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Zastavnyi operators and positive definite radial functions

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ABSTRACT

we consider a new operator acting on rescaled weighted differences between two members of the class Φ_d of positive definite radial functions. In particular, we study the positive definiteness of the operator for the Matérn, Generalized Cauchy and Wendland families.

1. Introduction

Positive definite functions are fundamental to many branches of mathematics as well as probability theory, statistics and machine learning amongst others. There has been an increasing interest in positive definite functions in d-dimensional Euclidean spaces (d is a positive integer throughout), and the reader is referred to Daley and Porcu (2014), Schoenberg (1938), Schaback (2011), Wu (1995) and Wendland (1995).

This paper is concerned with the class Φ_d of continuous functions $\phi : [0,\infty) \mapsto \mathbb{R}$ such that $\phi(0) = 1$ and the function $\boldsymbol{x} \mapsto \phi(\|\boldsymbol{x}\|)$ is positive definite in $\mathbb{R}^d$. The class Φ_d is nested, with the strict inclusion relation:

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_d \supset \cdots \supset \Phi_\infty := \bigcap_{d\geq 1} \Phi_d.$$

The classes Φ_d are convex cones that are closed under product, non-negative linear combinations, and pointwise convergence. Further, for a given member ϕ in Φ_d, the rescaled function $\phi(\cdot/\alpha)$ is still in Φ_d for any given α. We make explicit emphasis on this fact because it will be repeatedly used subsequently.

For any nonempty set $A \subseteq \mathbb{R}^d$, we call $C(A)$ the set of continuous functions from A into $\mathbb{R}$. For p a positive integer, let $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p$ and let $\{\phi(\cdot;\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Theta\}$ be a parametric family belonging to the class Φ_d. For $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$ and $0 < \beta_1 < \beta_2$ with β_i, $i = 1, 2$ two scaling parameters, we define the Zastavnyi operator $K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi] : \Phi_d \mapsto C(\mathbb{R})$ by

$$K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](t) = \frac{\beta_2^{\varepsilon}\phi\left(\frac{t}{\beta_2};\boldsymbol{\theta}\right) - \beta_1^{\varepsilon}\phi\left(\frac{t}{\beta_1};\boldsymbol{\theta}\right)}{\beta_2^{\varepsilon} - \beta_1^{\varepsilon}}, \qquad t \geq 0, \tag{1.1}$$

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with $K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](0)=1$. Here, by β_i^ε we mean β_i raised to the power of ε. It can be namely checked that

$$K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](0)=\frac{\beta_2^\varepsilon\phi\left(\frac{0}{\beta_2};\boldsymbol{\theta}\right)-\beta_1^\varepsilon\phi\left(\frac{0}{\beta_1};\boldsymbol{\theta}\right)}{\beta_2^\varepsilon-\beta_1^\varepsilon}=\frac{\beta_2^\varepsilon-\beta_1^\varepsilon}{\beta_2^\varepsilon-\beta_1^\varepsilon}=1.$$

A motivation for studying positive definiteness of the radial functions $\mathbb{R}^d\ni\mathbf{x}\mapsto K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](\|\mathbf{x}\|)$ comes from the problem of monotonicity of the so-called microergodic parameter of specific parametric families (Bevilacqua et al., 2018; Bevilacqua and Faouzi, 2018) when studying the asymptotic properties of the maximum likelihood estimation under fixed domain asymptotics. The operator (1.1) is a generalization of the operator proposed in Porcu et al. (2017) where ε is assumed to be positive. Our problem can be formulated as follows:

Problem 1.1. Let d and q be positive integers. Let $\{\phi(\cdot;\boldsymbol{\theta}),\ \boldsymbol{\theta}\in\Theta\subset\mathbb{R}^q\}$ be a family of functions belonging to the class Φ_d. Find the conditions on $\varepsilon\in\mathbb{R}$, $\varepsilon\neq0$ and $\boldsymbol{\theta}$, such that $K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi]$ as defined through (1.1) belongs to the class Φ_n for some $n=1,2,\ldots$ for given $0<\beta_1<\beta_2$.

