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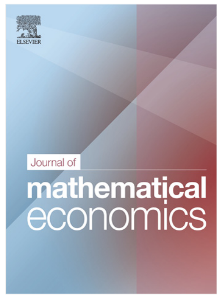

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Discounted Subjective Expected Utility in Continuous Time

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Abstract

By embedding uncertainty into time, we obtain a conjoint axiomatic characterization of both Exponential Discounting and Subjective Expected Utility that accommodates arbitrary state and outcome spaces. In doing so, we provide a novel and simple time-interpretation of subjective probability. The subjective probability of an event is calibrated using time discounting.

Keywords: Subjective Probability · Subjective Expected Utility · Exponential Discounting · Stationarity · Time Equivalents · Continuous Time.

JEL classification: D81.

1 Introduction

Consider the following bet. You get a constant and infinite stream of income of $10 if an event E obtains, and a $0 stream otherwise. How much would you be willing to pay to take this bet? Your answer may depend on your subjective probability of the event E. However, it is not always an easy task to come up with a specific value. In contrast, you may sometimes find it easier to analyze deterministic streams of outcomes over time. In this case, you can ask yourself the following question: For how long should you receive a sure stream of $10 (and then nothing forever) to remain indifferent to the initial bet involving uncertainty? If that value is given by t, then your evaluation of the stream yielding $10 up to t and $0 afterward encodes your subjective probability for the uncertain event. Suppose, for instance, that the central bank interest rate is $r > 0$. Then, assuming continuous compounding, the event's subjective probability is simply given by $1 - e^{-rt}$. From there, you can evaluate your willingness to bet on the uncertain event and make the right decision should you need to. More generally, the discount rate could be subjective and given by any parameter $\lambda > 0$.

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Proceeding along this line, this paper embeds decisions under uncertainty into continuous time and identifies behavioral conditions under which every event and, more generally, every alternative (henceforth: act) admits a deterministic time-equivalent flow as illustrated in the previous paragraph. It also identifies conditions under which these time equivalents lead indeed to meaningful and well-defined probabilities. In doing so, it provides an axiomatic characterization of both Exponential Discounting and Subjective Expected Utility (SEU) in continuous time. Thus, our paper brings together the seminal ideas of Samuelson (1937), Savage (1954), and Koopmans (1960) and axiomatizes Discounted Subjective Expected Utility in continuous time:

$$V(f) \;=\; \int_{\mathcal{S}} \left(\int_{\mathcal{T}} e^{-\lambda t} u[f(s,t)] dt \right) d\mu(s) \;=\; \int_{\mathcal{T}} \left(\int_{\mathcal{S}} e^{-\lambda t} u[f(s,t)] d\mu(s) \right) dt.$$

Despite their obvious importance in economic and financial applications, and to the best of our knowledge, Exponential Discounting and SEU have not received so far any conjoint axiomatic characterization in the context of continuous time.

In this axiomatic characterization, the key axioms are Monotone Continuity, Stationarity and Dominance. First, Monotone Continuity requires a form of continuity of preference with respect to "sufficiently small" events as in Villegas (1964) and Arrow (1970), but also time periods. In fact, its application to continuous time is key for the existence of time equivalents for every act. Second, Stationarity extends the original axiom of Koopmans (1960) from discrete to continuous time and accommodates the presence of uncertainty. In particular, it requires the invariance over time of preference on purely uncertain acts. Finally, Dominance requires time preferences to be independent from the state of the world which obtains. It can be traced back to the Anscombe and Aumann (1963) axiom of Monotonicity in the context of a second source of objective uncertainty instead of time.

Related literature. Our contribution is closely related to several papers that also embed decisions under uncertainty in richer frameworks. First, the sixth postulate of Savage (also known as Small Event Continuity) provides arbitrarily fine uniform partitions of the state space, which Savage uses to approximate subjective probabilities. Savage (1954)[p. 33] justifies this postulate by invoking the presence of a second source of uncertainty in the form of a fair coin. But this plays no role at all in his formal analysis. To define ambiguity aversion with respect to an urn of unknown composition, the Ellsberg (1961) two-urn experiment makes use of a second urn of known composition. Ambiguity aversion is then defined as a preference for betting on the latter rather than the former. Raiffa (1961) introduces explicitly a fair coin in the Ellsberg (1961) one-urn experiment and uses it for randomizing uncertain decisions. This allows him to obtain a defense of the Savage postulates and a critique of the Ellsberg pattern of choice. In our view, the Anscombe and Aumann (1963) (AA) theorem can be understood as a fully-fledged extension of Raiffa's argument into an axiomatic characterization of SEU. Indeed, they postulate the existence of an infinitely rich second source of uncertainty equipped with objective probabilities. Such richness makes sure that each act f_1 on the first (uncertain) source has an equivalent act f_2 on the second (objective) source and can hence be evaluated as the expected utility of f_2 with respect to objective probabilities, see Bastianello and Vergopoulos (2023). The

AA framework serves as a starting point for the axiomatization of many decision theories generalizing SEU and explaining the Ellsberg choices. These include the Choquet model of Schmeidler (1989) and the maxmin one of Gilboa and Schmeidler (1989). These authors need a second source to randomize acts depending on the first source. For instance, they explain the Ellsberg choices through a preference for randomizing uncertain decisions on objective probabilities and, in this way, smoothing outcomes on uncertain events.

Despite its success in simplifying the characterization of SEU and explaining the Ellsberg choices, the AA assumption of objective probabilities on the second source of uncertainty is largely criticized in the literature. Grabisch et al. (2022) show that it is possible to dispense with this assumption and reformulate the AA and Schmeidler theorems in a purely subjective way. But the purely subjective formulation of the axioms rely on behavioral notions of stochastic independence that remain somewhat unnatural and may undermine the normative appeal of the theory. See also Ghirardato et al. (2003), Ergin and Gul (2009), Mongin and Pivato (2015), Mongin (2020) and Ghirardato and Pennesi (2020) for other purely subjective versions of AA-type frameworks and theorems.

More recently, Kochov (2015) and Bastianello and Faro (2023) embed decisions under uncertainty in a temporal framework instead of postulating a second source of uncertainty. They obtain versions of the maxmin and Choquet models respectively. In their approach, ambiguity aversion is the expression of a preference for smoothing outcomes across the state space rather than across time. In fact, they postulate discrete time, a topological structure for the outcome space, and a restricted domain of acts. This allows them to construct a time equivalent for each act and, from there, axiomatize their representations by invoking AA-type arguments. Our approach differs in that we postulate continuous time which allows us in exchange to have an arbitrary outcome space and still get time equivalents. Furthermore, our argument involves the construction of preferences on state-contingent distribution of outcomes over time. Despite its time interpretation, such domain is formally identical to that of AA, and we obtain our representation through a direct application of the AA theorem. Hence, a merit of our approach is to provide a novel, clear-cut, and purely subjective interpretation of the AA framework and axioms with respect to time and exponential discounting instead of a second source of uncertainty and objective probabilities. In our view, such a temporal interpretation of the AA framework and axioms is even more natural than that in terms of a second source of uncertainty because it does not involve at all notions of stochastic independence. STATIONARITY and DOMINANCE provide indeed all the independence between time and uncertainty that one needs.

Finally, it is important to note that the focus on axiomatic intertemporal choice in continuous time is only quite recent. Ours is the first axiomatization of time discounting in continuous time under uncertainty dealing with measurable functions from an arbitrary state space to an arbitrary outcome space. Building on Debreu (1960), Harvey and Østerdal (2012) and Hara (2016) obtain versions of exponential discounting on a domain of piecewise continuous and *cadlag* deterministic acts respectively. In a recent paper, Song (2023) considered piecewise continuous trajectories over the simplex. Likewise, Pivato (2021) assumes topological structure on the outcome spaces and obtains in particular a form of exponential discounting on continuous deterministic acts. In contrast, Kopylov (2010) in his application to time preferences (his Corollary 4) and Webb (2016) obtain respectively exponential and quasi-hyperbolic discounting over piecewise constant functions.[1] Continuous time allows them to employ Savage-style arguments and hence to accommodate arbitrary outcome spaces. Finally, the fact that SEU could be reduced to discounted utility when the state space can be interpreted as a time interval was already noticed by Wakker (1993). However, except for Hara (2016), who provides an axiomatization of Discounted Expected Utility with objective lotteries, and Song (2023), who consider functions with images on the simplex, the literature focuses on the deterministic framework.

To summarize, there are at least three features that make our approach interesting. First, we provide a novel interpretation of subjective probability in terms of time discounting: the probability of an event is gauged by the willingness to wait before receiving a certain payment (or before ceasing to receive it). Hence we employ (continuous) time for measuring probability.[2] Second, our axiomatization of Discounted Subjective Expected Utility naturally arises from previous seminal works that separately addressed time and uncertainty, and it supports our time-interpretation of probability. While our framework is a purely subjective one à la Savage (1954), our proof uses continuous time to construct a temporal version of the classical Anscombe and Aumann (1963) framework and invokes their theorem. In doing so, it provides a novel interpretation of the AA framework that makes no use of objective probabilities. Third, we show that our axioms of Stationarity and Dominance imply, within this interpretation of the AA framework, the classical AA Independence axiom. This is explicitly illustrated in Section 5.

The remainder of the paper is organized as follows. Section 2 introduces our framework and notation. Section 3 presents the axioms needed for our main result which, is presented and discussed in Section 4. Finally, Section 5 illustrates with an example how to obtain a probability measure through time equivalents. Our proof appears in the appendices.

2 Framework

Uncertainty is represented by a *state space* $\mathcal{S}$. Time is continuous and represented by $\mathcal{T} = [0, +\infty)$. Let $\mathcal{B}_\mathcal{S}$ denote a σ-algebra $\mathcal{B}_\mathcal{S}$ of subsets of $\mathcal{S}$ and let $\mathcal{B}_\mathcal{T}$ denote the Borel algebra of subsets of $\mathcal{T}$. The product set $\mathcal{S} \times \mathcal{T}$ is equipped with the product σ-algebra $\mathcal{B} = \mathcal{B}_\mathcal{S} \times \mathcal{B}_\mathcal{T}$. Let also $\mathcal{X}$ be an *outcome space* equipped with a σ-algebra $\mathcal{B}_\mathcal{X}$.

An *act* is any measurable function from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{X}$. The set of acts is denoted by $\mathcal{F}$. A decision-maker is endowed with a binary relation $\succsim$ on $\mathcal{F}$ representing her preferences. We will suppose throughout the paper that $\succsim$ is complete, transitive and nontrivial. The interpretation is that an agent is deciding over acts at time $t = 0$.[3]

[1] Kopylov (2010) also discusses ways to extend his representations to larger domains.

[2] For discrete time, please refer to our discussion on Kochov (2015) and Bastianello and Faro (2023).

[3] In future work, we plan to consider the situation in which a decision-maker has to choose acts also in periods $t > 0$. Acts would be adapted processes over a filtration $(\mathcal{B}^t_\mathcal{S})_{t\in\mathcal{T}}$, and the decision-maker would have a set of preferences $(\succsim_t)_{t\in\mathcal{T}}$. At this stage, we do not know which axioms deliver Exponential Discounted Subjective Expected Utility.

We suppose that $\mathcal{B}_\mathcal{X}$ contains all singletons. This means that the agent can always identify the outcome she obtains. A finitely-valued function f from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{X}$ is an act if and only if $f^{-1}(\{x\}) \in \mathcal{B}_\mathcal{S} \times \mathcal{B}_\mathcal{T}$ for all $x \in \mathcal{X}$.

Let $\mathcal{F}_\mathcal{T}$ denote the subset of $\mathcal{F}$ made of all acts $f \in \mathcal{F}$ such that $f(s,t) = f(s',t)$ for all $s, s' \in \mathcal{S}$ and $t \in \mathcal{T}$. This set collects all *deterministic acts*. Each $f \in \mathcal{F}_\mathcal{T}$ is identified with the measurable function $\mathbf{x}$ from $\mathcal{T}$ to $\mathcal{X}$ that it defines. The restriction of $\succsim$ to $\mathcal{F}_\mathcal{T}$ is denoted by $\succsim_\mathcal{T}$ and represents the decision-maker's time preferences.

Likewise, let $\mathcal{F}_\mathcal{S}$ denote the subset of $\mathcal{F}$ made of all acts $f \in \mathcal{F}$ such that $f(s,t) = f(s,t')$ for all $s \in \mathcal{S}$ and $t, t' \in \mathcal{T}$. Such acts are referred to as *stochastic acts*. We identify each $f \in \mathcal{F}_\mathcal{S}$ with the measurable function ϕ from $\mathcal{S}$ to $\mathcal{X}$ that it defines. The restriction of $\succsim$ to $\mathcal{F}_\mathcal{S}$ is denoted by $\succsim_\mathcal{S}$ and represents the decision-maker's uncertainty preferences.

We identify each outcome $x \in \mathcal{X}$ with the act in $\mathcal{F}$ that is constantly equal to x over $\mathcal{S} \times \mathcal{T}$, the deterministic act in $\mathcal{F}_\mathcal{T}$ that is constantly equal to x over $\mathcal{T}$ and the stochastic act in $\mathcal{F}_\mathcal{S}$ that is constantly equal to x over $\mathcal{S}$.

For all $f, g \in \mathcal{F}$ and $t \in \mathcal{T}$, let $f_t g \in \mathcal{F}$ be the act defined in the following way: For all $s \in \mathcal{S}$ and $t' \in \mathcal{T}$,

$$(f_t g)(s,t') = \begin{cases} f(s,t') & \text{if } t' < t, \\ g(s,t'-t) & \text{if } t' \geq t. \end{cases}$$

Therefore $f_t g$ is the act in which g is shifted until time t and it is replaced by act f in the interval $[0,t)$.

For all $f, g \in \mathcal{F}$ and $E \in \mathcal{B}$, let $f_E g \in \mathcal{F}$ be the act defined in the following way: For all $s \in \mathcal{S}$ and $t \in \mathcal{T}$,

$$(f_E g)(s,t) = \begin{cases} f(s,t) & \text{if } (s,t) \in E, \\ g(s,t) & \text{if } (s,t) \notin E. \end{cases}$$

Therefore $f_E g$ is the act in which g is replaced by act f on event $E \in \mathcal{B}$. Moreover, for all $E_\mathcal{S} \in \mathcal{B}_\mathcal{S}$, we will write $f_{E_\mathcal{S}} g$ instead of $f_{E_\mathcal{S} \times \mathcal{T}} g$. Likewise, we write $f_{E_\mathcal{T}} g$ instead of $f_{\mathcal{S} \times E_\mathcal{T}} g$ for all $E_\mathcal{T} \in \mathcal{B}_\mathcal{T}$. Furthermore, if $E, F \in \mathcal{B}$ are disjoint, then, for all $f, g, h \in \mathcal{F}$, we denote by $f_E g_F h$ the element of $\mathcal{F}$ defined by $f_E(g_F h)$.

