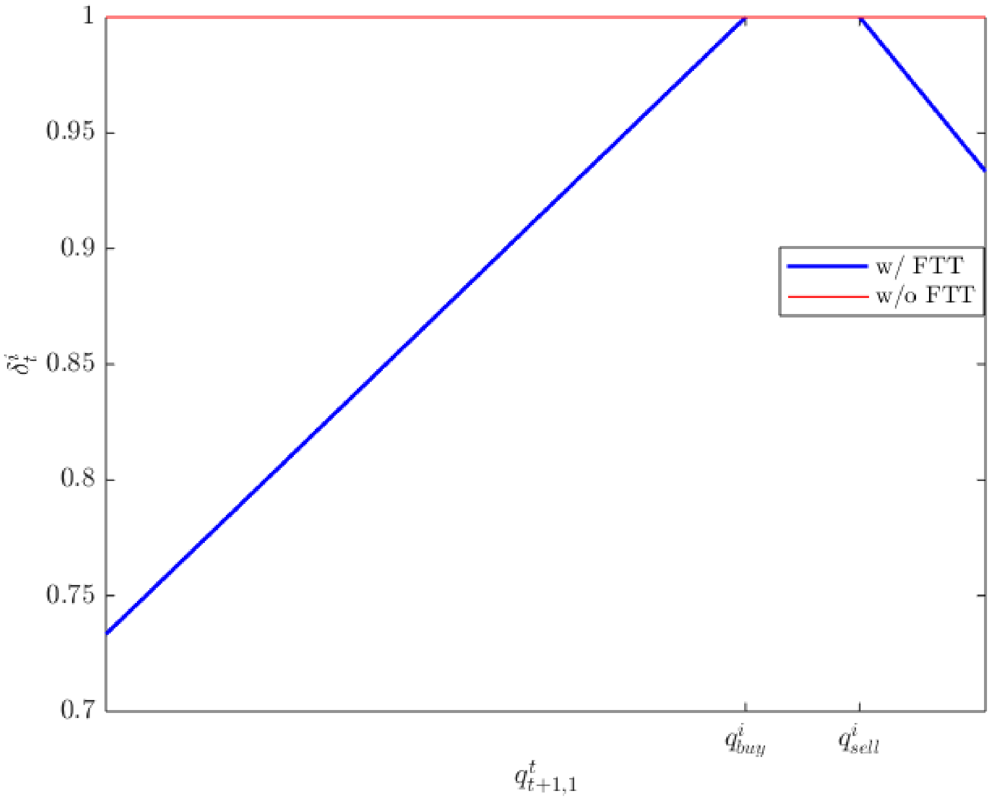


Figure 2.5: Distortion on the effective discount factor. In the benchmark case $\delta^i_t = 1$. Conversely, a positive FTT reduces the agent's effective discount factor $\delta^i_t = 1$, the lower is the fraction of the aggregate wealth owned (and thus the further is the equilibrium price from his reservation prices).

Figure 2.6 displays the overall tax distortion representing the value of the agent i relative asset demands

$$d^i_{t+1,s} = q^t_{t+1,1} \frac{a^i_{t+1,s}}{a^i_{t,s^t}}. \tag{2.13}$$

We first explain the shape of $d^i_{t+1,s}$ considering asset 1 (top); the same applies for asset 2 (bottom). The asset demand function is a piece-wise function, defined on three different intervals of the asset price. When $q^t_{t+1,1} < q^i_{buy}$, the agent eventually buys and $d^i_{t+1,1}$ is above the diagonal because $a^i_{t+1,1} > a^i_{t,s^t}$. Analogously, when $q^t_{t+1,1} > q^i_{sell}$, the agent eventually sell and $d^i_{t+1,1}$ is below the diagonal because $a^i_{t+1,1} < a^i_{t,s^t}$. Finally, the no-trading region lies on the diagonal given that $d^i_{t+1,s} = q^t_{t+1,s}$ because $a^i_{t+1,s} = a^i_{t,s^t}$.

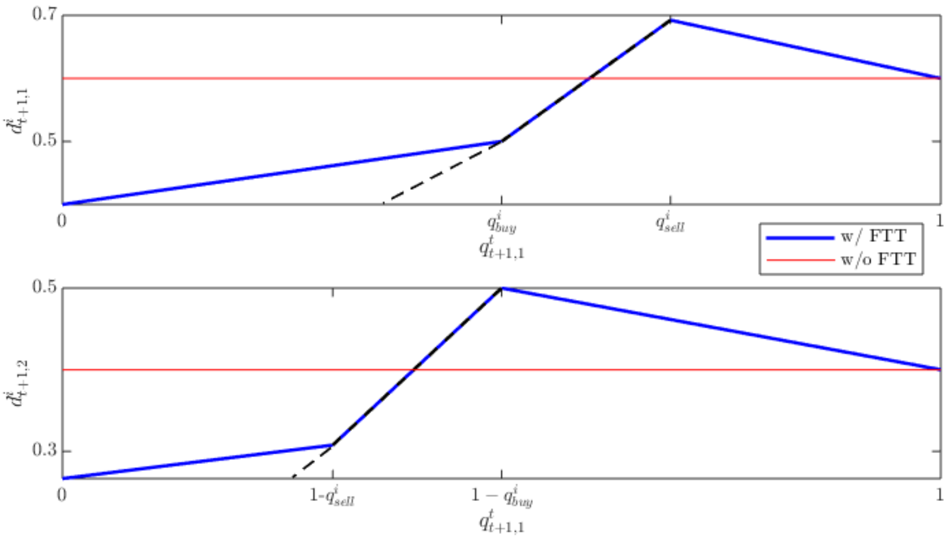


Figure 2.6: Value of the relative asset demands as a function of the state-price. For both asset 1 [top] and asset 2 [bottom] the diagonal is the no-trading region (dotted line). Above (below) the diagonal the agent is willing to buy (sell) the corresponding asset.

The overall distortion is due to change in the effective beliefs and discount factor explained above. The first is responsible for the size of the distance between the red and blue line. In this regards, it is worth to notice that in the “buy region" the distance is a negative function of the price and it progressively shrink up to the agent’s reservation price as a buyer. The explanation is that the FTT induces both a negative income and substitution effect in this case. Conversely, in the “sell region", the size of the the distortion is greater for prices close to q^i_{sell} and it progressively reduces for higher level of prices. This is due to income and substitution effect pushing in opposite direction. As a matter of fact, the FTT reduces the slope of the budget constraint above the kink point (see figure 2.3), making the asset 1 relatively less convenient to sell (positive substitution effect). However, the income effect is negative and the agent’s willingness to sell the asset increases with the new wealth level. The substitution effect is greater the higher is the price level since the agent would have been selling much more and it totally offset the income effect when $q^t_{t+1,1} = 1$.

Concerning the distortion on the agent’s discount factor, the values placed on the y-axis helps. It is possible to show that

$$\delta^i_t = d^i_{t+1,1} + d^i_{t+1,2}.$$

Therefore, when the agent owns most of the aggregate wealth and he buys asset 1 and sells asset 2, $q^t_{t+1,1} \to q^i_{buy}$ while $d^i_{t+1,1} = 0.50$ and $d^i_{t+1,2} = 0.50$, implying $\delta^i_t = 1$. Conversely, when he sells asset 1 and buys asset 2, $q^t_{t+1,1} \to q^i_{sell}$ and $d^i_{t+1,1} = 0.70$ and $d^i_{t+1,2} = 0.30$, still implying $\delta^i_t = 1$. Finally, consistently to Corollary 2.1, δ^i_t reduces as $q^t_{t+1,1}$ moves away from both q^i_{buy} and q^i_{sell}.

2.3.1 Competitive equilibrium

Having studied the features of the individual problem, we outline the competitive equilibrium of the economy in this section.

Definition 2.2. *A sequential trading competitive equilibrium is an allocation* $\left(c_T^i\right)_{i,s^T}$*, asset demands* $\left(a_{t+1,s}^i\right)_{i,t,s^t,s}$ *and a sequence of prices* $\left(q_{t+1,s}^t\right)_{t,s^t,s}$ *such that, given the tax rate and the equilibrium prices, traders solve the problem in* (2.1) *subject to the budget constraint* (.20) *faced in any* s^t. *Moreover, if they decide to trade in* s^t*, then the financial market clears as follows*

$$\begin{aligned}\sum_{i\in\{1,2\}} a_{t+1,1}^i + r_{t,1} = 1\\ \sum_{i\in\{1,2\}} a_{t+1,2}^i + r_{t,2} = 1\end{aligned} \tag{2.14}$$

where $r_{t,s} = \tau \sum_{i\in\{1,2\}} \left(\left|a_{t+1,s}^i - a_{t,s^t}^i\right|\right)$ *is the amount levied on the asset s trades,* $s \in \{1,2\}$. *Otherwise, if they do not trade in* s^t,

$$\sum_{i\in\{1,2\}} a_{t+1,s}^i = 1$$

for all $s \in \{1,2\}$.

The FTT changes, not only the agents' asset demands, but also the supply since some resources are withdrawn from the market. The total amount of each available asset will be reduced as a result.

Figure 2.7 shows $d_{t+1,s}^1$ and $d_{t+1,s}^2$ for different levels of $q_{t+1,1}^t$ and $s \in \{1,2\}$. Subjective beliefs affect both the agent' reservation prices and the shape of asset demands. Consistently with the assumption in Section 2.2.1, $\pi^1 > \pi^2$ and agent 2 is willing to buy (sell) asset 1 for a set of lower (higher) equilibrium prices relative to agent 1 (see figure on the top). The same mechanism applies for asset 2 (figure on the bottom). Moreover, agent 2 is the most accurate in this case and his asset demands are relatively closer to the ones implied by the truth as a consequence.

Focusing on the equilibrium features, trading occurs as long as the equilibrium price is halfway between the seller and buyer reservation prices. As depicted in Figure 2.7, $q_{sell}^2 < q_{buy}^1$ implying that, for prices included in $\left(q_{sell}^2, q_{buy}^1\right)$, agent 1 eventually buys asset 1 from agent 2. At the same time, the former must also sell the asset 2 to the latter. The next Lemma formalizes the condition for which trading occurs in the economy

Lemma 2.1. Consider a two-agent economy with terminal consumption in $t = T$, dynamically complete markets and a transaction tax $\tau > 0$. Assume that $\pi^1 > \pi^2$ and, without loss of generality, that $\pi^1 = \pi^2 + \Delta$, where $\Delta > 0$. Then $q_{sell}^1 > q_{sell}^2$ and $q_{buy}^1 > q_{buy}^2$.

If, moreover, $\tau \in (0, \tau')$ where

$$\tau' = \tau'\left(\pi^2, \Delta\right) = 1 - 2\frac{\sqrt{\pi^2\left(1-\pi^2-\Delta\right)\left[\pi^2\left(1-\pi^2-\Delta\right)+\Delta\right]} - \pi^2\left(1-\pi^2-\Delta\right)}{\Delta}$$

then $q_{buy}^1 > q_{sell}^2$ and trading occurs in every node s^t. Specifically, agent 1 buys asset 1 and sells asset 2 and agent 2 takes the opposite position in trading. The equilibrium price is such that $q_{t+1,1}^t \in \left(q_{sell}^2, q_{buy}^1\right)$.

Otherwise, trade does not occur in every s^t and $q_{t+1,1}^t$ is indeterminate in the set $\left(q_{buy}^1, q_{sell}^2\right)$.

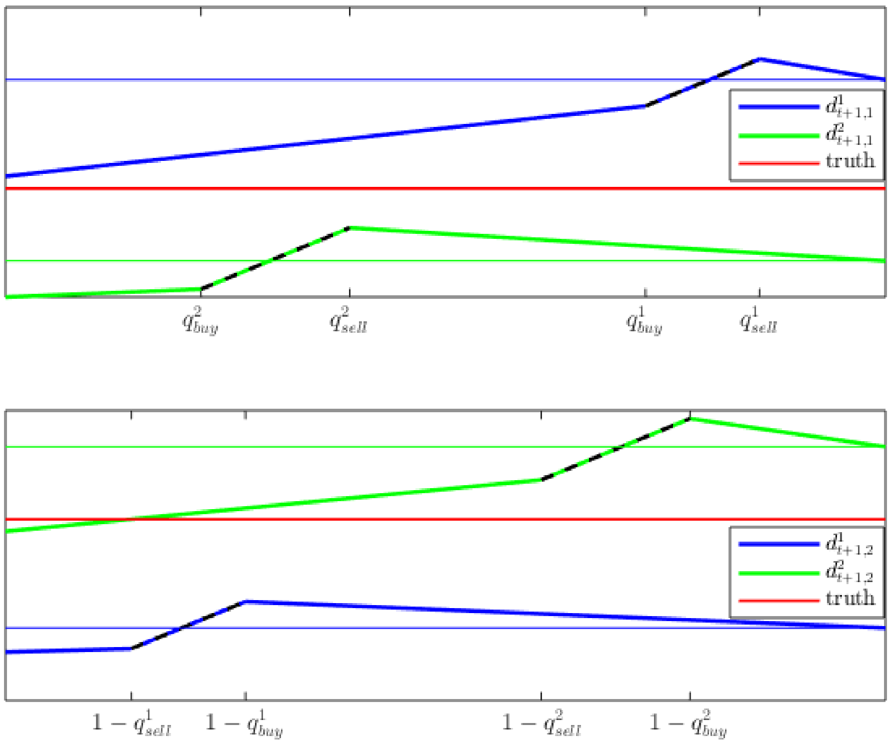


Figure 2.7: Agents $i \in \{1, 2\}$ value of relative asset demands as a function of the state-price. In this case, $\pi^1 = 0.8$, $\pi^2 = 0.3$ and $\pi = 0.5$.

Proof. in Appendix B.

An important implication follows the above Lemma: if the tax rate is sufficiently low, trading occurs in every period and consistently with the agents' beliefs ($\Delta a^1_{t+1,1} > 0 \wedge \Delta a^2_{t+1,1} < 0$ and $\Delta a^2_{t+1,1} < 0 \wedge \Delta a^2_{t+1,2} > 0$). If the FTT is instead greater than a certain threshold τ', agents decide to not trade in every period and just consume their given endowment in T. As expected, τ' depends on the level of disagreement characterizing the economy and it increases the larger is the distance between traders' beliefs, thus $\frac{\partial \tau'}{\partial \Delta} > 0$. This happens because, when agents strongly disagree, they are willing to trade even paying an high transaction cost, since they expect larger profits at the expense of the other. Figure 2.8 displays τ' as a function of the agents' belief distance Δ.

Lastly, since τ is constant over time, transitions between trading and no-trading regions are not possible over time. In other words, if $\tau \geq \tau'$ traders are stuck in the no-trading region for all t. While, if $\tau < \tau'$, then trading occurs in every possible node before the terminal one[7].

Given the optimal asset demands (2.8) and the result stated in Lemma 2.1, the agents' trading positions are summarized as follows

[7]This is not true when the FTT is dynamically set and the tax rate is allowed to change over time. We do not account for this possibility since we have concerns about the practical feasibility of this policy design.

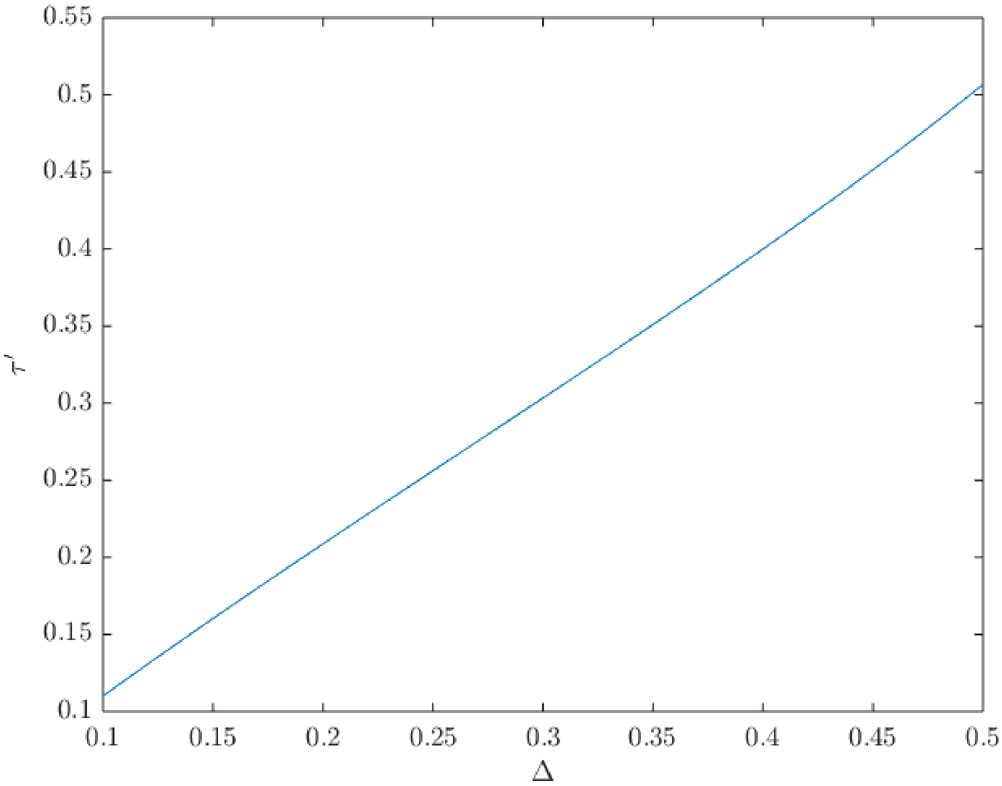


Figure 2.8: τ' as a function of the agents' belief distance $\Delta \in (0, 0.5]$. Here $\pi^2 = 0.3$, hence $\pi^1 \in (0.3, 0.8]$.

Agent 1:

$$\begin{cases} a^1_{t+1,1} = \frac{\tilde{\pi}^1_{t,1}\delta^1_t a^1_{t,s^t}}{q^t_{t+1,1}} & \text{if} \quad q^t_{t+1,1} \in \left(q^2_{sell}, q^1_{buy}\right) \\ a^1_{t+1,1} = a^1_{t,s^t} & \text{otherwise} \end{cases} \qquad \begin{cases} a^1_{t+1,2} = \frac{(1-\tilde{\pi}^1_{t,1})\delta^1_t a^1_{t,s^t}}{q^t_{t+1,2}} & \text{if} \quad q^t_{t+1,1} \in \left(q^2_{sell}, q^1_{buy}\right) \\ a^1_{t+1,2} = a^1_{t,s^t} & \text{otherwise} \end{cases} \tag{2.15}$$

Agent 2:

$$\begin{cases} a^2_{t+1,1} = \frac{\tilde{\pi}^2_{t,1}\delta^2_t a^2_{t,s^t}}{q^t_{t+1,1}} & \text{if} \quad q^t_{t+1,1} \in \left(q^2_{sell}, q^1_{buy}\right) \\ a^2_{t+1,1} = a^2_{t,s^t} & \text{otherwise} \end{cases} \qquad \begin{cases} a^2_{t+1,2} = \frac{(1-\tilde{\pi}^2_{t,1})\delta^2_t a^2_{t,s^t}}{q^t_{t+1,2}} & \text{if} \quad q^t_{t+1,1} \in \left(q^2_{sell}, q^1_{buy}\right) \\ a^2_{t+1,2} = a^2_{t,s^t} & \text{otherwise} \end{cases} \tag{2.16}$$

where $\tilde{\pi}^1_{t,1}$ and δ^1_t ($\tilde{\pi}^2_{t,1}$ and δ^2_t) are the first (second) rows of (2.9) and (2.12). The equilibrium prices[8] are determined provided that $\tau \in [0, \tau')$

$$\begin{aligned} q^t_{t+1,1} &= \frac{(1-\tau)\,\pi^1 a^1_{t,s^t} + (1+\tau)\,\pi^2 a^2_{t,s^t}}{(1-\tau\,(2\pi^1-1))\,a^1_{t,s^t} + (1+\tau\,(2\pi^2-1))\,a^2_{t,s^t}} \\ q^t_{t+1,2} &= \frac{(1+\tau)\,\pi^1 a^1_{t,s^t} + (1-\tau)\,\pi^2 a^2_{t,s^t}}{(1-\tau\,(2\pi^1-1))\,a^1_{t,s^t} + (1+\tau\,(2\pi^2-1))\,a^2_{t,s^t}}. \end{aligned} \tag{2.17}$$

Consistently with the benchmark, equilibrium prices are a convex combination of the agents' beliefs where weights are given by the agents' distribution of wealth. In addiction, they are also affected by the FTT rate as well.

[8]Recall that the prices are normalized such that the sum up to 1, $\sum_{s\in\{1,2\}} q^t_{t+1,s} = 1$.

Figure 2.9 shows the asset 1 market equilibrium for the set of prices for which trading takes place. The red line is the aggregate demand $D_{t+1,1} = \sum_{i \in \{1,2\}} d^i_{t+1,1} \frac{a^i_{t,s^t}}{\sum_{i \in \{1,2\}} a^i_{t,s^t}}$, while the black line is the supply function (see the financial clearing condition in Definition 2.2).

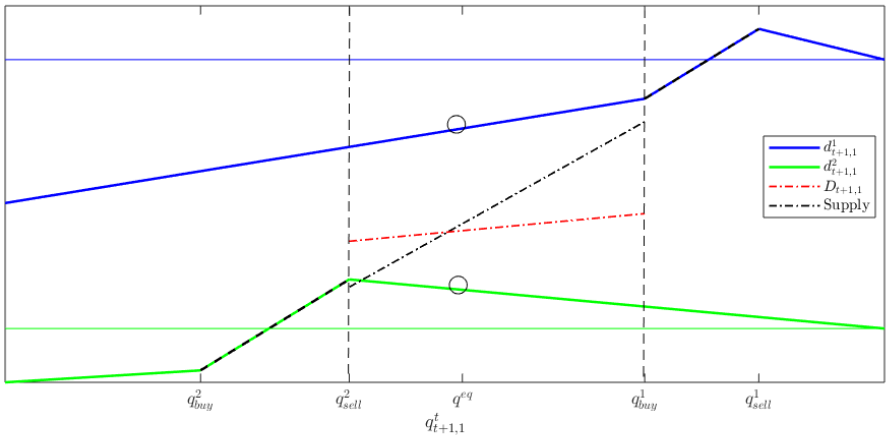


Figure 2.9: Agents $i \in \{1, 2\}$ asset 1 value of relative demands as a function of the price. The red and the black line are the aggregate demands and supply, while the equilibrium price q^{eq} originates from their intersection. The markers in the blue and green line are the equilibrium $d^i_{t+1,1}$. In this example, $\pi^1 = 0.8$, $\pi^2 = 0.3$, $\pi = 0.5$ and $\frac{a^1_{t,s^t}}{\sum_{i \in \{1,2\}} a^i_{t,s^t}} = 0.5$.

2.4 Long-run equilibrium features

A FTT reduces (up to preventing) the speculative trading that arises in a heterogeneous belief economy when financial markets are complete. This section assess the implications of this tax on the long-run equilibrium features. It is well known that, without policy intervention, the MSH supporting the most accurate agent holds. Therefore, heterogeneity is a transient feature of the market that, sooner or later, selects the most accurate type. Studying the long-run equilibrium of a time finite economy might look counter-intuitive, given that market selection may not occur when T is arbitrarily low. However, studying the asymptotic behavior of the economy, provides information about agents' wealth distribution in previous dates. Although we do not characterize convergence rates, finding that one agent dominates (vanishes) means that his wealth is increasing (decreasing) over time on the most likely histories. Moreover, note that every period actually represents one round of trading and, given that trades are empirically frequents in financial markets, a large T may corresponds to a relatively short time horizon. Finally, we assume here two states of the Nature for every period t. More realistically, the same consumption distribution can be achieved considering more states of the Nature at each point of time t and a shorter final period T.

We first introduce the standard notion of asymptotic dominance

Definition 2.3. *Agent i dominates if*

$$\lim_{t \to \infty} \frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}} = \infty \quad a.s.$$

while he vanishes if

$$\lim_{t \to \infty} \frac{a^i_{t,s^t}}{a^{-i}_{t,s^t}} = 0 \quad a.s.$$

.

Thereafter, we consider the logarithm of the agents' wealth ratio

$$z_{t+1} = \log\left(\frac{a^1_{t+1,s^{t+1}}}{a^2_{t+1,s^{t+1}}}\right).$$

that, replacing the optimal asset demands in (2.8), may be rewritten as

$$z_{t+1} = \xi_{t+1} + z_t. \tag{2.18}$$

Study the sign of the drift $\mathbb{E}\left[\xi_{t+1}\right]$ conditional on the process $z_{(t,s)}$, provides sufficient condition to assess market dominance [9], given that $z_{t+1} \to \infty \Leftrightarrow \frac{a^1_{t,s^{t+1}}}{a^2_{t,s^{t+1}}} \to \infty$ and $z_{t+1} \to -\infty \Leftrightarrow \frac{a^1_{t,s^{t+1}}}{a^2_{t,s^{t+1}}} \to 0$.

In the benchmark case, where $\tau = 0$, the drift is constant

$$\mathbb{E}\left[\xi_{t+1}\right] = D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right)$$

and only related to the difference between the agents' relative entropies with respect to the truth[10] (i.e. $D_{KL}\left(\pi^2||\pi\right) - D_{KL}\left(\pi^1||\pi\right) < 0$ if agent 2 is the most accurate). Therefore, it is positive if π^2 is more distant from the truth relative to π^i and negative otherwise. Hence, the drift is equal to zero, meaning that both the traders accumulate the same amount of wealth on average, when subjective probabilities are equidistant from π. Market dominance is purely determined by the agents' accuracy as a result.

When $\tau > 0$, the drift component is also affected by the agents' relative wealth, the tax rate and the true probability

$$\mathbb{E}\left[\xi_{t+1}\right] = D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + f\left(\frac{a^1_{t,s^t}}{a^2_{t,s^t}}, \tau, \pi\right). \tag{2.19}$$

For this reason, one needs to study the sign of (2.19) as the agents' wealth distribution approaches both its limits

$$\begin{aligned}
\mu_+ &= \lim_{z\to\infty} \mathbb{E}\left[\xi_{t+1}|z_t = z\right] \\
&= D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\left(1-\tau^2\right)}{(1+\tau)^2 - 4\tau\pi^1}\right) \\
\mu_- &= \lim_{z\to-\infty} \mathbb{E}\left[\xi_{t+1}|z_t = z\right] \\
& D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{(1-\tau)^2 + 4\tau\pi^2}{\left(1-\tau^2\right)}\right).
\end{aligned} \tag{2.20}$$

If both μ_+ and μ_- have the same sign, then heterogeneity has a transient nature and the economy quickly converges to the most accurate type. This happens because the sign of (2.19) is always the same (positive if agent 1 dominates and negative otherwise), regardless the distribution of wealth.

Conversely, the long-run equilibrium is non-degenerate when μ_+ is negative and μ_- is positive. Intuitively, that happens because the mean value of the wealth ratio process favors agent i when agent $-i$ owns most of the aggregate wealth. Hence, both the investors survive in the long-run, even when one of them is more accurate than the other.

The last case is when μ_+ is positive and μ_- is negative. Here heterogeneity is still transient, however, the belief selection process is path- dependent and thus related to the history of shocks.