We first note that Problem 1.1 has at least two possible solutions. Indeed, direct inspection shows

$$\lim_{\varepsilon\to+\infty}K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](t)=\phi\left(\frac{t}{\beta_1};\boldsymbol{\theta}\right),\quad \lim_{\varepsilon\to-\infty}K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](t)=\phi\left(\frac{t}{\beta_2};\boldsymbol{\theta}\right),\qquad t\geq0,$$

where the convergence is pointwise in t.

The positive definiteness of (1.1), assuming $\varepsilon>0$, has been studied in Porcu et al. (2017) when ϕ belongs to the Buhmann class (Buhmann, 2001). An important special case of the Buhmann class is the Generalized Wendland family (Gneiting, 2002). For $\kappa>0$, we define the class $\mathcal{GW}:[0,\infty)\to\mathbb{R}$ as:

$$\mathcal{GW}(t;\kappa,\mu)=\begin{cases}\frac{\int_t^1 u(u^2-t^2)^{\kappa-1}(1-u)^\mu\,\mathrm{d}u}{B(2\kappa,\mu+1)}, & 0\leq t<1,\\ 0, & t\geq1,\end{cases}\tag{1.2}$$

and, for $\kappa=0$, by continuity we have

$$\mathcal{GW}(t;0,\mu)=\begin{cases}(1-t)^\mu, & 0\leq t<1,\\ 0, & t\geq1.\end{cases}\tag{1.3}$$

The function $\mathcal{GW}(t;\kappa,\mu)$ is a member of the class Φ_d if and only if $\mu\geq0.5(d+1)+\kappa$ (Zastavnyi and Trigub, 2002). Porcu et al. (2017) found that if $\phi(\cdot;\boldsymbol{\theta})=\mathcal{GW}(\cdot;\kappa,\mu)$ and $\varepsilon>0$ then $K_{\varepsilon;\kappa,\mu;\beta_2,\beta_1}[\mathcal{GW}](t)$ is positive definite if $\mu\geq(d+7)/2+\kappa$ and $\epsilon\geq2\kappa+1$.

This paper is especially interested to the solution of Problem 1.1 when considering two celebrated parametric families:

The Matérn family. In this case, $\phi(\cdot;\boldsymbol{\theta})=\mathcal{M}(\cdot;\nu)$, so that the $\boldsymbol{\theta}=\nu$, a scalar and $\Theta=(0,\infty)$, with

$$\mathcal{M}(t;\nu)=\frac{2^{1-\nu}}{\Gamma(\nu)}t^\nu\mathcal{K}_\nu(t),\qquad t\geq0,\tag{1.4}$$

where $\mathcal{K}_\nu$ is the modified Bessel function of the second kind of order $\nu>0$ (Abramowitz and Stegun, 1970). The functions $\mathcal{M}(\cdot;\nu)$, $\nu>0$, belong to the class Φ_∞ (Stein, 1999).

The Generalized Cauchy family. In this case $\phi(\cdot;\boldsymbol{\theta})=\mathcal{C}(\cdot;\delta,\lambda)$, so that $\boldsymbol{\theta}=(\delta,\lambda)^\top$, with $\top$ denoting the transpose operator. Here, $\Theta=(0,2]\times(0,\infty)$, and

$$\mathcal{C}(t;\delta,\lambda)=\left(1+t^\delta\right)^{-\lambda/\delta},\qquad t\geq0.\tag{1.5}$$

The functions $\mathcal{C}(\cdot;\delta,\lambda)$ belong to the class Φ_∞ (Gneiting and Schlather, 2004).

Additionally, we provide a solution to Problem 1.1 when $\phi(\cdot;\boldsymbol{\theta})=\mathcal{GW}(\cdot;\kappa,\mu)$, $\boldsymbol{\theta}=(\kappa,\mu)^\top$, $\Theta=[0,\infty)\times(0,\infty)$, assuming $\varepsilon<0$.