A subset $E \in \mathcal{B}$ is said to be *null* if $f \sim g$ for all $f, g \in \mathcal{F}$ such that $f(s) = g(s)$ for all $s \in \mathcal{S} \times \mathcal{T} \setminus E$. A sequence $\{E_n,\ n \geq 1\}$ of subsets in $\mathcal{B}$ is said to be decreasing if $E_{n+1} \subseteq E_n$ for all $n \geq 1$. A decreasing sequence of subsets in $\mathcal{B}$ is said to be *vanishing* if its intersection is empty and *almost-vanishing* if its intersection is null.

3 Axioms

We now present the six axioms that our main result invokes. The first one is a version of Machina and Schmeidler's (1992) P4* (Strong Comparative Probability) that applies to time preferences $\succsim_\mathcal{T}$. In this context, it requires the comparison of two disjoint time periods to be independent not only of the stream of outcomes obtained outside the two time periods, but also of the outcomes obtained on these time periods.

$\mathcal{T}$-SEPARABILITY: For all disjoint $E_{\mathcal{T}}, F_{\mathcal{T}} \in \mathcal{B}_{\mathcal{T}}$, all $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\mathcal{T}}$ and all $x^*, x, y^*, y \in \mathcal{X}$ with $x^* \succ_{\mathcal{T}} x$ and $y^* \succ_{\mathcal{T}} y$, $x^*_{E_{\mathcal{T}}} x_{F_{\mathcal{T}}} \mathbf{x} \succsim_{\mathcal{T}} x_{E_{\mathcal{T}}} x^*_{F_{\mathcal{T}}} \mathbf{x}$ if and only if $y^*_{E_{\mathcal{T}}} y_{F_{\mathcal{T}}} \mathbf{y} \succsim_{\mathcal{T}} y_{E_{\mathcal{T}}} y^*_{F_{\mathcal{T}}} \mathbf{y}$.

Our next axiom is a fairly standard monotonicity condition for time preferences $\succsim_{\mathcal{T}}$. If a deterministic act yields a better outcome than another one at every time, then the first one is preferred. In addition, the latter preference is strict whenever the former one is strict for every time in some non-null time period. Note that such an axiom would not be needed if time were discrete. This is because the assumption of discrete time allows one to derive inductively the monotonicity of $\succsim_{\mathcal{T}}$ from the axiom of stationarity. Under continuous time, we rather need to postulate $\mathcal{T}$-MONOTONICITY explicitly.

$\mathcal{T}$-MONOTONICITY: For all $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\mathcal{T}}$, if $\mathbf{x}(t) \succsim_{\mathcal{T}} \mathbf{y}(t)$ for all $t \in \mathcal{T}$, then $\mathbf{x} \succsim_{\mathcal{T}} \mathbf{y}$; if additionally, $\mathbf{x}(t) \succ_{\mathcal{T}} \mathbf{y}(t)$ for all t in some non-null subset in $\mathcal{B}_{\mathcal{T}}$, then $\mathbf{x} \succ_{\mathcal{T}} \mathbf{y}$.

The next axiom imposes a form of measurability of time preference. It requires the agent to always be able to determine whether or not the outcome she obtains is preferred to any given deterministic act. Axiom $\mathcal{T}$-MEASURABILITY is not needed if $\mathcal{F}$ is restricted by finiteness. In contrast, countable additivity is an important feature of the representation we obtain in the next section. This feature forces us to restrict the domain $\mathcal{F}$ of preference by measurability and to commit to the measurability axiom for dealing adequately with infinitely valued acts.

$\mathcal{T}$-MEASURABILITY: For all $\mathbf{x} \in \mathcal{F}_{\mathcal{T}}$, the subsets $\{x \in \mathcal{X},\ x \succ_{\mathcal{T}} \mathbf{x}\}$ and $\{x \in \mathcal{X},\ \mathbf{x} \succ_{\mathcal{T}} x\}$ are measurable.

MONOTONE CONTINUITY requires a strict preference between two acts to continue to hold when their outcomes are changed on sufficiently small subsets of $\mathcal{S} \times \mathcal{T}$. It is a version of the classic axiom of Villegas (1964) and Arrow (1970) that applies here to both uncertainty and time, see also Kopylov (2010). Its application to uncertainty yields the countable additivity of subjective probability while its application to (continuous) time provides the existence of a time equivalent for every act. (See Lemma A3 in Appendix A.) In contrast, in the discrete time case, time equivalents can be obtained by assuming topological structure for the outcome space and an adequate axiom of continuity of preference.

We acknowledge that MONOTONE CONTINUITY is difficult to test empirically, for instance through revealed-preference approaches. Nonetheless, it carries clear normative appeal, capturing a form of robustness of preferences to sufficiently small changes in probability. Several authors relate MONOTONE CONTINUITY and countable additivity to immunity to generalized Dutch books. See, for instance, Seidenfeld and Schervish (1983) and Williamson (1999). In the words of Arrow (1970), "The assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem."

MONOTONE CONTINUITY: For all $f, g \in \mathcal{F}$ such that $f \succ g$, all $x \in \mathcal{X}$ and all vanishing sequences $\{E_n,\ n \geq 1\}$ of subsets in $\mathcal{B}$, there is $N \geq 1$ such that $x_{E_N} f \succ g$ and $f \succ x_{E_N} g$.

The next axiom, STATIONARITY, requires that a preference between two acts be preserved when their payments are delayed until any time $t \in \mathcal{T}$ and the payments up to t are kept the same. In doing so, it extends the logic of the original axiom of Koopmans (1960) from discrete to continuous time and accommodates the presence of uncertainty, see also Hara (2016).

STATIONARITY: For all $t \in \mathcal{T}$ and $f, g, h \in \mathcal{F}$, $f \succsim g$ if and only if $h_t f \succsim h_t g$.

STATIONARITY has far-reaching implications in our framework. First, it implies the independence of uncertainty preferences from time in the following sense: For all $t \in \mathcal{T}$ and $\phi, \chi, \psi \in \mathcal{F}_{\mathcal{S}}$,

$$\phi \succsim_{\mathcal{S}} \chi \iff \psi_t \phi \succsim \psi_t \chi.$$

The ranking $\psi_t \phi \succsim \psi_t \chi$ is between acts that only differ from each other after time t and can be understood as a preference for ϕ over χ at time t. Hence, it implies that preferences over stochastic acts are invariant over time. In addition, STATIONARITY makes sure that the uncertainty preferences, once adequately reformulated in the AA framework, satisfy independence (Lemma B2). Finally, it also implies exponential discounting of time preferences and hence rules out other (plausible) alternatives (Proposition A2). We do not know at this stage if it is possible to adapt it to weaker forms of stationarity.

Our final axiom, DOMINANCE, requires a preference for an act over a second one when the first one yields a better deterministic act at every state. It also complements this requirement with a strict version. This axiom can be seen as a version of the Monotonicity axiom that Anscombe and Aumann (1963) use in the context of a second source of uncertainty instead of time.

DOMINANCE: For all $f, g \in \mathcal{F}$, if $f(s, \cdot) \succsim_{\mathcal{T}} g(s, \cdot)$ for all $s \in \mathcal{S}$, then $f \succsim g$; if, additionally, $f(s, \cdot) \succ_{\mathcal{T}} g(s, \cdot)$ for all s in some non-null subset in $\mathcal{B}_{\mathcal{S}}$, then $f \succ g$.

Furthermore, DOMINANCE implies the independence of time preferences from uncertainty in the following sense: For all non-null $E_{\mathcal{S}} \in \mathcal{B}_{\mathcal{S}}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{F}_{\mathcal{T}}$,

$$\mathbf{x} \succsim_{\mathcal{T}} \mathbf{y} \iff \mathbf{x}_{E_{\mathcal{S}}}\mathbf{z} \succsim \mathbf{y}_{E_{\mathcal{S}}}\mathbf{z}.$$

In this expression, the ranking $\mathbf{x}_{E_{\mathcal{S}}}\mathbf{z} \succsim \mathbf{y}_{E_{\mathcal{S}}}\mathbf{z}$ involves deterministic acts that only differ from each other on $E_{\mathcal{S}}$ and can be understood as a preference for $\mathbf{x}$ over $\mathbf{y}$ conditional upon observing that $E_{\mathcal{S}}$ holds. The preference over deterministic acts is hence independent from the information on the state space that the agent may acquire. Note that we can state a similar axiom in which states and time are interchanged, with the same normative interpretation. Such an axiom is necessary, as it is implied by the representation. However, it is an open question whether it could replace some of the $\mathcal{T}$-axioms in order to prove sufficiency.

Finally, we will show in Section 5 that DOMINANCE and STATIONARITY play a key role in implying neutrality to ambiguity. It is hence possible in principle to accommodate the Ellsberg (1961) pattern of choice and, more generally, ambiguity aversion, by modifying these axioms. For instance, Bastianello and Faro (2023) use a version that is restricted by comonotonicity à la Schmeidler (1989). Moreover, while we formulated the DOMINANCE axiom state-wise, the same normative justifications could be used to formulate a time-wise dominance axiom. However, it is an open question of how one should change the other axioms to obtain the same representation. See Monet and Vergopoulos (2024) for an implementation of this alternative in the context of a second source of uncertainty instead of time.

4 Main result

For all $\lambda > 0$, let F_λ be the function from $\mathcal{T}$ to $[0,1]$ defined by $F_\lambda(t) = 1 - e^{-\lambda t}$ for all $t \in \mathcal{T}$. Therefore there exists a unique countably additive measure ϵ_λ on $\mathcal{B}_\mathcal{T}$ such that $\epsilon_\lambda[0,t] = F_\lambda(t)$ for all $t \in \mathcal{T}$.

Fix $\lambda > 0$ and a countably additive probability measure μ on $\mathcal{B}_\mathcal{S}$. Then, there exists a unique countably additive probability measure on $\mathcal{B}$, denoted by $\mu \times \epsilon_\lambda$, such that $(\mu \times \epsilon_\lambda)(E_\mathcal{S} \times E_\mathcal{T}) = \mu(E_\mathcal{S}) \cdot \epsilon_\lambda(E_\mathcal{T})$ for all $E_\mathcal{S} \in \mathcal{B}_\mathcal{S}$ and $E_\mathcal{T} \in \mathcal{B}_\mathcal{T}$.

Theorem 1 *$\succsim$ satisfies $\mathcal{T}$-Separability, $\mathcal{T}$-Monotonicity, $\mathcal{T}$-Measurability, Monotone Continuity, Stationarity and Dominance if and only if there exist $\lambda > 0$, a nonconstant, bounded and measurable function u from $\mathcal{X}$ to $\mathbb{R}$ and a countably additive probability measure μ on $\mathcal{B}_\mathcal{S}$ such that, for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff \int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)]d(\mu \times \epsilon_\lambda)(s,t) \geq \int_{\mathcal{S}\times\mathcal{T}} u[g(s,t)]d(\mu \times \epsilon_\lambda)(s,t).$$

Moreover, λ and μ are unique, and u is unique up to positive affine transformation.

Theorem 1 characterizes representations of preferences on acts where the agent evaluates outcomes through a utility function u representing her tastes, discounts future utility levels according to the exponential rule with respect to parameter λ and evaluates the likelihood of uncertain events through a probability measure μ representing her subjective beliefs. Therefore, we obtain an axiomatic characterization of both Exponential Discounting and Subjective Expected Utility. In greater detail, suppose the triple (λ, u, μ) provides a representation of $\succsim$ as in Theorem 1. Then, time preferences have the following representation: For all $\mathbf{x}, \mathbf{y} \in \mathcal{F}_\mathcal{T}$,

$$\mathbf{x} \succsim_\mathcal{T} \mathbf{y} \iff \int_\mathcal{T} u[\mathbf{x}(t)]d\epsilon_\lambda(t) \geq \int_\mathcal{T} u[\mathbf{y}(t)]d\epsilon_\lambda(t).$$

In the particular case where $u \circ \mathbf{x}$ and $u \circ \mathbf{y}$ are Riemann integrable functions from $\mathcal{T}$ to $\mathbb{R}$, we obtain the following more familiar representation which makes explicit the exponential discounting of future utility levels

$$\mathbf{x} \succsim_\mathcal{T} \mathbf{y} \iff \int_\mathcal{T} e^{-\lambda t} u[\mathbf{x}(t)]dt \geq \int_\mathcal{T} e^{-\lambda t} u[\mathbf{y}(t)]dt.$$

The classic axiomatization of exponential discounting of Koopmans (1960) uses discrete time and topological structure on the outcome space. (See also Bleichrodt et al. (2008).) Such structure leads to the representation by invoking the Debreu (1960) theorem for additive separability.

As mentioned in Section 1, axiomatizations of intertemporal preferences over acts on a continuous time domain are recent. Moreover, most of the papers impose restrictions on the domain of acts (Harvey and Østerdal (2012) focus on piecewise continuous functions, Hara (2016) on cadlad functions, Kopylov (2010) and Webb (2016) on piecewise constant functions). In light of this literature, it may appear that our domain $\mathcal{F}_\mathcal{T}$ for time preferences is excessively rich. Harvey and Østerdal (2012) and Pivato (2021) elaborate arguments for restricting the domain of preference to acts that are truly feasible or, at least, easy to understand and visualize. This leads to their restrictions of continuity. However, the fact that $\mathcal{F}_\mathcal{T}$ includes much more complicated deterministic acts is not key for our result. What is necessary for the construction of subjective probability in Theorem 1 is only the inclusion of all piecewise constant deterministic acts in the domain of time preferences. Furthermore, since our domain includes infinitely-valued deterministic acts, it covers all of the class of continuous and piecewise continuous deterministic acts.

Next, and still in the context of Theorem 1, uncertainty preferences admit the following representation: For all $\phi, \chi \in \mathcal{F}_\mathcal{S}$,

$$\phi \ \succsim_\mathcal{S} \ \chi \quad \iff \quad \int_\mathcal{S} u[\phi(s)]d\mu(s) \ \geq \ \int_\mathcal{S} u[\chi(s)]d\mu(s).$$

Hence, Theorem 1 provides a fairly standard SEU representation of uncertainty preferences. A remarkable feature of this representation is that the state and outcome spaces are left totally arbitrary. Indeed, unlike that of Savage (1954), it does not require the state space to be uncountable and subjective probability to be nonatomic. Unlike those of Anscombe and Aumann (1963) and Wakker (1989), it does not assume objective probabilities or topological structure on the outcome space. As our proof sketch below clarifies, what allows us to dispense with such richness conditions is truly the assumption of continuous time.