[9] A similar approach is adopted in Dindo (2019).

[10] The agent i relative entropy is also the Kullback-Leibler divergence of π^i with respect to π.

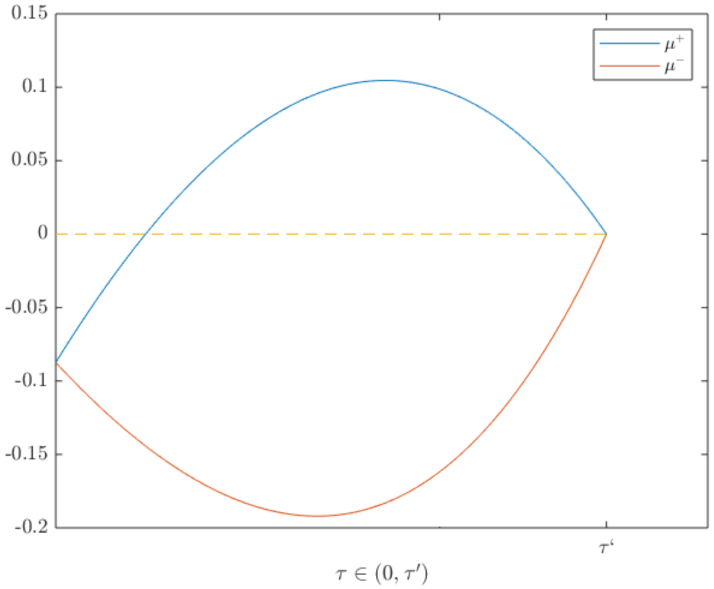


Figure 2.10: Limiting values of the drift for different levels of $\tau \in (0, \tau')$. In this example, agent 2 is the most accurate and he survives for lower τ. Parameters: $\pi = 0.5$, $\pi^1 = 0.8$ and $\pi^2 = 0.3$.

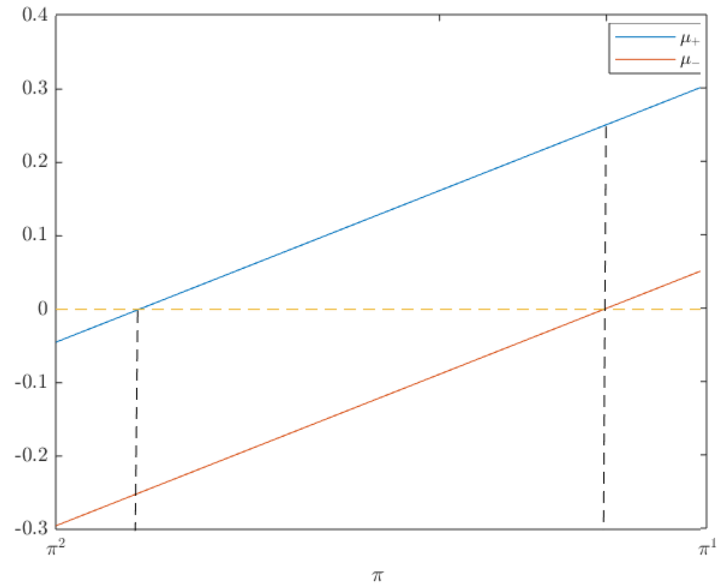


Figure 2.11: Limiting values of the drift for different levels of $\pi \in (\pi^2, \pi^1)$. Parameters: $\tau = 0.2$, $\pi^1 = 0.8$ and $\pi^2 = 0.3$. Hence, the truth π moves from 0.2 to 0.8.

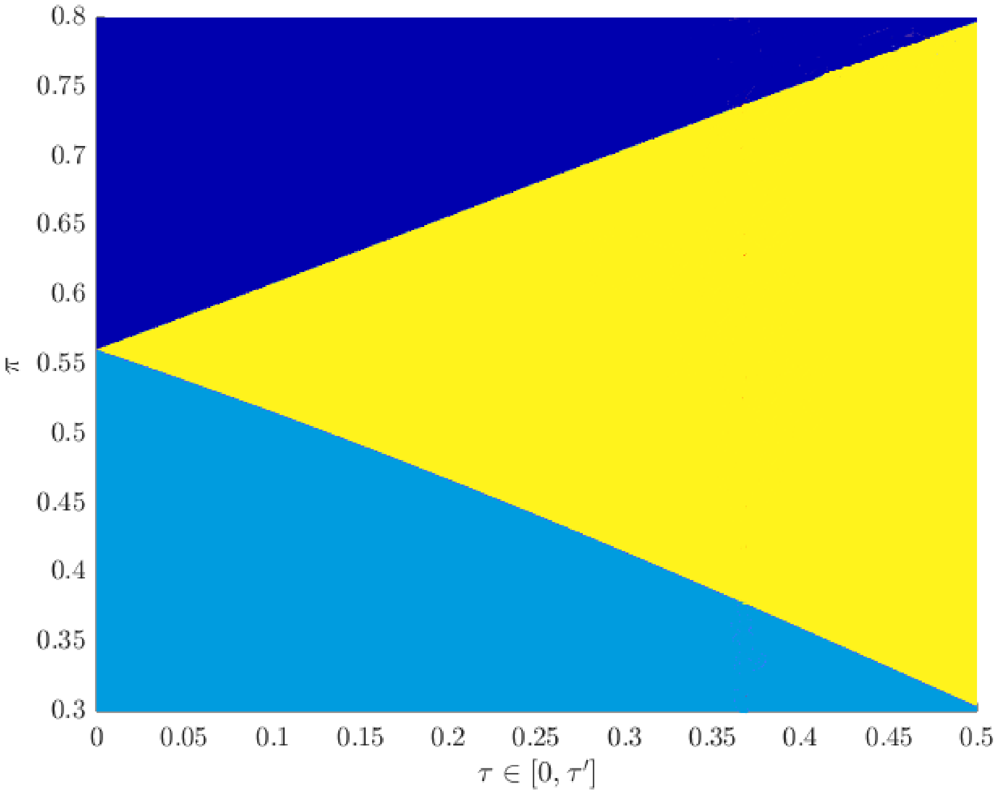


Figure 2.12: Path-dependency (yellow area) arises when the truth is in the between of the agents' beliefs ($\pi^1 = 0.8$ and $\pi^2 = 0.3$). Conversely, when one of the agent is much more accurate than the other, the MSH holds and he is the only one who survives in the long-run.

When this happens, the market always favors the wealthiest, regardless his relative accuracy. In fact, the drift is positive when agent 1 ends up owning most of the aggregate wealth and negative vice versa.

Figure 2.11 displays the limiting values of the drift (2.20) in such a case. Given the truth π, the figure on the top shows the set of τ under which the long-run equilibrium is path-dependent (μ_+ positive and μ_- negative). We assume here that agent 2 is the most accurate; he is hereby the only survivor when the FTT is low enough. When the tax rate is higher than a certain threshold, the long-run equilibrium is instead path-dependent and determined by the history of shocks. Conversely, in the figure in the bottom the limiting values of the drift are showed as a function of the truth π, and for a given level of taxation. Path-dependency arises provided that the truth lies in the middle of the agents' beliefs. Intuitively, when π approaches π^2 (π^1), agent 2 (1) asymptotically dominates the market.

Market dominance depends on both the tax rate and the location of the truth with respect to the agents beliefs. Figure 2.12 displays the combinations of π and τ giving rise to the path-dependent equilibrium (yellow) and the MSH argument (agent 2 (light blue) and agent 1 (blue) long-run dominance). Consistently with the previous figures, the MSH holds whenever one of the agents is much more accurate than the other. Instead, the equilibrium turns out to be path-dependent for a large set of τ, whenever the truth lies in the between of the trader's subjective probability measures.

The MSH failure derives from the result stated in Corollary 2.1. The tax impact is negligible for the agent who gained most of the aggregate wealth, while it relatively penalizes the other. It may happen that, for a matter of luck, the less accurate accumulates most of the wealth and comes to dominate over certain paths.

The results of this section are summarized in the next Proposition

Proposition 2.2. *Let i be the most accurate agent. If $\tau = 0$, then the MSH holds a.s. $\phi_t^i \to 1$ and $\phi_t^{-i} \to 0$. If $\tau \in (0, \tau')$ market dominance depends on the history of shocks s^t, and $\mathbb{P}\left(\lim_{t\to\infty} \phi_t^i = 0\right) > 0$. Finally, if $\tau > \tau'$, then $a_t^h = a_{t-1}^h = ... = a_0^h$ for $h \in \{i, -i\}$ and both the agents survive in the long-run a.s. $\phi_t^i \nrightarrow 0$ and $\phi_t^{-i} \nrightarrow 0$.*

Proof. in Appendix B.

2.5 Government problem

We characterize in this Section the Government problem aimed at setting the optimal tax rate. Besides speculation reduction, proponents of the FTT also claim about its ability to raise revenues. As a consequence, we assume the policy be driven by both revenue raising and welfare maximizing purposes. Fiscal revenues may be used as a proxy of the welfare gain that would result from public expenditure, even thought the latter is not explicitly modeled.

The total revenue function is the sum of all the future fiscal withdrawals $R_t\left(s^t\right)$, each discounted using the dated 0 node s^t equilibrium price[11]

$$TR = \sum_{t=0}^{T-1} \sum_{s^t} q_t^0\left(s^t\right) R_t\left(s^t\right) = \tau \left[\sum_{t=0}^{T-1} \sum_{s^t} q_t^0\left(s^t\right) \left[\underbrace{\sum_{s\in\{1,2\}} q_{t+1,s}^t \left(\sum_{i\in N} a_{t+1,s}^i \right)}_{R_t} \right] \right] \tag{2.21}$$

[11]Time 0 prices are computed using the no-arbitrage condition $q_t^0\left(s^t\right) = q_{t-1}^0\left(s^{t-1}\right) q_t^{t-1}\left(s_t|s^{t-1}\right)$, with $q_1^0\left(s_1 = 1\right) = q_{1,1}^0 = \frac{(1-\tau)\pi^1 + (1+\tau)\pi^2}{\left(1-\tau\left(2\pi^1-1\right)\right)+\left(1-\tau\left(1-2\pi^2\right)\right)}$ and $q_1^0\left(s_1 = 2\right) = q_{1,2}^0 = 1 - q_{1,1}^0$.

Concerning the welfare maximizing motive, we assume that the Government maximizes a weighted average of agents' utilities

$$W = \mathbb{E}^g \left[\sum_{i \in \{1,2\}} \gamma^i \log\left(c_T^i\right) \right] \tag{2.22}$$

where Pareto weights are proportional to the traders' initial endowment

$$\gamma^i = \frac{a_0^i}{\sum_{i \in \{1,2\}} a_0^i} = y^i \tag{2.23}$$

while the Government's probability measure is a convex combination of the agents' subjective beliefs

$$\pi^g = \sum_{i \in \{1,2\}} \lambda^i \pi^i \tag{2.24}$$

with $\lambda^i \in [0,1]$, $\forall i \in \{1,2\}$.

Ruling out the availability of lump sum taxes, the Government essentially solves a Ramsey problem by choosing the distortive tax τ that maximizes his objective function subject to the traders' optimal choices and the market clearing conditions.

Definition 2.4. *The Ramsey problem is to maximize* (2.25)*, choosing* τ *subject to* (2.15),(2.16), (2.17) *and the financial market clearing conditions established in Definition 2.2.*
The Government's problem is therefore given by

$$\begin{gathered} \max_{\tau} \left(1-\alpha\right) W + \alpha TR \\ \text{subject to} \\ \text{Agents' porfolio rules (2.15)-(2.16)} \\ \text{Equilibrium prices} \quad (2.17) \\ \text{Asset clearing conditions (Def. 2.2)} \end{gathered} \tag{2.25}$$

where α denotes the relative weight given to the revenue motive.

To better investigate the optimal taxation features when the planner's aim is twofold, we separately study the cases welfare maximizing ($\alpha = 0$), revenue raising ($\alpha = 1$) and convex combination of the two ($\alpha \in (0,1)$).

Welfare maximizing motive In this section, we analyze the optimal taxation problem when the planner has purely welfare maximizing purposes and $\alpha = 0$. As already assumed in (2.23), Pareto weights are proportional to the agents' initial endowment. Moreover, the ex-ante welfare maximization is under the probability measure defined in (2.24), that is the convex hull of the agents' beliefs.

In our specific framework, where trading arises just for a speculative reason, the allocation that maximizes social welfare is the one that shuts the market down. As a consequence, when the planner has a pure welfare maximizing motive he sets $\tau^* \geq \tau'$, so that agents do not speculate in any $t < T$ and $c_T^i = y^i$. This has to do with the fact that, despite resulting from the agents' utility maximization, trading would never arise in an homogeneous economy. As a consequence, it is not even optimal for the policy maker that solves the problem using its own probability. Speculation is bad in this environment because agents are guaranteed to consume the same endowment in every possible final state. The result would have been different if traders had needed to hedge against idiosyncratic risks.

Looking at the long-run, the asymptotic equilibrium is trivially non-degenerate in such a case. By discouraging any trades before the final period T, the optimal tax rate ensures survival for both the agents' type.

Revenue motive In the same spirit, we characterize the optimal FTT rate when the policy purpose is to raise revenues. In this case, the Government essentially solves the Ramsey problem by maximizing the objective function in (2.25) under the assumption that $\alpha = 1$.

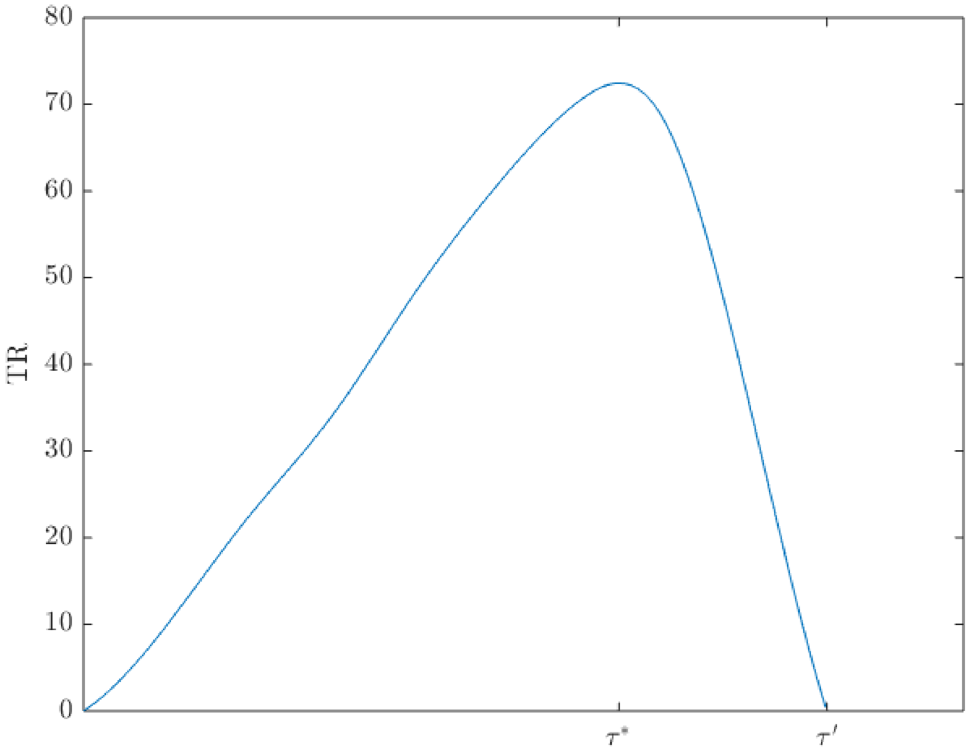


Figure 2.13: Simulated Laffer curve. The total revenue function is an average of 100000 simulations of a 100 periods economy using, at first, a grid of τ made up of 10 equidistant point in the set $[0, \tau']$. Therefore, the function is interpolated using a finer grid of 50 points located in the same interval. Parameters: $\pi^1 = 0.8$, $\pi^2 = 0.3$ and $\pi = 0.4$.

By replacing the equilibrium prices (2.17) and the traders' investment rules (2.15) and (2.16) in the node s^t revenue R_t as defined by (2.21), we get the following expression

$$R_t = \frac{4\tau}{1-\tau^2}\left[\frac{(1-\tau)^2\pi^1 - \pi^2(1+\tau)^2 + 4\tau\pi^1\pi^2}{\left[a^1_{t,s^t}\left(1+\tau\left(2\pi^2-1\right)\right) + a^2_{t,s^t}\left(1-\tau\left(2\pi^1-1\right)\right)\right]}\right] \tag{2.26}$$

Figure 2.13 is the simulated Laffer curve capturing the relationship between the rates of taxation and the resulting Government revenue. The function has a global maximizer τ^* that is always lower than τ'. As a matter of fact, when the tax rate is larger than τ' financial transactions are shutted down and so does fiscal revenues[12].

[12] The revenue maximizing problem is solved numerically through the following steps: simulate the total revenue function (2.21) using a coarse grid of points for τ, spline interpolate the function over a finer grid and choose the value of the grid, τ^*, that maximizes the interpolated function.

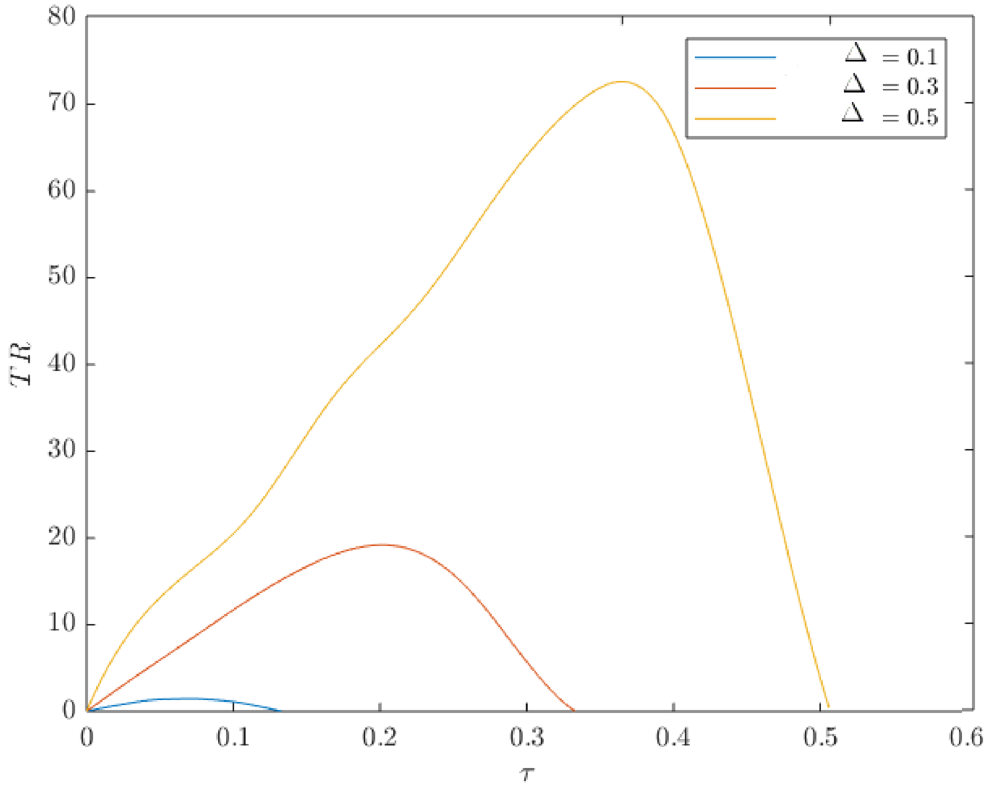


Figure 2.14: Laffer curves in economies with different level of heterogeneity. In this example, $\pi^2 = 0.3$ while $\pi^1 = \pi^2 + \Delta$ (Δ is equal to 0.1 (blue line), 0.3 (red line) and 0.5 (yellow line). The true value is $\pi = 0.4$. TR are average values of 100000 simulations of a 100 periods economy.

Moreover, as can be deduced from (2.26), the optimal tax rate depends on the traders' subjective probabilities. Simulating equation (2.21) in economies characterized by different degrees of heterogeneity, we find out that both the overall revenue raised and the optimal tax rate are positive functions of the distance of the traders' beliefs (see Figure 2.14). Intuitively, this happens because trading is more intense when agents strongly disagree and their willingness to trade persist even paying high trading costs.

Looking at the long-run, the red line crossing Figure 2.15 is the optimal tax rate set when the Government maximizes total fiscal revenues. Consistently with the result in Section 2.4, the latter gives rise to a path-dependent equilibrium for a large set of $\pi \in \left(\pi^2, \pi^1\right)$.

Convex combination of revenue and welfare maximizing motive We finally describe the optimal taxation problem when the policy pursues two objectives at the same time and $\alpha \in (0, 1)$. Figure 2.16 shows the optimal tax rate for different combinations of $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$. Regardless the Government's probability, the optimal tax rate is $\tau^* = \tau'$ when $\alpha = 0$ and $\tau^* < \tau'$ when $\alpha = 1$. As one would expect, the optimal FTT ranges between these two extreme values when $\alpha \in (0, 1)$.

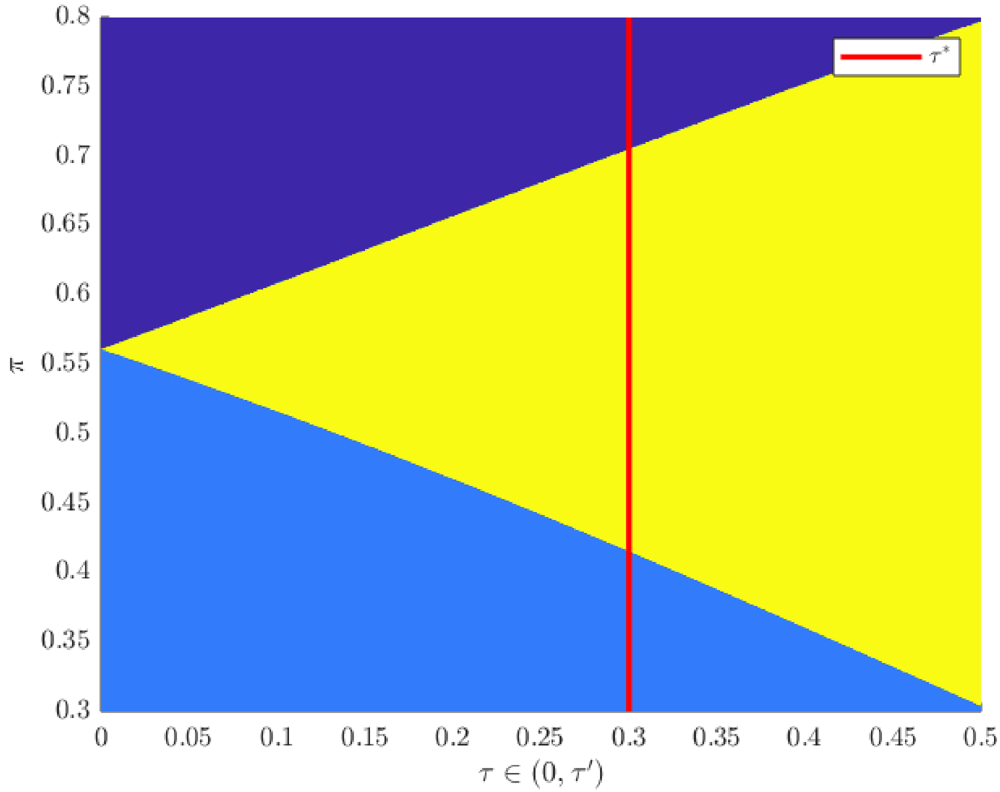


Figure 2.15: Optimal tax rate when the Government purely aims at raising revenues. Path-dependency (yellow area) arises when the truth is in the between of the agents' beliefs ($\pi^1 = 0.8$ and $\pi^2 = 0.3$). Conversely, when one of the agent is much more accurate than the other, the MSH holds and he is the only one who survives in the long-run. The optimal tax rate results from an average of 100000 simulations of the total revenue maximization problem of a 100 periods economy.

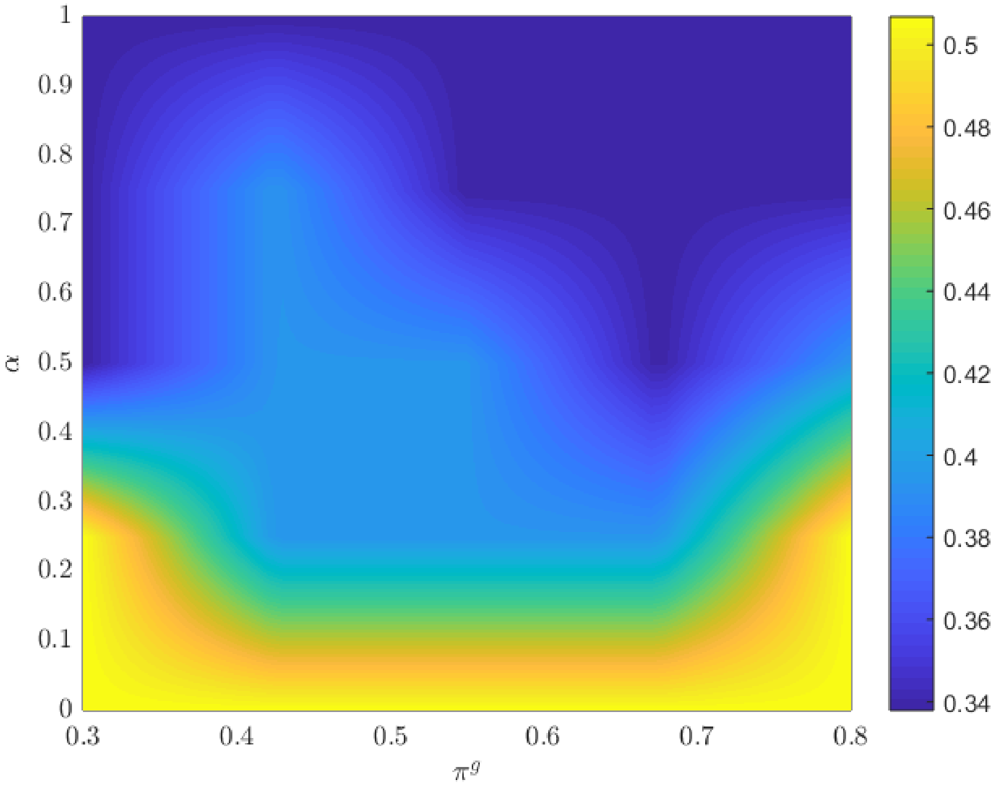


Figure 2.16: Optimal tax rate for different combinations of α and π^g. Average value of 100000 simulations of a 100 periods economy with parameters $\pi^1 = 0.8$ and $\pi^2 = 0.3$.