To give an idea of how the operator $K_{\varepsilon;\boldsymbol{\theta};\beta_2,\beta_1}[\phi](t)$ acts on $\phi(t,\boldsymbol{\theta})$ for given $0<\beta_1<\beta_2$ when ϕ is the Matérn family, Fig. 1(A) compares $K_{\varepsilon;0.5;\beta_2,\beta_1}[\mathcal{M}](t)$ with $\mathcal{M}(t/\beta_1;0.5)$ and $\mathcal{M}(t/\beta_2;0.5)$ when $\beta_1=0.075$, $\beta_2=0.15$, $\varepsilon=1$ (red line) and $\varepsilon=-2$ (blue line). We note that the behaviour at the origin of $K_{1;0.5;\beta_2,\beta_1}[\mathcal{M}](t)$ changes drastically with respect to the behaviour at the origin of $\mathcal{M}(\cdot;0.5)$. Moreover, $K_{-2;0.5;\beta_2,\beta_1}[\mathcal{M}](t)$ can attain negative values. It turns out from Theorem 3.1 that $K_{1;0.5;\beta_2,\beta_1}[\mathcal{M}](t)\in\Phi_\infty$ and $K_{-2;0.5;\beta_2,\beta_1}[\mathcal{M}](t)\in\Phi_2$.

A similar graphical representation is given in Fig. 1(B) when $\phi\equiv\mathcal{C}$, the Cauchy family in (1.5). In this case, we consider $\varepsilon=-0.7,1.25$, $\delta=0.6$, $\lambda=2.5$, $\beta_1=0.2$, $\beta_2=0.3$. Note that,under this setting, $K_{-1.25;0.6,2.5;\beta_2,\beta_1}[\mathcal{C}](t)$ attains negative values as well. It turns out from Theorem 3.3 $K_{\varepsilon;2.5,0.6;\beta_2,\beta_1}[\mathcal{C}](t)\in\Phi_\infty$ for $\varepsilon=-0.7$ and 1.25. Finally, Fig. 1(C) compares $K_{\varepsilon;0,4.5;\beta_2,\beta_1}[\mathcal{GW}](t)$ with $\mathcal{GW}(t/\beta_1;0,4.5)$ and $\mathcal{GW}(t/\beta_2;0,4.5)$ with $\beta_2=0.6$, $\beta_1=0.4$ when $\varepsilon=1$ (red line) and $\varepsilon=-2$ (blue line). As for the Matérn case, the behaviour at the origin of $K_{\varepsilon;0,4.5;\beta_2,\beta_1}[\mathcal{GW}](t)$ changes neatly with respect