Yet an unusual feature of this representation is the countable additivity of subjective probability. The literature offers several arguments both in favor and against this property. In our case where preferences apply to functions on the Cartesian product $\mathcal{S} \times \mathcal{T}$, we think of countable additivity as a desirable feature. Indeed, letting $V(f)$ denote the expectation of $u \circ f$ under $\mu \times \epsilon_\lambda$ for all $f \in \mathcal{F}$, we obtain by the Fubini theorem that preferences can equivalently be represented by the following functionals:

$$V(f) \ = \ \int_\mathcal{S} \left(\int_\mathcal{T} u[f(s,t)]d\epsilon_\lambda(t) \right) d\mu(s) \ = \ \int_\mathcal{T} \left(\int_\mathcal{S} u[f(s,t)]d\mu(s) \right) d\epsilon_\lambda(t).$$

Hence, our agent analyzes every act $f \in \mathcal{F}$ both in terms of the stochastic deterministic act and the deterministic stochastic act it yields. Equivalently, the representation obtained in Theorem 1 can be understood as both Discounted Subjective Expected Utility and Subjective Expected Discounted Utility. This has some normative appeal. Indeed, though Theorem 1 only explicitly requires Dominance with respect to $\mathcal{S}$, this reformulation of the representation shows that a dual form of dominance with respect to $\mathcal{T}$ also holds.

Furthermore, the axioms used in Theorem 1 are all standard and "nontechnical" in the sense that they all admit a normative interpretation. For instance, the theorem dispenses with Savage's P6 and P7. What makes this possible is again the assumption of continuous time. Note here that our point is not to make a philosophical claim on the nature of time. We merely require an agent to be sophisticated enough to imagine time as a continuum and claim that doing so will help him quantify the uncertainty she faces.

We now briefly sketch the proof of Theorem 1 and explain the organization of the appendix. Appendix A constructs the discount rate λ. In particular, it uses $\mathcal{T}$-Separability to define a "comparative discounting relation" on $\mathcal{B}_\mathcal{T}$ similar to Savage's comparative likelihood relation. stationarity implies first that the comparative discounting relation has no atoms. This allows us to invoke a theorem of Villegas (1964) and obtain a numerical representation in the form of a measure, on $\mathcal{B}_\mathcal{T}$. From there, stationarity further implies that this measure is of the exponential type with respect to some λ, i.e. for all $t \in \mathcal{T}$, $\epsilon_\lambda[0,t] = 1 - e^{-\lambda t}$. Appendix B studies the restriction of preferences to the subdomain $\mathcal{F}_0$ collecting all $f \in \mathcal{F}$ for which there exist a finite measurable partition $\Pi_\mathcal{S}$ of $\mathcal{S}$ and a finite measurable partition $\Pi_\mathcal{T}$ of $\mathcal{T}$ such that f is constant on $E_\mathcal{S} \times E_\mathcal{T}$ for all $E_\mathcal{S} \in \Pi_\mathcal{S}$ and $E_\mathcal{T} \in \Pi_\mathcal{T}$. Thanks to the exponential measure obtained in Appendix A, each act in $f \in \mathcal{F}_0$ induces a finitely-valued and measurable function $\varphi(f)$ from $\mathcal{S}$ to the set of finitely-supported probabilities over outcomes: for outcome $x \in \mathcal{X}$, lottery $\varphi(f)(s)$ associates the probability $\epsilon_\lambda\{t \in \mathcal{T} | f(s,t) = x\}$. The collection $\mathcal{A}$ of such induced functions $\varphi(f)$ forms a domain that is technically identical to that of Anscombe and Aumann (1963) (AA). In fact, Dominance implies the existence and the AA Monotonicity of preferences over $\mathcal{A}$. Stationarity and Monotone Continuity further imply their AA Independence and AA Continuity respectively. Then, an application of the AA theorem yields a bounded and measurable utility function u and a subjective probability μ, thereby establishing our representation on $\mathcal{F}_0$. Appendix C first extends the representation to all bounded acts. The key is to construct, for each act $f \in \mathcal{F}$, a time equivalent $\mathbf{x} \in \mathcal{F}_\mathcal{T}$ such that $f \sim \mathbf{x}$ and show that f and $\mathbf{x}$ have necessarily the same value. From there, the representation is extended to arbitrary acts. In the two extension stages, the key axioms are $\mathcal{T}$-Monotonicity, Monotone Continuity and Dominance. Finally, Appendix D shows the necessity of the axioms and uniqueness of the representation.

This proof sketch shows how Theorem 1 is truly a purely subjective formulation of the AA theorem that appeals to continuous time and endogenous discounting in order to eschew objective probabilities on a second source of uncertainty. Hence, Theorem 1 provides a novel temporal interpretation of the AA framework (and, in particular, of the von Neumann and Morgenstern (1947) one), as well as a novel interpretation of AA Independence in terms of Stationarity. Finally, we conjecture that Theorem 1 lends itself easily, just like the AA theorem, to generalizations accommodating ambiguity aversion by appealing to weak versions of Stationarity à la Kochov (2015) and Bastianello and Faro (2023) or of Dominance.

5 Stationarity, dominance and subjective probability

This section illustrates in the simplest possible way the use of time equivalents to quantify uncertainty with emphasis on the role that Stationarity and Dominance play in delivering a genuine probability measure on the state space.

To simplify the argument, we assume from the outset that time equivalents exist, see Lemma A3. In particular, for all event E in the state space, there exists a number $t_E \geq 0$ such that the agent is indifferent between a bet on E that pays \$10 forever if the event obtains and \$0 otherwise and a deterministic stream of outcomes that yields \$10 up to t_E

and \$0 from t_E and onward.

Second, we assume a discounted expected utility representation of time preferences on binary deterministic acts, see Proposition A2. In this representation, the discount factor is denoted by $\lambda > 0$. Alternatively, one may think of λ as an objective discount rate. Importantly, we do not commit to any assumption on how uncertain outcomes are discounted. The discount factor λ only serves in the evaluation of deterministic acts.

In this context, we may define a function μ from the collection of events in the state space to $[0, 1]$ by setting

$$\mu(E) \; = \; 1 - e^{-\lambda t_E} \tag{1}$$

for every event E. Clearly, by construction, μ provides a representation of betting preferences in the following sense: For all events E and F, the agent prefers a bet on E to a bet on F if and only if $\mu(E) \geq \mu(F)$. This function μ has the flavor of a probability measure. However, at this stage, whether μ is additive remains unclear. Its additivity is in fact implied by Stationarity and Dominance as we now explain.

To illustrate, suppose two agents, denoted by 1 and 2, with same discount rate λ. Suppose also that E is an event in the state space such that $t^1_E \geq t^2_E$. This means that agent 1 needs to receive the payment of \$10 for a longer time to be indifferent to a bet on E. Hence, a bet on E is of higher value to him. This suggests that his subjective probability for E will be larger, as captured by Formula (1).

Our main result invokes the axioms of Stationarity and Dominance. Supposing that the agent is initially indifferent between two acts, Stationarity says in particular that she remains indifferent if the payments are delayed until any time t and the payments up to t are identical. Dominance says in particular that the agent is indifferent between two acts whenever they yield deterministic streams of outcomes that are indifferent to each other at every state. Suppose E and F are two disjoint events. We will show that the two axioms imply $\mu(E \cup F) = \mu(E) + \mu(F)$ for all disjoint events E, F and hence lead to standard probability measures.

It is instructive to consider first the particular and simpler case where E and F are complementary events, and the agent is indifferent between the bets on E and F. In this case, E and F have the same time equivalent, and we necessarily have $\mu(E) = \mu(F)$. There still remains to show $\mu(E) = \mu(F) = 1/2$. Let t be the value such that $e^{-\lambda t} = 1/2$. The agent is hence indifferent between $[0, t)$ and $[t, +\infty)$. We represent the various acts by matrices where the first and second column describe the outcomes obtained on E and F respectively, and the timeline is as indicated. (For instance, the second matrix in the formula below represents an act that pays 10 up to t if E obtains and 10 after t if F obtains.) Stationarity and Dominance then yield respectively the first and second indifferences below:

$$\begin{array}{cc|l} 10 & 0 & [0,t) \\ 10 & 0 & [t,+\infty) \end{array} \sim \begin{array}{cc|l} 10 & 0 & [0,t) \\ 0 & 10 & [t,+\infty) \end{array} \quad \text{and} \quad \begin{array}{cc|l} 10 & 0 & [0,t) \\ 0 & 10 & [t,+\infty) \end{array} \sim \begin{array}{cc|l} 10 & 10 & [0,t) \\ 0 & 0 & [t,+\infty) \end{array}$$

This shows that t is the time equivalent of a bet on E and hence leads to $\mu(E) = 1/2$. For instance, in the Ellsberg two-urn experiment, this argument implies an indifference between bets on the ambiguous urn and ones on the unambiguous one and hence leads to neutrality to ambiguity.

We now treat the case of general disjoint events E and F and still consider that t is specifically the value such that $e^{-\lambda t} = 1/2$. The third column in the matrices below then describes the outcomes obtained on the complement of $E \cup F$. (For instance, the third matrix in Formula (2) below represents an act that pays 10 after t and up to $t + t_{E\cup F}$ at every state.) First, applying Dominance and then Stationarity to the definition of the time equivalent of $E \cup F$ yields

$$\begin{array}{ccc|l} 10 & 10 & 0 & [0,t) \\ 0 & 0 & 0 & [t, t+t_{E\cup F}) \\ 0 & 0 & 0 & [t+t_{E\cup F}, \infty) \end{array} \sim \begin{array}{ccc|l} 0 & 0 & 0 & [0,t) \\ 10 & 10 & 0 & [t, t+t_{E\cup F}) \\ 10 & 10 & 0 & [t+t_{E\cup F}, \infty) \end{array} \sim \begin{array}{ccc|l} 0 & 0 & 0 & [0,t) \\ 10 & 10 & 10 & [t, t+t_{E\cup F}) \\ 0 & 0 & 0 & [t+t_{E\cup F}, \infty) \end{array} \tag{2}$$

Formula (2) provides a first way to eliminate uncertainty in the first act it features while maintaining a constant utility. Indeed, it suggests that the agent accepts to delay the gains of \$10 on $E \cup F$ from $[0,t)$ to $[t, t+t_{E\cup F})$ if, in exchange, the gain is delivered at every state including those not in $E \cup F$.

Applying next Dominance and then Stationarity to the definition of the time equivalent of F yields

$$\begin{array}{ccc|l} 10 & 10 & 0 & [0,t) \\ 0 & 0 & 0 & [t, t+t_F) \\ 0 & 0 & 0 & [t+t_F, \infty) \end{array} \sim \begin{array}{ccc|l} 10 & 0 & 0 & [0,t) \\ 0 & 10 & 0 & [t, t+t_F) \\ 0 & 10 & 0 & [t+t_F, \infty) \end{array} \sim \begin{array}{ccc|l} 10 & 0 & 0 & [0,t) \\ 10 & 10 & 10 & [t, t+t_F) \\ 0 & 0 & 0 & [t+t_F, \infty) \end{array} \tag{3}$$

Finally, let $t'_F \leq t$ be such that $1 - e^{-\lambda t'_F} = e^{-\lambda t}(1 - e^{-\lambda t_F})$. Hence, the value of the time interval $[0, t'_F)$ is half that of $[0, t_F)$. Applying again Dominance and then Stationarity to the definition of the time equivalent of E yields

$$\begin{array}{ccc|l} 10 & 0 & 0 & [0,t) \\ 10 & 10 & 10 & [t, t+t_F) \\ 0 & 0 & 0 & [t+t_F, \infty) \end{array} \sim \begin{array}{ccc|l} 10 & 10 & 10 & [0,t'_F) \\ 0 & 0 & 0 & [t'_F, t) \\ 10 & 0 & 0 & [t, t+t_E) \\ 10 & 0 & 0 & [t+t_E, \infty) \end{array} \sim \begin{array}{ccc|l} 10 & 10 & 10 & [0,t'_F) \\ 0 & 0 & 0 & [t'_F, t) \\ 10 & 10 & 10 & [t, t+t_E) \\ 0 & 0 & 0 & [t+t_E, \infty) \end{array} \tag{4}$$

Formulae (3) and (4) provide together another way to eliminate uncertainty in the same initial act. Indeed, Formula 3 shows that the agent accepts to delay the gain of \$10 on F from $[0,t)$ to $[t, t+t_F)$ if, in exchange, the gain is delivered at every state, which reduces partially her exposure to uncertainty. In addition, by Formula (4), she also accepts to advance the sure gain of \$10 obtained on $[t, t+t_F)$ to $[0, t'_F)$ and delay the gain of \$10 on E from $[0, t_E)$ into a sure gain of \$10 on $[t, t+t_E)$, which this time eliminates uncertainty completely.

Hence, the two axioms imply overall the indifference between the last acts of Formulae (2) and (4). As these two are purely deterministic streams of outcomes, the assumption of exponential discounting yields

$$e^{-\lambda t} - e^{-\lambda(t+t_{E\cup F})} \;=\; 1 - e^{-\lambda t'_F} \;+\; e^{-\lambda t} - e^{-\lambda(t+t_E)}$$

which, by Formula (1) and the definitions of t and t'_F, simplifies finally into $\mu(E \cup F) = \mu(E) + \mu(F)$ and establishes our claim.

Therefore, the argument here shows that Stationarity and Dominance are sufficient for the additivity of the set function on events defined by the exponential discounting of their time equivalents. Our main result, Theorem 1, extends this example into a full axiomatic characterization of Exponential Discounting and Subjective Expected Utility. Finally, as mentioned earlier, Stationarity and Dominance may be too restrictive to accommodate the Ellsberg (1961) pattern of choice and, more generally, ambiguity aversion. One may consider restricting Stationarity to comonotonic acts as in Bastianello and Faro (2023). Note that Formulae (2) and (4) already apply Stationarity to comonotonic acts. Hence, the restricted version of Stationarity only possibly yields nonindifference in Formula (3).

Appendix

A Discounting

In Appendix A, we construct a monotonely continuous and atomless qualitative probability on $\mathcal{B}_\mathcal{T}$ in the sense of Villegas (1964). Using his theorem, we derive a countably additive, nonatomic probability measure ϵ_λ on $\mathcal{B}_\mathcal{T}$ such that $\forall t \in \mathcal{T}$, $\epsilon[t, +\infty) = e^{-\lambda t}$.

By nontriviality, there exist $x^*, x_* \in \mathcal{X}$ such that $x^* \succ x_*$. Consider the binary relation $\succsim$ on $\mathcal{B}_\mathcal{T}$ defined as follows: For all $A, B \in \mathcal{B}_\mathcal{T}$,

$$A \succsim B \iff x^*_A x_* \succsim x^*_B x_*.$$

A subset $A \in \mathcal{B}_\mathcal{T}$ is called an *atom* if there exists no $B \in \mathcal{B}_\mathcal{T}$ such that $A \succ B \succ \varnothing$. We say that $\succsim$ is *atomless* if there are no atoms.