2.6 Conclusion

This paper studies the long-run effect and welfare implications of taxing financial transactions in a general equilibrium dynamic model, where trading is purely driven by speculative motives. We characterize the optimal taxation that maximizes both fiscal revenues and agents' welfare. Specifically, when the Government' aim is purely revenue raising, the optimal taxation turns out to be a positive function of the agents' belief distance. Intuitively, this is due to the fact that the trading volume is enhanced by the level of disagreement characterizing the economy. Moreover, it is always lower than the threshold that shuts down any market trade. Conversely, the planner always sets a tax rate above this thresholds when he exclusively attempts to maximize the agents' welfare. The main contribution of this paper entails the long-run analysis outlined in Section 2.4. Specifically, we find that, when the Government aims at raising revenues, agents' survival may be path-dependent and not related to belief accuracy as the MSH predicts. By contrast, the allocation that maximizes social welfare trivially guarantees persistent heterogeneity in the economy, since agents do not trade and they just consume their given endowment.

Chapter 3

Welfare effects of Tobin tax in a production economy with speculation

Abstract

In this paper, I consider a production economy with complete markets where investors hold different opinions about the probability of a technology shock affecting the aggregate production. The existence of a complete contingent-claim market creates an incentive for them to speculate, thereby affecting the real sector through an increase of volatility of the aggregate capital, consumption and output. Without policy intervention, the market selection argument holds implying prices, individual and aggregate consumption allocation reflect the most accurate belief in the long-run. Due to the progressive impoverishment faced by more inaccurate agents, this result is not compelling under the standpoint of a benevolent policy maker. To this end, I explore the potential benefit of a Tobin tax on the agents' speculative transactions. When agents are biased in opposite directions, a positive tax rate corrects the individual and aggregate consumption pattern toward the ones implied by the truth. However, inaccurate agents may dominate the market in the long-run. By contrast, when agents are biased to the same direction, the tax worse the most accurate type's decision rules although it does not prevent him to dominate the market in the long-run.

Keywords: Heterogeneous Beliefs, Speculative Markets, Tobin Tax, Market Selection Hypothesis, Real Investments, Growth.
JEL Classification: E21, G11, H30

3.1 Introduction

In principle, there is a positive relationship between a well-developed financial market and economic growth. Providing a variety of investment instrument, financial markets are avenues for firms to raise capital and for investors to satisfy their saving needs. However, the recent financial crises has emphasized the negative spillover from financial to the real sector, drawing the attention to potential regulatory measures aimed at mitigating the distortions in macroeconomic outcomes.

Within a standard production economy, I study the impact of the *Financial Transaction Tax* (FTT) as one of the policy measure proposed by regulators to reduce the effect of speculation in financial market, and thus set the correct incentives for firms improving the overall social welfare. In this model, speculation arises from traders having different opinions about the probability of high and low productivity states affecting the aggregate production level. Under the common prior assumption, the representative agent would simply invest in the productive unit, giving rise to the endogenous aggregate consumption process as a result. On top of that, belief heterogeneity creates an incentive for traders to entertain a speculative trading. These trades are speculative because they are purely driven by the investors' disagreement: traders despise the others' opinions and foresee a welfare-improvement from placing bets on future states.

Without policy intervention, disagreement enhances the fluctuations experienced by aggregate variables in the real market. However, when agents are subjective utility maximizers and markets are complete, disagreement has a transient nature since the market selects the most accurate driving to others out of the economy (see Sandroni (2000) and Blume and Easley (2006)). Most of the mainstream economics build on this assumption, called *Market Selection Hypothesis* (MSH), to justify the use of rational expectations behind financial and output decision making processes. Although being a desirable feature of the market, the MSH may be not fully satisfactory from the standpoint of a benevolent policy maker, due to the realized losses incurred by the less accurate types during the selection process. Moreover, a recent literature argues that Pareto optimality is not a persuasive criterion to address normative analysis in heterogeneous belief economies (among the others, Brunnermeier et al. (2014) and Gilboa et al. (2014)). Referring to this argument, this article provides a better understanding of the trade-off arising between speculation and accuracy reductions as induced by a trading restrictive policy as the FTT.

Provided that the true probability is somewhere in the middle of the agents' beliefs (i.e. traders are biased in opposite directions), this tax may undermine the validity of the market selection argument, implying the less accurate agent eventually dominates the market in the long-run. If that happens, long-run prices, individual and aggregate consumption patterns are less accurate than they would have been without policy intervention. Nevertheless, there are some potential benefits in the real sector until heterogeneity is preserved. Due to its distortive nature, the FTT corrects, to some extent, the individual and the aggregate decision rules toward the ones implied by the truth. Specifically, it reduces (increase) the marginal propensity to consume of those agents who would over(under)-consume without any regulatory measure. As a result, the effect of expectation biases on the aggregate variables is dampened as well.

However, when the truth lies elsewhere, the tax amplifies the distortion induced by the existence of biased beliefs. In this case, the FTT further compromise the most accurate type's decisions in favor of the less accurate agents, whose behaviors are instead partially corrected. Nevertheless, this mechanism does not prevent the most accurate to accumulate most of the aggregate wealth, implying the validity of the MSH under this scenario.

The economy tax rate is endogenously set by a Government pursuing a dual purpose: maximize the agents' utilities and the amount of fiscal revenues. In this regard, the major challenge is the lack of a unique probability used to evaluate the planner's objective and, therefore, discuss about the policy implications. Following Brunnermeier et al. (2014), social welfare is evaluated using those probabilities that are included in the set of reasonable beliefs, that is the convex hull of all the investors' subjective probabilities. While the Government's probability does not affect the tax rate that maximizes fiscal revenue, it becomes relevant for the ex-ante welfare maximization problem. In the latter case, the optimal FTT may either prevent or just mitigate the speculative trading, depending on the Government's probability.

Related Literature This paper hooks up to the argument raised by Brunnermeier et al. (2014) and Gilboa et al. (2014), focusing on regulatory trading measures aimed at improving the decentralized Pareto optimal result. In this spirit, it shares some common purposes with Blume et al. (2018), that likewise evaluates the impact on welfare of some trading regulatory measures. However, I deviate from the latter on at least three dimensions. First, I focus more on the FTT instead of considering other alternative policy tools. Second, I study the taxation impact on a production rather than an endowment economy. Third, instead of the true probability, I evaluate social welfare using the set of reasonable beliefs as introduced by Brunnermeier et al. (2014).

To the best of my knowledge, the most related work is Buss et al. (2016). This paper also considers heterogeneous belief agents in a production framework, however, it assumes trading arises from both speculative and hedging reasons. With a similar objective, this work investigates the effect on real economy and social welfare of three alternatives trading restricting measures: stock holding constraints, borrowing constraints and Tobin tax. Consistent to my results, they find a positive relation between the tax rate and the agents' expected utilities. However, the positive impact of the FTT is less significant than in this paper due to the trade-off arising between speculation and risk-sharing reduction. In this vein, I contribute by characterizing the optimal taxation problem when the Government's purpose is to raise fiscal revenues as well.

There are similarities with Walden and Heyerdahl-Larsen (2015), that outlines a heterogeneous belief production economy using a two-period framework. The paper shows that all competitive equilibria emerging in disagreement economies are welfare inefficient, if evaluated under any homogeneous probability measure. As a policy implication, they compare a Tobin and a real investment tax. The find the latter more appealing since it does not prevent the agents' ability to hedge against idiosyncratic risks.

More widely, this work also contributes to the growing literature studying the optimal taxation problem with cognitive biased agents (Gabaix and Farhi (2017), O'Donoghue and Rabin (2006), Chetty et al. (2009), Westerhoff and Dieci (2006)). In contrast to these papers, this model considers subjective utility maximizing traders in the sense of Savage (1951). The same assumption is made by Davila (2014), that derives the analytical properties of the optimal FTT in a static framework. It shows that the optimal tax rate can be expressed as a function of investors' beliefs and assets demands sensitivities. The optimal corrective tax is positive, provided that hedging and speculative trading motives are orthogonally distributed among the population of traders. However, as a static model, it cannot address the long-term effect of this policy measure.

3.2 Decentralized economy

In the absence of any policy measure, I characterize the competitive equilibrium of the economy that I refer as the benchmark.

3.2.1 The model

In this section, I outline the baseline framework of the model. Time is discrete and indexed by $t \in \mathbb{N}_+$. A sequence of random variables (θ_t) following an i.i.d. process is the date $t = 0, 1, ...,$ exogenous shock, while $\theta^t = \{\theta_0, \theta_1, ..., \theta_t\}$ is the vector collecting the shock realizations up to t. Without loss of generality, I assume the existence of two states of the Nature, $\Theta \in \{\theta^h, \theta^l\}$ and define π^i and $1 - \pi^i$ the state θ^h and θ^l objective probabilities, respectively[1]. As will be explained in Section 3.2.2, the technology shock is related to the output elasticity of capital that, in Cobb-Douglas productions, is equivalent to the capital share. Therefore, the superscripts on θ stands for the high and the low productivity state and $0 < \theta^l < \theta^h < 1$.

Heterogeneity concerns the households sector. The economy is made up of two equal-sized groups of agents, endowed with different subjective probability measure π^i, for $i \in \{1, 2\}$ and $\pi^1 \neq \pi^2$. Agents know both the i.i.d. nature of the stochastic process and the possible shock

[1]Since there are two states of the world and probabilities sum to one, I simply denote: $\pi^i = Pr(\theta_t = \theta^h)$ and $1 - \pi^i = Pr(\theta_t = \theta^l)$.

realizations, however they disagree on the occurrences of them. On top of that, I assume $\pi^1 > \pi^2$ from here on out (agent 1 attributes more probability to the high productivity state relative to agent 2). Therefore $\mathbb{E}^1(\theta_{t+1}) > \mathbb{E}^2(\theta_{t+1})$ for any node θ^t. Agents do not update their opinions when they observe new shock realizations but keep their beliefs all along. The assumption of persistent disagreement can be thought of as a proxy of an economy where, convergence to one single model, is prevented by agents having disjoint probability supports among themselves.

Households solve an inter-temporal consumption-investment problem by maximizing the ex-ante expected utility under the subjective probability they are endowed with. The investment opportunities consist of two assets, a stock s and a security b. The pay-off matrix is given by

$$\begin{bmatrix} R_t^h(\theta^t) & 1 \\ R_t^l(\theta^t) & 0 \end{bmatrix} \tag{3.1}$$

where the assets (states) pay-off are represented by the columns (rows) vectors. In each node θ^t, the stock s pays a stochastic return $R_t(\theta^t) \in \{R_t^h(\theta^t), R_t^l(\theta^t)\}$ that is related to the firm's stochastic productivity (see Section 3.2.2). By contrast, the security b is essentially an Arrow security paying one unit of consumption if state θ^h realizes. As a consequence, the existence of b completes the financial markets allowing the agents replicate any possible future cash flow.

Finally, to simplify the notation, I drop the reference to the node θ^t when it is not necessary, so that x_t is used in place of $x_t(\theta^t)$. We denote by q_{t+1,θ^r}^t the dated t price of the Arrow security delivering consumption in the next period, conditional on the realization $\theta_{t+1} = \theta^r$. Consistently, we also use the same notation to specify the time t state θ^r production y_{t,θ^r}, when that is necessary.

3.2.2 Firm

The final good is produced by a representative firm whose production between t and $t+1$ depends on the following technology

$$y_t = Ak_t^{\theta_t} \tag{3.2}$$

where A is the *Total Factor Productivity* (TFP) term and θ_t the time t technology shock realization. Production is part of the Mirman and Zilcha (1975) class of stochastic growth model that, together with the log utility assumption, ensures analytical tractability of the model. At time t, the representative firm observes θ_t and decides on k_{t+1} which, together with θ_{t+1}, determines the next period production. Capital is the only production factor and it is rented in a perfectly competitive market so that profits are maximized, that is, the capital choice solves

$$\begin{gathered} \max_{k_{t+1}} P_t = \max_{k_{t+1}} \left[\sum_{r \in \{h,l\}} q_{t+1,\theta^r}^t y_{t+1,\theta^r} \right] - k_{t+1} \\ \text{s.to} \\ y_{t+1} \leq Ak_{t+1}^{\theta_{t+1}}. \end{gathered} \tag{3.3}$$

The node θ^t optimal capital determines, together with the $t+1$ shock realization $\theta_{t+1} \in \{\theta^h, \theta^l\}$, the next period production. Profit is, therefore, the next period contingent outputs, discounted by the corresponding state-prices q_{t+1,θ^r}^t, minus the current cost of the capital[2]. The optimality condition of the above maximization problem is given by

$$\sum_{r \in \{h,l\}} q_{t+1,\theta^r}^t A\theta^r k_{t+1}^{\theta^r - 1} = 1 \tag{3.4}$$

[2]Consumption, capital and production are the same good whose price is normalized to 1 in every t.

Finally, given that capital fully depreciates at the end of any period, the terms aggregate capital and aggregate investment are used interchangeably throughout the paper.

3.2.3 Households

As mentioned in Section 3.2.1, the economy is populated by two equal-sized groups of households indexed by $i \in \{1, 2\}$. Households hold heterogeneous opinions about the occurrences of high and low productivity states and they solve the same consumption-investment problem accordingly. The latter is represented as follows

$$\begin{aligned} &\max_{c_t^i, s_{t+1}^i, b_{t+1}^i} \quad \mathbb{E}^i \left[\sum_{t=0}^{\infty} \beta^t \log c_t^i \right] \\ \text{s.to} \quad & c_t^i + p_t s_{t+1}^i + q_{t+1,\theta^h}^t b_{t+1}^i = w_t^i + \frac{P_t}{2} \\ & w_t^i = \begin{cases} s_t^i y_{t,\theta^h} + b_t^i & \text{if} \quad \theta_t = \theta^h \\ s_t^i y_{t,\theta^l} & \text{if} \quad \theta_t = \theta^l \end{cases} \end{aligned} \tag{3.5}$$

where the initial endowment w_0^i is given. In addiction to the share in the firm's profit, that is uniformly granted between the two population types, the node θ^t disposable income also includes the financial wealth w_t^i. The latter depends on both the portfolio choice made in $t-1$, that is the vector $\left(s_t^i, b_t^i\right)$, and the time t shock realization, θ_t. The stock s yields a stochastic return that is equal to the time t aggregate production. Therefore, the financial wealth is greater if the high productivity state realizes at t, either because $y_{t,\theta^h} > y_{t,\theta^l}$ and because the Arrow securities b_t^i delivers consumption in state θ^h.

The optimality conditions consist of the usual equality between the agent's MRS between future and current consumption and their price ratio

$$\begin{aligned} p_t &= \beta \left[\frac{c_t^i \pi^i}{c_{t+1}^i} A k_{t+1}^{\theta^h} + \frac{c_t^i \left(1 - \pi^i\right)}{c_{t+1}^i} A k_{t+1}^{\theta^l} \right] \\ q_{t+1,\theta^h}^t &= \beta \frac{c_t^i \pi^i}{c_{t+1}^i} \end{aligned} \tag{3.6}$$

3.2.4 Competitive equilibrium

Definition 3.1. *Given an initial capital distribution* $\left(w_0^i\right)_i$, *a sequential trading competitive equilibrium is a sequence of prices* $\left(p_t, q_{t+1,\theta^h}^t\right)_{t,\theta^t}$, *an allocation* $\left(c_t^i\right)_{i,t,\theta^t}$, *portfolio rules* $\left(s_{t+1}^i, b_{t+1}^i\right)_{i,t,\theta^t}$ *and output decisions* $\left(k_{t+1}, y_t\right)_{t,\theta^t}$ *such that, given the equilibrium prices, the firm maximizes profit as in* (3.3), *households solve the problem in* (3.5) *and markets clear as follows*

$$\begin{aligned} \sum_{i \in \{1,2\}} c_t^i + k_{t+1} &= A k_t^{\theta_t} \\ \sum_{i \in \{1,2\}} s_{t+1}^i &= 1 \\ \sum_{i \in \{1,2\}} b_{t+1}^i &= 0 \end{aligned} \tag{3.7}$$

The first clearing condition is the feasibility constraint of the economy and it requires that all the resources produced are allocated either as consumption or as investment. The other two equalities state the clearing conditions in the financial market. Specifically, each agent owns a fraction of the firm-related stock, which is available in unitary supply, while they are allowed to exchange a zero-sum security to adjust the investment and consumption strategy according to their preferences.

The individual consumption rule is given by[3]

$$c_t^i = \left(1 - \beta \mathbb{E}^i\left(\theta_{t+1}\right)\right) w_t^i. \tag{3.8}$$

Consumption is a fixed fraction of the individual wealth and it is negatively affected by the expected value of the exogenous shock, $\mathbb{E}^i\left(\theta_{t+1}\right) = \pi^i \theta^h + \left(1 - \pi^i\right)\theta^l$. As one would expect, the larger is the probability attributes to θ^h the less is the agent's propensity to consume in order to increase investments.

Using the economy resource constraint, I figure out the capital stock's law of motion

$$k_{t+1} = \beta \left(\sum_{i \in \{1,2\}} \mathbb{E}^i\left(\theta_{t+1}\right) \phi_t^i \right) A k_t^{\theta_t} \tag{3.9}$$

The fraction invested in the productive unit is a convex combination of the agents' expected values, whose weights are given by the individual relative fraction of wealth owned in t

$$\phi_t^i = \frac{w_t^i}{A k_t^{\theta_t}}. \tag{3.10}$$

Disagreement enhances capital, and thus output, volatility due to the endogenous distribution of the agents' wealth $\{w_t^i\}_{i \in \{1,2\}}$. During high productivity states, the aggregate investment increases not only because of the positive shock realization, but also because ϕ_t^1 rises at the expense of ϕ_t^2 and $\mathbb{E}^1\left(\theta_{t+1}\right) > \mathbb{E}^2\left(\theta_{t+1}\right)$. According to (3.9), the fraction of wealth invested increases as a result.

Rewriting the stock price as $p_t = \sum_{r \in \{h,l\}} q^t_{t+1,\theta^r} A k_{t+1}^{\theta^r}$, we can derive the individual portfolio composition just in terms of state-contingent prices

$$s_{t+1}^i = \beta \frac{\left(1 - \pi^i\right) w_t^i}{\left(q^t_{t+1,\theta^l}\right) A k_{t+1}^{\theta^l}} \qquad b_{t+1}^i = \beta \left(\frac{\pi^i}{q^t_{t+1,\theta^h}} - \frac{1 - \pi^i}{q^t_{t+1,\theta^l}} k_{t+1}^{\theta^h - \theta^l} \right) w_t^i. \tag{3.11}$$

while equilibrium state-prices comes from the equality between the assets' aggregate demands and supplies

$$\begin{aligned} q^t_{t+1,\theta^h} &= \beta \frac{\sum_{i \in \{1,2\}} \pi^i w_t^i}{A k_{t+1}^{\theta^h}} \\ q^t_{t+1,\theta^l} &= \beta \frac{\sum_{i \in \{1,2\}} \left(1 - \pi^i\right) w_t^i}{A k_{t+1}^{\theta^l}} \end{aligned} \tag{3.12}$$

Replacing the equilibrium contingent-prices, the agent's asset demands are given by

$$\begin{aligned} b_{t+1}^i &= \left(\frac{\pi^i}{\sum_{i \in 1,2} \pi^i w_t^i} - \frac{1 - \pi^i}{\sum_{i \in 1,2} \left(1 - \pi^i\right) w_t^i} \right) w_t^i A k_{t+1}^{\theta^h} \\ s_{t+1}^i &= \beta \frac{1 - \pi^i}{\sum_{i \in \{1,2\}} \left(1 - \pi^i\right) w_t^i} w_t^i. \end{aligned} \tag{3.13}$$

Unsurprisingly, the agent is willing to take a long position on b as long as the difference between the state θ^h and θ^l individual over aggregate belief ratio is positive. Loosely speaking, he holds a

[3] The individual optimal consumption rule (3.8) is proved in the Appendix C (proof of Proposition 3.1), setting $\tau = 0$. The aggregate investment rule (3.9) is obtained by replacing (3.8) for $i \in \{1,2\}$ in the consumption good clearing condition displayed in Definition 3.1.

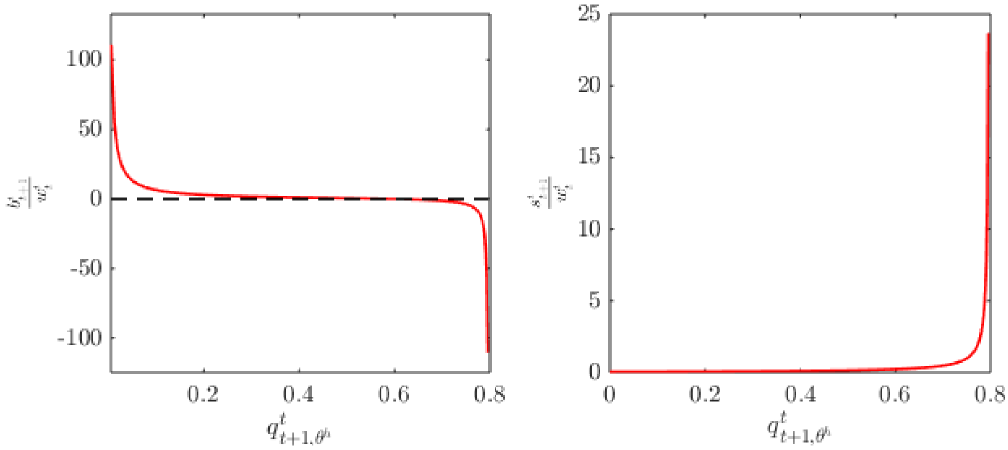


Figure 3.1: Agent i relative assets' demands as functions of the state θ^h price $q^t_{t+1,\theta^h} \in \left(0, \sup(q^t_{t+1,\theta^h})\right)$ and for a given level of k_{t+1}.

long position on b provided that he attributes more probability to θ^h relative to the other agent. Otherwise, he sells short the security purchasing a larger fraction of the stock. Moreover, it is worth to notice that $b^i_{t+1} = 0$ and $s^i_{t+1} = 1$ when the equilibrium prices only reflect the agent i beliefs, that is what happens in a homogeneous economy. In our model, optimal asset demands are instead functions of the distance between personal and market evaluation. The individual financial strategy is thus affected by the others' beliefs by means of the equilibrium prices. This is exactly the speculative channel of the economy: prices turn out to be over(under)-evaluated in the eyes of some traders creating the conditions for the securities exchange.

Using (3.4) to derive the optimal relation between state-prices, the individual asset demands are expressed just in terms of q_{t+1,θ^h} from here on out

$$\begin{aligned} s^i_{t+1} &= \beta \frac{\left(1-\pi^i\right)\theta^l k^{-1}_{t+1}}{\left(1 - q^t_{t+1,\theta^h} A\theta^h k^{\theta^h - 1}_{t+1}\right)} w^i_t \\ b^i_{t+1} &= \beta \left[\frac{\pi^i}{q^t_{t+1,\theta^h}} - \frac{\left(1-\pi^i\right) A\theta^l k^{\theta^h - 1}_{t+1}}{\left(1 - q^t_{t+1,\theta^h} A\theta^h k^{\theta^h - 1}_{t+1}\right)} \right] w^i_t \end{aligned} \tag{3.14}$$

Figure 3.1 displays the agent i relative assets' demands

$$\frac{b^i_{t+1}}{w^i_t} \qquad \frac{s^i_{t+1}}{w^i_t}$$

as a function of q_{t+1,θ^h} and for a given level of aggregate investment k_{t+1}. Clearly, the two investment rules are linked by a negative relationship. If the equilibrium state θ^h price is lower than a certain threshold, the agent takes a long position on b buying a lower fraction of the stock. The s demand increases for higher levels of q^t_{t+1,θ^h}, since the agent is willing to short sell the security in such a case.

3.3 Policy intervention

Trading possibilities are limited by means of a linear transaction tax paid on the market value of the Arrow security traded

$$\tau q^t_{t+1,\theta^h} |b^i_{t+1}|$$

and $\tau \in (0,1)$. The FTT is a trading cost either when the agent decides to buy and short-sell the Arrow security. The use of the absolute value is thus justified by the fact that households face no constraints in trading: they can loosely long and short sell the Arrow security. Moreover, as discussed in the previous section, the tax only applies to the asset b's trades since it is the channel of speculation in this economy.

Policy intervention does not affect the firm's optimization problem in (3.3) so that the optimality condition (3.4) is still valid under this framework. However, the output decisions are influenced by the FTT through changes in the state-prices resulting from the households' optimization problem. The latter modifies as

$$\max_{c_t^i, s_{t+1}^i, b_{t+1}^i} \mathbb{E}^i \left[\sum_{t=0}^{\infty} \beta^t \log c_t^i \right]$$
$$\text{s.to} \tag{3.15}$$
$$\begin{cases} c_t^i + p_t s_{t+1}^i + q_{t+1,\theta^h}^t b_{t+1}^i = w_t^i + \frac{P_t}{2} & \text{if } b_{t+1}^i = 0 \\ c_t^i + p_t s_{t+1}^i + (1+\tau)\, q_{t+1,\theta^h}^t b_{t+1}^i = w_t^i + \frac{P_t}{2} & \text{if } b_{t+1}^i > 0 \\ c_t^i + p_t s_{t+1}^i + (1-\tau)\, q_{t+1,\theta^h}^t b_{t+1}^i = w_t^i + \frac{P_t}{2} & \text{if } b_{t+1}^i < 0 \end{cases}$$

where the first (second) line of the agent's budget constraint binds if he takes a long (short) trading position on the security b. The transaction tax distorts the households' F.O.C. with respect to b_{t+1}^i (second line in (3.6)) in the following way

$$\begin{cases} q_{t+1,\theta^h}^t (1+\tau) = \beta \frac{c_t^i \pi^i}{c_{t+1}^i} & \text{if} \quad b_{t+1}^i > 0 \\ q_{t+1,\theta^h}^t (1-\tau) = \beta \frac{c_t^i \pi^i}{c_{t+1}^i} & \text{if} \quad b_{t+1}^i < 0 \end{cases}$$

The FTT modifies the security buyer (seller) behaviour by increasing (decreasing) the units of node $t+1, \theta^h$ consumption that the agent is willing to give up in favour of c_t^i. In this way, it mitigates the agents' position on the security trades.