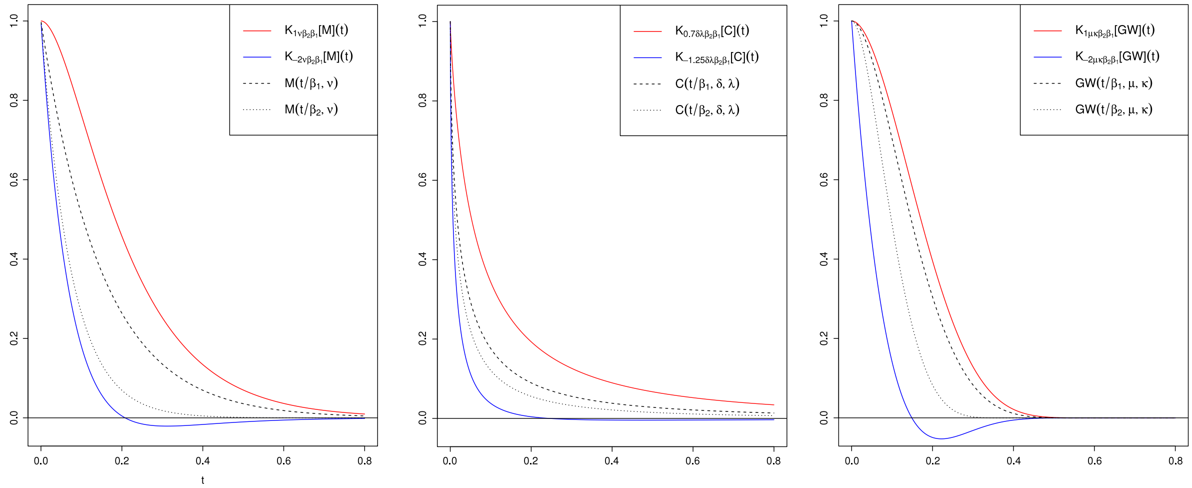

Fig. 1. From left to right: (A) comparison of $K_{\varepsilon;\nu;\beta_2,\beta_1}[\mathcal{M}](t)$ when $\varepsilon = 1$ (red line) and $\varepsilon = -2$ (blue line) with $\mathcal{M}(t/\beta_1; \nu)$ and $\mathcal{M}(t/\beta_2; \nu)$ when $\nu = 0.5$, $\beta_1 = 0.075$, $\beta_2 = 0.15$. (B) comparison of $K_{\varepsilon;\delta,\lambda;\beta_2,\beta_1}[\mathcal{C}](t)$ when $\varepsilon = 0.7$ (red line) and $\varepsilon = -1.25$ (blue line) with $\mathcal{C}(t/\beta_1; \delta, \lambda)$ and $\mathcal{C}(t/\beta_2; \delta, \lambda)$ when $\delta = 0.6$, $\lambda = 2.5$, $\beta_1 = 0.2$, $\beta_2 = 0.3$. (C) comparison of $K_{\varepsilon;\mu,\kappa;\beta_2,\beta_1}[\mathcal{GW}](t)$ when $\varepsilon = 1$ (red line) and $\varepsilon = -2$ (blue line) with $\mathcal{GW}(t/\beta_1; \mu, \kappa)$ and $\mathcal{GW}(t/\beta_2; \mu, \kappa)$ when $\mu = 4.5$, $\kappa = 0$, $\beta_1 = 0.4$, $\beta_2 = 0.6$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

to the behaviour at the origin of $\mathcal{GW}(\cdot; 0, 4.5)$. Moreover $K_{-2;0,4.5;\beta_2,\beta_1}[\mathcal{GW}](t)$ can reach negative values. It turns out from Theorem 3.2 that $K_{-2;0,4.5;\beta_2,\beta_1}[\mathcal{GW}](t)$ belongs to Φ_2.

These three examples show that operator (1.1) can change substantially the features of given families ϕ in terms of both differentiability at the origin and negative correlations. The solution of Problem 1.1 for the Matérn and Cauchy families, passes necessarily through the specification of the properties of the radial Fourier transforms of the radially symmetric functions $\mathcal{M}(\|\cdot\|; \nu)$ and $\mathcal{C}(\|\cdot\|; \delta, \lambda)$ in $\mathbb{R}^d$. For the Matérn family, such a Fourier transform is available in closed form. For the Generalized Cauchy family we obtain the Fourier transform as series expansions generalizing a recent result obtained by Lim and Teo (2010). The plan of the paper is the following: Section 2 contains the necessary preliminaries and background. Section 3 gives the main results of this paper.

2. Preliminaries

We start with some expository material. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called positive definite if, for any collection $\{a_k\}_{k=1}^N \subset \mathbb{R}$ and points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \mathbb{R}^d$, the following holds:

$$\sum_{k=1}^{N}\sum_{h=1}^{N} a_k f(\boldsymbol{x}_k - \boldsymbol{x}_h) a_h \geq 0.$$

By Bochner's theorem, continuous positive definite functions are the Fourier transforms of positive and bounded measures, that is

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{x}, \boldsymbol{z}\rangle} F(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{x} \in \mathbb{R}^d, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes inner product and where i is the complex number such that $\mathrm{i}^2 = -1$. Additionally, if $f(\boldsymbol{x}) = \phi(\|\boldsymbol{x}\|)$ for some continuous function defined on the positive real line, Schoenberg's theorem (Daley and Porcu, 2014, with the references therein) shows that f is positive definite if and only if its *radial part* ϕ can be written as

$$\phi(t) = \int_{[0,\infty)} \Omega_d(\xi t) G_d(\mathrm{d}\xi), \qquad t \geq 0, \tag{2.2}$$

where G_d is a positive and bounded measure, and where

$$\Omega_d(t) = t^{-(d-2)/2} J_{(d-2)/2}(t), \qquad t \geq 0,$$

where J_ν defines a Bessel function of order ν. If $\phi(0) = 1$, then G_d is a probability measure (Daley and Porcu, 2014). Classical Fourier inversion in concert with Bochner's theorem shows that the function ϕ belongs to the class Φ_d if and only if it admits the representation (2.2), and this in turn happens if and only if the function $\widehat{\phi}_d : [0,\infty) \to [0,\infty)$, defined through