Lemma A1 *The following hold:*

(i) $\succsim$ *is complete and transitive,*

(ii) $\mathcal{T} \succ \varnothing$ *and* $A \succsim \varnothing$ *for all* $A \in \mathcal{B}_\mathcal{T}$*,*

(iii) *For all* $A, B, C \in \mathcal{B}_\mathcal{T}$ *such that* $A \cap C = B \cap C = \varnothing$*,* $A \succsim B$ *if and only if* $A \cup C \succsim B \cup C$*,*

(iv) *For all* $A, B \in \mathcal{B}_\mathcal{T}$ *and all monotone increasing sequences* $\{A_n,\ n \geq 1\}$ *of subsets in* $\mathcal{B}_\mathcal{T}$ *converging to* A*, if* $B \succsim A_n$ *for all* $n \geq 1$*, then* $B \succsim A$*.*

(v) $\succsim$ *is atomless.*

Proof. (i) Obvious.

(ii) Note that by $\mathcal{T}$-Monotonicity, we have $A \succsim \varnothing$ for all $A \in \mathcal{B}_\mathcal{T}$. We also have $\mathcal{T} \succ \varnothing$ because $x^* \succ x_*$.

(iii) It follows from $\mathcal{T}$-SEPARABILITY. See Machina and Schmeidler's (1992), Section 4.2.

(iv) Remark that $\{A \setminus A_n,\, n \geq 1\}$ is a vanishing sequence. Suppose now $A \succ B$. Then, by MONOTONE CONTINUITY, we obtain $A \cap (A \setminus A_N)^c \succ B$ and hence $A_N \succ B$ for some $N \geq 1$, a contradiction.

(v) We prove that $\succsim$ is atomless in several steps.

Step 1. For all $A, B \in \mathcal{B}_\mathcal{T}$ and $t \in \mathcal{T}$, $A \succsim B$ if and only if $t + A \succsim t + B$.
Consider $A \in \mathcal{B}_\mathcal{T}$ and set $\mathbf{x} = x^*_A x_* \in \mathcal{F}_\mathcal{T}$. Then, $x_{*t}\mathbf{x} = x^*_{t+A} x_*$. The result follows from this construction and STATIONARITY.

Step 2. For all $t \in \mathcal{T}$, $\{t\} \sim \varnothing$ and $\{t\}$ is null. Suppose $\{t\} \succ \varnothing$. By Step 1, we must have $\{t'\} \succ \varnothing$ and $\{t'\}$ is an atom for all $t' \geq t$. Such a continuum of atoms contradicts Lemma 4 of Villegas (1964). Suppose now that $\mathbf{x}, \mathbf{y} \in \mathcal{F}_\mathcal{T}$ are equal to each other on $\mathcal{T} \setminus \{t\}$. Since $\{t\} \sim \varnothing$, $\mathcal{T}$-SEPARABILITY implies $x_{\{t\}}\mathbf{x} \sim y_{\{t\}}\mathbf{x}$ for all $x, y \in \mathcal{X}$ such that $x \not\sim y$. By $\mathcal{T}$-MONOTONICITY, this also holds true if $x \sim y$. By applying this to $x = \mathbf{x}(t)$ and $y = \mathbf{y}(t)$, we obtain $\mathbf{x} \sim \mathbf{y}$. Suppose now that $f, g \in \mathcal{F}$ are equal to each other on $\mathcal{S} \times (\mathcal{T} \setminus \{t\})$. By the previous point,we must have $f(s, \cdot) \sim g(s, \cdot)$ for all $s \in \mathcal{S}$ and obtain $f \sim g$ by DOMINANCE. Hence $\{t\}$ is null.

Step 3. For all $t \in \mathcal{T}$, $\{[t - 1/n, t),\, n \geq 1\}$ and $\{[t, t + 1/n),\, n \geq 1\}$ are respectively vanishing and almost-vanishing. The intersection of $\{[t - 1/n, t),\, n \geq 1\}$ is empty while that of $\{[t, t + 1/n),\, n \geq 1\}$ is equal to $\{t\}$ and hence null by Step 1.

Step 4. For all $B, C \in \mathcal{B}_\mathcal{T}$ such that $B \succ C$ and all almost-vanishing sequences $\{A_n, n \geq 1\}$, there exists $N \geq 1$ such that $B \succ C \cup A_N$. Let $A \in \mathcal{B}_\mathcal{T}$ denote the intersection of $\{A_n,\, n \geq 1\}$. Then, A is null. For all $n \geq 1$, let $A'_n = A_n \setminus A$. Then, $\{A'_n,\, n \geq 1\}$ is vanishing and, by MONOTONE CONTINUITY, there exists $N \geq 1$ such that $x^*_B x_* \succ x^*_{C \cup A_N \setminus A} x_*$. However, note that $x^*_{C \cup A_N \setminus A} x_*$ and $x^*_{C \cup A_N} x_*$ are equal to each other on the complement of $\mathcal{S} \times A$. Since the latter set is null, we obtain $x^*_B x_* \succ x^*_{C \cup A_N} x_*$ and, finally, $B \succ C \cup A_N$.

Step 5. $\succsim$ is atomless. Fix $A \in \mathcal{B}_\mathcal{T}$ such that $A \succ \varnothing$. Let $I^+ = \{t \in \mathcal{T} | A \succ A \cap [0, t)\}$ and $I^- = \{t \in \mathcal{T} | A \cap [0, t) \succ \varnothing\}$. First, note that I^+ and I^- are nonempty. Indeed, $\{A \cap [0, 1/n),\, n \geq 1\}$ is almost-vanishing. By Step 4, we obtain $A \succ A \cap [0, 1/N)$ for some $N \geq 1$. Then, $1/N \in I^+$. Likewise, $\{[n, +\infty),\, n \geq 1\}$ is vanishing. By MONOTONE CONTINUITY, we obtain $A \cap [0, M) \succ \varnothing$ for some $M \geq 1$. Then, $M \in I^-$. Second, I^+ and I^- are open in $\mathcal{T}$. Indeed, fix $t \in I^+$. Then, $A \succ A \cap [0, t)$. Since $\{A \cap [t, t + 1/n),\, n \geq 1\}$ is almost-vanishing, there exists $N \geq 1$ such that $A \succ A \cap [0, t + 1/N)$ by Step 4. Then, $t + 1/N \in I^+$ and, by $\mathcal{T}$-MONOTONICITY, $(t - 1/N, t + 1/N) \subseteq I^+$. Likewise, fix $t \in I^-$ so that $A \cap [0, t) \succ \varnothing$. Since $\{[t - 1/n, t),\, n \geq 1\}$ is vanishing, MONOTONE CONTINUITY yields the existence of $M \geq 1$ such that $A \cap [0, t - 1/M) \succ \varnothing$. Then, $t - 1/M \in I^-$ and, by $\mathcal{T}$-MONOTONICITY, $(t - 1/M, t + 1/M) \subseteq I^-$. Finally, suppose first that I^+ and I^- overlap. Then, let $t \in I^+ \cap I^-$. We have $A \succ A \cap [0, t) \succ \varnothing$ so that A cannot be an atom. If I^+ and I^- are disjoint, then, since $\mathcal{T}$ is connected, there must exist $t \in \mathcal{T}$ such that $t \notin I^+$ and $t \notin I^-$. By $\mathcal{T}$-MONOTONICITY and (i), this implies $A \cap [0, t) \sim A$ and $A \cap [0, t) \sim \varnothing$, a contradiction. □

Proposition A2 *There exists a unique countably additive and nonatomic probability measure ϵ_λ on $\mathcal{B}_\mathcal{T}$ with $\epsilon_\lambda[0,t) = 1 - e^{-\lambda t} = F_\lambda(t)$ such that $A \succsim B$ if and only if $\epsilon_\lambda(A) \geq \epsilon_\lambda(B)$ for all $A, B \in \mathcal{B}_\mathcal{T}$.*

Proof. By Lemma A1, $\succsim$ is a monotonely continuous and atomless qualitative probability on $\mathcal{B}_\mathcal{T}$ in the sense of Villegas (1964). By his Theorem 3, Section 4, there exists a unique countably additive and nonatomic probability measure ϵ on $\mathcal{B}_\mathcal{T}$ providing a representation of $\succsim$. By Step 1 in the proof of Lemma A1(v), for all $A, B \in \mathcal{B}_\mathcal{T}$ and $t \in \mathcal{T}$, $A \succsim B$ if and only if $t + A \succsim t + B$. By the uniqueness of the representation of $\succsim$ on $\mathcal{B}_\mathcal{T}$, we obtain: For all $A \in \mathcal{B}_\mathcal{T}$ and $t \in \mathcal{T}$,

$$\epsilon(A) = \frac{\epsilon(t + A)}{\epsilon[t, +\infty)}. \tag{A1}$$

In particular, for $A = [t', +\infty)$, we obtain $\epsilon[t + t', +\infty) = \epsilon[t, +\infty) \cdot \epsilon[t', +\infty)$ for all $t, t' \in \mathcal{T}$. By standard arguments, we must have $\epsilon[t, +\infty) = e^{-\lambda t}$ for all $t \in \mathcal{T}$ and some $\lambda > 0$. Then, by countable additivity and uniqueness in the Caratheodory extension theorem, we have $\epsilon = \epsilon_\lambda$. □

We conclude Appendix A showing how to construct time equivalents for acts $f \in \mathcal{F}$ bounded by two outcomes.

Lemma A3 *For all $f \in \mathcal{F}$ and $x, y \in \mathcal{X}$ such that $x \succ f \succ y$, there exists $A \in \mathcal{B}_\mathcal{T}$ such that $f \sim x_A y$. Moreover, we may assume $A = [0, t)$ for some $t \in \mathcal{T}$.*

Proof. Remark that for all $A, B \in \mathcal{B}_\mathcal{T}$ and $x, y \in \mathcal{X}$ such that $x \succ y$, $A \succsim B$ if and only if $x_A y \succsim x_B y$. This follows noting that $\mathcal{T}$-Separability implies the following form of Savage's (1954) P4: For all $A, B \in \mathcal{B}_\mathcal{T}$ and $x, y, x', y' \in \mathcal{X}$ such that $x \succ y$ and $x' \succ y'$, $x_A y \succsim x_B y$ if and only if $x'_A y' \succsim x'_B y'$. See Machina and Schmeidler's (1992) Section 4.2.

Now, let $I^- = \{\epsilon_\lambda(A) | f \succ x_A y,\ A \in \mathcal{B}_\mathcal{T}\}$. Clearly, $0 \in I^-$ and $1 \notin I^-$. Moreover, by $\mathcal{T}$-Monotonicity and the nonatomicity of ϵ_λ, if $q \in I^-$ and $q' \leq q$, then $q' \in I^-$. Now, fix $q \in I^-$. We will construct some $q' \in I^-$ such that $q' > q$. Since $q \in I^-$, there exists $A \in \mathcal{B}_\mathcal{T}$ such that $q = \epsilon_\lambda(A)$ and $f \succ x_A y$. Moreover, we must have $q < 1$. Then, there exists $t \in \mathcal{T}$ such that $F_\lambda(t) = \epsilon_\lambda(A)$. Then, set $B = [0, t) \in \mathcal{B}_\mathcal{T}$. We have $\epsilon_\lambda(B) = \epsilon_\lambda(A)$. By Proposition A2 and the first part of the proof, we obtain $x_A y \sim x_B y$ and hence $f \succ x_B y$. By Monotone Continuity, there exists $N \geq 1$ such that $f \succ x_{[N,+\infty)}(x_B y) = x_{[N,+\infty) \cup B} y$. Therefore, we have $q' \in I^-$ where $q' = \epsilon_\lambda([N, +\infty) \cup B)$. Now, if $[N, +\infty)$ and B are disjoint we have $q' > \epsilon_\lambda(B) = q$. If the two sets are not disjoint, then $[N, +\infty) \cup B = \mathcal{T}$ and $1 = q' \in I^-$, a contradiction. This shows that $I^- = [0, \underline{q})$ for some $\underline{q} \in (0, 1)$.

Let $I^+ = \{\epsilon_\lambda(A) | x_A y \succ f,\ A \in \mathcal{B}_\mathcal{T}\}$. Proceeding as in the previous paragraph, we obtain $\overline{q} \in (0, 1)$ such that $I^+ = (\overline{q}, 1]$.

Now, we must have $\underline{q} \leq \overline{q}$. Otherwise, consider any $q \in [0, 1]$ such that $\underline{q} > q > \overline{q}$. Then, there exist $A, B \in \mathcal{B}_\mathcal{T}$ such that $\epsilon_\lambda(A) = \epsilon_\lambda(B) = q$ with $f \succ x_A y$ and $x_B y \succ f$.

However, by Proposition A2 and the first part of the proof, $\epsilon_\lambda(A) = \epsilon_\lambda(B)$ implies $x_A y \sim x_B y$, a contradiction. Finally, take any $q \in [0,1]$ such that $\underline{q} \leq q \leq \bar{q}$. Then, $q \notin I^-$ and $q \notin I^+$. By nonatomicity, there exists $A \in \mathcal{B}_\mathcal{T}$ such that $\epsilon_\lambda(A) = q$. Then, $f \succsim x_A y$ and $x_A y \succsim f$. This implies $f \sim x_A y$. □

B Utility and probability

In Appendix B we construct an Anscombe and Aumann (1963) setup. This allows us to derive a countably additive probability measure μ on $\mathcal{B}_\mathcal{S}$ and a nonconstant function u from $\mathcal{X}$ to $\mathbb{R}$ using Anscombe and Aumann (1963). Moreover, we prove our representation result for finitely valued acts.

We start by introducing some notation.