The next Proposition characterize the households' optimal decision rules

Proposition 3.1. *The individual optimal decision rules resulting from the household's maximization problem in* (3.15) *are given by*[4]

$$c_t^i = \begin{cases} \left(1 - \beta \frac{\mathbb{E}^i(\theta_{t+1}) + \tau\theta^l}{1+\tau}\right) w_t^i & \text{if } q_{t+1,\theta^h}^t < q_{buy}^i \\ \left(1 - \beta \frac{\mathbb{E}^i(\theta_{t+1}) - \tau\theta^l}{1-\tau}\right) w_t^i & \text{if } q_{t+1,\theta^h}^t > q_{sell}^i \\ \left(1 - \beta \mathbb{E}^i(\theta_{t+1})\right) w_t^i & \text{if } q_{t+1,\theta^h}^t \in \left[q_{buy}^i, q_{sell}^i\right] \end{cases}$$

$$s_{t+1}^i = \begin{cases} \beta \frac{(1-\pi^i)\theta^l k_{t+1}^{-1}}{\left(1 - q_{t+1,\theta^h}^t \left[\theta^h + \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}\right)} w_t^i & \text{if } q_{t+1,\theta^h}^t < q_{buy}^i \\ \beta \frac{(1-\pi^i)\theta^l k_{t+1}^{-1}}{\left(1 - q_{t+1,\theta^h}^t \left[\theta^h - \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}\right)} w_t^i & \text{if } q_{t+1,\theta^h}^t > q_{sell}^i \\ \beta \frac{\theta^l k_{t+1}^{-1} w_t^i}{1 - q_{t+1,\theta^h}^t (\theta^h - \theta^l) A k_{t+1}^{\theta^h - 1}} & \text{if } q_{t+1,\theta^h}^t \in \left[q_{buy}^i, q_{sell}^i\right] \end{cases} \tag{3.16}$$

$$b_{t+1}^i = \begin{cases} \beta \left[\frac{\pi^i}{q_{t+1,\theta^h}^t (1+\tau)} - \frac{(1-\pi^i)\theta^l A k_{t+1}^{\theta^h - 1}}{\left(1 - q_{t+1,\theta^h}^t \left[\theta^h + \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}\right)} \right] w_t^i & \text{if } q_{t+1,\theta^h}^t < q_{buy}^i \\ \beta \left[\frac{\pi^i}{q_{t+1,\theta^h}^t (1-\tau)} - \frac{(1-\pi^i)\theta^l A k_{t+1}^{\theta^h - 1}}{\left(1 - q_{t+1,\theta^h}^t \left[\theta^h - \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}\right)} \right] w_t^i & \text{if } q_{t+1,\theta^h}^t > q_{sell}^i \\ 0 & \text{if } q_{t+1,\theta^h}^t \in \left[q_{buy}^i, q_{sell}^i\right] \end{cases}$$

[4]Again, having derived the optimal relation between the state-prices from (3.4), all the individual decision rules are expressed just in terms of q_{t+1,θ^h}^t, for all θ^t.

where

$$q^i_{buy} = \frac{\pi^i}{\left[\mathbb{E}^i\left[\theta_{t+1}\right] + \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}} \quad q^i_{sell} = \frac{\pi^i}{\left[\mathbb{E}^i\left[\theta_{t+1}\right] - \tau\theta^l\right] A k_{t+1}^{\theta^h - 1}}$$

are the agent's reservation prices[5] *as a buyer and seller of the security b, respectively.*

Proof. in Appendix C.

Comparative analysis with the case $\tau = 0$ is not immediate since the agent's reservation prices are state dependent and related to the time t aggregate investment level. However, conditional on the same k_{t+1}, I compare the agents' portfolio rules in Proposition 3.1 with the ones outlined in the benchmark case (3.14).

The FTT challenges the agents' portfolio strategies along two different dimensions. First, it reduces the set of prices for which the agents are willing to trade. Compared to the benchmark, where $q^i_{buy} = q^i_{sell} = \frac{\pi^i}{\mathbb{E}^i[\theta_{t+1}] A k_{t+1}^{\theta^h - 1}}$, the marginal price at which the agents are willing to buy (short-sell) the Arrow security is lower (higher). On top of that, given that $q^i_{buy} \neq q^i_{sell}$, a positive FTT creates an inactive region where $b^i_{t+1} = 0$ and the agents only invest in the firm-related stock. Second, it mitigates the size of trading. Figures 3.2-3.3 show the relative asset demands when $\tau = 0$ (red) and $\tau > 0$ (blue). In general, the FTT mitigates the agent's trading position on both b and s. However, the size of the distortion is greater for the set of prices for which the agent short-sell the Arrow security.

3.3.1 Competitive equilibrium

Having outlined the individual best replies, I characterize the competitive equilibrium of this economy.

Definition 3.2. *Given an initial distribution of capital* $\left(w^i_0\right)_i$*, a sequential trading competitive equilibrium is a price vector* $\left(q^t_{t+1,\theta^h}\right)_{t,\theta^t}$*, an allocation* $\left(c^i_t\right)_{i,t,\theta^t}$*, portfolio rules* $\left(s^i_{t+1}, b^i_{t+1}\right)_{i,t,\theta^t}$ *and output decisions* $\left(k_{t+1}, y_t\right)_{t,\theta^t}$ *such that, given the tax rate* τ *and the equilibrium price, the representative firm solves the problem in* (3.3)*, households face the problem* (3.15) *and the financial market clears*

$$\sum_{i\in\{1,2\}} s^i_{t+1} = 1 \qquad\qquad \sum_{i\in\{1,2\}} b^i_{t+1} = 0 \tag{3.17}$$

Moreover, if in node θ^t *agents take a non-zero position in the Arrow security trades, then the market of consumption good clears as follows*

$$\sum_{i\in\{1,2\}} c^i_t + k_{t+1} + R_t = A k_t^{\theta_t} \tag{3.18}$$

where $R_t = \tau q^t_{t+1,\theta^h} \sum_{i\in\{1,2\}} |b^i_{t+1}|$ *is the node* θ^t *Government's tax revenue. Otherwise, if agents do not trade the Arrow security, the feasibility constraint of the economy is equivalent to the one without taxation*

$$\sum_{i\in\{1,2\}} c^i_t + k_{t+1} = A k_t^{\theta_t} \tag{3.19}$$

[5] Using the agent i optimal demand of b stated in the Proposition, q^i_{buy} and q^i_{sell} are the marginal prices for which $b^i_{t+1} > 0$ and $b^i_{t+1} < 0$, respectively.

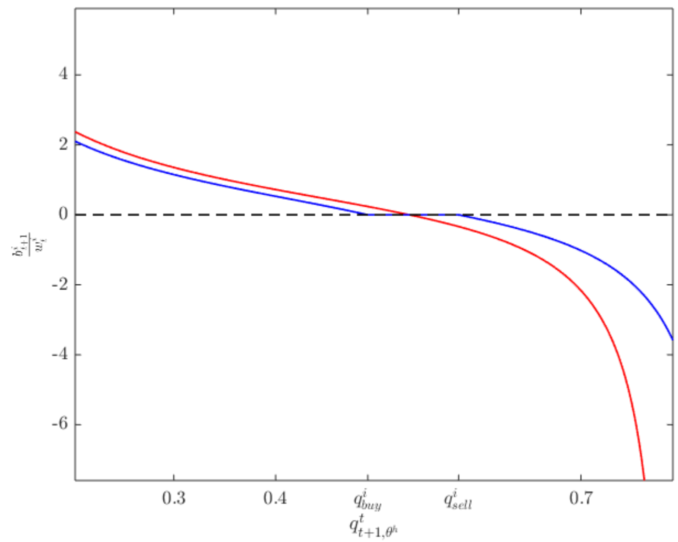


Figure 3.2: Individual Arrow security relative demand as a function of its price q^t_{t+1,θ^h} when $\tau = 0$ (red line) and $\tau > 0$ (blue line) and conditional to the same k_{t+1}.

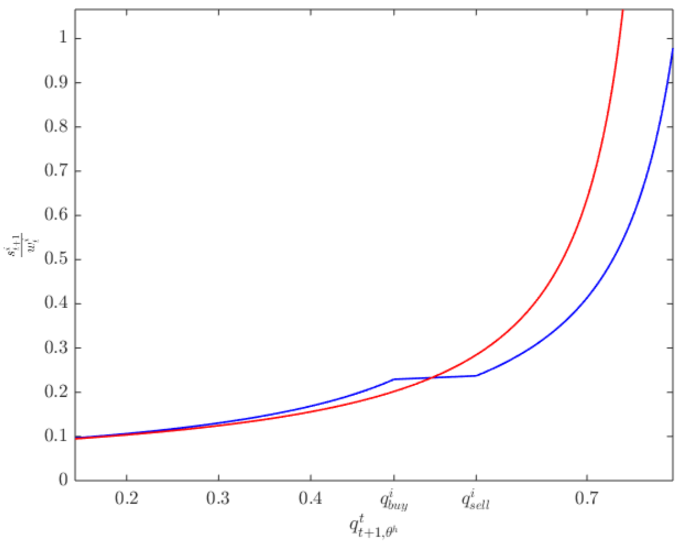


Figure 3.3: Individual stock relative demand as a function of the Arrow security price q^t_{t+1,θ^h} when $\tau = 0$ (red line) and $\tau > 0$ (blue line) and conditional to the same k_{t+1}. A FTT smooths the stock demand curve due to reduction in the size of trading affecting the other asset.

It is important to emphasize that policy intervention affects, not only the agents' asset demands, but also the total supply of consumption good, given that some resources are withdrawn from the economy. The next Lemma states the conditions under which agents speculate on the Arrow security exchange

Lemma 3.1. In a two-agent economy where subjective probabilities are such that $\pi^1 > \pi^2$, then $q^2_{sell} < q^1_{sell}$ and $q^1_{buy} > q^2_{buy}$. Moreover, there exist a limit value of the tax rate

$$\tau' = \frac{\pi^1 - \pi^2}{\pi^1 + \pi^2} \tag{3.20}$$

over which agents decide to not trade b at all. Therefore, if $\tau \in [0, \tau')$, then $q^2_{sell} < q^1_{buy}$ and $b^i_{t+1} \neq 0$ for all $i \in \{1, 2\}$ and in every period t. Specifically, agent 1 (2) takes a long (short) trading position on b.

Otherwise, $b^i_{t+1} = 0$ for all $i \in \{1, 2\}$ and agents only invest in the stock in every period t. In this case, the equilibrium state θ^h price is indeterminate in the set $\left(q^1_{buy}, q^2_{sell}\right)$ and such that equation (3.4) holds.

Proof. in Appendix C.

Whether the policy mitigates or eliminate speculative trading, depends on the size of the FTT rate. Specifically, Lemma 3.1 states the existence of a limit value τ' over which speculative trades are turned off. Since speculation originates from the agents' different opinions, this threshold is a positive function of the size of disagreement characterizing the economy. Unsurprisingly, the larger is the distance between the investors' beliefs the higher is the tax rate that turn off their willingness to trade. Finally, given that τ' is just a function of the fixed parameters of the economy and the tax rate is constant over time, therefore, if $\tau > \tau'$, agents will not trade in every period. In other words, depending on the tax rate, agents may decide to trade in every t or only invest in the firm-related stock.

Using the results in Lemma 3.1, together with Definition 3.2, the equilibrium allocation is characterized in the next Proposition

Proposition 3.2. The individual optimal choices are given by
Agent 1:

- if $\tau \in [0, \tau')$

$$\begin{aligned} c^1_t &= \left(1 - \beta \frac{\mathbb{E}^1(\theta_{t+1}) + \tau\theta^l}{(1+\tau)}\right) w^1_t \\ s^1_{t+1} &= \beta \frac{\left(1-\pi^1\right) A\theta^l k^{\theta^l - 1}_{t+1}}{\left(1 - q^t_{t+1,\theta^h}\left[\theta^h + \tau\theta^l\right] A k^{\theta^h - 1}_{t+1}\right)} w^1_t \\ b^1_{t+1} &= \beta \left[\frac{\pi^1}{q^t_{t+1,\theta^h}(1+\tau)} - \frac{\left(1-\pi^1\right)\theta^l A k^{\theta^h - 1}_{t+1}}{\left(1 - q^t_{t+1,\theta^h}\left[\theta^h + \tau\theta^l\right] A k^{\theta^h - 1}_{t+1}\right)}\right] w^1_t \end{aligned} \tag{3.21}$$

- if $\tau \geq \tau'$

$$\begin{aligned} c^1_t &= \left(1 - \beta\mathbb{E}^1(\theta_{t+1})\right) s^1_t A k^{\theta_t}_t \\ s^1_{t+1} &= \beta \frac{\theta^l k^{-1}_{t+1} s^1_t A k^{\theta_t}_t}{1 - q^t_{t+1,\theta^h}\left(\theta^h - \theta^l\right) A k^{\theta^h - 1}_{t+1}} \\ b^1_{t+1} &= 0 \end{aligned} \tag{3.22}$$

Agent 2:

- if $\tau \in [0, \tau')$

$$c_t^2 = \left(1 - \beta \frac{\mathbb{E}^2(\theta_{t+1}) - \tau\theta^l}{(1-\tau)}\right) w_t^2$$

$$s_{t+1}^2 = \beta \frac{(1-\pi^2) A\theta^l k_{t+1}^{\theta^l - 1}}{\left(1 - q_{t+1,\theta^h}^t [\theta^h - \tau\theta^l] A k_{t+1}^{\theta^h - 1}\right)} w_t^2 \tag{3.23}$$

$$b_{t+1}^2 = \beta \left[\frac{\pi^2}{q_{t+1,\theta^h}^t (1-\tau)} - \frac{(1-\pi^2)\theta^l A k_{t+1}^{\theta^h - 1}}{\left(1 - q_{t+1,\theta^h}^t [\theta^h - \tau\theta^l] A k_{t+1}^{\theta^h - 1}\right)}\right] w_t^2$$

- if $\tau \geq \tau'$

$$c_t^2 = \left(1 - \beta\mathbb{E}^2(\theta_{t+1})\right) s_t^2 A k_t^{\theta_t}$$

$$s_{t+1}^2 = \beta \frac{\theta^l k_{t+1}^{-1} s_t^2 A k_t^{\theta_t}}{1 - q_{t+1,\theta^h}^t (\theta^h - \theta^l) A k_{t+1}^{\theta^h - 1}} \tag{3.24}$$

$$b_{t+1}^2 = 0$$

The aggregate capital investment evolves as

$$k_{t+1} = \begin{cases} f(\mathbf{X}) & \text{if} \quad \tau \in [0, \tau') \\ \beta \sum_{i \in \{1,2\}} \left(\mathbb{E}^i [\theta_{t+1} | \theta^t] s_t^i\right) A k_t^{\theta_t} & \text{otherwise} \end{cases} \tag{3.25}$$

where $\mathbf{X}$ is a vector including the economy time discount factor β, the shock realizations θ^r ($r \in \{h, l\}$), the FTT rate τ, the agents' subjective probabilities π^i and incomes w_t^i ($i \in \{1, 2\}$). Finally, the equilibrium state θ^h price is given by

$$q_{t+1,\theta^h}^t = \beta \frac{\pi^1 (1-\tau) w_t^1 + \pi^2 (1+\tau) w_t^2}{(1-\tau^2) A k_{t+1}^{\theta^h}} \quad \text{if } \tau \in [0, \tau') \tag{3.26}$$

while q_{t+1,θ^h}^t is indeterminate in the set $\left(q_{buy}^1, q_{sell}^2\right)$ otherwise.

Proof. in Appendix C.

Consistent with the benchmark, each investor consumes a fixed fraction of his wealth. However, provided that $\tau \in [0, \tau')$, the fraction allocated as aggregate investment is a non-linear function of some parameters characterizing the agents' preferences and the firm's technology[6]. By contrast, when $\tau \geq \tau'$, the capital stock evolves as in (3.9) with s_t^i in place of ϕ_t^i. It is worth to notice that s_t^i is constant over time and fully determined by the initial capital distribution, $s_0^i = \frac{w_0^i}{\sum_{i \in \{1,2\}} w_0^i}$. Therefore, setting a tax rate greater than τ' eliminates the additional macroeconomic volatility induced by the agents' disagreement. In the benchmark economy, the latter results from the endogenous distribution of the agents' wealth $\{w_t^i\}_i$ evolving over time and states.

3.3.2 Long-run equilibrium analysis

Without any Government intervention, it is well known that the market selects rational against irrational agents, eventually eliminating the effect of inaccurate beliefs from the economy dynamics. The MSH holds leading the economy to converge to an homogeneous framework entirely inhabited by the most accurate type. Does the FTT prevent or slow down the market selection process? To answer this question, I study the impact of the FTT on the long-run equilibrium properties, using the standard definition of market dominance

[6]See Appendix C for the close-form solution.

Definition 3.3. *Agent i dominates if*

$$\lim_{t \to \infty} \phi_t^i = 1 \quad a.s.$$

while he vanishes if

$$\lim_{t \to \infty} \phi_t^i = 0 \quad a.s.$$

As discussed in the previous section, depending on the size of the policy intervention, a FTT may mitigate up to preventing speculative trades. For this reason, the asymptotic analysis is outlined under these two possible scenarios.

Define

$$z_t = \log\left(\frac{w_t^1}{w_t^2}\right)$$

as the logarithm of the agents wealth ratio, so that $z_t \to \infty \Leftrightarrow \phi_t^1 \to 1$, the time $t+1$ realization may be rewritten as

$$z_{t+1} = \xi_{t+1} + z_t. \tag{3.27}$$

Study the sign of the drift $\mathbb{E}\left[\xi_{t+1}\right]$ of the process $(z_t)_{t,\theta^t}$ provides sufficient condition to state the agents' long-run market dominance (see Dindo (2019)). The process z_t describes the evolution of the agents' wealth ratio, that favors agent 1 (2), provided that its drift is positive (negative). The latter is given by

$$\mathbb{E}\left[\xi_{t+1}|z_t\right] = D_{KL}\left(\pi^2||\pi\right) - D_{KL}\left(\pi^1||\pi\right) + g\left(\pi, \tau, A, \theta^h, \theta^l, q^t_{t+1,\theta^h}, k_{t+1}\right) \tag{3.28}$$

where $g\left(\pi, \tau, A, \theta^h, \theta^l, q^t_{t+1,\theta^h}, k_{t+1}\right) = \pi \log\left(\frac{1-\tau}{1+\tau}\right) + (1-\pi)\log\left(\frac{\left(1-q^t_{t+1,\theta^h}\left[\theta^h - \tau\theta^l\right]Ak_{t+1}^{\theta^h-1}\right)}{\left(1-q^t_{t+1,\theta^h}\left[\theta^h + \tau\theta^l\right]Ak_{t+1}^{\theta^h-1}\right)}\right)$. The drift is made up of two separate components: the first related to the difference between the agents' relative entropies[7] and the second related to some of the parameters and the economy state as well. It is easy to see that the second term cancels out whenever $\tau = 0$. This is consistent with the MSH: the market favors the investor with the lowest relative entropy in a friction-less market. By contrast, it is different from zero when $\tau = 0$, reducing the role played by belief accuracy during the selection process as a result.

Apart from the economy parameters, the function $g\left(\pi, \tau, A, \theta^h, \theta^l, q^t_{t+1,\theta^h}, k_{t+1}\right)$ depends on the node θ^t capital and equilibrium price. According to the rules in (3.25) and (3.26), these are in turn functions of the agents distribution of wealth w_t^i, $i \in \{1,2\}$. For this reason, I study the sign of (3.28) when the aggregate wealth distribution approaches its limits and one of the two agents ends up dominating the market, $w_t^i \to \infty \Longrightarrow \phi_t^i \to 1$ for $i \in \{1,2\}$. Define

$$\begin{aligned} \mu_+ &= \lim_{z \to +\infty} \mathbb{E}\left[\xi_{t+1}|z_t = z\right] \\ \mu_- &= \lim_{z \to -\infty} \mathbb{E}\left[\xi_{t+1}|z_t = z\right] \end{aligned} \tag{3.29}$$

the limiting values of (3.28) as the distribution of the individual wealth favors agent 1 and 2, respectively. Studying the sign of μ_+ and μ_- provides sufficient condition for market dominance: when they are both positives, agent 1 dominates and agent 2 vanishes in the long-run. The reason is that the process z_t has a positive drift not only when the wealth distribution favors agent ($\mu_+ > 0$), but also in those paths where the wealth distribution favors agent 2 ($\mu_+ < 0$). Following the same idea, agent 2 dominate and agent 1 vanishes when both μ_+ and μ_- are negative.

[7] The individual relative entropy is also the Kullback-Leibler divergence, that measures the distance between the π^i and π, for $i \in \{1,2\}$.

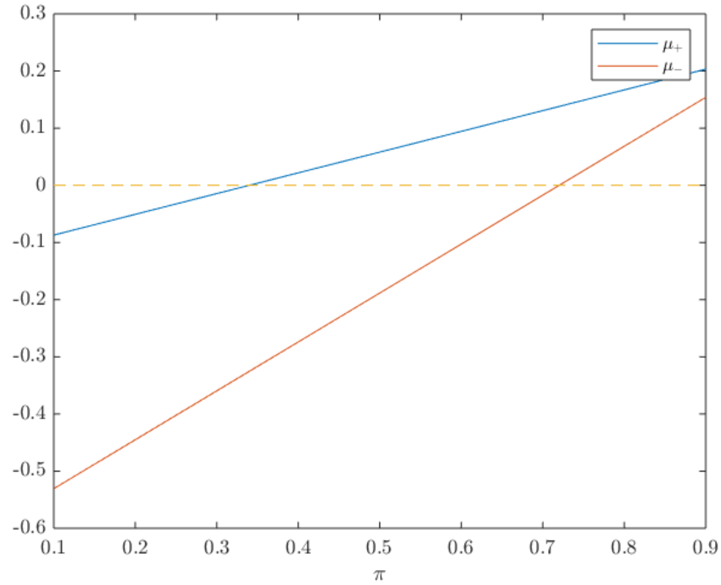


Figure 3.4: Limiting values of (3.28) as the aggregate wealth favors agent 1 (μ_+) and 2 (μ_-) and for different levels of the truth π. Path-dependency arises whenever the truth is in the middle of the agents' beliefs while the MSH holds when both are biased in the same direction (i.e. when $\pi^1 = 0.8$, $\pi^2 = 0.3$ and $\pi = 0.9$, both the agents are under-estimating the state 1 probability and only agent 1 survives). Parameters: $A = 4$, $\theta^h = 0.4, \theta^l = 0.3$, $\pi^1 = 0.8$, $\pi^2 = 0.3$, $\tau = 0.35$.

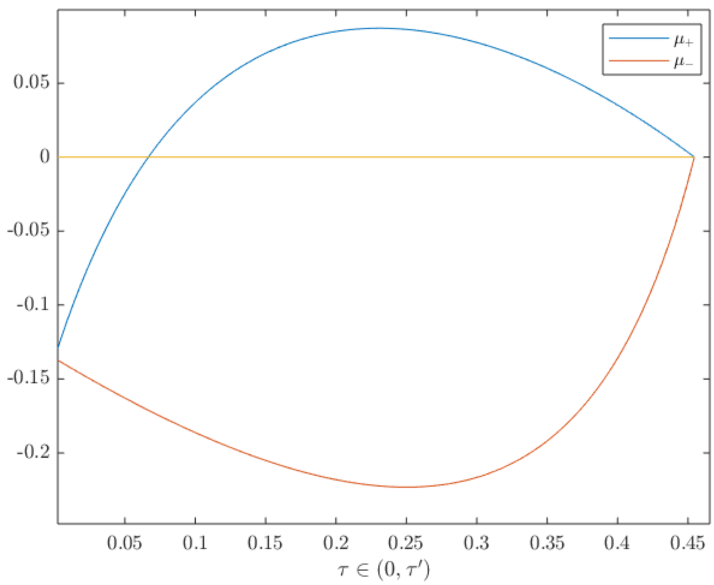


Figure 3.5: Limiting values of (3.28) as the aggregate wealth favours agent 1 (μ_+) and 2 (μ_-) for different $\tau \in [0, \tau')$. Parameters: $A = 4$, $\theta^h = 0.4, \theta^l = 0.3$, $\pi^1 = 0.8$, $\pi^2 = 0.3$, $\pi = 0.5$.

By contrast, two different situations may arise when μ_+ and μ_- have opposite signs. First, when $\mu_+ < 0$ and $\mu_- > 0$, it means that heterogeneity is preserved in the long run. Intuitively, this happens because the process z_t has a positive (negative) drift in those paths where the wealth distribution favors agent 2 (1). Second, when $\mu_+ > 0$ and $\mu_- < 0$, heterogeneity is still a transient feature of the market, however, the outcome of the selection process depends on the history of shocks rather than on the agents' accuracy. In this case the long-run equilibrium is path-dependent, given that the drift of the wealth ratio process z_t is positive in those path where agent 1 is wealthier and negative otherwise.

Everything else equal, Figure 3.4 displays μ_+ and μ_- for different values of the truth and for a given tax rate. Assuming that $\pi^1 = 0.8$ e $\pi^2 = 0.3$, when π approaches one of these values, both μ_+ and μ_- share the same sign (positive when $\pi \to \pi^1$ and negative when $\pi \to \pi^2$). This happens because one of the agent is much more accurate than the other and comes to dominate regardless the fact that the FTT is limiting trades. By contrast, when π is somewhere in the between of π^1 and π^2, the long run equilibrium is path-dependent ($\mu_+ > 0$ and $\mu_- < 0$). Figure 3.5 shows the drift limiting values for a set of tax rates $\tau \in (0, \tau')$ and given the true probability, that has been arbitrarily set at 0.5. In this case, agent 2 is more accurate and he dominates provided that τ is low enough (μ_+ and μ_- are both negative for τ approximately lower than 0.07). The equilibrium is path-dependent for tax rates highest than this threshold. It follows that the MSH holds (both μ_+ and μ_- have the same sign) provided that either is accurate enough (Figure 3.4) and the tax rate is low enough (Figure 3.5).

To understand the source of path-dependency, I define

$$T_t^i = \tau q_{t+1,\theta^h}^t |b_{t+1}^i|$$

the individual tax levy in period t. The budget constraint in the agent i's problem (3.15) can be equivalently rewritten as

$$c_t^i + p_t s_{t+1}^i + q_{t+1,\theta^h}^t b_{t+1}^i = w_t^i \left(1 - \frac{T_t^i}{w_t^i}\right) + \frac{P_t}{2}$$

and the smaller is $\frac{T_t^i}{w_t^i}$ the lower is the tax impact on the agent's resources. Define

$$\lim_{\phi_t^i \to 1} \frac{T_t^i}{w_t^i} = 0 \qquad \lim_{\phi_t^i \to 0} \frac{T_t^i}{w_t^i} > 0 \tag{3.30}$$

the limit values of the agent i tax levy as the distribution of the agents' wealth approaches its limits, $\phi_t^i \to 1$ for $i \in \{1, 2\}$.

Within our setting, path-dependency arises because the impact of the FTT reduces as the agent accumulates most of the aggregate consumption, hence implying $\mu_+ > 0$ and $\mu_- < 0$.

Finally, I turn to the case $\tau \geq \tau'$. According to Proposition (3.2), the agents' wealth ratio is determined by the initial distribution of wealth and, therefore

$$z_{t+1} = z_0.$$

The drift component is zero letting both the agents survive in the long-run.

The analytical results of this section are summarized in the next Proposition

Proposition 3.3. *Let agent i be the most accurate type implying $D_{KL}\left(\pi^i||\pi\right) - D_{KL}\left(\pi^{-i}||\pi\right) < 0$. If $\tau = 0$, then $g\left(\tau, \theta^h, \theta^l, A, q_{t+1,\theta^h}^t, k_{t+1}\right)$ in (3.28) is zero implying the MSH holds a.s. $\phi_t^i \to 1$ and $\phi_t^{-i} \to 0$. If $\tau > \tau'$, then $\mathbb{E}\left[\xi_{t+1}|z_t\right] = 0$ and both the agents survive a.s. $\phi_t^i \nrightarrow 0$ and $\phi_t^{-i} \nrightarrow 0$. Finally, if $\tau \in (0, \tau')$ market dominance may be path-dependent and $\mathbb{P}\left(\lim_{t\to\infty} \phi_t^i = 0\right) > 0$.*

Proof. in Appendix C.