$$\widehat{\phi}_d(z) := \mathcal{F}_d[\phi(t)](z) = \frac{z^{1-d/2}}{(2\pi)^d} \int_0^\infty t^{d/2} J_{d/2-1}(tz)\phi(t)\mathrm{d}t, \qquad z \geq 0, \tag{2.3}$$

is nonnegative and such that $\int_{[0,\infty)} \widehat{\phi}_d(z) z^{d-1} \mathrm{d}z < \infty$. Note that we intentionally put a subscript d into G_d and $\widehat{\phi}_d$ to emphasize the dependence on the dimension d corresponding to the class Φ_d where ϕ is defined. This is explicitly stated in Daley and Porcu (2014), where it is explained that for any member of the class Φ_d there exists at least $G_1, \ldots, G_d$ non negative bounded measures in the representation (2.2). Hence the term d-Schoenberg measures proposed therein. Finally, a convergence argument as much as in Schoenberg (1938) shows that $\phi \in \Phi_\infty$ if and only if

$$\phi(\sqrt{t}) = \int_{[0,\infty)} \mathrm{e}^{-\xi t} G(\mathrm{d}\xi), \qquad t \geq 0, \tag{2.4}$$

for G positive, nondecreasing and bounded. Thus, $\psi := \phi(\sqrt{\cdot})$ is the Laplace transform of G, which shows, in concert with Bernstein's theorem, that the function ψ is completely monotonic on the positive real line, that is ψ is infinitely often differentiable and $(-1)^k \psi^{(k)}(t) \geq 0$, $t > 0$, $k \in \mathbb{N}$. Here, $\psi^{(k)}$ denotes the kth order derivative of ψ, with $\psi^{(0)} = \psi$ by abuse of notation.

For a given $d \in \mathbb{N}$, direct inspection shows that for any scale parameter $\beta > 0$, the Fourier transform (2.3) of $\phi(\cdot/\beta)$ is identically equal to $\beta^d \widehat{\phi}_d(\beta \cdot) =: \widehat{\phi}_{d,\beta}(\cdot)$, and we shall repeatedly make use of this fact for the results following subsequently. It is well known that the Fourier transform of the Matérn covariance function $\mathcal{M}$ in Eq. (1.4) is given by Stein (1999)

$$\widehat{\mathcal{M}}_{d,\beta}(z;\nu) = \frac{\Gamma(\nu+d/2)}{\pi^{d/2}\Gamma(\nu)} \frac{\beta^d}{(1+\beta^2 z^2)^{\nu+d/2}}, \qquad z \geq 0. \tag{2.5}$$

An ingenious approach in Lim and Teo (2009) shows that the Fourier transform of Generalized Cauchy covariance function $\mathcal{C}$ in Eq. (1.5) can be written as

$$\widehat{\mathcal{C}}_{d,\beta}(z;\delta,\lambda) = -\frac{\beta^{d/2+1} z^{-d}}{2^{d/2-1}\pi^{d/2+1}} \Im \left(\int_0^\infty \frac{\mathcal{K}_{(d-2)/2}(\beta t)}{(1+e^{i\frac{\pi\delta}{2}}(t/z)^\delta)^{\lambda/\delta}} t^{d/2} \mathrm{d}t \right), \qquad z > 0, \tag{2.6}$$

where $\Im$ denotes the imaginary part of a complex argument, $\mathcal{K}_{(d-2)/2}$ is the modified Bessel function of the second kind of order $(d-2)/2$ and β is a scale parameter. A closed form expression for $\widehat{\mathcal{C}}_{d,\beta}$ has been elusive for longtime. Lim and Teo (2010) shed some light for this problem giving a infinite series representation of the spectral density under some specific restriction of the parameters. The result following below shows that the series representation given in Lim and Teo (2010) is valid without any restriction on the parameters.