- $\mathcal{F}^0_\mathcal{T}$ denotes the set of all finitely-valued and measurable deterministic acts.
- $\mathcal{L}$ denotes the set of all (finitely-supported) lotteries on $\mathcal{X}$.
- $\lambda > 0$ is as in Proposition A2.
- $\phi : \mathcal{F}^0_\mathcal{T} \to \mathcal{L}$ is defined for all $\mathbf{x} \in \mathcal{F}^0_\mathcal{T}$ and $x \in \mathcal{X}$ by $\phi(\mathbf{x})(x) = \epsilon_\lambda\{\mathbf{x}(\cdot) = x\}$. Note that, by the nonatomicity of ϵ_λ, this mapping is surjective.
- $\mathcal{A} = \{\alpha : \mathcal{S} \to \mathcal{L} | \alpha \text{ is finitely-valued and measurable}\}$. Measurability of $\alpha \in \mathcal{A}$ means that $\alpha^{-1}(\{l\}) \in \mathcal{B}_\mathcal{S}$ for all $l \in \mathcal{L}$.
- $\mathcal{F}_0$ denotes the set of all acts $f \in \mathcal{F}$ such that there exist a finite measurable partition $\Pi_\mathcal{S}$ of $\mathcal{S}$ and a finite measurable partition $\Pi_\mathcal{T}$ of $\mathcal{T}$ such that f is constant on $E_\mathcal{S} \times E_\mathcal{T}$ for all $E_\mathcal{S} \in \Pi_\mathcal{S}$ and $E_\mathcal{T} \in \Pi_\mathcal{T}$. Note that $\mathcal{F}^0_\mathcal{T} \subset \mathcal{F}_0$.
- For all $f \in \mathcal{F}_0$, $\varphi(f) : \mathcal{S} \to \mathcal{L}$ is defined by $\varphi(f)(s)(x) = \epsilon_\lambda\{f(s,\cdot) = x\}$ for all $s \in \mathcal{S}$ and $x \in \mathcal{X}$. In words, $\varphi(f)(s)$ is the probability distribution induced by $f(s,\cdot)$ under ϵ_λ.
- $\varphi : \mathcal{F}_0 \to \mathcal{A}$ is surjective function from $\mathcal{F}_0$ to $\mathcal{A}$.
- Finally we remark that for all $\mathbf{x} \in \mathcal{F}^0_\mathcal{T}$, $f \in \mathcal{F}_0$ and $s \in \mathcal{S}$

$$\varphi(\mathbf{x})(s) = \phi(\mathbf{x}) \quad \text{and} \quad \varphi(f)(s) = \phi(f(s,\cdot)). \tag{B1}$$

Note that $\mathcal{A}$ is the standard Anscombe and Aumann (1963) framework. For $l, m \in \mathcal{L}$ and $\mu \in [0,1]$, we define the mixture $\mu l + (1-\mu)m \in \mathcal{L}$ in the usual way by setting, for all $x \in \mathcal{X}$,

$$(\mu l + (1-\mu)m)(x) = \mu l(x) + (1-\mu)m(x).$$

This mixture operation extends readily to $\mathcal{A}$. For $\alpha, \beta \in \mathcal{A}$ and $\mu \in [0,1]$, let $\mu\alpha + (1-\mu)\beta \in \mathcal{A}$ by defined by, for all $s \in \mathcal{S}$,

$$(\mu\alpha + (1-\mu)\beta)(s) = \mu\alpha(s) + (1-\mu)\beta(s).$$

We will construct now a preference relation $\succsim_{\mathcal{A}}$ on $\mathcal{A}$ satisfying Anscombe and Aumann (1963) axioms. A binary relation $\succsim_{\mathcal{A}}$ on $\mathcal{A}$ is *monotonic* if, for all $\alpha, \beta \in \mathcal{A}$ such that $\alpha(s) \succsim_{\mathcal{A}} \beta(s)$ for all $s \in \mathcal{S}$, we have $\alpha \succsim_{\mathcal{A}} \beta$. We say that $\succsim_{\mathcal{A}}$ satisfies *Independence* if, for all $\alpha, \beta, \gamma \in \mathcal{A}$ and $\mu \in (0,1)$, $\alpha \succsim_{\mathcal{A}} \beta$ holds if and only if $\mu\alpha + (1-\mu)\gamma \succsim_{\mathcal{A}} \mu\beta + (1-\mu)\gamma$ holds. We say that $\succsim_{\mathcal{A}}$ satisfies *Continuity* if, for all $\alpha \in \mathcal{A}$ and $x, y \in \mathcal{X}$ such that $x \succsim_{\mathcal{A}} \alpha \succsim_{\mathcal{A}} y$, there exists $\mu \in [0,1]$ such that $\alpha \sim_{\mathcal{A}} \mu x + (1-\mu)y$.

Lemma B1 *There exists a nontrivial, complete, transitive and monotonic binary relation $\succsim_{\mathcal{A}}$ on $\mathcal{A}$ such that, for all $f, g \in \mathcal{F}_0$, $f \succsim g$ if and only if $\varphi(f) \succsim_{\mathcal{A}} \varphi(g)$.*

Proof. *Step 1. There exists a nontrivial, complete and transitive binary relation $\succsim_{\mathcal{L}}$ on $\mathcal{L}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\mathcal{T}}^0$, $\mathbf{x} \succsim_{\mathcal{T}} \mathbf{y}$ if and only if $\phi(\mathbf{x}) \succsim_{\mathcal{L}} \phi(\mathbf{y})$.* The proof is similar to that of Machina and Schmeidler's (1992) Theorem 1. However, note that they assume P6 while we do not. In fact, they only use P6 in their Step 1 to construct the probability measure. We have already constructed the measure in Proposition A2 and do not need P6. The rest is identical.

Step 2. For all $f, g \in \mathcal{F}_0$, $\varphi(f) = \varphi(g)$ implies $f \sim g$. Take $f, g \in \mathcal{F}_0$ such that $\varphi(f) = \varphi(g)$. By the second equality in Formula (B1), we have $\phi(f(s,\cdot)) = \phi(g(s,\cdot))$ for all $s \in \mathcal{S}$. By Step 1, we obtain $f(s,\cdot) \sim_{\mathcal{T}} g(s,\cdot)$ for all $s \in \mathcal{S}$. Dominance finally yields $f \sim g$.

We define $\succsim_{\mathcal{A}}$ as follows: For all $\alpha, \beta \in \mathcal{A}$, we set $\alpha \succsim_{\mathcal{A}} \beta$ if and only if $f \succsim g$ for some $f, g \in \mathcal{F}_0$ such that $\varphi(f) = \alpha$ and $\varphi(g) = \beta$. Preference $\succsim_{\mathcal{A}}$ is well defined by Step 2.

Step 3. $\succsim_{\mathcal{A}}$ is nontrivial, complete, transitive and monotonic. Nontriviality, completeness and transitivity of $\succsim_{\mathcal{A}}$ follow from standard arguments. As for monotonicity, suppose $\alpha, \beta \in \mathcal{A}$ are such that $\alpha(s) \succsim_{\mathcal{A}} \beta(s)$ for all $s \in \mathcal{S}$. The first equality in Formula (B1) implies that $\succsim_{\mathcal{L}}$ and $\succsim_{\mathcal{A}}$ agree on $\mathcal{L}$. So we have $\alpha(s) \succsim_{\mathcal{L}} \beta(s)$ for all $s \in \mathcal{S}$. The second equality in Formula (B1) and Step 1 then imply $f(s,\cdot) \succsim_{\mathcal{T}} g(s,\cdot)$ for all $s \in \mathcal{S}$ where $f, g \in \mathcal{F}_0$ are such that $\varphi(f) = \alpha$ and $\varphi(g) = \beta$. By Dominance, we obtain $f \succsim g$ and finally $\alpha \succsim_{\mathcal{A}} \beta$. □

Lemma B2 *$\succsim_{\mathcal{A}}$ satisfies Independence and Continuity.*

Proof. *Step 1. For all $t \in \mathcal{T}$ and $n \in \mathcal{L}$, there exists $\mathbf{z} \in \mathcal{F}_{\mathcal{T}}^0$ such that, for all $x \in \mathcal{X}$, $\epsilon_\lambda[\{\mathbf{z}(\cdot) = x\} \cap [0,t)]] = n(x) \cdot \epsilon_\lambda[0,t)$.* Let $\{x_1, \ldots, x_N\} \subseteq \mathcal{X}$ be the support of n and set $p_i = n(x_i)$ for all $i \in [1 \ldots N]$. By the continuity of F_λ, we can partition $[0,t)$ into intervals $[t_i, t_{i+1})$ for $i \in [0 \ldots N]$ with $t_0 = 0$ and $t_N = t$ and also $F_\lambda(t_{i+1}) - F_\lambda(t_i) = p_i \cdot \epsilon_\lambda[0,t)$ for all $i \in [1 \ldots N]$. Then, it is sufficient to take any $\mathbf{z} \in \mathcal{F}_{\mathcal{T}}^0$ constantly equal to x_i on each $[t_i, t_{i+1})$.

Step 2. For all $t \in \mathcal{T}$ and $\gamma \in \mathcal{A}$, there exists $h \in \mathcal{F}_0$ such that, for all $s \in \mathcal{S}$ and $x \in \mathcal{X}$, $\epsilon_\lambda[\{h(s,\cdot) = x\} \cap [0,t)]] = \gamma(s)(x) \cdot \epsilon_\lambda[0,t)$. Let $\Pi_{\mathcal{S}} = \{E_i, i \in [1 \ldots n]\}$ be a measurable partition of $\mathcal{S}$ such that γ is constantly equal to some $l_i \in \mathcal{L}$ for all $i \in [1 \ldots n]$. By Step 1, we can find $\mathbf{z}_i \in \mathcal{F}^0_{\mathcal{T}}$ such that, for all $x \in \mathcal{X}$,

$$\epsilon_\lambda[\{\mathbf{z}_i(\cdot) = x\} \cap [0,t)] \;=\; l_i(x) \cdot \epsilon_\lambda[0,t),$$

for all $i \in [1 \ldots n]$. Now, each $\mathbf{z}_i$ is finitely-valued. So there exists a finite measurable partition $\Pi_{\mathcal{T}}$ of $\mathcal{T}$ that is adapted to every $\mathbf{z}_i$. We define a function h from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{X}$ by setting $h(s,t) = \mathbf{z}_i(t)$ where $i \in [1 \ldots n]$ is such that $s \in E_i$. Clearly, h is finitely-valued and adapted to $\Pi_{\mathcal{S}} \times \Pi_{\mathcal{T}}$. Hence, h lies in $\mathcal{F}_0$ and has the desired property.

Step 3. $\succsim_{\mathcal{A}}$ satisfies Independence. Fix $\kappa \in (0,1)$ and $\alpha, \beta, \gamma \in \mathcal{A}$. By the continuity of F_λ, we can find $t \in \mathcal{T}$ such that $1 - \kappa = \epsilon_\lambda[0,t)$. Let $h \in \mathcal{F}_0$ be as in the Lemma Step 2. Since φ is surjective, we can find $f, g \in \mathcal{F}_0$ such that $\varphi(f) = \alpha$ and $\varphi(g) = \beta$. Next, for all $s \in \mathcal{S}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
\varphi(h_t f)(s)(x) &= \epsilon_\lambda[\{h_t f(s,\cdot) = x\}] \\
&= (1-\kappa) \cdot \epsilon_\lambda[\{h(s,\cdot) = x\} | [0,t)] \;+\; \kappa \cdot \epsilon_\lambda[t + \{f(s,\cdot) = x\} | [t, +\infty)] \\
&= (1-\kappa) \cdot \gamma(s)(x) \;+\; \kappa \cdot \epsilon_\lambda[\{f(s,\cdot) = x\}] \\
&= (1-\kappa) \cdot \gamma(s)(x) \;+\; \kappa \cdot \alpha(s)(x),
\end{aligned}$$

where the third equality is by Step 2 and Formula (A1). We obtain $\varphi(h_t f) = (1-\kappa)\gamma + \kappa\alpha$. A similar argument provides $\varphi(h_t g) = (1-\kappa)\gamma + \kappa\beta$. Finally, we have

$$\alpha \succsim_{\mathcal{A}} \beta \iff f \succsim g \iff h_t f \succsim h_t g,$$

where the first equivalence is by the definition of $\succsim_{\mathcal{A}}$ in Lemma B1, and the second one is by Stationarity. Then, Lemma B1 and the previous paragraph provide

$$\alpha \succsim_{\mathcal{A}} \beta \iff \varphi(h_t f) \succsim_{\mathcal{A}} \varphi(h_t g) \iff (1-\kappa)\gamma + \kappa\alpha \succsim_{\mathcal{A}} (1-\kappa)\gamma + \kappa\beta.$$

Step 4. $\succsim_{\mathcal{A}}$ satisfies Continuity. Suppose $\alpha \in \mathcal{A}$ and $x, y \in \mathcal{X}$ are such that $x \succsim_{\mathcal{A}} \alpha \succsim_{\mathcal{A}} y$. If $\alpha \sim_{\mathcal{A}} x$ or $\alpha \sim_{\mathcal{A}} y$, we are done. So we may suppose $x \succ_{\mathcal{A}} \alpha \succ_{\mathcal{A}} y$. Let $f \in \mathcal{F}_0$ be such that $\varphi(f) = \alpha$. Then, $x \succ f \succ y$. By Lemma A3, there exists $A \in \mathcal{B}_{\mathcal{T}}$ such that $f \sim x_A y$. Then, $\varphi(x_A y) = \kappa x + (1-\kappa) y$ with $\kappa = \epsilon_\lambda(A)$. Hence, we have $\alpha \sim_{\mathcal{A}} \kappa x + (1-\kappa) y$. □

Proposition B3 *There exists a countably additive probability measure μ on $\mathcal{B}_{\mathcal{S}}$ and a nonconstant, measurable and bounded function u from $\mathcal{X}$ to $\mathbb{R}$ such that, for all $f, g \in \mathcal{F}_0$,*

$$f \succsim g \iff \int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)] d(\mu \times \epsilon_\lambda)(s,t) \;\geq\; \int_{\mathcal{S}\times\mathcal{T}} u[g(s,t)] d(\mu \times \epsilon_\lambda)(s,t).$$

Moreover, μ is unique and u is unique up to positive affine transformation.

Proof. Step 1. Representation of $\succsim$. By Lemmata B1 and B2, we can apply the Schmeidler (1989) version of the Anscombe and Aumann (1963) theorem and obtain a nonconstant mixture-linear function v from $\mathcal{L}$ to $\mathbb{R}$ and a finitely-additive probability measure μ on $\mathcal{B}_\mathcal{S}$ such that, for all $\alpha, \beta \in \mathcal{A}$,

$$\alpha \succsim_\mathcal{A} \beta \iff E_\mu[v \circ \alpha] \geq E_\mu[v \circ \beta].$$

Monotone Continuity implies the countable additivity of μ. For instance, see Arrow (1970). Moreover, let u denote the function from $\mathcal{X}$ to $\mathbb{R}$ obtained as the restriction of v to $\mathcal{X}$. By mixture-linearity, we have $v(l) = E_l[u]$ for all $l \in \mathcal{L}$. Then, for all $f, g \in \mathcal{F}_0$, we have

$$f \succsim g \iff E_\mu[E_{\varphi(f)(\cdot)}[u]] \geq E_\mu[E_{\varphi(g)(\cdot)}[u]].$$

From there, the representation of $\succsim$ on $\mathcal{F}_0$ follows from the remark that, for all $f \in \mathcal{F}_0$

$$E_\mu[E_{\varphi(f)(\cdot)}[u]] = \int_\mathcal{S}\left(\int_\mathcal{T} u[f(s,t)]d\epsilon_\lambda(t)\right)d\mu(s) = \int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)]d(\mu \times \epsilon_\lambda)(s,t),$$

where the second equality invokes the Fubini theorem.