3.3.3 Real effect of the FTT

One of the major aims of the paper is to investigate the impact of the FTT in the real sector. The optimal production level turns out to be affected by the policy measure through the capital investment rule that, according to Proposition 3.2, is a function of the financial transaction tax rate as well. By its own structure, the tax affects the economy dynamics as long as heterogeneity is preserved and agents are involved in a speculative trading. As investigated in the previous section, different scenario arises in this regard. First, the FTT reduces but allows speculative trading, $\tau \in (0, \tau')$. Heterogeneity is always a transient phenomenon in such a case since one of the agent is destined to vanish in the long-run. Therefore, the policy has an impact on the real sector only during the transition path to the corresponding homogeneous economy. Second, the tax rate is high enough to prevent speculative trading, $\tau \geq \tau'$. In this situation, disagreement has a persistent effect on the macroeconomic variables.

Back to the first scenario, the MSH is no longer guaranteed provided that either the tax rate is high enough and the truth is somewhere in the middle of the agents' beliefs (see Figures 3.4-3.5). What might happen is that the most biased belief persistently affects the long-run features of the capital accumulation path and thus the economy production level.

Within this environment, I have already proved the existence of a unique invariant distribution of capital in Chapter 1. Aggregate capital, and so does consumption, always moves in a defined recurrent set and it converges to the invariant distribution implied by the most accurate type when the MSH holds. By contrast, an invariant distribution for capital does not exist when the long-run equilibrium is path dependent. Depending on the history of shocks, the capital pattern will converge to the invariant distribution implied by the agent who asymptotically dominates.

Looking at the short-term, disagreement enhances macroeconomic volatility due to the endogenous wealth distribution of agents. In the absence of policy intervention, the economy is characterized by periods of over/under-investment depending on the type owning the majority of wealth over time (see equation (3.9)). In this regard, the FTT affects the real sector in a way that depends on the position of the truth with respect to the agents' subjective probabilities. When $\pi \in \left[\pi^2, \pi^1\right]$, the FTT partially corrects the benchmark result, affecting both the aggregate and the individual consumption allocations. To some extent, trading costs increases (decreases) the agent 1 (2) marginal propensity to consume, $\text{mpc}^i = \frac{\Delta c^i_{t+1}}{\Delta w^i_{t+1}}$

$$\frac{\partial \text{mpc}^1}{\partial \tau} > 0 \qquad \frac{\partial \text{mpc}^2}{\partial \tau} < 0. \tag{3.31}$$

Since agent 1 (2) under (over) consumes under the true probability, the FTT has a corrective nature in that sense.

The impact on the capital-output ratio, that is the fraction of the aggregate wealth invested, is state-dependent and related to the agents' wealth distribution. However, as the the latter approaches its limits, the latter is given by

$$\lim_{\phi^1_t \to 1} \frac{\partial \frac{k_{t+1}}{y_t}}{\partial \tau} < 0 \qquad \lim_{\phi^2_t \to 1} \frac{\partial \frac{k_{t+1}}{y_t}}{\partial \tau} > 0 \tag{3.32}$$

As showed in Figure 3.6, a positive tax rate reduces the aggregate investment when agent 1 owns the majority of wealth. The effect is positive for the other limit. Over/under-investment characterizing an economy without policy intervention is mitigated as a result.

Proposition 3.4. *Provided that $\pi \in \left[\pi^2, \pi^1\right]$, then the individual marginal propensity to consume mpc^i is closer to the one implied by rational expectations for $i \in \{1, 2\}$. Capital, and therefore out-*

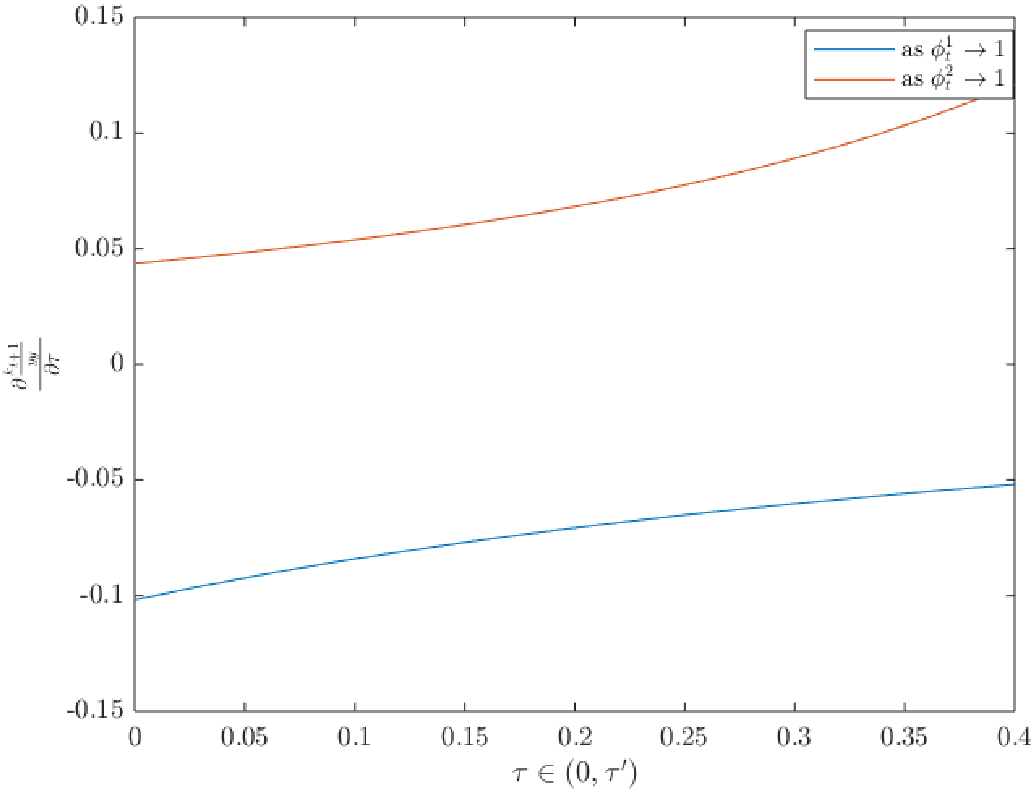


Figure 3.6: Marginal impact of the FTT on the capital-output ratio, for different level of τ as the agents' wealth distribution approaches its limit values.

put, choices are corrected in the same direction as the distribution of the agents' wealth approaches both its limits $\phi^i \to 1$ for $i \in \{1,2\}$.

Proof. in Appendix C.

When the true lies elsewhere than in $\left[\pi^2, \pi^1\right]$, the FTT amplifies the distortions induced by expectation biases even though the market always select the most accurate type. Suppose that both the agents over-estimates the probability of the high productivity states, $\pi < \pi^2 < \pi^1$. In this case, agent 2 is the most accurate and he dominates the market in the long-run. However, according to (3.31), the FTT further reduces his propensity to consume despite he is already under-consuming under the true probability.

In short, the tax pushes the dominant agent's optimal decisions to the wrong direction, thereby increasing the size of the distortion at the aggregate level.

Lastly, I consider the implications on the real economy when $\tau \geq \tau'$ and all the investors' resources are employed in the firm. Without policy intervention, the capital-output ratio is a convex-combination of the households' beliefs whose weights are given by the agents' wealth positions (see 3.9). Therefore, disagreement enhances the macroeconomic volatility due to the endogenous distribution of wealth among traders. As stated by Proposition 3.2, $\frac{k_{t+1}}{y_t}$ is still a convex combination of the traders' shock expected value, however weights are constant and fully determined by the initial distribution of capital. As a result, the economy dynamics is equivalent to one that characterize an homogeneous economy, with common prior $w_0^1 \pi^1 + w_0^2 \pi^2$.

3.4 The Government

Having outlined the competitive equilibrium under an exogenous τ, I characterize the Government' problem aimed at setting the optimal tax rate. I assume the policy purpose be both to raise revenues and maximize social welfare. Although this tax has been conceived as a regulatory measure for speculative financial markets, there is a consensus from the existing literature about its capability to raise fiscal revenues as well (see McCulloch and Pacillo (2010)).

The fiscal revenue function is the discounted sum of all the future histories θ^t fiscal withdrawals

$$TR = \sum_{t=0,\theta^t}^{\infty} q_t^0\left(\theta^t\right) R_t\left(\theta^t\right) = \sum_{t=0,\theta^t}^{\infty} q_t^0\left(\theta^t\right) \underbrace{\tau\left[q_{t+1}^t\left(\theta^h|\theta^t\right) \sum_{i\in\{1,2\}} \left(|b_{t+1}^i\left(\theta^t\right)\right)|\right]}_{R_t(\theta^t)}. \tag{3.33}$$

The tax rate is set in time 0, therefore future withdrawals are discounted using the time 0 price[8]. Agents' utilities are represented by the following social welfare function

$$W = \mathbb{E}^g\left[\sum_{t=0}^{\infty} \beta^t \left[\sum_{i\in\{1,2\}} \gamma^i \log\left(c_t^i\right)\right]\right] \tag{3.34}$$

where Pareto weights are proportional to the households' initial endowment

$$\gamma^i = \frac{w_0^i}{\sum_{i\in\{1,2\}} w_0^i} \tag{3.35}$$

while the Government's probability measure is the convex hull of the agents' subjective beliefs (see Brunnermeier et al. (2014))

$$\pi^g = \lambda\pi^1 + (1-\lambda)\pi^2 \tag{3.36}$$

[8] $q_t^0\left(\theta^t\right)$ is the dated 0 price of the asset delivering consumption in node θ^t. Given the no-arbitrage condition $q_{t+1}^0\left(\theta^h|\theta^t\right) = q_t^0\left(\theta^t\right) q_{t+1}^t\left(\theta^h|\theta^t\right)$.

where $\lambda \in [0, 1]$. The Government's Ramsey problem is defined as follows

Definition 3.4. The Ramsey problem is to maximize (3.37), choosing τ such that the private sector's optimality conditions included in Proposition 3.2 are satisfied

$$\begin{gathered}\max_{\tau} \alpha W + (1-\alpha) TR \\ \text{s.to} \\ \text{Firm's optimality condition (3.4)} \\ \text{Households' consumption and porfolio composition rules (3.21)-(3.22)-(3.23)-(3.24)} \\ \text{Aggregate capital investment (3.25)} \quad \text{equilibrium price (3.26)} \\ \text{Market clearings (Def. 3.2)}\end{gathered} \tag{3.37}$$

where α is the relative weight given to the welfare maximizing motive.

Among the competitive equilibria, the Ramsey problem pinpoints the one that maximizes the Government's objective function. To better investigate the optimal taxation problem, I separately study the cases that originate by challenging α in the Government's objective function (equation (3.37)).

Specifically, I find τ^* when the planner's motive is purely welfare maximizing ($\alpha = 1$), revenue raising ($\alpha = 0$), and a middle way of the two ones ($\alpha \in (0, 1)$).

3.4.1 Welfare maximizing motive

I first assume that the Government's aim is purely welfare maximizing. Figure 3.7 displays the numerical values[9] of the social welfare function for different levels of $\tau \in [0, \tau']$ and Governments' probabilities.

The optimal tax rate depends on the probability measure used in the ex-ante welfare maximization problem, π^g. Specifically, when π^g is sufficiently close to π^2, the Government optimally set a tax rate equal to τ' (in the figure this happens for $\lambda \in [0, 0.5]$). Therefore, speculative trading is shut down whenever the Government attributes more probability to the low rather than the high productivity state. In this case, households only invest in the firm and the aggregate output is split up in proportion to the distribution of the initial endowment.

Instead, the optimal tax rate reduces as the Government's probability approaches π^1. The reason is that high productivity states are expected to be more likely and the firm's technology more productive as a result. The chosen tax rate is zero or a smaller value than τ' since the ex-ante expected welfare function is maximized even though only agent 1 survives in such a case. In fact, the high level of consumption resulting from the agent 1 over-investment overweights the welfare losses due to the agent 2 resource depletion.

3.4.2 Revenue motive

I now discuss the case $\alpha = 0$ in the Government's objective function (3.37). Fiscal revenues may be here interpreted as a proxy for welfare improvements, even though public expenditure is not explicitly modeled. In this case, the optimal tax rate is never τ' that would imply $R_t = 0$ for all θ^t. Moreover, by its own structure, the optimal FTT is not affected by the Government's

[9] The problem (3.37) is numerically solved through the following steps: simulate the welfare function (3.34) using a coarse grid of points for τ, spline interpolate the objective function over a finer grid and choose the value of the grid, τ^*, that maximizes the interpolated function.

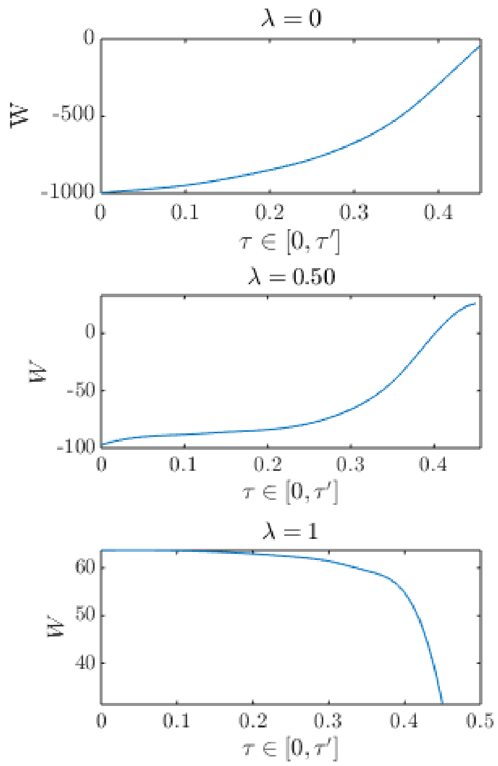


Figure 3.7: Social welfare function for different levels of $\tau \in [0, \tau']$ and λs. Agents beliefs are $\pi^1 = 0.8$ and $\pi^2 = 0.3$. The Government's beliefs, under which the function is maximized, ranges from $\pi^g = 0.3$ [top-left corner] to $\pi^g = 0.8$ [bottom-left corner]. The ex-ante welfare maximization problem is solved numerically as described in Footnote 9. For each panel, W is an average of 100000 simulations of a 100 periods economy using, at first, grid of τ made up of 20 equidistant points in the interval $[0, \tau']$. Therefore, the function is interpolated using a finer grid of 50 points located in the same interval. Parameters: $A = 4$, $\theta^h = 0.4$, $\theta^l = 0.3$, $\beta = 0.97$.

probability π^g since all the future fiscal withdrawals in (3.33) are discounted by the stochastic discount factor m_t. Therefore, given the economy features (agents' preferences, firm' technology and exogenous stochastic process), there exist a unique

$$\tau^* = \arg\max TR \tag{3.38}$$

for any π^g defined in (3.36). Now, it is worth understanding what are the implication of the optimal tax rate on both the economy long-run equilibrium and the real sector. Following the discussion in Section 3.3.2, Figure 3.8 displays the values of the tax rate $\tau \in (0, \tau')$ and the true probability π giving rise to the agent 1 (blue), the agent 2 (light blue) and the path-dependent (yellow) market dominance. Subjective probabilities are coherent with the other simulated result

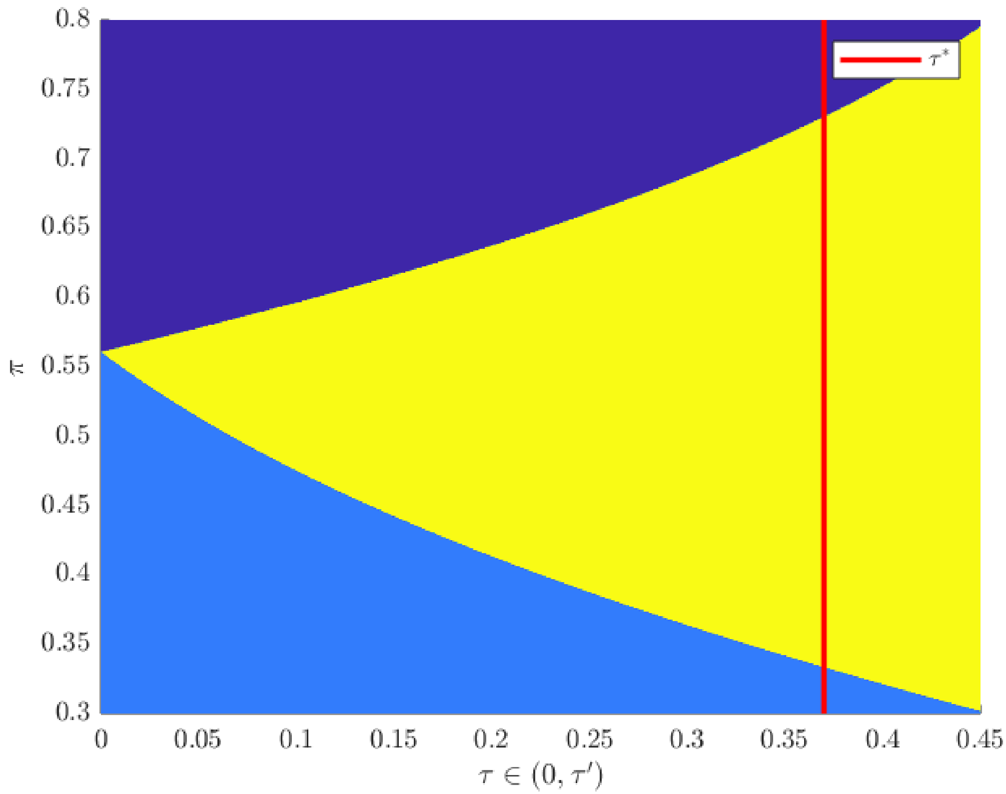


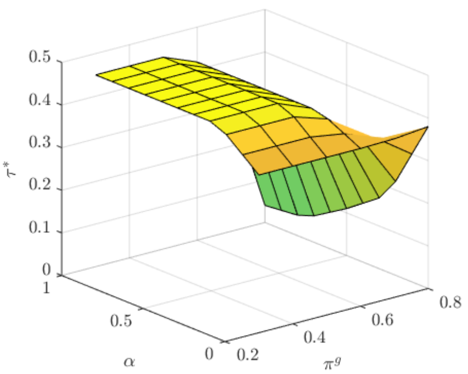
Figure 3.8: Agent 1 (blue), agent 2 (light blue) market dominance and path dependency (yellow) for different values of $\tau \in (0, \tau')$ and of $\pi \in [\pi^1, \pi^2]$. The red line is optimal tax rate (3.38), that is constant over π^g. The total revenue function is maximized numerically following the procedure described in Footnote 9 and in Figure 3.7. Parameters: $A = 4$, $\theta^h = 0.4$, $\theta^l = 0.3$, $\pi^1 = 0.8$, $\pi^2 = 0.3$, $\beta = 0.97$.

showed throughout the paper ($\pi^1 = 0.8$ and $\pi^2 = 0.3$). The red line $\tau = 0.37$ is the optimal tax rate (3.38), that is constant for any π^g. Policy intervention implies a path-dependent long-run equilibrium for a large set of true probabilities, approximately $\pi \in (0.35, 0.75)$. Under this scenario, it does not exist an invariant distribution of capital, although individual and aggregate decision rules are corrected toward the ones implied by the truth (see the marginal impact 3.31-3.32).

Finally, if agents are biased in the same direction, the MSH holds ensuring the less accurate vanish from the market. However, the FTT worsen the dominant type's decision rules in favour of the less accurate type.

3.4.3 Convex combination of welfare maximizing and revenue motives

Lastly, I consider the case $\alpha \in (0,1)$. In this context, the Government's aim is two-fold and devoted to both the outlined policy goals. As emerge from the model simulation, the larger is α the closer is the optimal tax to the one solving the welfare rather than the revenue maximization problems. Figure 3.9 displays the optimal tax under different combinations of α and π^g. Given the set of parameters adopted throughout the paper (see Figure 3.8), $\tau' = 0.45$ and it is the tax rate optimally set when both α and π^g are low enough. Consistent with the discussion in Section 3.4.1, the Government prefers shut down speculative markets whenever he thinks that low productivity states are more likely than the other. The tax drastically reduces as π^g increases. Finally, consistent with Section 3.4.2, the optimal tax is unaffected by π^g when the policy aim is purely revenue raising and $\alpha = 0$.



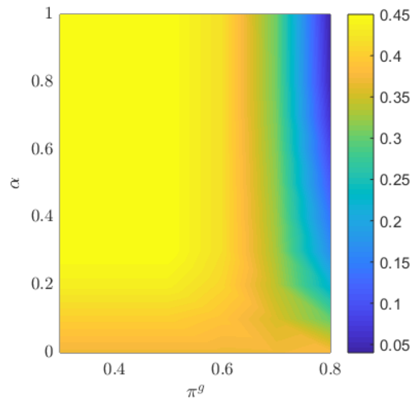


Figure 3.9: Optimal tax when the Government's objective function is a convex combination of welfare-maximizing and revenue-raising motives. The solution is displayed for different combinations of π^g and α. The Government's objective function in (3.37) is an average value of 100000 simulations of a 100 period economy with parameters as in Figure 3.8.

3.5 Conclusion

This paper entails a general equilibrium analysis of a production economy where investors disagree about the occurrences of high and low productivity states, affecting the overall production level. Apart from financing the aggregate consumption process, complete markets provides conditions for agents to speculate on such opinions divergence enhancing the volatility experienced by the real market. The growing role of speculation in financial markets creates room for a normative analysis aimed at preventing the resulting destabilizing effects in the real market.

In this vein, I study the implications of a FTT as a regulatory measure aimed at mitigating speculative trades among investors. The impact on the overall economy depends on both the size of the tax rate and the position of the truth with respect to the agents' subjective probabilities. When the truth is in the between of the investors' opinions and the FTT sufficiently high, the long-run equilibrium is path-dependent implying a failure of the MSH. The less accurate type dominates the market in those path he accumulates most of the aggregate wealth. On the one hand, this creates significant price distortion. On the other hand, there are benefits in the real sector since the FTT partially corrects the individual decision rules, making the aggregate consumption process approaches the one implied by rational expectations. However, the FTT further distorts real outcomes when the truth lies elsewhere. In this case, the MSH holds but the most accurate agent's decisions are even worse compared to the ones characterizing the economy without intervention. Thus, provided that the Government does not observe the true probability, the impact of this regulatory measure is ambiguous and impossible to predict in advance.

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Appendix A

Remark .1. Consider the following Cobb-Douglas production characterized by a multiplicative technology shock

$$y_t = \theta_t k_t^{\alpha}$$

where $\alpha < 1$. In a logarithm economies, the optimal consumption and capital investment rules are given by[10]

$$\begin{aligned} c_t &= (1-\alpha\beta)\, A\theta_t k_t^{\alpha} \\ k_{t+1} &= \alpha\beta A\theta_t k_t^{\alpha} \end{aligned} \tag{.1}$$

The consumption and capital accumulation path is free from the agent's expectations about the shock. Therefore, having biased beliefs π does not affect the economy dynamic and the optimal growth path.

To understand the mechanism which lead to this result, we briefly present a two-period model. Assume that the household faces the same problem as in (1.4)

$$\begin{aligned} &\max_{c_0, c_1(\theta^h), c_1(\theta^l)} \log(c_0) + \beta\left[\log\left(c_1\left(\theta^h\right)\right)\pi\left(\theta^h\right) + \log\left(c_1\left(\theta^d\right)\right)\pi\left(\theta^d\right)\right] \\ &\text{subject to} \quad c_0 + q_1^0(\theta^h)c_1(\theta^h) + q_1^0(\theta^l)c_1(\theta^l) = k_0 + P_0 \end{aligned} \tag{.2}$$

where variables has been already defined in Section 1.3.3.

Euler Equation:

$$c_1(\theta^h) = \frac{\beta c_0 \pi(\theta^h)}{q_1^0(\theta^h)} \tag{.3}$$

for all $\theta^h|\theta_0$. Moreover, using the agent budget constraint, we derive the state-contingent prices

$$q_1^0(\theta^h) = \frac{\beta\pi(\theta^h)(k_0+P_0)}{(1+\beta)A\theta^h k_1^{\alpha}}; \qquad q_1^0(\theta^l) \;\; = \frac{\beta\pi(\theta^l)(k_0+P_0)}{(1+\beta)A\theta^l k_1^{\alpha}} \tag{.4}$$

At $t = 0$ a perfectly competitive firm demands capital

$$P_0 = \sum_{\theta^h} q_1^0(\theta^h)\left(\alpha\theta^h k_1^{\alpha-1}\right) - k_1 \tag{.5}$$

F.O.C wrt k_1:

$$q_1^0(\theta^h)\alpha\theta^h k_1^{\alpha-1} + q_1^0(\theta^l)\alpha\theta^l k_1^{\alpha-1} = 1 \tag{.6}$$

Now, replace the state-prices in (.4)

$$\frac{\beta\pi(\theta^h)(k_0+P_0)}{(1+\beta)A\theta^h k_1^{\alpha}}\alpha\theta^h k_1^{\alpha-1} + \frac{\beta\pi(\theta^l)(k_0+P_0)}{(1+\beta)A\theta^l k_1^{\alpha}}\alpha\theta^l k_1^{\alpha-1} = 1$$

both θ^h and θ^l cancel out making the competitive capital demand be free of the expectation term.

[10] See Ljungqvist and Sargent (2004)

Proof of Proposition 1.1. We first derive the economy consumption and investment optimal decision rules as defined in (1.7). Guess the following consumption and capital investment policy rules

$$\begin{aligned} c_t &= h_0 A k_t^{\theta_t} \\ k_{t+1} &= (1-h_0) A k_t^{\theta_t} \end{aligned} \tag{.7}$$

for any constant $h_0 \in (0,1)$. Backward iterate the logarithm of the guessed consumption policy function up to the initial period

$$\log(c_t) = \log(1-h_0) + (1+\theta_t+...+\theta_t...\theta_0)\log(A) + (\theta_t+...+\theta_t...\theta_0)\log(h_0) + \theta_t\theta_{t-1}....\theta_0\log(k_0) \tag{.8}$$

Define the agent's value function

$$V^0(k_0,\theta_0) = \mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t \log(c_t)\right]$$

replace (.8)

$$\begin{aligned} V^0(k_0,\theta_0) &= \mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t\left(\log(1-h_0) + (1+\theta_t+...+\theta_t...\theta_0)\log(A) + \right.\right. \\ &\quad \left.\left. + (\theta_t+...+\theta_t...\theta_0)\log(h_0) + \theta_t\theta_{t-1}....\theta_0\log(k_0)\right)\right] \\ &= \mathbb{E}^i\left[\sum_{t=0}^{\infty}\beta^t\left(H_0 + \theta_t\theta_{t-1}....\theta_0\log(k_0)\right)\right] \\ &= \frac{H_0}{1-\beta} + \frac{\mathbb{E}[\theta_1|\theta_0]\log(k_0)}{1-\beta\mathbb{E}[\theta_1|\theta_0]} \end{aligned} \tag{.9}$$

where H_0 is a constant that does not depend on the state variables. Next, choose a policy h_1 that maximizes the following dynamic problem

$$\begin{aligned} &\max_{k'}\left\{\log\left(Ak^\theta - k'\right) + \beta V^0(k',\theta')\right\} \\ &\max_{k'}\left\{\log\left(Ak^\theta - k'\right) + \beta\left[\frac{H_0}{1-\beta} + \frac{\mathbb{E}[\theta'|\theta]\log(k')}{1-\beta\mathbb{E}[\theta'|\theta]}\right]\right\} \end{aligned} \tag{.10}$$

F.O.C wrt k':

$$k' = \beta\mathbb{E}[\theta'|\theta] A k^\theta$$

where $h_1 = \beta\mathbb{E}[\theta'|\theta]$. Plug the derived policy function $k' = \beta\mathbb{E}[\theta'|\theta] A k^\theta$ into (.8) to derive $V^1(k_0,\theta_0)$. Outline the recursive problem as in (.10) using the updated value function and verify that the maximizer policy $h_2 \equiv h_1$, so that the algorithm has converged.