Theorem 2.1. *Let $\mathcal{C}(\cdot;\delta,\lambda)$ be the Generalized Cauchy covariance function as defined in Eq. (1.5). Then, it is true that*

$$\begin{aligned} \widehat{\mathcal{C}}_{d,\beta}(z;\delta,\lambda) = & \frac{z^{-d}}{\pi^{d/2}} \frac{1}{\Gamma(\frac{\lambda}{\delta})} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(\frac{\lambda}{\delta}+n)\Gamma(d/2-(\frac{\lambda}{\delta}+n)\delta/2)}{\Gamma((\frac{\lambda}{\delta}+n)\delta/2)} \left(\frac{z\beta}{2}\right)^{(\frac{\lambda}{\delta}+n)\delta} \\ & + \frac{z^{-d}}{\pi^{d/2}\delta} \frac{2}{\Gamma(\frac{\lambda}{\delta})} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma\left(\frac{2n+d}{\delta}\right)\Gamma\left(\frac{\lambda}{\delta}-\frac{2n+d}{\delta}\right)}{\Gamma(n+d/2)} \left(\frac{z\beta}{2}\right)^{2n+d}, \end{aligned} \tag{2.7}$$

with $z > 0$, where $\delta \in (0,2)$ and $\lambda > 0$.

Proof. The Mellin–Barnes transform is defined through the identity which is given by

$$\frac{1}{(1+x)^\alpha} = \frac{1}{2\pi\mathrm{i}} \frac{1}{\Gamma(\alpha)} \oint_\Lambda x^u \Gamma(-u)\Gamma(\alpha+u)\mathrm{d}u, \tag{2.8}$$

here $\Gamma(\cdot)$ denotes the Gamma function. This representation is valid for any $x \in \mathbb{R}$. The contour Λ contains the vertical line which passes between left and right poles in the complex plane u from negative to positive imaginary infinity, and should be closed to the left in case $x > 1$, and to the right complex infinity if $0 < x < 1$.

We now proceed to compute $\widehat{\mathcal{C}}_{d,\beta}$ as follows:

$$\begin{aligned} \widehat{\mathcal{C}}_{d,\beta}(\|\mathbf{z}\|;\delta,\lambda) = & \mathcal{F}_d[\mathcal{C}_{d,\beta}(t;\delta,\lambda)](\|\mathbf{z}\|) \\ = & \beta^d \mathcal{F}_d[\mathcal{C}_d(t;\delta,\lambda)](\beta\|\mathbf{z}\|), \qquad \mathbf{z} \in \mathbb{R}^d. \end{aligned}$$

Applying Eq. (2.8), we obtain

$$\widehat{\mathcal{C}}_{d,\beta}(\|\boldsymbol{z}\|\,;\delta,\lambda) = \frac{\beta^d}{(2\pi)^d}\frac{1}{\Gamma(\frac{\lambda}{\delta})}\int_{\mathbb{R}^d} e^{i\beta\langle \boldsymbol{z},\boldsymbol{x}\rangle}\oint_{\Lambda}\Gamma(-u)\Gamma(u+\frac{\lambda}{\delta})\,\|\boldsymbol{x}\|^{u\delta}\,du\,d\boldsymbol{x}$$
$$= \frac{\beta^d}{(2\pi)^d}\frac{1}{\Gamma(\frac{\lambda}{\delta})}\oint_{\Lambda}\Gamma(-u)\Gamma(u+\frac{\lambda}{\delta})\int_{\mathbb{R}^d} e^{i\beta\langle \boldsymbol{z},\boldsymbol{x}\rangle}\,\|\boldsymbol{x}\|^{u\delta}\,d\boldsymbol{x}du\,.$$