Step 2. u is measurable. For every $q \in \mathbb{R}$, we show that $U := u^{-1}(q, +\infty) \in \mathcal{B}_\mathcal{X}$. If $u(x) > q$ for all $x \in \mathcal{X}$, then $U = \mathcal{X} \in \mathcal{B}_\mathcal{X}$. If $u(x) \leq q$ for all $x \in \mathcal{X}$, then $U = \varnothing \in \mathcal{B}_\mathcal{X}$. In the remaining case, we have $u(x) \leq q < u(y)$ for some $x, y \in \mathcal{X}$. We can then find $\alpha \in [0,1)$ such that $q = \alpha u(y) + (1-\alpha)u(x)$. Let $t \in \mathcal{T}$ be such that $F_\lambda(t) = \alpha$, and set $\mathbf{x} = y_t x \in \mathcal{F}^0_\mathcal{T}$. We have $\int_\mathcal{T} u[\mathbf{x}(t)]d\epsilon_\lambda(t) = q$ and, by Step 1, $U = \{z \in \mathcal{X},\, z \succ \mathbf{x}\}$. Then, $U \in \mathcal{B}_\mathcal{X}$ by $\mathcal{T}$-Measurability. The measurability of $u^{-1}(-\infty, q)$ can be proved likewise.

Step 3. u is bounded. Suppose by way of contradiction that u is unbounded from above. Consider a partition $\{A_n, n \geq 1\}$ of $\mathcal{T}$ such that the A_n are successive intervals with A_1 starting from $t = 0$ and such that $\epsilon_\lambda(A_n) = 1/2^n$ for all $n \geq 1$. Let $\mathbf{x} \in \mathcal{F}_\mathcal{T}$ be such that the value of $\mathbf{x}$ over A_n is an outcome $x_n \in \mathcal{X}$ such that $u(x_n) \geq 2^n$. Such an outcome exists since u is unbounded from above. We may suppose that $u(x_{n+1}) > u(x_n)$ for all $n \geq 1$ without loss of generality.

Note that we have for all $N \geq 1$

$$\sum_{n=1}^{+\infty} \min(u(x_n), u(x_N)) \cdot \epsilon(A_n) \geq \sum_{n=1}^{N} u(x_n) \cdot \epsilon(A_n) \geq N$$

We prove that $\mathbf{x} \succsim z$ for all $z \in \mathcal{X}$. Fix $z \in \mathcal{X}$. By the previous formula, there exists $N \geq 1$ such that

$$\sum_{n=1}^{+\infty} \min(u(x_n), u(x_N)) \cdot \epsilon(A_n) \geq u(z).$$

Then, set $y = x_N$ and let $\mathbf{y} \in \mathcal{F}_\mathcal{T}$ be such that, for all $t \in \mathcal{T}$, $\mathbf{y}(t) = \mathbf{x}(t)$ if $y \succsim \mathbf{x}(t)$ and $\mathbf{y}(t) = y$ if $\mathbf{x}(t) \succ y$. Since u and $\mathbf{x}$ are measurable, so is $\mathbf{y}$. Furthermore, by $\mathcal{T}$-Monotonicity, $\mathbf{x} \succsim \mathbf{y}$ Furthermore, $\mathbf{y} \in \mathcal{F}^0_\mathcal{T}$. The previous formula and the representation on $\mathcal{F}^0_\mathcal{T}$ obtained in Step 1 yield $\mathbf{y} \succsim z$ and, finally, $\mathbf{x} \succsim z$.

Define now $\mathbf{z} \in \mathcal{F}_\mathcal{T}$ by $\mathbf{z}(t) = \mathbf{x}(t)$ for all $t \in \mathcal{T} \setminus A_1$ and $\mathbf{z}(t) = x_2$ for all $t \in A_1$. Since u and $\mathbf{x}$ are measurable, so is $\mathbf{z}$. We have $\epsilon_\lambda(A_1) > 0$ and A_1 is hence non-null by Step 1.

Then, $\mathcal{T}$-Monotonicity yields $\mathbf{z} \succ \mathbf{x}$. Since $\{\cup_{i\geq n} A_i,\, n \geq 1\}$ is a vanishing sequence, by Monotone Continuity there is $n \geq 2$ such that $\mathbf{z}_{t_n} x_1 \succ \mathbf{x}$, where $t_n \in \mathcal{T}$ denotes the lower bound of A_n. Note that $\mathbf{z}_{t_n} x_1$ lies in $\mathcal{F}^0_{\mathcal{T}}$. The preferred outcome in its range is x_{n-1}. By $\mathcal{T}$-Monotonicity, $x_{n-1} \succsim \mathbf{z}_{t_n} x_1$ and, therefore, $x_{n-1} \succ \mathbf{x}$, which contradicts the previous paragraph. Therefore u must be bounded from above. A similar argument shows that it is also bounded from below. □

C Representation

In Appendix C we prove sufficiency of the axioms for the representation in Theorem 1. First, we prove the representation for bounded deterministic acts in $\mathcal{F}_{\mathcal{T}}$ (Lemma C3), then for bounded acts in $\mathcal{F}$ (Lemma C4), and finally for general acts in $\mathcal{F}$ (Proposition C7).

We say that $f \in \mathcal{F}$ is *bounded* if there exist $x_0, x_1 \in \mathcal{X}$ such that $x_1 \succsim f(s,t) \succsim x_0$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$. Proposition B3 shows that u is bounded and measurable. We may then define functions $V : \mathcal{F} \to \mathbb{R}$ and $U : \mathcal{F}_{\mathcal{T}} \to \mathbb{R}$ by

$$V(f) \;=\; \int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)]d(\mu \times \epsilon_\lambda)(s,t) \quad \text{and} \quad U(\mathbf{x}) \;=\; \int_{\mathcal{T}} u[\mathbf{x}(t)]d\epsilon_\lambda(t).$$

Lemma C1 *For all bounded $f \in \mathcal{F}$, there exists $\mathbf{x} \in \mathcal{F}^0_{\mathcal{T}}$ such that $f \sim \mathbf{x}$.*

Proof. Suppose that $f \in \mathcal{F}$ is such that $x \succsim f(s,t) \succsim y$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$ and for some $x, y \in \mathcal{X}$. Then, by $\mathcal{T}$-Monotonicity, $x \succsim_{\mathcal{T}} f(s,\cdot) \succsim_{\mathcal{T}} y$ for all $s \in \mathcal{S}$. By Dominance, $x \succsim f \succsim y$. If $f \sim x$ or $f \sim y$, we are done. So we may suppose $x \succ f \succ y$. Then, the result follows from Lemma A3. □

For $A, B \in \mathcal{B}_{\mathcal{T}}$ and $\mathbf{x} \in \mathcal{F}_{\mathcal{T}}$, we write $A \perp B$ if $\epsilon_\lambda(A \cap B) = \epsilon_\lambda(A) \cdot \epsilon_\lambda(B)$ and $\mathbf{x} \perp A$ if $\{t \in \mathcal{T},\, \mathbf{x}(t) = x\} \perp A$ for all $x \in \mathcal{X}$. Moreover, for a given finite measurable partition $\{A_1, \ldots, A_N\}$ of $\mathcal{T}$ and sequence $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of elements of $\mathcal{F}^0_{\mathcal{T}}$, we denote by $\sum_{n=1}^N \mathbf{1}_{A_n}\mathbf{x}_n$ the element of $\mathcal{F}^0_{\mathcal{T}}$ that is equal to $\mathbf{x}_n$ on A_n for all $n \in [1 \ldots N]$.

Lemma C2 *For all $\mathbf{x} \in \mathcal{F}_{\mathcal{T}}$, all finite measurable partitions $\{A_1, \ldots, A_N\}$ of $\mathcal{T}$ and sequences $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of elements of $\mathcal{F}^0_{\mathcal{T}}$ such that $\mathbf{x}_n \perp A_m$ for all $n, m \in [1 \ldots N]$,*

(i) *If $\mathbf{x}(t) \succsim \mathbf{x}_n$ for all $t \in A_n$ and all $n \in [1 \ldots N]$, then $\mathbf{x} \succsim \sum_{n=1}^N \mathbf{1}_{A_n}\mathbf{x}_n$,*

(ii) *If $\mathbf{x}_n \succsim \mathbf{x}(t)$ for all $t \in A_n$ and all $n \in [1 \ldots N]$, then $\sum_{n=1}^N \mathbf{1}_{A_n}\mathbf{x}_n \succsim \mathbf{x}$.*

Proof. We only show (i). Consider first the case where $\mathbf{x}$ is constant on each cell of the partition, i.e. $\mathbf{x} = \sum_{n=1}^N \mathbf{1}_{A_n} x_n$ with $x_n \in \mathcal{X}$ for all $n \in [1 \ldots N]$. Then, by assumption, we have $x_n \succsim \mathbf{x}_n$ and Proposition B3 implies $u(x_n) \geq U(\mathbf{x}_n)$ for all $n \in [1 \ldots N]$. We obtain

$$U(\mathbf{x}) \;=\; \sum_{n=1}^N \epsilon_\lambda(A_n) \cdot u(x_n) \;\geq \sum_{n=1}^N \epsilon_\lambda(A_n) \cdot U(\mathbf{x}_n) \;=\; U(\sum_{n=1}^N \mathbf{1}_{A_n}\mathbf{x}_n),$$

where the last equality is because $\mathbf{x}_n \perp A_n$ for all $n \in [1 \dots N]$. Since $\mathbf{x}$ is an element of $\mathcal{F}_{\mathcal{T}}^0$, Proposition B3 yields the desired ranking.

Consider next the case where $\mathbf{x}$ has a minimum on each A_n, i.e. for each $n \in [1 \dots N]$, there exists $t_n \in A_n$ such that $\mathbf{x}(t) \succsim \mathbf{x}(t_n)$ for all $t \in A_n$ and set $x_n = \mathbf{x}(t_n)$. Then, by $\mathcal{T}$-MONOTONICITY, we have $\mathbf{x} \succsim \sum_{n=1}^N \mathbf{1}_{A_n} x_n$. Since $x_n \succsim \mathbf{x}_n$ for all $n \in [1 \dots N]$, we can apply the previous paragraph and obtain $\sum_{n=1}^N \mathbf{1}_{A_n} x_n \succsim \sum_{n=1}^N \mathbf{1}_{A_n} \mathbf{x}_n$. Transitivity allows to conclude.

Consider finally the general case and suppose by contradiction that $\sum_{n=1}^N \mathbf{1}_{A_n} \mathbf{x}_n \succ \mathbf{x}$. Let I collect all integers $n \in [1 \dots N]$ such that $\mathbf{x}$ has no minimum on A_n. Fix any $m \in I$. Let $\{t_n^m,\, n \geq 1\}$ be a sequence of points in $\mathcal{T}$ such that $\{u[\mathbf{x}(t_n^m)],\, n \geq 1\}$ is decreasing and converges to $\inf_{A_m} u \circ \mathbf{x}$. For all $n \geq 1$, define $B_n^m = \{t \in A_m | u[\mathbf{x}(t)] < u[\mathbf{x}(t_n^m)]\}$. Since $\mathbf{x}$ has no minimum on A_m, the sequence $\{B_n^m,\, n \geq 1\}$ is vanishing. Then, the sequence $\{C_n,\, n \geq 1\}$ defined by $C_n = \cup_{m \in I} B_n^m$ for all $n \geq 1$ is also vanishing. MONOTONE CONTINUITY then yields $\sum_{n=1}^N \mathbf{1}_{A_n} \mathbf{x}_n \succ x_{C_N} \mathbf{x}$ for some $N \geq 1$ where $x \in \mathcal{X}$ is a preferred outcome in the collection $\{\mathbf{x}(t_1^m),\, m \in I\}$. Then, set $\mathbf{x}' = x_{C_N} \mathbf{x}$. Observe first that, by construction, we have $\mathbf{x}'(t) \succsim \mathbf{x}(t)$ for all $t \in \mathcal{T}$ and therefore obtain $\mathbf{x}'(t) \succsim \mathbf{x}_n$ for all $t \in A_n$ and $n \in [1 \dots N]$. Indeed, if $t \in C_N$, then $t \in B_N^m$ for some $m \in I$ so that $u(\mathbf{x}(t)) \leq u[\mathbf{x}(t_N^m)] \leq u(x) = u(\mathbf{x}'(t))$ and therefore $\mathbf{x}'(t) \succsim \mathbf{x}(t)$ by Proposition B3. Observe also that $\mathbf{x}'$ has a minimum on each cell A_m. Indeed, if $m \notin I$, then $\mathbf{x}' = \mathbf{x}$ on A_m with $\mathbf{x}$ presenting a minimum on A_m by definition of I. If $m \in I$, consider any $t \in A_m$. If $t \notin C_N$, then it must be that $t \notin B_N^m$ and $u[\mathbf{x}'(t)] = u[\mathbf{x}(t)] \geq u[\mathbf{x}(t_N^m)]$ which by Proposition B3 yields $\mathbf{x}'(t) \succsim \mathbf{x}(t_N^m)$. Moreover if $t \in C_N$, $\mathbf{x}'(t) = x \succsim \mathbf{x}(t_1^m) \succsim \mathbf{x}(t_N^m)$. Overall, we may apply the previous paragraph to $\mathbf{x}'$ and obtain $\mathbf{x}' \succsim \sum_{n=1}^N \mathbf{1}_{A_n} \mathbf{x}_n$ in contradiction with the ranking implied by MONOTONE CONTINUITY above. □

Lemma C3 *For all bounded $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\mathcal{T}}$, $\mathbf{x} \succsim_{\mathcal{T}} \mathbf{y}$ if and only if $U(\mathbf{x}) \geq U(\mathbf{y})$.*

Proof. Consider first $\mathbf{x} \in \mathcal{F}_{\mathcal{T}}$ and $\mathbf{x}_0 \in \mathcal{F}_{\mathcal{T}}^0$ such that $\mathbf{x} \sim_{\mathcal{T}} \mathbf{x}_0$ with $\mathbf{x}$ bounded. Existence of such an $\mathbf{x}_0$ is guaranteed by Lemma C1. We will show

$$\int_{\mathcal{T}} u[\mathbf{x}(t)] d\epsilon_\lambda(t) \;=\; \int_{\mathcal{T}} u[\mathbf{x}_0(t)] d\epsilon_\lambda(t). \tag{C1}$$

Let $x_0, x_1 \in \mathcal{X}$ be such that $x_1 \succsim \mathbf{x}(t) \succsim x_0$ for all $t \in \mathcal{T}$. By applying positive affine transformations if necessary, we may assume $u(x_1) = 1$ and $u(x_0) = 0$ without loss of generality. Fix also $N \geq 1$.