Having derived the optimal decisions rules, we prove the properties stated in Proposition 1.1. Using the first derivative of $H(k,\theta) = \beta\mathbb{E}[\theta]Ak^\theta$, we figure out whether the policy function is increasing or decreasing in both its arguments

$$H_k(k,\theta) = \beta\mathbb{E}[\theta]A\theta k^{\theta-1}$$

$$H_\theta(k,\theta) = \beta\mathbb{E}[\theta]Ak^\theta\log(k)$$

the first is always positive for any k and θ, while the second is increasing when $k>1$ and decreasing when $k\in(0,1)$.

Moreover, $H(k,\theta)$ is continuous on the positive real line since

$$\lim_{k\to k'} H(k,\theta) = H(k',\theta)$$

for any $k' \in \mathbb{R}_+$. □

Proof of Proposition1.2. We outline the existence of a stationary distribution for capital in the case $\beta\mathbb{E}[\theta]A > 1$ (configuration A in Figure 1.2). The same procedure may be used also for the remaining cases.

To state the validity of the Proposition we need to show that $[k_m, k_M]$ is the smallest $\pi-$ invariant set as characterized in Definition 1.2 and that once entered, the process remains there with probability one, regardless the position of the initial state. To do so, we consider the three possible situations: $k_0 \in [k_m, k_M]$, $k_0 \in (0, k_m)$ and $k_0 \in (k_m, \infty)$.

First, assume that $k_0 \in [k_m, k_M]$, we have to prove that $H^n(k_0, \theta) \in [k_m, k_M]$ where

$$H^n(k_0, \theta^n) = H(H(...(H(k_0, \theta_0), \theta^h)..., \theta_n)$$

is the n-th iteration of the investment function $H(k_0, \theta)$, with $n \geq 0$.

Since $H(k_0, \theta)$ is increasing in both the arguments, for $k \geq k_m$ we have that $H(k_0, \theta) \geq k_m$ and specifically

$$H(k_0, \theta) > H_m(k) > H_m(k_m) = k_m$$

Similarly, for $k \leq k_M$ we have that $H(k_0, \theta) \leq k_M$.

Hence

$$H^n(k_0, \theta) \geq k_m \quad \text{for} \quad k \geq k_m; \qquad H^n(k_0, \theta) \leq k_M \quad \text{for} \quad k \leq k_M$$

The "splitting condition" established by Dubins and Freedman holds in the space $[k_m, k_M]$ if for any $n' \geq 0$ there exists a $z \in [k_m, k_M]$ such that

$$\pi^{n'}\{\theta^{n'}|H^{n'}(k, \theta^{n'}) \leq z\} \geq 0;$$

$$\pi^{n'}\{\theta^{n'}|H^{n'}(k, \theta^{n'}) \geq z\} \geq 0;$$

Now, we know that

$$\lim_{n\to\infty} H_m^n(k_M) = (\beta\mathbb{E}[\theta]A)^{\frac{1-(\theta^d)^{n+1}}{1-\theta^l}} k_M^{(\theta^l)^{n+1}} = k_m$$

and

$$\lim_{n\to\infty} H_M^n(k_m) = k_M$$

Therefore, there must exists a $n' \geq 0$ and a $z \in [k_m, k_M]$ such that

$$H_m^{n'}(k_M) < z < H_M^{n'}(k_m)$$

Second, consider the case with an initial state $k_0 \in (0, k_m)$. We set a point $(k_m - m)$ such that $(k_m - m) > 1$ in order to prove that

$$\lim \inf_{n\to\infty} H^n(k, \theta) > k_m$$

and that the system enters in the recurrent set.

By construction, $H(k, \theta) \geq H_m(k) > (k_m - m)$ and, since we are studying the inf $H^n(k, \theta)$ we choose the realization of θ which implies the minimum distance between the transition function $H(k, \theta)$ and its minimum envelope $H_m(k)$.

$$\begin{aligned} \min\big(H(k, \theta) - H_m(k), \quad &: \theta \in [\theta^l + \eta, \theta^h], k \in (0, k_m - \epsilon)\big) \\ &= \beta\mathbb{E}[\theta]A(k_m - \epsilon)^{\theta^l}\left((k_m - \epsilon)^\eta - 1\right) \end{aligned} \tag{.11}$$

where $\epsilon > 0$ such that $(k_m - \epsilon) > 1$ and $\eta = \frac{\theta^h - \theta^l}{2}$. Since for $n \to \infty$ the $\pi\left(\{\theta^n \in [\theta^l + \eta, \theta^h]\}\right) = 1$, then

$$\lim \inf_{n\to\infty} H^n(k, \theta) > k_m - m + \eta$$

setting an $m < \eta$, the system will exceed k_m with probability one.

Using the same procedure, we can prove that for $k \in (k_M, \infty)$ the following hold

$$\lim \sup_{n\to\infty} H^n(k,\theta) < k_M$$

Dubins and Freedman show that, provided that this condition holds, there exists a unique invariant distribution F_t associated to the measure μ_t such that $F_t(k) = P(k_t \leq k) = \mu_t([0,k])$ is the distribution function of the $\pi-$invariant measure μ_t and

$$F_t(k) \to F(k)$$

on (k_m, k_M). Moreover, since $\mathbb{R}_+ - (k_m, k_M)$ is transient and (k_m, k_M) is the smallest $\pi-$invariant set on $\mathbb{R}_+$, then

$$F_t(k) \to F(k)$$

on $\mathbb{R}_+$. The latter statement ensure the independence from initial condition of the unique invariant distribution $F(k)$. □

Proof of Proposition 1.3. Conditional on the same k_t, the expected value of k_{t+1} in the representative agent i economy is given by

$$\mathbb{E}[k^i_{t+1}|k_t] = \beta A\left(\pi(\theta^h)\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^h]k_t^{\theta^h} + \pi(\theta^l)\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^l]k_t^{\theta^l}\right) \tag{.12}$$

Now, suppose the above be lower in the optimist rather than the benchmark economy

$$\begin{gathered}\mathbb{E}[k^o_{t+1}|k_t] < \mathbb{E}[k_{t+1}|k_t]\\ \pi(\theta^h)k_t^{\theta^h}\left(\mathbb{E}^o[\theta_{t+1}|\theta^h] - \mathbb{E}[\theta_{t+1}|\theta^h]\right) + \pi(\theta^l)k_t^{\theta^l}\left(\mathbb{E}^o[\theta_{t+1}|\theta^l] - \mathbb{E}[\theta_{t+1}|\theta^l]\right) < 0\\ b(\theta^h - \theta^l)\left(k_t^{\theta^h} + k_t^{\theta^l}\right) < 0\end{gathered} \tag{.13}$$

that is a contradiction $\forall k \in (0,\infty)$. Using the same procedure, we find the conditional average of capital be decreasing in the level of pessimism.

The conditional volatility of k_{t+1} is given by

$$Var[k^i_{t+1}|k_t] = \mathbb{E}\left[\left(k^i_{t+1}|k_t\right)^2\right] - \left(\mathbb{E}\left[k^i_{t+1}|k_t\right]\right)^2$$

After some algebra, we end up with the following

$$Var[k^i_{t+1}|k_t] = \begin{cases}(\beta A)^2\,\pi(\theta^h)\pi(\theta^l)\left(\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^h]k_t^{\theta^h} - \mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^l]k_t^{\theta^l}\right)^2, \text{if } k_t > 1\ \forall t\\ (\beta A)^2\,\pi(\theta^h)\pi(\theta^l)\left(\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^l]k_t^{\theta^l} - \mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^h]k_t^{\theta^h}\right)^2, \text{if } k_t \in (0,1)\ \forall t\end{cases} \tag{.14}$$

Clearly, the two branches are equivalent since

$$\left(\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^h]k_t^{\theta^h} - \mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^l]k_t^{\theta^l}\right)^2 = \left(\mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^l]k_t^{\theta^l} - \mathbb{E}^i_t[\theta_{t+1}|\theta_t = \theta^h]k_t^{\theta^h}\right)^2$$

however, we use the first for $k_t > 1$ and the second for $k_t \in (0,1)$ since we need the term inside the bracket to be positive.

Consider first the case $k_t > 1$ and assume

$$Var[k^o_{t+1}|k_t] < Var[k_{t+1}|k_t]$$

by substituting the subjective transition probability matrices into the first branch of (.14)

$$
\begin{aligned}
k_t^{\theta^h}\left(\mathbb{E}^o[\theta_{t+1}|\theta^h]-\mathbb{E}[\theta_{t+1}|\theta^h]\right) &< k_t^{\theta^l}\left(\mathbb{E}^o[\theta_{t+1}|\theta^l]-\mathbb{E}[\theta_{t+1}|\theta^l]\right)\\
k_t^{\theta^h} &< k_t^{\theta^l}
\end{aligned} \tag{.15}
$$

which is a contradiction for $k_t > 1$. Second, use the second branch of (.14) to outline the case where $k_t \in (0,1)$ for any t

$$
\begin{aligned}
k_t^{\theta^l}\left(\mathbb{E}^o[\theta_{t+1}|\theta^l]-\mathbb{E}[\theta_{t+1}|\theta^l]\right) &< k_t^{\theta^h}\left(\mathbb{E}^o[\theta_{t+1}|\theta^h]-\mathbb{E}[\theta_{t+1}|\theta^h]\right)\\
k_t^{\theta^l} &< k_t^{\theta^h}
\end{aligned} \tag{.16}
$$

that is, again, a contradiction for $k_t \in (0,1)$.

Using the same procedure, we can easily show that

$$
Var[k_{t+1}^p|k_t] < Var[k_{t+1}|k_t] < Var[k_{t+1}^o|k_t]
$$

$$
Var[k_{t+1}^T|k_t] > Var[k_{t+1}^o|k_t], \quad k_t > 1 \qquad Var[k_{t+1}^o|k_t] > Var[k_{t+1}^T|k_t], \quad k_t \in (0,1)
$$

Lastly, we compare the capital volatility of the trend-follower with the one characterizing the benchmark

$$
Var[k_{t+1}^T|k_t] > Var[k_{t+1}|k_t]
$$

again, we need to make a distinction between the cases $k_t > 1$ and $k_t \in (0,1)$ in any t. Consider the first, we use the first branch of (.14)

$$
\begin{aligned}
\left(\mathbb{E}_t^T[\theta_{t+1}|\theta_t=\theta^h]k_t^{\theta^h}-\mathbb{E}_t^T[\theta_{t+1}|\theta_t=\theta^l]k_t^{\theta^l}\right)^2 &> \left(\mathbb{E}_t[\theta_{t+1}|\theta_t=\theta^h]k_t^{\theta^h}-\mathbb{E}_t[\theta_{t+1}|\theta_t=\theta^l]k_t^{\theta^l}\right)^2\\
k_t^{\theta^h}\left(\mathbb{E}_t^T[\theta_{t+1}|\theta_t=\theta^h]-\mathbb{E}_t[\theta_{t+1}|\theta_t=\theta^h]\right) &> k_t^{\theta^l}\left(\mathbb{E}_t^T[\theta_{t+1}|\theta_t=\theta^l]k_t^{\theta^l}-\mathbb{E}_t[\theta_{t+1}|\theta_t=\theta^l]k_t^{\theta^l}\right)
\end{aligned} \tag{.17}
$$

we find it holds for every $k_t > 1$. By contrast, $Var[k_{t+1}^T|k_t] < Var[k_{t+1}|k_t]$ when $k_t \in (0,1)$. □

Proof of Lemma 1.1. First, consider the case where the optimistic and pessimistic economy have both a capital recurrent set entirely placed above than one. The recurrent sets are disjoints provided that

$$
k_m^o > k_M^p \tag{.18}
$$

the lower extreme of the recurrent set of the optimist is greater than the upper extreme of the set of the pessimist. Moreover, it derives from (1.9)

$$
k_m^o = (\beta A\mathbb{E}^o[\theta])^{\frac{1}{1-\theta^l}} \quad k_M^p = (\beta A\mathbb{E}^p[\theta])^{\frac{1}{1-\theta^h}}
$$

as a consequence, the two economies' capital recurrent sets are disjoints if

$$
(\beta A\mathbb{E}^o[\theta])^{\frac{1}{1-\theta^l}} > (\beta A\mathbb{E}^p[\theta])^{\frac{1}{1-\theta^h}}
$$

The parametric area for which the above holds is displayed in Figure 1.5.

Thus, we turn to the case where the capital recurrent sets are located below one in both the optimistic and pessimistic economies. As showed in Figure 1.3, the upper and lower extreme of the recurrent set are, in this case, reversed since θ^l represents the "good" state. Therefore

$$
k_m^o = (\beta A\mathbb{E}^o[\theta])^{\frac{1}{1-\theta^h}}; \quad k_M^p = (\beta A\mathbb{E}^p[\theta])^{\frac{1}{1-\theta^l}}
$$

the two sets are disjoint if

$$
(\beta A\mathbb{E}^o[\theta])^{\frac{1}{1-\theta^h}} > (\beta A\mathbb{E}^p[\theta])^{\frac{1}{1-\theta^l}}
$$

Consider now the unconditional first and second moments of capital, regarding the second part of the Proposition. Define

$$\mathbb{E}[k] = \beta\mathbb{E}^i[\theta]A\mathbb{E}[k^\theta]$$

the unconditional mean of capital. Now, if the conditions above are satisfied, the capital recurrent set in the optimistic and pessimistic economy are disjoint, with $k_m^o > k_M^p$. As a consequence, it is easy to verity that the unconditional mean of capital is greater (lower) in the optimistic (pessimistic) economy compared to the benchmark, $\mathbb{E}^p[k^\theta] < \mathbb{E}[k^\theta] < \mathbb{E}^o[k^\theta]$.

Regarding the unconditional variance, we need to make a distinction between the cases where k_t moves above or below one for all t. In the first case, the variance is increasing in expectations. The variance of k is given by

$$Var[k] = (\beta\mathbb{E}[\theta]A)^2 \left(\mathbb{E}[k^{2\theta}] - \left(\mathbb{E}[k^\theta]\right)^2\right)$$

and when $k_t > 1$ for all t

$$\frac{\partial\left(\mathbb{E}[k^{2\theta}] - \left(\mathbb{E}[k^\theta]\right)^2\right)}{\partial k} > 0$$

meaning that $Var[k]$ is monotonically increasing in k and, as a consequence, it is increasing in the degree of optimism.

Instead, when $k_t \in (0,1)$ for all t, the function $\left(\mathbb{E}[k^{2\theta}] - \left(\mathbb{E}[k^\theta]\right)^2\right)$ is not monotonic in k. Thus, the unconditional variance of capital is increasing (decreasing) in the degree of optimism (pessimism) provided that the following inequality holds

$$(\mathbb{E}^o[\theta])^2 \left((k_o)^{\theta^h-\theta^l} + (k_o)^{\theta^l-\theta^h}\right) + (\mathbb{E}[\theta])^2 \left((k_p)^{\theta^h-\theta^l} + (k_p)^{\theta^l-\theta^h}\right) > 2\left((\mathbb{E}^o[\theta])^2 - (\mathbb{E}^p[\theta])^2\right)$$

where $k_o \in [k_m^o, k_M^o]$ and $k_p \in [k_m^p, k_M^p]$. □

Proof of Competitive Equilibrium in Section 1.4. Define the Bellman equation of the individual problem in (1.15) as

$$V^t\left(a_t^i\right) = \max_{c_t^i, \left(a_{t+1}^i(\theta_{t+1}|\theta_t)\right)_{\theta_{t+1}\in\{\theta^h,\theta^l\}}} \log c_t^i + \beta\mathbb{E}^i\left[V^{t+1}\left(a_{t+1}^i\right)\right] \tag{.19}$$

where, using the agent budget constraint in (1.15)

$$c_t^i = a_t^i + \frac{P_t}{2} - \sum_{\theta_{t+1}|\theta_t} q_{t+1}^t\left(\theta_{t+1}|\theta_t\right) a_{t+1}^i\left(\theta_{t+1}|\theta_t\right). \tag{.20}$$

Start guessing $V^{t+1} = 0$ and and solve a one-period problem: this involves choosing the vector $\left(c_t^i, \left(a_{t+1}^i(\theta_{t+1}|\theta_t)\right)_{\theta_{t+1}\in\{\theta^h,\theta^l\}}\right)$ that maximize (.3) subject to (.20) under the assumption that $\left(a_{t+1}^i(\theta_{t+1}|\theta_t)\right)_{\theta_{t+1}\in\{\theta^h,\theta^l\}} = 0$. The trivial result is given by $c_t^i = a_t^i$ and, therefore

$$V^t\left(a_t^i\right) = \log a_t^i.$$

backward iterating, find the vector $\left(c_{t-1}^i, \left(a_t^i(\theta_t|\theta_{t-1})\right)_{\theta_t\in\{\theta^h,\theta^l\}}\right)$ that maximizes

$$V^{t-1}\left(a_{t-1}^i\right) = \max_{c_{t-1}^i(\theta^{t-1}), \left(a_t^i(\theta_t|\theta_{t-1})\right)_{\theta_t\in\{\theta^h,\theta^l\}}} \log c_{t-1}^i + \beta\mathbb{E}^i\left[V^t\left(a_t^i\right)\right]$$

Keeping going and using the properties of geometric series, we find the optimal decision rules

$$\begin{aligned} c_t^i\left(\theta^t\right) &= (1-\beta)\left[a_t^i + P_t^{ei}\right] \\ a_{t+1}^i\left(\theta^h|\theta_t\right) &= \beta\frac{\pi^i\left(\theta^h|\theta_t\right)\left[a_t^i + P_t^{ei}\right]}{q_{t+1}^t\left(\theta^h|\theta_t\right)} \\ a_{t+1}^i\left(\theta^l|\theta_t\right) &= \beta\frac{\pi^i\left(\theta^l|\theta_t\right)\left[a_t^i + P_t^{ei}\right]}{q_{t+1}^t\left(\theta^l|\theta_t\right)} \end{aligned} \tag{.21}$$

where $P_t^{ei} = \sum_{\tau=t}^{\infty}\sum_{\theta^\tau\in\Theta^\tau}\left[\prod_{s=t+1}^{\tau} q_s^{s-1}\left(\theta_s|\theta^{s-1}\right)\frac{P\left(\theta_s|\theta^{s-1}\right)}{2}\right]$ is the discounted sum of all the agent i future profit, for all $\tau > t, \theta^\tau|\theta^t$. Since agents are not endowed with a system of second-order beliefs, they compute P^{ei} as if they were alone in the economy, using as state-prices their own MRS showed in (1.5). Replacing (1.3) into the firm's profit function, we derive the node θ^τ maximized profit

$$P_\tau = \beta\sum_{\theta_{\tau+1}|\theta_\tau}\left[\frac{c_\tau^i\pi^i\left(\theta^{\tau+1}|\theta_\tau\right)}{c_{t+1}^i}A\left(1-\theta_{t+1}\right)k_{t+1}^{\theta_{\tau+1}}\right]$$

Since the individual consumption is always a fixed fraction of wealth in log-economies, the above becomes

$$P_\tau = \beta\left(1-\mathbb{E}^i\left[\theta_{\tau+1}|\theta_\tau\right]\right)Ak_\tau^{\theta_\tau}$$

implying

$$P_t^{ei} = \frac{P_t}{1-\beta}$$

Replace the above in (.4), we obtain the individual policy functions (1.16). Finally, the aggregate capital investment rule (1.17) derives from (1.16) and the feasibility constraint of the economy displayed in Definition 1.1. □

Proof of Proposition 1.4. Replace the law of motion of the individual wealth in (1.16) in the conditional drift

$$\begin{aligned} \mathbb{E}\left[\xi_{t+1}\left(\theta_{t+1}|\theta_t\right)\right] &= \pi\left(\theta^h|\theta_t\right)\log\left(\frac{\pi^o\left(\theta^h|\theta_t\right)}{\pi^p\left(\theta^h|\theta_t\right)}\right) + \pi\left(\theta^l|\theta_t\right)\log\left(\frac{\pi^o\left(\theta^l|\theta_t\right)}{\pi^p\left(\theta^l|\theta_t\right)}\right) \\ &= \pi\left(\theta^h|\theta_t\right)\log\left(\frac{\pi^o\left(\theta^h|\theta_t\right)}{\pi^p\left(\theta^h|\theta_t\right)}\frac{\pi\left(\theta^h|\theta_t\right)}{\pi\left(\theta^h|\theta_t\right)}\right) + \pi\left(\theta^l|\theta_t\right)\log\left(\frac{\pi^o\left(\theta^l|\theta_t\right)}{\pi^p\left(\theta^l|\theta_t\right)}\frac{\pi\left(\theta^l|\theta_t\right)}{\pi\left(\theta^l|\theta_t\right)}\right) \\ &= D_{KL}^p\left(\pi^i\left(\theta_{t+1}|\theta_t\right)||\pi\left(\theta_{t+1}|\theta_t\right)\right) - D_{KL}^o\left(\pi^i\left(\theta_{t+1}|\theta_t\right)||\pi\left(\theta_{t+1}|\theta_t\right)\right) \\ &= \mathcal{E}_{t+1}^p(\theta_{t+1}|\theta_t) - \mathcal{E}_{t+1}^o(\theta_{t+1}|\theta_t) \end{aligned} \tag{.22}$$

Using the S.L.L.N.,

$$\mathbb{P}\left(\lim_{T\to\infty}\sum_{t=0}^{T}\frac{\mathcal{E}_{t+1}^p(\theta_{t+1}|\theta_t) - \mathcal{E}_{t+1}^o(\theta_{t+1}|\theta_t)}{T}\to\xi\right) = 1$$

by the ergodic theorem

$$\mathcal{E}^i = \pi\left(\theta^h\right)\mathcal{E}^i\left(\theta^h\right) + \pi\left(\theta^l\right)\mathcal{E}^i\left(\theta^l\right)$$

for $i\in\{o,p\}$, and

$$\xi = \mathcal{E}^p - \mathcal{E}^o.$$

□

Proof of Proposition (1.5). Consider the approximated negative growth rate of a^B in (.23)

$$\begin{aligned} g_{a^B} &\approx g_{a^{B*}} + g'_{a^{B*}}(a^B - a^{B*}) \\ &\approx 0 + \left.\frac{\partial g_{a_t^B}}{\partial a_t^B}\right|_{a_t^B = a_t^{B*} = 0} \left(a^B - 0\right) \\ &\approx -\left(\frac{\pi^B}{\pi^b}\right)^2 \frac{a^B}{y^*} \end{aligned} \tag{.23}$$

Taking logs and adding and subtracting the true probability measure π, we get equation (1.21). □

Proof of Proposition 1.6. The Bellman equation of the individual problem (1.24) is given by

$$V^t\left(s_t^i, \theta_t\right) = \max_{c_t^i, s_{t+1}^i} \log c_t^i + \beta \mathbb{E}^i\left[V^{t+1}\left(s_{t+1}^i, \theta_{t+1}\right)\right]$$

Iterating the value function as we showed for the dynamic problem (.3), we find that consumption is equivalent to (.4) while the fraction of the stock owned is given by

$$s_{t+1}^i = \frac{\beta s_t^i y_t}{p_t}$$

with $s_0^i = \frac{k_0^i}{\sum_{i \in \{1,2\}} k_0^i}$. Since the stock has unitary supply, therefore

$$p_t = \beta y_t.$$

□

Appendix B

Appendix

Proof of competitive equilibrium without FTT (Section 2.2). The agent i optimization problem in (2.1) is solved using the method of Lagrangian multipliers. The Lagrangian function is given by

$$L^i = \log\left(a^i_{T,1}\right)\pi^i + \log\left(a^i_{T,2}\right)\left(1-\pi^i\right) + \sum_{t=0,s^t}^{T-1} \lambda_t\left[a^i_{t,s^t} - q^t_{t+11}a^i_{t+11} - q^t_{t+12}a^i_{t+12}\right] \tag{.1}$$

Starting from the final period T, we derive the agent's F.O.C wrt to $a^i_{T,1}$ and $a^i_{T,2}$

$$\begin{aligned}\frac{\pi^i}{a^i_{T,1}} &= \lambda^i_{T-1}q^{T-1}_{T,1}\\ \frac{\left(1-\pi^i\right)}{a^i_{T,2}} &= \lambda^i_{T-1}q^{T-1}_{T,2}\end{aligned} \tag{.2}$$

Equating the agents' MRS with the assets price ratio resulting from the node s^T budget constraint, we derive the last period asset demands

$$\begin{aligned}a^i_{T,1} &= \frac{\pi^i a^i_{T-1,s^{T-1}}}{q^{T-1}_{T,1}}\\ a^i_{T,2} &= \frac{\left(1-\pi^i\right)a^i_{T-1,s^{T-1}}}{q^{T-1}_{T,2}}\end{aligned} \tag{.3}$$

and, according to (.2), the node s^{T-1} Lagrangian multiplier

$$\lambda^i_{T-1} = \frac{1}{a^i_{T-1,s^{T-1}}}$$

The above is replaced in the previous period F.O.C. wrt $a^i_{T-1,1}$ and $a^i_{T-1,2}$

$$\begin{aligned}\lambda^i_{T-1}\pi^i &= \lambda^i_{T-2}q^{T-2}_{T-1,1}\\ \lambda^i_{T-1}\left(1-\pi^i\right) &= \lambda^i_{T-2}q^{T-2}_{T-1,2}\end{aligned} \tag{.4}$$

Asset demands are equivalently derived equating the agent's MRS with the price ratio that results from the budget constraint

$$\begin{aligned}a^i_{T-1,1} &= \frac{\pi^i a^i_{T-2,s^{T-2}}}{q^{T-2}_{t-1,1}}\\ a^i_{T-1,2} &= \frac{\left(1-\pi^i\right)a^i_{T-2,s^{T-2}}}{q^{T-2}_{T-1,2}}\end{aligned} \tag{.5}$$

Therefore, the agent i asset demands for any $t \in [0, T-1]$ are given by

$$\begin{aligned} a^1_{t+1,1} &= \frac{\pi^i a^i_{t,s^t}}{q^t_{t+1,1}} \\ a^1_{t+1,2} &= \frac{\left(1-\pi^i\right) a^i_{t,s^t}}{q^t_{t+1,2}} \end{aligned} \tag{.6}$$

□

Proof of Corollary 2.1. We first show that

$$1 - \delta^i_t = \frac{T^i_t}{a^i_{t,s^t}}$$

considering the case where the agent buys asset 1 and sells asset 2 (first row of 2.12).

$$1 - \delta^i_t = \frac{\tau \left[q^t_{t+11}\left(a^i_{t+11} - a^i_{t,s^t}\right) + q^t_{t+12}\left(a^i_{t,s^t} - a^i_{t+12}\right)\right]}{a^i_{t,s^t}} \tag{.7}$$

Replacing (2.8) and (2.12), after some algebra, (.7) reduces to an identity

$$\begin{aligned} 1 - \left[\frac{1-\tau\left(2\pi^i - 1\right)}{1-\tau^2}\right]\left(2\tau q^t_{t+11} + 1 - \tau\right) &= \tau\left[\frac{1-\tau\left(2\pi^i - 1\right)}{1-\tau^2}\left(2\tau q^t_{t+11} + 1 - \tau\right) + 1 - 2q^t_{t+11}\right] \\ 1 - \left[\frac{\pi^i}{1+\tau}(1+\tau) + \frac{\left(1-\pi^i\right)}{1-\tau}(1-\tau)\right]\left(2\tau q^t_{t+11} + 1 - \tau\right) &= \tau\left[1 - 2q^t_{t+11}\right] \\ 1 - \left(2\tau q^t_{t+11} + 1 - \tau\right) &= \tau\left[1 - 2q^t_{t+11}\right] \\ \tau\left(1 - 2q^t_{t+11}\right) &= \tau\left[1 - 2q^t_{t+11}\right] \end{aligned}$$

Using the same procedure, one can easily show that $1 - \delta^i_t = \frac{\tau\left[q^t_{t+11}\left(a^i_{t,s^t} - a^i_{t+11}\right) + q^t_{t+12}\left(a^i_{t+12} - a^i_{t,s^t}\right)\right]}{a^i_{t,s^t}}$ holds as well. Therefore, we prove the following limits

$$\lim_{\frac{a^1_t}{a^2_t}\to\infty} 1 - \delta^1_t = 0 \qquad \lim_{\frac{a^1_t}{a^2_t}\to\infty} 1 - \delta^2_t > 0$$

$$\lim_{\frac{a^1_t}{a^2_t}\to 0} 1 - \delta^2_t = 0 \qquad \lim_{\frac{a^1_t}{a^2_t}\to 0} 1 - \delta^2_t > 0$$

where the following holds, under the assumption that agent 1 buys asset 1 and sell asset 2 [11]

$$\begin{aligned} \lim_{\frac{a^1_t}{a^2_t}\to\infty} q^t_{t+11} = q^1_{buy} &= \frac{(1-\tau)\pi^1}{1-\tau\left(\pi^1 - \left(1-\pi^1\right)\right)} \\ \lim_{\frac{a^1_t}{a^2_t}\to 0} q^t_{t+11} = q^2_{sell} &= \frac{(1+\tau)\pi^2}{1+\tau\left(2\pi^2 - 1\right)} \end{aligned} \tag{.8}$$

First, consider the limits

$$\lim_{\frac{a^1_t}{a^2_t}\to\infty} \delta^1_t = 1 \qquad \lim_{\frac{a^1_t}{a^2_t}\to 0} \delta^2_t = 1$$

[11]According to Definition 3.2 this is actually the only case when trading takes place in this economy. However, it can be equivalently shown that the above holds even when agent 1 buys asset 2 and sells asset 1. In this case, replace $\lim_{\frac{a^1_t}{a^2_t}\to\infty} q^t_{t+1,1} = q^1_{sell}$ and $\lim_{\frac{a^1_t}{a^2_t}\to 0} q^t_{t+1,1} = q^2_{buy}$ in (.8).

by replacing (2.12), it easy to see that the above limits are always true. Second, consider the limits

$$\begin{aligned} \lim_{\frac{a_t^1}{a_t^2}\to\infty} \delta_t^2 &< 1 \\ q_{buy}^1 > \frac{\pi^2\left(1+\tau\right)}{\left(1+\tau\left(2\pi^2-1\right)\right)} &= q_{sell}^2 \end{aligned} \tag{.9}$$

and

$$\begin{aligned} \lim_{\frac{a_t^1}{a_t^2}\to 0} \delta_t^1 &< 1 \\ q_{sell}^2 < \frac{\pi^1\left(1-\tau\right)}{1-\tau\left(2\pi^1-1\right)} &= q_{buy}^1 \end{aligned} \tag{.10}$$

that holds for any $\tau \in (0, \tau')$ as defined in Definition 2.2. □

Proof of Proposition 2.1. The Lagrangian of the investors' maximization problem in (2.1) subject to the budget constraints (.20) are given by

$$\begin{aligned} L^i = \log\left(a_{T,1}^i\right)\pi^i + \log\left(a_{T,2}^i\right)\left(1-\pi^i\right) + \sum_{t=0,s^t}^{T-1} \lambda_{t,s^t}^i \left[a_{t,s^t}^i - \overline{q}_{t+1,1}^t a_{t+1,1}^i - \underline{q}_{t+1,2}^t a_{t+1,2}^i\right] + \\ + \sum_{t=0,s^t}^{T-1} \mu_t^i \left[a_t^1 - \underline{q}_{t+1,1}^t a_{t+1,1}^i - \overline{q}_{t+1,2}^t a_{t+1,2}^i\right] \end{aligned} \tag{.11}$$

where the sequence of multipliers $\{\lambda_t^i\}_{t,s^t}$ and $\{\mu_t^i\}_{t,s^t}$ are all greater or equal than zero depending on whether the first or the second segment of the budget constraint (.20) binds.

Last period T K.K.T. conditions

$$\begin{aligned} \frac{\pi^i}{a_{T,1}^i} &= \lambda_{T-1}^i \overline{q}_{T,1}^{T-1} + \mu_{T-1}^i \underline{q}_{T,1}^{T-1} \\ \frac{\left(1-\pi^i\right)}{a_{T,2}^i} &= \lambda_{T-1}^i \underline{q}_{T,2}^{T-1} + \mu_{T-1}^i \overline{q}_{T,2}^{T-1} \\ \lambda_{T-1}^i\left[a_{T-1,s^{T-1}}^i - \overline{q}_{T,1}^{T-1} a_{T,1}^i - \underline{q}_{T,2}^{T-1} a_{T,2}^i\right] &= 0 \\ \mu_{T-1}^i\left[a_{T-1,s^{T-1}}^i - \underline{q}_{T,1}^{T-1} a_{T,1}^i - \overline{q}_{T,2}^{T-1} a_{T,2}^i\right] &= 0 \end{aligned} \tag{.12}$$

1. Case 1 (Agent i buys asset 1 and sells asset 2)

$$\mu^i_{T-1} = 0$$
$$\underline{q}^{T-1}_{T,1} a^i_{T,1} + \overline{q}^{T-1}_{T,2} a^i_{T,2} \leq a^i_{T-1,s^{T-1}}$$
$$\lambda^i_{T-1} \geq 0$$
$$\overline{q}^{T-1}_{T,1} a^i_{T,1} + \underline{q}^{T-1}_{T,2} a^i_{T,2} = a^i_{T-1,s^{T-1}}$$
$$\frac{\pi^i}{a^i_{T,1}} = \lambda^i_{T-1} \overline{q}^{T-1}_{T,1}$$
$$\frac{\left(1-\pi^1\right)}{a^i_{T,2}} = \lambda^i_{T-1} \underline{q}^{T-1}_{T,2}$$

sol.1

$$a^i_{T,1} = \frac{\pi^1 a^i_{T-1,s^{T-1}}}{\overline{q}^{T-1}_{T,1}}$$
$$a^i_{T,2} = \frac{\left(1-\pi^i\right) a^i_{T-1,s^{T-1}}}{\underline{q}^{T-1}_{T,2}}$$

By replacing the derived asset demands, verify the other complementary slackness

$$\underline{q}^{T-1}_{T,1} a^i_{T,1} + \overline{q}^{T-1}_{T,2} a^i_{T,2} \leq a^i_{T-1,s^{T-1}}$$
$$\frac{(1-\tau)\,\pi^i}{(1+\tau)} \frac{2\tau q^{T-1}_{T,1} + 1 - \tau}{-2\tau q^{T-1}_{T,1} + 1 + \tau} + \frac{(1+\tau)\left(1-\pi^i\right)}{(1-\tau)} \frac{-2\tau q^{T-1}_{T,2} + 1 + \tau}{2\tau q^{T-1}_{T,2} + 1 - \tau} \leq 1$$
$$\frac{(1-\tau)\,\pi^i}{(1+\tau)} \frac{2\tau q^{T-1}_{T,1} + 1 - \tau}{-2\tau q^{T-1}_{T,1} + 1 + \tau} + \frac{(1+\tau)\left(1-\pi^i\right)}{(1-\tau)} \frac{2\tau q^{T-1}_{T,1} + 1 - \tau}{-2\tau q^{T-1}_{T,1} + 1 + \tau} \leq 1$$
$$q^{T-1}_{T,1} \leq \frac{4\tau\pi^i\,(1-\tau)}{4\tau\left[1-\tau\left(2\pi^i-1\right)\right]}$$

according to (2.7), the rhs is the agent's reservation price as a buyer of asset 1. Therefore

$$q^{T-1}_{T,1} \leq q^1_{buy} \tag{.13}$$

The inequality is consistent with the assumption $\lambda^i_{T-1} \geq 0$ and $\overline{q}^{T-1}_{T,1} a^i_{T,1} + \underline{q}^{T-1}_{T,2} a^i_{T,2} = a^i_{T-1,s^{T-1}}$ stated above. In fact,

$$\begin{cases} q^{T-1}_{T,1} < q^i_{buy} & \text{if agent } i \text{ buys asset 1} \\ q^{T-1}_{T,1} = q^i_{buy} & \text{if agent } i \text{ does not trade} \end{cases}$$

2. Case 2 (Agent i sells asset 1 and buys asset 2)

$$\lambda^i_{T-1} = 0$$
$$\overline{q}^{T-1}_{T,1} a^i_{T,1} + \underline{q}^{T-1}_{T,2} a^i_{T,2} \leq a^i_{T-1,s^{T-1}}$$
$$\mu^i_{T-1} \geq 0$$
$$\underline{q}^{T-1}_{T,1} a^i_{T,1} + \overline{q}^{T-1}_{T,2} a^i_{T,2} = a^i_{T-1,s^{T-1}}$$
$$\frac{\pi^i}{q^{T-1}_{T,1-} a^i_{T,1}} = \mu^i_{T-1} > 0$$
$$q^{T-1}_{T,1-} a^i_{T,1} = \frac{\pi^i q^{T-1}_{T,2-} a^i_{T,2}}{(1-\pi^i)}$$

sol. 2

$$a^i_{T,1} = \frac{\pi^i a^i_{T-1,s^{T-1}}}{\underline{q}^{T-1}_{T,1}}$$
$$a^i_{T,2} = \frac{\left(1-\pi^i\right) a^i_{T-1,s^{T-1}}}{\overline{q}^{T-1}_{T,2}}$$

By replacing the derived asset demands, verify the other complementary slackness

$$\overline{q}^{T-1}_{T,1} a^i_{T,1} + \underline{q}^{T-1}_{T,2} a^i_{T,2} \leq a^i_{T-1,s^{T-1}}$$
$$\overline{q}^{T-1}_{T,1} \frac{\pi^i a^i_{T-1}}{\underline{q}^{T-1}_{T,1}} + \underline{q}^{T-1}_{T,2} \frac{\left(1-\pi^i\right) a^i_{T-1}}{\overline{q}^{T-1}_{T,2}} \leq a^i_{T-1,s^{T-1}} \tag{.14}$$
$$\frac{(1+\tau)\,\pi^i}{(1-\tau)} \frac{-2\tau q^{T-1}_{T1} + 1 + \tau}{2\tau q^{T-1}_{T1} + 1 - \tau} + \frac{(1-\tau)\left(1-\pi^i\right)}{(1+\tau)} \frac{2\tau q^{T-1}_{T2} + 1 - \tau}{-2\tau q^{T-1}_{T2} + 1 + \tau} \leq 1$$
$$\frac{(1+\tau)\,\pi^i}{(1-\tau)} \frac{-2\tau q^{T-1}_{T1} + 1 + \tau}{2\tau q^{T-1}_{T1} + 1 - \tau} + \frac{(1-\tau)\left(1-\pi^i\right)}{(1+\tau)} \frac{-2\tau q^{T-1}_{T1} + 1 + \tau}{2\tau q^{T-1}_{T1} + 1 - \tau} \leq 1$$
$$\frac{(1+\tau)\,\pi^i}{(1-\tau)} + \frac{(1-\tau)\left(1-\pi^i\right)}{(1+\tau)} \leq \frac{2\tau q^{T-1}_{T1} + 1 - \tau}{-2\tau q^{T-1}_{T1} + 1 + \tau}$$
$$\frac{(1+\tau)^2\,\pi^i + (1-\tau)^2\left(1-\pi^i\right)}{(1-\tau^2)} \leq \frac{2\tau q^{T-1}_{T1} + 1 - \tau}{-2\tau q^{T-1}_{T1} + 1 + \tau}$$
$$\frac{\left((1+\tau)^2\,\pi^i + (1-\tau)^2\left(1-\pi^i\right)\right)(1+\tau) - (1-\tau)\left(1-\tau^2\right)}{\left(1-\tau^2 + (1+\tau)^2\,\pi^i + (1-\tau)^2\,(1-\pi^i)\right)} \leq 2\tau q^{T-1}_{T1}$$
$$\frac{\pi^i\,(1+\tau)}{(1+\tau\,(2\pi^i-1))} \leq q^{T-1}_{T1}$$
$$q^{T-1}_{T1} \geq \frac{\pi^i\,(1+\tau)}{(1+\tau\,(2\pi^i-1))}$$
$$q^{T-1}_{T1} \geq q^1_{sell}$$

Again, the last inequality is consistent with the assumption related to Case 2. In fact,

$$\begin{cases} q^{T-1}_{T,1} > q^i_{sell} & \text{if agent } i \text{ sells asset 1} \\ q^{T-1}_{T,1} = q^i_{sell} & \text{if agent } i \text{ does not trade} \end{cases}$$

To verify that sol.1 and sol.2 are the policy functions of the dynamic optimization problem, and thus they hold in any s^t, one needs to replace sol.1 and sol.2 in λ^i_{T-1} and μ^i_{T-1}, respectively. Thereafter, replace the latter in the period $T-1$ K.K.T. conditions (as in Proof of asset demands without FTT). Since the slackness conditions hold in Case 1 (2) provided that the equilibrium price is lower (greater) than q^i_{buy} (q^i_{sell}) and reservation prices are constant over time, therefore

$$\lambda^i_{T-1} > 0 \text{ and } \mu^i_{T-1} = 0 \quad (\lambda^i_{T-1} = 0 \text{ and } \mu^i_{T-1} > 0)$$

is valid in any node s^t. As a consequence, the agents' asset demands are derived as in the Proof without FTT

$$\begin{cases} a^i_{t+1,1} = \frac{\pi^i a^i_{t,s^t}}{\overline{q}^t_{t+1,1}} \\ \\ a^i_{t+1,2} = \frac{(1-\pi^i)a^i_{t,s^t}}{\underline{q}^t_{t+1,2}} \end{cases} \text{if } q^t_{t+1,1} < q^i_{buy} \qquad \begin{cases} a^i_{t+1,1} = \frac{\pi^i a^i_{t,s^t}}{\underline{q}^t_{t+1,1}} \\ \\ a^i_{t+1,2} = \frac{(1-\pi^i)a^i_{t,s^t}}{\overline{q}^t_{t+1,2}} \end{cases} \text{if } q^t_{t+1,1} > q^i_{sell} \tag{.15}$$

Finally, if both the branches of (.20) are bindings, then $\mu^i_{T-1} = 0$ and $\lambda^i_{T-1} = 0$ and

$$\begin{cases} a^i_{t+1,1} = \frac{\pi^i a^i_{t,s^t}}{q^t_{t+1,1}} \\ a^i_{t+1,2} = \frac{(1-\pi^i)a^i_{t,s^t}}{q^t_{t+1,2}} \end{cases} \tag{.16}$$

implying $q^t_{t+1,1} = q^i_{sell} = q^i_{buy}$ that is also the case of $\tau = 0$.

The asset demands in (2.8) has been rewritten in terms of $\tilde{\pi}^i$, δ^i_{t,s^t} and the pre-tax prices $q^t_{t+1,s}$ to study the double distortion induced by the FTT. However, given a state s

$$\overline{q}^t_{t+1,s} = \frac{(1+\tau)\, q^t_{t+1,s}}{(1+\tau)\, q^t_{t+1,s} + (1-\tau)\, q^t_{t+1,-s}}; \quad \underline{q}^t_{t+1,s} = \frac{(1-\tau)\, q^t_{t+1,s}}{(1-\tau)\, q^t_{t+1,s} + (1+\tau)\, q^t_{t+1,-s}}$$

and, relying on price normalization, the other state $-s$ prices are given by

$$\underline{q}^t_{t+1,-s} = 1 - \overline{q}^t_{t+1,s}; \quad \overline{q}^t_{t+1,-s} = 1 - \underline{q}^t_{t+1,s}.$$

It is thus possible to show that

$$\overline{q}^t_{t+1,1} = \frac{\pi^i q^t_{t+1,1}}{\tilde{\pi}^i_{t,s}\delta^i_t}; \quad \underline{q}^t_{t+1,1} = \frac{\pi^i q^t_{t+1,1}}{\tilde{\pi}^i_{t,s}\delta^i_t}$$

where $\tilde{\pi}^i_{t,s}$ and δ^i_t when agent i buys (left) and sells (right) asset 1 are the first and second rows of both (2.9) and (2.12), respectively.

□

Proof of Lemma 2.1. Agents are willing to trade asset 1 in both the two following cases

$$\begin{cases} q^t_{t+1,1} \in \left(q^2_{sell}, q^1_{buy}\right) & \text{Agent 1 buys and agent 2 sells} \\ q^t_{t+1,1} \in \left(q^1_{sell}, q^2_{buy}\right) & \text{Agent 2 buys and agent 1 sells} \end{cases}$$

It is possible to show that $\nexists \tau \in (0,1)$ such that $q^1_{sell} < q^2_{buy}$, ruling out the exchange possibility described in the second row (agent 2 buys and agent 1 sells asset 1). Suppose, by contrast, that the following inequality holds

$$q^1_{sell} < q^2_{buy} \tag{.17}$$

Replace the agents' reservation prices (2.7)

$$\begin{gathered} \frac{\pi^1(1+\tau)}{(1+\tau(2\pi^1-1))} < \frac{\pi^2(1-\tau)}{(1-\tau(2\pi^2-1))} \\ \frac{\pi^1(1+\tau)}{(1+\tau(\pi^1-(1-\pi^1)))} < \frac{\pi^2(1-\tau)}{(1+\tau((1-\pi^2)-\pi^2))} \\ \left(\pi^1-\pi^2\right) + 2\tau\left(\pi^1-\pi^2+2\pi^2\left(1-\pi^1\right)\right) + \tau^2\left(\pi^1-\pi^2\right) < 0 \end{gathered} \tag{.18}$$

It is a quadratic inequality with solution[12]

$$\tau_i = -\frac{\left(\pi^1 - \pi^2 + 2\pi^2\left(1-\pi^1\right)\right)}{\left(\pi^1 - \pi^2\right)} \pm \sqrt{\frac{\left(\pi^1 - \pi^2 + 2\pi^2\left(1-\pi^1\right)\right)^2}{\left(\pi^1 - \pi^2\right)^2} - 1}$$

it is easy to prove that the solutions are never positive π^i, $i \in \{1,2\}$.

We show now that

$$\exists \tau \in (0,1) : q^2_{sell} < q^1_{buy}$$

and agent 1 buys asset 1 from agent 2.

$$\begin{aligned}
q^2_{sell} &< q^1_{buy} \\
\frac{\pi^2(1+\tau)}{\left(1+\tau\left(2\pi^2-1\right)\right)} &< \frac{\pi^1(1-\tau)}{\left(1-\tau\left(2\pi^1-1\right)\right)} \\
\frac{\pi^2(1+\tau)}{\left(1-\tau\left(\left(1-\pi^2\right)-\pi^2\right)\right)} &< \frac{\pi^1(1-\tau)}{\left(1-\tau\left(\pi^1-\left(1-\pi^1\right)\right)\right)} \\
\left(\pi^2-\pi^1\right) + 2\tau\left(\pi^2+\pi^1-2\pi^2\pi^1\right) + \tau^2\left(\pi^2-\pi^1\right) &< 0 \\
\tau = \frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)}{\left(\pi^1-\pi^2\right)} &\pm \sqrt{\frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)^2}{\left(\pi^1-\pi^2\right)^2} - 1}
\end{aligned} \tag{.19}$$

Define

$$\begin{aligned}
\tau_1 &= \frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)}{\left(\pi^1-\pi^2\right)} + \sqrt{\frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)^2}{\left(\pi^1-\pi^2\right)^2} - 1} \\
\tau_2 &= \frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)}{\left(\pi^1-\pi^2\right)} - \sqrt{\frac{\left(\pi^2+\pi^1-2\pi^2\pi^1\right)^2}{\left(\pi^1-\pi^2\right)^2} - 1}
\end{aligned} \tag{.20}$$

The inequality holds for

$$\tau < \tau_2 \wedge \tau > \tau_1$$

It is possible to show that $\tau_1 > 1$, therefore, since $\tau \in (0,1)$, the sufficient condition for trading is given by

$$\tau < \tau'$$

where $\tau' = \tau_2$. ☐

Proof of Proposition 2.2. Replacing the optimal portfolio rules (2.8), the conditional drift of the process $z_{(t,s)}$ is given by

$$\begin{aligned}
\mathbb{E}\left[\xi_{t+1}|z_t\right] =& \pi \log\left(\frac{a^1_{t+1,1}}{a^2_{t+1,1}}\right) + (1-\pi)\log\left(\frac{a^1_{t+1,2}}{a^2_{t+1,2}}\right) \\
=& \pi \log\left(\frac{\pi^1_1 \underline{q}^t_{t+1,1}}{\pi^2_1 \overline{q}^t_{t+1,1}}\right) + (1-\pi)\log\left(\frac{\pi^1_2 \overline{q}^t_{t+1,2}}{\pi^2_2 \underline{q}^t_{t+1,2}}\right)
\end{aligned} \tag{.21}$$

Rewrite the after-tax normalized prices in terms of $q^t_{t+1,1}$

$$\overline{q}^t_{t+1,1} = \frac{(1+\tau)\,q^t_{t+1,1}}{2\tau q^t_{t+1,1} + 1 - \tau}; \quad \underline{q}^t_{t+1,1} = \frac{(1-\tau)\,q^t_{t+1,1}}{-2\tau q^t_{t+1,1} + 1 + \tau}$$

where, $\overline{q}^t_{t+1,2} = 1 - \underline{q}^t_{t+1,1}$ and $\underline{q}^t_{t+1,2} = 1 - \overline{q}^t_{t+1,1}$, due to price normalization. Equation (.21) becomes

$$\mathbb{E}\left[\xi_{t+1}|z_t\right] = \pi \log\left(\frac{\pi^1(1-\tau)}{\pi^2(1+\tau)} \frac{1+\tau\left[2q^t_{t+1,1}-1\right]}{1+\tau\left[1-2q^t_{t+1,1}\right]}\right) + (1-\pi)\log\left(\frac{\left(1-\pi^1\right)(1+\tau)}{\left(1-\pi^2\right)(1-\tau)} \frac{1+\tau\left[2q^t_{t+1,1}-1\right]}{1+\tau\left[1-2q^t_{t+1,1}\right]}\right). \tag{.22}$$

[12]The discriminant of the quadratic equation is always positive $\forall \pi^i, i \in \{1,2\}$.

Adding and subtracting $\log(\pi)$, the rhs may be further rewritten as

$$D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{1+\tau}{1-\tau}\frac{1+\tau\left[2q^t_{t+1,1}-1\right]}{1+\tau\left[1-2q^t_{t+1,1}\right]}\right)$$

and, replacing the equilibrium price in (2.17), it becomes

$$D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\left(1-\tau^2\right)a^1_{t,s^t} + \left(-2\tau\left(1-2\pi^2\right)+\tau^2+1\right)a^2_{t,s^t}}{\left(\tau^2+1-2\tau\left(2\pi^1-1\right)\right)a^1_{t,s^t} + \left(1-\tau^2\right)a^2_{t,s^t}}\right).$$

Consider the two limits

$$\begin{aligned}
\mu_+ &= \lim_{z\to\infty}\mathbb{E}\left[\xi_{t+1}|z_t=z\right] \\
&= \lim_{z\to\infty}\left(D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\frac{a^1_{t,s^t}}{a^2_{t,s^t}}\left(1-\tau^2\right)+\left(\tau^2+1-2\tau\left(1-2\pi^2\right)\right)}{\left(1-\tau^2\right)+\frac{a^1_{t,s^t}}{a^2_{t,s^t}}\left(\tau^2+1-2\tau\left(2\pi^1-1\right)\right)}\right)\right) \\
&= \left(D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\left(1-\tau^2\right)}{\left(\tau^2+1-2\tau\left(2\pi^1_1-1\right)\right)}\right)\right. \\
&= \left(D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\left(1-\tau^2\right)}{\left(1+\tau\right)^2-4\tau\pi^1_1}\right)\right.
\end{aligned} \quad (.23)$$

$$\begin{aligned}
\mu_- &= \lim_{z\to-\infty}\mathbb{E}\left[\xi_{t+1}|z_t=z\right] \\
&= \lim_{z\to-\infty}\left(D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\frac{a^1_{t,s^t}}{a^2_{t,s^t}}\left(1-\tau^2\right)+\left(\tau^2+1-2\tau\left(1-2\pi^2\right)\right)}{\left(1-\tau^2\right)+\frac{a^1_{t,s^t}}{a^2_{t,s^t}}\left(\tau^2+1-2\tau\left(2\pi^1-1\right)\right)}\right)\right) \\
&= D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{1+\tau^2-2\tau\left(1-2\pi^2\right)}{\left(1-\tau^2\right)}\right) \\
&= D_{KL}\left(\pi||\pi^2\right) - D_{KL}\left(\pi||\pi^1\right) + (1-2\pi)\log\left(\frac{1+\tau}{1-\tau}\right) + \log\left(\frac{\left(1-\tau\right)^2+4\tau\pi^2_1}{\left(1-\tau^2\right)}\right)
\end{aligned} \quad (.24)$$

□

Appendix C

Proof of Proposition 3.1. The individual investment opportunities in (3.15) may be equivalently rewritten in terms Arrow securities. In this regard, the node $t+1, \theta^{t+1}$ contingent-claims available in θ^t are given by

$$\begin{aligned} a^i_{t+1,\theta^h} &= s^i_{t+1} A k^{\theta^h}_{t+1} + b^i_{t+1} \\ a^i_{t+1,\theta^l} &= s^i_{t+1} A k^{\theta^l}_{t+1} \end{aligned} \tag{.1}$$

where a^i_{t+1,θ^s} is the Arrow security delivering consumption provided that state $s \in \{h, l\}$ realizes at $t+1$. Moreover,

$$b^i_{t+1} = a^i_{t+1,\theta^h} - a^i_{t+1,\theta^l} k^{\theta^h - \theta^l}_{t+1}.$$

Thus, the agent i problem in (3.15) can be rewritten in terms of contingent-assets as

$$\begin{gathered} \max_{c^i_t, a^i_{t+1,\theta^h}, a^i_{t+1,\theta^l}} \mathbb{E}^i \left[\sum_{t=0}^{\infty} \beta^t \log(c^i_t) \right] \\ \text{s.to} \\ c^i_t + \sum_{s\in\{h,l\}} q^t_{t+1,\theta^s} a^i_{t+1,\theta^s} + \tau q_{t+1,\theta^h} |a^i_{t+1,\theta^h} - a^i_{t+1,\theta^l} k^{\theta^h-\theta^l}_{t+1}| = a^i_t + \frac{P_t}{2} \end{gathered} \tag{.2}$$

Policy functions are found for a^i_{t+1,θ^s} so that s^i_{t+1} and b^i_{t+1} may be derived from (.1).