For all $t \in \mathcal{T}$, we have $1 \geq u[\mathbf{x}(t)] \geq 0$. Let $\Pi_{\mathcal{T}} = \{A_1, \dots, A_N\}$ be the partition of $\mathcal{T}$ defined for all $n \in [1 \dots N]$ by

$$A_n \;=\; \left\{ t \in \mathcal{T},\ \frac{n-1}{N} \;\leq\; u[\mathbf{x}(t)] \;<\; \frac{n}{N} \right\}$$

By the measurability of u and $\mathbf{x}$, $\Pi_{\mathcal{T}}$ forms a finite measurable partition of $\mathcal{T}$ possibly with empty subsets. Then, by the monotonicity of the integral, we further obtain

$$\sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n-1}{N} \quad \leq \quad \int_{\mathcal{T}} u[\mathbf{x}(t)]\ d\epsilon_\lambda(t) \quad \leq \quad \sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n}{N}. \tag{C2}$$

Fix any $n \in [0 \dots N]$. Since ϵ_λ is countably additive and nonatomic, there exists $B_n \in \mathcal{B}_{\mathcal{T}}$ such that

$$\frac{n}{N} \quad = \quad \epsilon_\lambda(B_n) \quad = \quad \int_{\mathcal{T}} u[\mathbf{x}_n]\ d\epsilon_\lambda,$$

where $\mathbf{x}_n = x 1_{B_n} x_0 \in \mathcal{F}^0_{\mathcal{T}}$. We can assume $A \perp B_n$ for all $A \in \Pi_{\mathcal{T}}$ without loss of generality. Indeed, for all $A \in \Pi_{\mathcal{T}}$, the nonatomicity of ϵ_λ provides $B_n^A \in \mathcal{B}_{\mathcal{T}}$ such that $B_n^A \subseteq A$ and $\epsilon_\lambda(B_n^A) = (n/N) \cdot \epsilon_\lambda(A)$. Then, set $B_n = \cup_{A \in \Pi_{\mathcal{T}}} B_n^A$. We have $B_n \in \mathcal{B}_{\mathcal{T}}$ and $\epsilon_\lambda(B_n) = n/N$. Furthermore, for all $A \in \Pi_{\mathcal{T}}$, we have $B_n \cap A = B_n^A$ and, therefore, $\epsilon_\lambda(B_n \cap A) = \epsilon_\lambda(B_n) \cdot \epsilon_\lambda(A)$.

Next, we define $\mathbf{y}_0, \mathbf{z}_0 \in \mathcal{F}^0_{\mathcal{T}}$ in the following way:

$$\mathbf{y}_0 \ = \ \sum_{n=1}^{N} \mathbf{1}_{A_n} \mathbf{x}_{n-1} \ \text{ and } \ \mathbf{z}_0 \ = \ \sum_{n=1}^{N} \mathbf{1}_{A_n} \mathbf{x}_n$$

Now, we have

$$\int_{\mathcal{T}} u[\mathbf{y}_0(t)] d\epsilon_\lambda(t) \ = \ \sum_{n=1}^{N} \epsilon_\lambda(A_n \cap B_{n-1}) \ = \ \sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n-1}{N},$$

and likewise for $\mathbf{z}_0$. Furthermore, we have by construction $\mathbf{x}(t) \succsim \mathbf{x}_{n-1} = x 1_{B_{n-1}} x_0$ for all $t \in A_n$ and $n \in [1 \dots N]$, and $\mathbf{x}_{n-1} \perp A$ for all $n \in [1 \dots N]$ and $A \in \Pi_{\mathcal{T}}$. By Lemma C2, we obtain $\mathbf{x} \succsim \mathbf{y}_0$. Similarly, we can prove that $\mathbf{z}_0 \succsim \mathbf{x}$. Since $\mathbf{x} \sim \mathbf{x}_0$, we obtain $\mathbf{z}_0 \succsim \mathbf{x}_0 \succsim \mathbf{y}_0$, which implies by Proposition B3

$$\sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n-1}{N} \quad \leq \quad \int_{\mathcal{T}} u[\mathbf{x}_0(t)]\ d\epsilon_\lambda(t) \quad \leq \quad \sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n}{N}. \tag{C3}$$

Combining Formulae (C2) and (C3) gives

$$\Big| \int_{\mathcal{T}} u[\mathbf{x}(t)] d\epsilon_\lambda(t) \ - \int_{\mathcal{T}} u[\mathbf{x}_0(t)] d\epsilon_\lambda \ \Big| \ \leq \ \sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n}{N} \ - \ \sum_{n=1}^{N} \epsilon_\lambda(A_n) \cdot \frac{n-1}{N} \ = \ \frac{1}{N}.$$

We finally obtain Formula (C1) by taking the limit as N goes to ∞.

Consider next any bounded $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\mathcal{T}}$. By Lemma C1, there exist $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{F}^0_{\mathcal{T}}$ such that $\mathbf{x} \sim_{\mathcal{T}} \mathbf{x}_0$ and $\mathbf{y} \sim_{\mathcal{T}} \mathbf{y}_0$. Finally, the result follows by Formula (C1) and Proposition B3.
□

Lemma C4 *For all bounded $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $V(f) \geq V(g)$.*

Proof. Suppose that $f \in \mathcal{F}$ is bounded. By Lemma C1 there exists $\mathbf{x} \in \mathcal{F}^0_{\mathcal{T}}$ such that $f \sim \mathbf{x}$. We will show

$$\int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)] d(\mu \times \epsilon_\lambda)(s,t) \quad = \quad \int_{\mathcal{T}} u[\mathbf{x}(t)] d\epsilon_\lambda(t). \tag{C4}$$

Let $x_0, x_1 \in \mathcal{X}$ be such that $x_1 \succsim f(s,t) \succsim x_0$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$. By applying positive affine transformations if necessary, we may assume $u(x_1) = 1$ and $u(x_0) = 0$ without loss of generality. Fix also $N \geq 1$.

For all $s \in \mathcal{S}$ and $t \in \mathcal{T}$, we have $1 \geq u[f(s,t)] \geq 0$. Let $\Pi_{\mathcal{S}} = \{E_1, \ldots, E_N\}$ be the partition of $\mathcal{S}$ defined for all $n \in [1 \ldots N]$ by

$$E_n \;=\; \left\{ s \in \mathcal{S},\; \frac{n-1}{N} \;\leq\; \int_{\mathcal{T}} u[f(s,t)]\; d\epsilon_\lambda(t) \;<\; \frac{n}{N} \right\}$$

Note that $\Pi_{\mathcal{S}}$ forms a finite measurable partition of $\mathcal{S}$ possibly with empty subsets. By the monotonicity of the integral, we further obtain

$$\sum_{n=1}^{N} \mu(E_n) \cdot \frac{n-1}{N} \quad \leq \quad \int_{\mathcal{S}\times\mathcal{T}} u[f(s,t)]\; d(\mu \times \epsilon_\lambda)(s,t) \quad \leq \quad \sum_{n=1}^{N} \mu(E_n) \cdot \frac{n}{N} \tag{C5}$$

Now fix $n \in [0 \ldots N]$. By the nonatomicity of ϵ_λ, there exists $A_n \in \mathcal{B}_{\mathcal{T}}$ such that

$$\frac{n}{N} \quad = \quad \epsilon_\lambda(A_n) \quad = \quad \int_{\mathcal{T}} u(\mathbf{x}_n)\; d\epsilon_\lambda,$$

where $\mathbf{x}_n = x_{1A_n} x_0 \in \mathcal{F}^0_{\mathcal{T}}$. Then, we define $f_0, g_0 \in \mathcal{F}_0$ in the following way:

$$f_0 \;=\; \sum_{n=1}^{N} \mathbf{1}_{E_n} \mathbf{x}_{n-1} \quad \text{and} \quad g_0 \;=\; \sum_{n=1}^{N} \mathbf{1}_{E_n} \mathbf{x}_n$$

Now, fix $s \in \mathcal{S}$ and let $n \in [1 \ldots N]$ be such that $s \in E_n$. Then, $f_0(s,\cdot) = \mathbf{x}_{n-1}$ and $g_0(s,\cdot) = \mathbf{x}_n$ and therefore

$$\int_{\mathcal{T}} u[f_0(s,\cdot)] d\epsilon_\lambda \;=\; \frac{n-1}{N} \;\leq\; \int_{\mathcal{T}} u[f(s,\cdot)]\; d\epsilon_\lambda \;\leq\; \frac{n}{N} \;=\; \int_{\mathcal{T}} u[g_0(s,\cdot)] d\epsilon_\lambda.$$

Since $f(s,\cdot) \in \mathcal{F}_{\mathcal{T}}$ is bounded, Lemma C3 yields $g_0(s,\cdot) \succsim_{\mathcal{T}} f(s,\cdot) \succsim_{\mathcal{T}} f_0(s,\cdot)$, and this holds for all $s \in \mathcal{S}$. Then, Dominance further yields $g_0 \succsim f \succsim f_0$. Since $f \sim \mathbf{x}$, we obtain $g_0 \succsim \mathbf{x} \succsim f_0$. By Proposition B3 , we have:

$$\sum_{n=1}^{N} \mu(E_n) \cdot \frac{n-1}{N} \quad \leq \quad \int_{\mathcal{T}} u(\mathbf{x})\; d\epsilon_\lambda \quad \leq \quad \sum_{n=1}^{N} \mu(E_n) \cdot \frac{n}{N}. \tag{C6}$$

Combining Formulae (C5) and (C6) gives

$$\left| \int_{\mathcal{S}\times\mathcal{T}} u(f) d\mu \times \epsilon_\lambda \;-\; \int_{\mathcal{T}} u(\mathbf{x}) d\epsilon_\lambda \right| \;\leq\; \sum_{n=1}^{N} \mu(E_n) \cdot \frac{n}{N} \;-\; \sum_{n=1}^{N} \mu(E_n) \cdot \frac{n-1}{N} \;=\; \frac{1}{N}.$$

We obtain Formula (C4) taking the limit as N goes to ∞.

Finally, consider bounded $f, g \in \mathcal{F}$. By Lemma C1, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{F}^0_{\mathcal{T}}$ such that $f \sim_{\mathcal{T}} \mathbf{x}$ and $g \sim_{\mathcal{T}} \mathbf{y}$. The result follows by Formula (C4) and Proposition B3. □

We say that $f \in \mathcal{F}$ is *bounded from above* if there exists $x \in \mathcal{X}$ such that $x \succsim f(s,t)$ for all $(s,t) \in \mathcal{S} \times \mathcal{T}$. We say it is *bounded from below* if there exists $y \in \mathcal{X}$ such that $f(s,t) \succsim y$ for all $(s,t) \in \mathcal{S} \times \mathcal{T}$.

Lemma C5 *For all $E \in \mathcal{B}$, E is null if and only if $(\mu \times \epsilon_\lambda)(E) = 0$.*

Proof. Suppose $(\mu \times \epsilon_\lambda)(E) = 0$. Let $f, g \in \mathcal{F}$ be such that $f = g$ on the complement of E. If f and g are bounded, then $V(f) = V(g)$, and $f \sim g$ follows from Lemma C4.

Suppose now f and g are bounded from above, and not bounded from below. Let $\{x_n,\ n \geq 1\}$ be a sequence in $\mathcal{X}$ such that $\{u(x_n),\ n \geq 1\}$ is decreasing and converges to $\inf u$. For all $n \geq 1$, let $E_n \in \mathcal{B}$ be the collection of all $(s,t) \in \mathcal{S} \times \mathcal{T}$ such that $u[f(s,t)] < u(x_n)$ and $u[g(s,t)] < u(x_n)$. Since f (or g) is not bounded from below, $\{E_n,\ n \geq 1\}$ is vanishing. For all $n \geq 1$, let $f_n = x_{nE_n}f$ and $g_n = x_{nE_n}g$. Finally, suppose f and g are not indifferent to each other. Without loss of generality, we may suppose $f \succ g$. Then, by Monotone Continuity, we have $f \succ x_{1E_N}g$ for some $N \geq 1$. By $\mathcal{T}$-Monotonicity and Dominance, we obtain $f \succ x_{NE_N}g$. Meanwhile, and still by $\mathcal{T}$-Monotonicity and Dominance, we have $x_{NE_N}f \succsim f$ and obtain $f_N \succ g_N$. However, note that f_N and g_N are bounded and equal to each other except on $E_N^c \cap E$ with $(\mu \times \epsilon_\lambda)(E_N^c \cap E) = 0$. Then, Lemma C4 gives $f_N \sim g_N$, hence a contradiction.

Suppose next f and g are not bounded from above. Let $\{x_n,\ n \geq 1\}$ be a sequence in $\mathcal{X}$ such that $\{u(x_n),\ n \geq 1\}$ is increasing and converges to $\sup u$. For all $n \geq 1$, let $E_n \in \mathcal{B}$ be the collection of all $(s,t) \in \mathcal{S} \times \mathcal{T}$ such that $u[f(s,t)] > u(x_n)$ and $u[g(s,t)] > u(x_n)$. Supposing again $f \succ g$, we obtain $x_{NE_N}f \succ x_{NE_N}g$ for some $N \geq 1$, which contradicts the two previous paragraphs since $x_{NE_N}f$ and $x_{NE_N}g$ are bounded from above and equal to each other on $E_N^c \cap E$ with $(\mu \times \epsilon_\lambda)(E_N^c \cap E) = 0$.

Suppose finally that E is null. Let $x, y \in \mathcal{X}$ be such that $x \succ y$. Then, $x_E y$ and y agree on the complement of E. Hence, $x_E y \sim y$. The two acts are bounded. By Lemma C4, we obtain $(\mu \times \epsilon_\lambda)(E) = 0$. □

Lemma C6 *For all $f, g \in \mathcal{F}$, if $f(s,t) \succsim g(s,t)$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$, then $f \succsim g$; if additionally, $f(s,t) \succ g(s,t)$ for all (s,t) in some non-null subset in $\mathcal{B}$, then $f \succ g$.*

Proof. Suppose first that $f, g \in \mathcal{F}$ are such that $f(s,t) \succsim g(s,t)$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$. Then, by $\mathcal{T}$-Monotonicity, we have $f(s,\cdot) \succsim_{\mathcal{T}} g(s,\cdot)$ for all $s \in \mathcal{S}$. By Dominance, we further obtain $f \succsim g$. Suppose now additionally that $f(s,t) \succ g(s,t)$ for all $(s,t) \in E$ with $E \in \mathcal{B}$ non-null. By Lemma C5, we have $(\mu \times \epsilon_\lambda)(E) > 0$. For all $s \in \mathcal{S}$, let $E_s = \{t \in \mathcal{T},\ (s,t) \in E\} \in \mathcal{B}_{\mathcal{T}}$. Since we have

$$0 < (\mu \times \epsilon_\lambda)(E) \;=\; \int_{\mathcal{S}} \epsilon_\lambda(E_s) d\mu(s)$$

it must be that the set $A = \{s \in \mathcal{S} | \epsilon_\lambda(E_s) > 0\} \in \mathcal{B}_{\mathcal{S}}$ satisfies $\mu(A) > 0$. Then, by Lemma C5, A is non-null. Furthermore, by definition of the set A and by Lemma C5, E_s is non-null for all $s \in A$. Now, fix $s \in A$. For all $t \in E_s$, we have $(s,t) \in E$ and therefore $f(s,t) \succ g(s,t)$. Then, since E_s is non-null, $\mathcal{T}$-Monotonicity implies $f(s,\cdot) \succ_{\mathcal{T}} g(s,\cdot)$. The latter holds for all $s \in A$ with A non-null. Then, Dominance yields $f \succ g$. □

Proposition C7 *For all $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $V(f) \geq V(g)$.*

Proof. Step 1. For all $E \in \mathcal{B}$ and $f \in \mathcal{F}$, there exist $x, y \in \mathcal{X}$ such that $x_E f \succsim f \succsim y_E f$. Note that the result is straightforward if E is null. Hence, we suppose that E is non-null.

We only show the existence of $y \in \mathcal{X}$ such that $f \succsim y_E f$. Suppose by contradiction that no such y exists. Then $y_E f \succ f$ for all $y \in \mathcal{X}$. If f has a minimum on E, in the sense that there exists $x \in \mathcal{X}$ such that $f(s,t) \succsim x$ for all $(s,t) \in E$ with $x = f(s,t)$ for some $(s,t) \in E$, then, by Lemma C6, we have $f \succsim x_E f$. However by our hypothesis we have $x_E f \succ f$, a contradiction.

Suppose now that f has no minimum on E in the previous sense. Let $\{(s_n, t_n),\, n \geq 1\}$ be a sequence in E such that $\{u[f(s_n, t_n)],\, n \geq 1\}$ is decreasing and converges to $\inf u(f)$. For all $n \geq 1$, let $E_n \in \mathcal{B}$ be the collection of all $(s,t) \in E$ such that $u[f(s,t)] \leq u[f(s_n, t_n)]$. Let also $F_n := E \setminus E_n$ and $x_n := f(s_n, t_n)$. Since f has no minimum on E, $\{E_n,\, n \geq 1\}$ is vanishing. Then, for some $N \geq 1$, F_N is non-null. Indeed, suppose that F_n is null for all $n \geq 1$. Consider $f, g \in \mathcal{F}$ such that $f = g$ on $\mathcal{S} \setminus E$. If $f \succ g$, Monotone Continuity yields $f \succ f_{E_N} g$ for some $N \geq 1$. Since F_N is null, we further have $f \succ f_E g$ with $f_E g = f$, a contradiction. This shows that $f \sim g$ and hence that E must be null, another contradiction. Then, we have $f(s,t) \succsim (x_{N F_N} f)(s,t)$ for all $(s,t) \in \mathcal{S} \times \mathcal{T}$ with $f(s,t) \succ (x_{N F_N} f)(s,t)$ for all $(s,t) \in F_N$ with F_N non-null. By Lemma C6, we obtain $f \succ x_{N F_N} f$. By Monotone Continuity there exists $M \geq N$ such that $f \succ f'$ where $f' := x_{M E_M}(x_{N F_N} f)$. Meanwhile, we also have $f'(s,t) \succsim x_M$ for all $(s,t) \in E$ with $f' = f$ on $\mathcal{S} \setminus E$ and hence obtain $f' \succsim x_{M E} f$ by Lemma C6. However, we assumed $y_E f \succ f$ for all $y \in \mathcal{X}$. Hence we obtain $f' \succ f$, a contradiction.

Step 2. For all $f \in \mathcal{F}$, sequences $\{f_n,\, n \geq 1\}$ in $\mathcal{F}$ and vanishing sequences $\{E_n,\, n \geq 1\}$ in $\mathcal{B}$, if $f = f_n$ on $\mathcal{S} \times \mathcal{T} \setminus E_n$ for all $n \geq 1$, then $\{V(f_n),\, n \geq 1\}$ converges to $V(f)$. For all $n \geq 1$, we have

$$\left|V(f_n) \;-\; V(f)\right| \;=\; \Big|\int_{\mathcal{S}\times\mathcal{T}} \mathbf{1}_{E_n}\,(u(f_n) - u(f))\, d(\mu \times \epsilon_\lambda)\Big| \;\leq\; 2 \cdot \sup |u| \cdot (\mu \times \epsilon_\lambda)(E_n).$$

Now, since $\{E_n,\, n \geq 1\}$ is vanishing and $\mu \times \epsilon_\lambda$ is countably additive, the sequence $\{(\mu \times \epsilon_\lambda)(E_n),\, n \geq 1\}$ converges to 0.

Step 3. Representation of $\succsim$. Fix $f \in \mathcal{F}$. By Step 1, we can find $x, y \in \mathcal{X}$ such that $x \succsim f \succsim y$. Then there exists $A \in \mathcal{B}_\mathcal{T}$ such that $f \sim x_A y$. Indeed, this is obvious if $f \sim x$ or $f \sim y$ and follows from Lemma Lemma A3 in the remaining cases.

We now show $V(f) = U(x_A y)$. This follows from Lemma C4 if f is bounded. Otherwise, consider the following exhaustive cases:

(Case 1) f is bounded from below, and not bounded from above. Since u is bounded, there exists a sequence $\{x_n,\, n \geq 1\}$ of elements of $\mathcal{X}$ such that $\{u(x_n),\, n \geq 1\}$ converges to $\sup u(f)$. For all $n \geq 1$, let $E_n^+ \in \mathcal{B}$ be the subset defined as the collection of all $(s,t) \in \mathcal{S} \times \mathcal{T}$ such that $u[f(s,t)] > u(x_n)$. Since f is not bounded from above, $\{E_n^+,\, n \geq 1\}$ is vanishing. By Step 1, for all $n \geq 1$, there exist $\overline{x}_n, \underline{x}_n \in \mathcal{X}$ such that $\overline{x}_{n E_n^+} f \succsim f \succsim \underline{x}_{n E_n^+} f$ and, therefore, $\overline{x}_{n E_n^+} f \succsim x_A y \succsim \underline{x}_{n E_n^+} f$. Since the three acts are bounded, Lemma C4 yields

$$V(\overline{x}_{n E_n^+} f) \;\geq\; U(x_A y) \;\geq\; V(\underline{x}_{n E_n^+} f).$$

By Step 2, taking limits gives $V(f) = U(x_A y)$.
(Case 2) f is bounded from above, and not bounded from below. Consider then a sequence $\{y_n, n \geq 1\}$ of elements of $\mathcal{X}$ such that $\{u(y_n), n \geq 1\}$ converges to $\inf u(f)$ and, for all $n \geq 1$, let $E_n^- \in \mathcal{B}$ be the collection of all $(s,t) \in \mathcal{S} \times \mathcal{T}$ such that $u[f(s,t)] < u(y_n)$. Since f is not bounded from below, $\{E_n^-, n \geq 1\}$ is vanishing. As in Case 1, we obtain $V(f) = U(x_A y)$ again.
(Case 3) f is neither bounded from below nor from above. Then, define $\{E_n^+, n \geq 1\}$ and $\{E_n^-, n \geq 1\}$ as in Cases 1 and 2. These are again vanishing sequences. For all $n \geq 1$, let $E_n = E_n^+ \cup E_n^-$. Then, $\{E_n, n \geq 1\}$ is another vanishing sequence. Proceeding as in Cases 1 and 2, we obtain $V(f) = U(x_A y)$ once more.

Now, consider $f, g \in \mathcal{F}$. By the previous paragraphs, there exist $x, y, x', y' \in \mathcal{X}$ and $A, B \in \mathcal{B}_\mathcal{T}$ such that $f \sim x_A y$ and $g \sim x'_B y'$ with $V(f) = U(x_A y)$ and $V(g) = U(x'_B y')$. Then,

$$f \succsim g \iff x_A y \succsim x'_B y' \iff U(x_A y) \geq U(x'_B y') \iff V(f) \geq V(g),$$

where the second equivalence is by Proposition B3. □

D Proof of Theorem 1

We now come to the proof of Theorem 1. Proposition C7 establishes the sufficiency of the axioms for the representation. Moreover, the uniqueness of λ is implied by Proposition A2 while the uniqueness of μ and u is implied by Proposition B3.

Finally, as for the necessity of the axioms, suppose (λ, u, μ) provides a representation as in Theorem 1. Let U and V be the representing functionals for $\succsim_\mathcal{T}$ and $\succsim$ as defined in Appendix C. $\mathcal{T}$-Separability follows from the remark that, for all disjoint $E_\mathcal{T}, F_\mathcal{T} \in \mathcal{B}_\mathcal{T}$, all $\mathbf{x} \in \mathcal{F}_\mathcal{T}$ and all $x^*, x \in \mathcal{X}$ with $x^* \succ_\mathcal{T} x$, we have $x^*_{E_\mathcal{T}} x_{F_\mathcal{T}} \mathbf{x} \succsim_\mathcal{T} x_{E_\mathcal{T}} x^*_{F_\mathcal{T}} \mathbf{x}$ if and only if $\epsilon_\lambda(E_\mathcal{T}) \geq \epsilon_\lambda(F_\mathcal{T})$, where we assume $u(x^*) = 1$ and $u(x) = 0$ without loss of generality. As for $\mathcal{T}$-Measurability, fix $\mathbf{x} \in \mathcal{F}_\mathcal{T}$. Then, we have

$$\{x \in \mathcal{X},\ x \succ_\mathcal{T} \mathbf{x}\} = u^{-1}(]\alpha, +\infty[) \quad \text{and} \quad \{x \in \mathcal{X},\ \mathbf{x} \succ_\mathcal{T} x\} = u^{-1}(]-\infty, \alpha[),$$

where $\alpha = U(\mathbf{x})$. Then, $\mathcal{T}$-Measurability follows from the measurability of u. To show Monotone Continuity, suppose $f, g \in \mathcal{F}$ are such that $f \succ g$, and consider $x \in \mathcal{X}$ and a vanishing sequence $\{E_n, n \geq 1\}$ of subsets in $\mathcal{B}$. By Step 2 of Proposition C7, $\{V(x_{E_n} f), n \geq 1\}$ and $\{V(x_{E_n} g), n \geq 1\}$ converge respectively to $V(f)$ and $V(g)$. By the representation, we have $V(f) > V(g)$. Hence, there exists $N \geq 1$ such that $V(x_{E_N} f) > V(g)$ and $V(f) > V(x_{E_N} g)$. Then, still by the representation, we obtain $x_{E_N} f \succ g$ and $f \succ x_{E_N} g$.

Lemma D1 *Let $(\Omega, \mathcal{A}, P)$ be a (countably additive) probability space. For all $E \in \mathcal{A}$ and all measurable real-valued function F on Ω such that $F(\omega) \geq 0$ for all $\omega \in \Omega$ and $F(\omega) > 0$ for all $\omega \in E$, if $P(E) > 0$, then $\int_\Omega F(\omega) dP(\omega) > 0$.*

Proof. A proof is given for the sake of completeness. For all $n \geq 1$, let $E_n \in \mathcal{B}$ be the subset of E collecting all $\omega \in \Omega$ such that $F(\omega) \geq 1/n$. Then, the union of $\{E_n, n \geq 1\}$ is equal to E. By countable additivity, the limit of $\{P(E_n), n \geq 1\}$ is equal to $P(E)$ so there exists $N \geq 1$ such that $P(E_N) > 0$. We obtain

$$\int_\Omega F(\omega)dP(\omega) \;\geq\; \int_{E_N} F(\omega)dP(\omega) \;\geq\; \frac{1}{N} \cdot P(E_N) \;>\; 0,$$

where the first inequality is by the (weak) monotonicity of the Lebesgue integral. □

Now, to show $\mathcal{T}$-Monotonicity, let $\mathbf{x}, \mathbf{y} \in \mathcal{F}_\mathcal{T}$ be such that $\mathbf{x}(t) \succsim_\mathcal{T} \mathbf{y}(t)$ for all $t \in \mathcal{T}$. Then, by the representation, we have $u(\mathbf{x}(t)) \geq u(\mathbf{y}(t))$ for all $t \in \mathcal{T}$ and obtain $U(\mathbf{x}) \geq U(\mathbf{y})$ by the (weak) monotonicity of the Lebesgue integral. The representation further yields $\mathbf{x} \succsim_\mathcal{T} \mathbf{y}$. If additionally, $\mathbf{x}(t) \succ_\mathcal{T} \mathbf{y}(t)$ for all t in some non-null subset $E_\mathcal{T} \in \mathcal{B}_\mathcal{T}$, then $u(\mathbf{x}(t)) > u(\mathbf{y}(t))$ for all $t \in E_\mathcal{T}$. As $E_\mathcal{T}$ is non-null, we have $(\mu \times \epsilon_\lambda)(\mathcal{S} \times E_\mathcal{T}) > 0$. Lemma D1 yields $U(\mathbf{x}) > U(\mathbf{y})$, and we obtain $\mathbf{x} \succ_\mathcal{T} \mathbf{y}$ by the representation. As for Dominance, note that, for all $f \in \mathcal{F}$, we have by the Fubini theorem

$$V(f) \;=\; \int_\mathcal{S} \int_\mathcal{T} u[f(s,t)]d\epsilon_\lambda(t)d\mu(s).$$

Then, Dominance follows from an argument similar to that yielding $\mathcal{T}$-Monotonicity. Finally, Stationarity follows from Lemma D2.

Lemma D2 *For all $t \in \mathcal{T}$ and $f, h \in \mathcal{F}$,*

$$V(h_t f) \;=\; \int_\mathcal{S} \int_\mathcal{T} \mathbf{1}_{[0,t)} \cdot u[h(s,t')]d\epsilon_\lambda(t')d\mu(s) \;+\; e^{-\lambda t} \cdot V(f).$$

Proof. By standard arguments, Formula (A1) extends into the following one: For all $t \in \mathcal{T}$ and $\mathbf{x} \in \mathcal{F}_\mathcal{T}$

$$\int_\mathcal{T} \mathbf{1}_{[t,+\infty[} \cdot u[\mathbf{x}(t'-t)]d\epsilon_\lambda(t') \;=\; e^{-\lambda t} \cdot U(\mathbf{x}).$$

Then, we have for all $t \in \mathcal{T}$ and $\mathbf{x}, \mathbf{z} \in \mathcal{F}_\mathcal{T}$

$$U(\mathbf{z}_t\mathbf{x}) \;=\; \int_\mathcal{T} \mathbf{1}_{[0,t)} \cdot u[\mathbf{z}(t')]d\epsilon_\lambda(t') \;+\; e^{-\lambda t} \cdot U(\mathbf{x}).$$

For all $t \in \mathcal{T}$ and $f, h \in \mathcal{F}$, applying the previous formula to $f(s,\cdot)$ and $h(s,\cdot)$ for all $s \in \mathcal{S}$ and integrating on $\mathcal{S}$ yields the result. □

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Declaration of interests

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