Since the individual optimal decision rules are piece-wise functions of the equilibrium state θ^h price, I show the proof of the equilibrium for the first rows of them, defined in $q_{t+1,\theta^h} < q^i_{buy}$. The proofs of other sub-functions may be derived following the same procedure.

Define the Bellman equation of the individual problem in (3.15) as

$$V^t\left(a^i_t\right) = \max_{c^i_t, a^i_{t+1,\theta^h}, a^i_{t+1,\theta^l}} \left\{ \log\left(c^i_t\left(\theta^t\right)\right) + \beta \mathbb{E}^i\left[V^{t+1}\left(a^i_{t+1}\right)\right] \right\} \tag{.3}$$

where $c^i_t = a^i_t + \frac{P_t}{2} - \sum_{s\in\{h,l\}} q^t_{t+1,\theta^s} a^i_{t+1,\theta^s} - \tau q_{t+1,\theta^h} |a^i_{t+1,\theta^h} - a^i_{t+1,\theta^l} k^{\theta^h-\theta^l}_{t+1}|$.

The dynamic problem is solved using value function iteration: start with an initial guess of the value function and then solve a one-period problem choosing the vector $\left(c^i_t, a^i_{t+1,\theta^h}, a^i_{t+1,\theta^l}\right)$ that maximizes (.3) subject to the binding budget constraint in (.2). I start with a guess $V^{t+1} = 0$ that implies $\left(s^i_{t+1}, b^i_{t+1}\right) = 0$. The trivial solution is $c^i_t = a^i_t$ and, therefore

$$V^t = \log\left(a^i_t\right).$$

Replace the above in the previous-period Bellman equation. Backward iterating and exploit the properties of geometric series, one ends up with

$$\begin{aligned} c_t^i &= (1-\beta)\left[w_t^i + P_t^{ei}\left(\theta^t\right)\right] \\ a_{t+1,\theta^h}^i &= \frac{\beta\pi^i\left[a_t + P_t^{ei}\left(\theta^t\right)\right]}{q_{t+1,\theta^h}^t(1+\tau)} - P_{t+1}^{ei}\left(\theta_{t+1}=\theta^h\right) \\ a_{t+1,\theta^l}^i &= \frac{\beta\left(1-\pi^i\right)\left[a_t^i + P_t^{ei}\left(\theta^t\right)\right]}{\left(q_{t+1,\theta^l}^t - q_{t+1,\theta^h}^t\tau k_{t+1}^{\theta^h-\theta^l}\right)} - P_{t+1}^{ei}\left(\theta_{t+1}=\theta^l\right) \end{aligned} \tag{.4}$$

where $P_t^{ei}(\theta^t)$ is the node θ^t sum of the discounted future expected profits. Define the after-tax state-prices as

$$\begin{aligned} Q_{t+1,\theta^h}^t &= q_{t+1,\theta^h}^t(1+\tau) \\ Q_{t+1,\theta^l}^t &= \left(q_{t+1,\theta^l}^t - q_{t+1,\theta^h}^t\tau k_{t+1}^{\theta^h-\theta^l}\right) \end{aligned} \tag{.5}$$

therefore, $P_r^{ei}\left(\theta^r\right)$ with $r \in t$ is given by

$$P_r^{ei}\left(\theta^r\right) = \sum_{t=r}^{\infty}\sum_{\theta^t|\theta^r}\left[\prod_{s=t}^{r-1} Q_{s+1}^s\left(\theta_{s+1}|\theta^s\right)\frac{P_s\left(\theta^s\right)}{2}\right]$$

Since agents are not endowed with a system of second-order beliefs, they compute $P_r^{ei}\left(\theta^r\right)$ as if they were alone in the economy, using as state-prices the individual MRS resulting from the household's optimization problem in (3.15). Therefore, the after-tax state-prices are given by

$$\begin{aligned} Q_{t+1,\theta^h}^t &= \frac{\pi^i c_t^i}{c_{t+1,\theta^h}^i} \\ Q_{t+1,\theta^l}^t &= \frac{\left(1-\pi^i\right)c_t^i}{c_{t+1,\theta^l}^i} \end{aligned} \tag{.6}$$

given that, the individual consumes a fixed fraction of the aggregate wealth for all θ^t in complete markets log economies, the above becomes

$$\begin{aligned} Q_{t+1,\theta^h}^t &= \frac{\pi^i A k_t^{\theta_t}}{A k_{t+1}^{\theta^h}} \\ Q_{t+1,\theta^l}^t &= \frac{\left(1-\pi^i\right)A k_t^{\theta_t}}{A k_{t+1}^{\theta^l}} \end{aligned} \tag{.7}$$

implying

$$P_t^{ei} = \beta A k_t^{\theta_t}\left[\frac{1-\mathbb{E}^i\left(\theta_{t+1}\right)+\tau\left(1-\theta^l\right)}{(1-\beta)(1+\tau)}\right]$$

for all θ^t. Replace the latter in (.4) to get

$$\begin{aligned} c_t^i &= \left(1-\beta\frac{\mathbb{E}^i\left(\theta_{t+1}\right)+\tau\theta^l}{1+\tau}\right)w_t^i \\ a_{t+1,\theta^h}^i &= \frac{\beta\pi^i w_t^i}{q_{t+1,\theta^h}^t(1+\tau)} \\ a_{t+1,\theta^l}^i &= \beta\frac{\left(1-\pi^i\right)w_t^i}{\left(q_{t+1,\theta^l}^t - q_{t+1,\theta^h}^t\tau k_{t+1}^{\theta^h-\theta^l}\right)} \end{aligned} \tag{.8}$$

therefore the policy functions for s_{t+1}^i and b_{t+1}^i is obtained by replacing (.8) in (.1). □

Proof of Lemma 3.1. Agents are willing to exchange the Arrow security b provided that the equilibrium state θ^h price is in one of the two following cases

$$\begin{cases} q^t_{t+1,\theta^h} \in \left(q^2_{sell}, q^1_{buy}\right) & \text{Agent 1 buys and agent 2 shorts-sell} \\ q^t_{t+1,\theta^h} \in \left(q^1_{sell}, q^2_{buy}\right) & \text{Agent 2 buys and agent 1 shorts-sell} \end{cases}$$

It is possible to show that $\nexists \tau \in (0,1)$ such that $q^1_{sell} < q^2_{buy}$ and the statement in the second row is true. Suppose, by contrast, that the following inequality holds

$$q^1_{sell} < q^2_{buy}. \tag{.9}$$

Replacing the agents' reservation prices as defined in Proposition 3.1, the above inequality holds provided that

$$\tau < \frac{\pi^2 - \pi^1}{(\pi^1 + \pi^2)}$$

and, therefore, for negatives τ given that $\pi^2 - \pi^1 < 0$ by the assumption made in Section 3.2.1.

The second is step is to show that We show now that

$$\exists \tau \in (0,1) : q^2_{sell} < q^1_{buy}$$

and the agent 1 buys while agent 2 shorts-sell the Arrow security b. Again, replacing the agents' reservation prices, the above holds provided that

$$\tau < \frac{\pi^1 - \pi^2}{(\pi^1 + \pi^2)}$$

that is always positive. □

Proof of Proposition 3.2. The individual optimal decision rules regarding both consumption and portfolio composition derives from the results stated in Proposition 3.1 together with Lemma 3.1.

The equilibrium state prices q^t_{t+1,θ^s} , in both the case $\tau \in [0, \tau')$, stems from the clearing conditions of the market of contingent-claims as defined in (.1)

$$\sum_{i\in\{1,2\}} a^i_{t+1,\theta^s} = Ak^{\theta^s}_{t+1}$$

for $s \in \{h, l\}$. Plugging the optimal a^i_{t+1,θ^s} as defined[13] in (.8) for all $i \in \{1, 2\}$, one can find

$$q^t_{t+1,\theta^h} = \frac{\beta\pi^1 (1-\tau) a^1_t + \pi^2 (1+\tau) a^2_t}{(1-\tau^2) Ak^{\theta^h}_{t+1}}$$

$$q^t_{t+1,\theta^l} = \beta \frac{\sqrt{\begin{array}{l}\left[\left(1-\pi^1\right) w^1_t + \left(1-\pi^2\right) w^2_t\right]^2 + \\ \quad + \tau \frac{\pi^1(1-\tau)w^1_t + \pi^2(1+\tau)w^2_t}{(1-\tau^2)} \left[\left(1-\pi^1\right) w^1_t - \left(1-\pi^2\right) w^2_t + \tau \frac{\pi^1(1-\tau)w^1_t + \pi^2(1+\tau)w^2_t}{(1-\tau^2)}\right]\end{array}}}{Ak^{\theta^l}_{t+1}}. \tag{.10}$$

Finally, the aggregate capital investment rule is obtained replacing (.10) in the firms' F.O.C. (3.4) and solving for k_{t+1}. □

[13]This actually displays the Arrow security demands of the agent buying b (agent 1 according to Lemma 3.1). The Arrow security demands for the agent that shorts-sell b (agent 2) are given by: $a^2_{t+1,\theta^h} = \frac{\beta\pi^2 w^2_t}{q^t_{t+1,\theta^h}(1-\tau)}$ $\qquad a^2_{t+1,\theta^l} = \beta \frac{(1-\pi^2) w^i_t}{\left(q^t_{t+1,\theta^l} + q^t_{t+1,\theta^h} \tau k^{\theta^h - \theta^l}_{t+1}\right)}$.

Proof of Proposition 3.3. The conditional drift of the process z_{t+1} is given by

$$\begin{aligned}\mathbb{E}\left[\xi_{t+1}|z_t\right] &= \pi\log\left(\frac{w^1_{t+1,\theta^h}}{w^2_{t+1,,\theta^h}}\right)+(1-\pi)\log\left(\frac{w^1_{t+1,\theta^l}}{w^2_{t+1,\theta^l}}\right)\\ &= \pi\log\left(\frac{b^1_{t+1}+z^1_{t+1}Ak^{\theta^h}_{t+1}}{b^2_{t+1}+z^2_{t+1}Ak^{\theta^h}_{t+1}}\right)+(1-\pi)\log\left(\frac{z^1_{t+1}Ak^{\theta^l}_{t+1}}{z^2_{t+1}Ak^{\theta^l}_{t+1}}\right)\end{aligned} \tag{.11}$$

Replacing the individual optimal portfolio rules (3.21) and (3.23), the above becomes

$$\mathbb{E}\left[\xi_{t+1}|z_t\right] = \pi\log\left(\frac{\pi^1(1-\tau)}{\pi^2(1+\tau)}\right)+(1-\pi)\log\left(\frac{\left(1-\pi^1\right)\left(1-q^t_{t+1,\theta^h}\left[\theta^h-\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}{\left(1-\pi^2\right)\left(1-q^t_{t+1,\theta^h}\left[\theta^h+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}\right) \tag{.12}$$

Adding and subtracting $\log(\pi)$ and $\log(1-\pi)$ the above may be rewritten in terms of the agents' relative entropies

$$\mathbb{E}\left[\xi_{t+1}|z_t\right] = D_{KL}\left(\pi^2||\pi\right)-D_{KL}\left(\pi^1||\pi\right)+g(\pi,\tau,A,\theta^h,\theta^l,q^t_{t+1,\theta^h},k_{t+1}) \tag{.13}$$

where $g(\pi,\tau,A,\theta^h,\theta^l,q^t_{t+1,\theta^h},k_{t+1}) = \pi\log\left(\frac{1-\tau}{1+\tau}\right)+(1-\pi)\log\left(\frac{\left(1-q^t_{t+1,\theta^h}\left[\theta^h-\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}{\left(1-q^t_{t+1,\theta^h}\left[\theta^h+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}\right)$. The latter cancels out when $\tau=0$, while it is positive or negative otherwise.

The second statement of the Proposition, posits persistent heterogeneity provided that $\tau\geq\tau'$. In this case, the agents' assets demand function changes and, according to (3.22) and (3.24)

$$z_{t+1}=z_t$$

for any θ^t and $\mathbb{E}\left[\xi_{t+1}\right]=0$. Therefore $\lim_{t\to\infty}z_{t+1}=0$ and $\lim_{t\to\infty}\frac{w^1_t}{w^2_t}=1$ as a result.

Lastly, I prove that the most accurate agent i may eventually be driven out of the economy in the long-run. This scenario, that I refer as path-dependent long-run equilibrium, is due to (3.30). The latter is proved just for agent 1 but the same procedure may be used also for agent 2. First, show that

$$\lim_{\phi^1_t\to1}\frac{T^1_t}{w^1_t}=0 \tag{.14}$$

where

$$\frac{T^1_t}{w^1_t}=\tau q_{t+1,\theta^h}|b^1_{t+1}|$$

and, replacing (3.21),

$$\frac{T^1_t}{w^1_t}=\tau q^t_{t+1,\theta^h}\left|\beta\left[\frac{\pi^i}{q^t_{t+1,\theta^h}(1+\tau)}-\frac{\left(1-\pi^i\right)\theta^lAk^{\theta^h-1}_{t+1}}{\left(1-q^1_{t+1,\theta^h}\left[\theta^h+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}\right]\right|$$

therefore,

$$\begin{aligned}\lim_{\phi^1_t\to1}\frac{T^1_t}{w^1_t} &= \tau q^1_{buy}\left|\beta\left[\frac{\pi^1}{q^1_{buy}(1+\tau)}-\frac{\left(1-\pi^1\right)\theta^lAk^{\theta^h-1}_{t+1}}{\left(1-q^1_{buy}\left[\theta^h+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}\right]\right|\\ &= \tau\left|\beta\left[\frac{\pi^1-q^1_{buy}\left[\mathbb{E}^i\left(\theta_{t+1}\right)+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}}{(1+\tau)\left(1-q^1_{buy}\left[\theta^h+\tau\theta^l\right]Ak^{\theta^h-1}_{t+1}\right)}\right]\right|.\end{aligned} \tag{.15}$$

Replacing q^1_{buy} as defined from Proposition 3.1, one can easily show that (.14) is true.

Second, prove that

$$\lim_{\phi^1_t\to0}\frac{T^1_t}{w^1_t}>0 \tag{.16}$$

again, replacing (3.21), the above can be rewritten as

$$\begin{aligned}\lim_{\phi_t^1\to 0}\frac{T_t^1}{w_t^1} &= \tau q_{sell}^2\left|\beta\left[\frac{\pi^1}{q_{sell}^2(1+\tau)}-\frac{\left(1-\pi^1\right)\theta^l Ak_{t+1}^{\theta^h-1}}{\left(1-q_{sell}^2\left[\theta^h+\tau\theta^l\right]Ak_{t+1}^{\theta^h-1}\right)}\right]\right| \\ &= \tau\left|\beta\left[\frac{\pi^1-q_{sell}^2\left[\mathbb{E}^1\left(\theta_{t+1}\right)+\tau\theta^l\right]Ak_{t+1}^{\theta^h-1}}{(1+\tau)\left(1-q_{sell}^2\left[\theta^h+\tau\theta^l\right]Ak_{t+1}^{\theta^h-1}\right)}\right]\right|\end{aligned} \tag{.17}$$

substitute q_{sell}^2 in Proposition 3.1, the latter is rearranged as

$$= \tau\left|\beta\left[\frac{\frac{\pi^1\left[\mathbb{E}^2(\theta_{t+1})-\tau\theta^l\right]-\pi^2\left[\mathbb{E}^1(\theta_{t+1})+\tau\theta^l\right]}{\left[\mathbb{E}^2(\theta_{t+1})-\tau\theta^l\right]}}{(1+\tau)\left(1-q_{sell}^2\left[\theta^h+\tau\theta^l\right]Ak_{t+1}^{\theta^h-1}\right)}\right]\right|$$

that is positive provided that

$$\frac{\pi^1-\pi^2}{\left(\pi^1+\pi^2\right)} > \tau = \tau'$$

and, according to Lemma 3.1, provided that trading occurs between the agents and $T_t^i > 0$ for all $i \in \{1,2\}$. □

Proof of Proposition 3.4. Define as

$$\text{mpc} = \left(1-\beta\mathbb{E}\left(\theta_{t+1}\right)\right)$$

the agent's marginal propensity to consume under the REH. Given that $\pi \in [\pi^2, \pi^1]$, without any policy intervention (see 3.8), the agent 1 (2) propensity to consume is lower (greater) than the one implied by rational expectations

$$\begin{aligned}&\left(1-\beta\mathbb{E}^1\left(\theta_{t+1}\right) < \left(1-\beta\mathbb{E}\left(\theta_{t+1}\right)\right)\right) \\ &\left(1-\beta\mathbb{E}^2\left(\theta_{t+1}\right)\right) > \left(1-\beta\mathbb{E}\left(\theta_{t+1}\right)\right)\end{aligned} \tag{.18}$$

The FTT changes the agents' mpc as follows

$$\begin{aligned}\text{mpc}^1 &= \left(1-\beta\frac{\mathbb{E}^1\left(\theta_{t+1}\right)+\tau\theta^l}{(1+\tau)}\right) \\ \text{mpc}^2 &= \left(1-\beta\frac{\mathbb{E}^2\left(\theta_{t+1}\right)-\tau\theta^l}{(1-\tau)}\right)\end{aligned} \tag{.19}$$

The marginal impact of the tax rate is positive for agent 1 and negative for agent 2

$$\begin{aligned}\frac{\partial\text{mpc}^1}{\partial\tau} &= \beta\frac{\mathbb{E}^1\left(\theta_{t+1}\right)-\theta^l}{(1+\tau)^2} = \beta\frac{\pi^1\left(\theta^h-\theta^l\right)}{(1+\tau)^2} > 0 \\ \frac{\partial\text{mpc}^2}{\partial\tau} &= \beta\frac{\theta^l-\mathbb{E}^2\left(\theta_{t+1}\right)}{(1-\tau)^2} = -\beta\frac{\pi^2\left(\theta^h-\theta^l\right)}{(1-\tau)^2} < 0\end{aligned} \tag{.20}$$

implying that

$$\text{mpc}^i \to \text{mpc} \tag{.21}$$

for all $i \in \{1,2\}$.

Lastly, the capital-output ratio as the distribution of the agents' wealth approaches its limits is given by

$$\begin{aligned}\lim_{\phi_t^1\to 1}\frac{k_{t+1}}{Ak_t^{\theta_t}} &= \beta\frac{1-\pi^1+\sqrt{\left[1-\pi^1\right]^2+4\frac{\pi_1^1}{(1+\tau)}\left(\frac{\tau(1-\pi^1)+\tau^2}{(1+\tau)}\right)}}{2}\theta^l+\frac{\beta\pi^1}{(1+\tau)}\theta^h \\ \lim_{\phi_t^2\to 1}\frac{k_{t+1}}{Ak_t^{\theta_t}} &= \beta\frac{1-\pi^2+\sqrt{\left[1-\pi^2\right]^2+4\frac{\pi^2}{(1-\tau)}\left(\frac{-\tau(1-\pi^2)+\tau^2}{(1-\tau)}\right)}}{2}\theta^l+\frac{\beta\pi^2}{(1-\tau)^2}\theta^h\end{aligned} \tag{.22}$$

The marginal effect of the FTT is given by

$$\lim_{\phi_t^1 \to 1} \frac{\partial \left(k_{t+1}/Ak_t^{\theta_t} \right)}{\partial \tau} = \frac{\beta \pi^1}{(1+\tau)^2} \frac{\theta^l \left(\left(1-\pi^1\right)(1-\tau) + 2\tau \right) - \theta^h \sqrt{\left[\left(1-\pi^1\right)\right]^2 (1+\tau)^2 + 4\pi^1 \left(\tau \left(1-\pi^1\right) + \tau^2\right)}}{\sqrt{\left[\left(1-\pi^1\right)\right]^2 (1+\tau)^2 + 4\pi^1 \left(\tau \left(1-\pi^1\right) + \tau^2\right)}}$$
$$\lim_{\phi_t^2 \to 1} \frac{\partial \left(k_{t+1}/Ak_t^{\theta_t} \right)}{\partial \tau} = \frac{\beta \pi^2}{(1-\tau)^2} \frac{\left[-\left(1-\pi^2\right)(1+\tau) + 2\tau \right] \theta^l + \sqrt{\left[\left(1-\pi^2\right)\right]^2 (1-\tau)^2 + 4\pi^2 \left[-\tau \left(1-\pi^2\right) + \tau^2\right]} \theta^h}{\sqrt{\left[\left(1-\pi^2\right)\right]^2 (1-\tau)^2 + 4\pi^2 \left[-\tau \left(1-\pi^2\right) + \tau^2\right]}}. \tag{.23}$$

First:

$$\lim_{\phi_t^1 \to 1} \frac{\partial \left(k_{t+1}/Ak_t^{\theta_t} \right)}{\partial \tau} < 0$$

provided that

$$\frac{\beta \pi^1}{(1+\tau)^2} \frac{\theta^l \left(\left(1-\pi^1\right)(1-\tau) + 2\tau \right) - \theta^h \sqrt{\left[\left(1-\pi^1\right)\right]^2 (1+\tau)^2 + 4\pi^1 \left(\tau \left(1-\pi^1\right) + \tau^2\right)}}{\sqrt{\left[\left(1-\pi^1\right)\right]^2 (1+\tau)^2 + 4\pi^1 \left(\tau \left(1-\pi^1\right) + \tau^2\right)}} < 0$$

the denominator is always positive, while the numerator can be rearranged as

$$\left[\left(1-\pi^1\right)(1-\tau) - 2\tau \right]^2 \left(\left(\theta^l\right)^2 - \left(\theta^h\right)^2 \right)$$

that is always negative given that $\left(\left(\theta^l\right)^2 - \left(\theta^h\right)^2 \right) < 0$. Second:

$$\lim_{\phi_t^2 \to 1} \frac{\partial \left(k_{t+1}/Ak_t^{\theta_t} \right)}{\partial \tau} > 0$$

provided that

$$\frac{\beta \pi^2}{(1-\tau)^2} \frac{\left[-\left(1-\pi^2\right)(1+\tau) + 2\tau \right] \theta^l + \sqrt{\left[\left(1-\pi^2\right)\right]^2 (1-\tau)^2 + 4\pi^2 \left[-\tau \left(1-\pi^2\right) + \tau^2\right]} \theta^h}{\sqrt{\left[\left(1-\pi^2\right)\right]^2 (1-\tau)^2 + 4\pi^2 \left[-\tau \left(1-\pi^2\right) + \tau^2\right]}}$$

again, the denominator is always positive, while the numerator can be rearranged as

$$\left[\left(1-\pi^2\right)(1+\tau) - 2\tau \right]^2 \left[\left(\theta^l\right)^2 + \left(\theta^h\right)^2 \right]$$

that is always positive.

□

Estratto per riassunto della tesi di dottorato

L'estratto (max. 1000 battute) deve essere redatto sia in lingua italiana che in lingua inglese e nella lingua straniera eventualmente indicata dal Collegio dei docenti.

L'estratto va firmato e rilegato come ultimo foglio della tesi.

Studente: Arianna Traini matricola: 956320

Dottorato: Economics

Ciclo: 32

Titolo della tesi[1] : Three essays on speculation and welfare in dynamic economies

Abstract:

This thesis analyses the impact of belief heterogeneity on individual and aggregate investment choices, focusing on welfare effects and policy implications. Using a theoretical structure, I consider economies where trading exclusively arises for speculative purposes: investors hold different opinions about states of the Nature and foresee a welfare gain from placing bets on future states. In production economies, disagreement affects not only the individual but also the aggregate consumption process, that is endogenously determined by the agents' investment strategies. In general, it enhances the macroeconomic volatility produced by the exogenous shocks. However, this effect has a transient nature since the Market Selection Hypothesis (MSH) holds draining inaccurate traders out of the economy. Despite ensuring long-run accuracy of economic outcomes, the MSH may be not fully appealing from the standpoint of a benevolent policy maker, due to the realized losses incurred by less accurate agents. Therefore, I focus on regulatory trading measures aimed at improving the decentralized Pareto optimal result. In particular, I investigate the effect of a linear financial transaction tax (FTT) placed on the agents' speculative trades. The tax is set by a Government with a dual purpose: maximize the social welfare and the amount of fiscal revenues. The overall impact of the FTT depends on both its magnitude and the position of the truth compared to the agents' subjective probabilities. In particular, when the truth lies in the middle of the agents' beliefs and the tax rate is high enough, this measure may undermine the market selection argument, implying the most accurate agent vanishes with positive probability and leading to severe miss-pricing in the long-run.

Abstract (Italiano):

Questa tesi studia l'impatto della differenza di opinioni sulle scelte di investimento individuali e aggregate, mettendo in luce gli effetti sul welfare e le potenziali implicazioni di policy. Utilizzando un framework teorico, considero economie in cui l'attività di trading tra gli agenti scaturisce da finalità puramente speculative: gli investitori hanno opinioni diverse sulle probabilità associate ai diversi stati del mondo e, per questo motivo, sono incentivati a scommettere sulla realizzazione degli stessi. Nelle economie di produzione, il disaccordo influenza non solo le scelte individuali, ma anche il consumo aggregato, il quale deriva dalle scelte di investimento poste in essere dagli agenti. In generale, questo crea un meccanismo di amplificazione della volatilità indotta dagli shock esogeni. Tuttavia, l'effetto è solo di natura transitoria poiché, secondo l'ipotesi di selezione del mercato (MSH), gli agenti irrazionali sono destinati a svanire dal mercato nel lungo termine. Nonostante questo garantisca l'accuratezza dei risultati economici, ciò potrebbe non essere completamente soddisfacente dal punto di vista di un policy maker benevolo, a causa delle perdite subite dagli agenti meno

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[1] Il titolo deve essere quello definitivo, uguale a quello che risulta stampato sulla copertina dell'elaborato consegnato.

precisi. Pertanto, dedico parte dell’elaborato ad analizzare l’effetto di potenziali misure volte a migliorare l’ottimo Paretiano. In particolare, studio l'effetto di un’imposta sulle transazioni finanziarie (FTT) applicata alle negoziazioni speculative tra gli agenti. L'imposta è definita da un governo che persegue una duplice finalità: massimizzare il benessere sociale e l’ammontare delle entrate fiscali. L'impatto complessivo della FTT dipende sia dall’entità dell’aliquota fiscale applicata che dalla posizione della verità rispetto alle probabilità soggettive degli agenti. In particolare, quando la verità sta nel mezzo delle convinzioni degli agenti e l'aliquota fiscale è abbastanza elevata, questa misura può compromettere la MSH: l'agente più accurato svanisce con probabilità positiva, conducendo ad una sostanziale distorsione dei prezzi di lungo periodo.

Firma dello studente