We now invoke the well known relationship (Allendes et al., 2013),

$$\int_{\mathbb{R}^d} e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle}\,\|\boldsymbol{x}\|^{u\delta}\,d\boldsymbol{x} = \frac{2^{d+u\delta}\pi^{d/2}\Gamma(d/2+u\delta/2)}{\Gamma(-u\delta/2)\,\|\boldsymbol{z}\|^{d+u\delta}},$$

and, by abuse of notation, we now write $z := \|\boldsymbol{z}\|$. We have

$$\widehat{\mathcal{C}}_{d,\beta}(z;\delta,\lambda) = \frac{\beta^d}{(2\pi)^d}\frac{1}{\Gamma(\frac{\lambda}{\delta})}\frac{1}{2\pi i}\oint_{\Lambda}\Gamma(-u)\Gamma(u+\frac{\lambda}{\delta})\frac{2^{d+u\delta}\pi^{d/2}\Gamma(d/2+u\delta/2)}{\Gamma(-u\delta/2)}\frac{1}{(\beta z)^{d+u\delta}}du$$
$$= \frac{z^{-d}}{\pi^{d/2}}\frac{1}{\Gamma(\frac{\lambda}{\delta})}\frac{1}{2\pi i}\oint_{\Lambda}\frac{\Gamma(-u)\Gamma(u+\frac{\lambda}{\delta})\Gamma(d/2+u\delta/2)}{\Gamma(-u\delta/2)}\left(\frac{2}{z\beta}\right)^{u\delta}du. \tag{2.9}$$

For any given value of $|2/z\beta|$, it does not matter whether it is smaller or greater than 1. In fact, we may close the contour to the left complex infinity. This is a convergent series for any values of the variable z because we have a situation when denominator suppresses numerator in the coefficient in front of powers of z. We now observe that the functions $u \mapsto \Gamma(\frac{\lambda}{\delta}+u)$ and $u \mapsto \Gamma\left(d/2+\frac{u\delta}{2}\right)$ contain poles in the complex plane, respectively when $\frac{\lambda}{\delta}+u=-n$, and when $d/2+\frac{u\delta}{2}=-n$, $n\in\mathbb{N}$. Using this fact and through direct inspection we obtain that the right hand side in (2.9) matches with (2.7). The proof is completed. □

3. Main results

We start by providing a solution to Problem 1.1 when $\phi(\cdot;\boldsymbol{\theta}) = \mathcal{M}(\cdot;\nu)$, so that $\boldsymbol{\theta}\equiv\nu$ and $\Theta=(0,\infty)$.

Theorem 3.1. *Let $\mathcal{M}(\cdot;\nu)$ be the Matérn function as defined in Eq. (1.4). Let $K_{\varepsilon;\nu;\beta_2,\beta_1}[\mathcal{M}]$ with $0<\beta_1<\beta_2$, be the Zastavnyi operator (1.1) related to the function $\mathcal{M}(\cdot;\nu)$.*

1. *Let $\varepsilon>0$. Then, $K_{\varepsilon;\nu;\beta_2,\beta_1}[\mathcal{M}]\in\Phi_\infty$ if and only if $\varepsilon\ge 2\nu>0$;*
2. *For a given $d\in\mathbb{N}$, let $\varepsilon<0$. Then, $K_{\varepsilon;\nu;\beta_2,\beta_1}[\mathcal{M}]\in\Phi_d$ if and only if $\varepsilon\le -d<0$.*

The following result gives some conditions for the solution of Problem 1.1 when $\phi(\cdot;\boldsymbol{\theta}) = \mathcal{GW}(\cdot;\mu,\kappa)$, so that $\boldsymbol{\theta}=(\kappa,\mu)^\top$, $\Theta=[0,\infty)\times(0,\infty)$.

Theorem 3.2. *Let d be a positive integer. Let $\mathcal{GW}(\cdot;\mu,\kappa)$ be the function defined through Eqs. (1.2) and (1.3), for $\kappa>0$ and $\kappa=0$ respectively. Let $K_{\varepsilon;\mu,\kappa,\beta_2,\beta_1}[\mathcal{GW}]$ with $0<\beta_1<\beta_2$, be the Zastavnyi operator (1.1) related to the function $\mathcal{GW}(\cdot;\mu,\kappa)$. Then:*

1. *If $\varepsilon\ge 2\kappa+1>0$ and $\mu\ge(d+7)/2+\kappa$, then $K_{\varepsilon;\mu,\kappa,\beta_2,\beta_1}[\mathcal{GW}]\in\Phi_d$.*
2. *$K_{\varepsilon;\mu,\kappa,\beta_2,\beta_1}[\mathcal{GW}]\in\Phi_d$ if and only if $\varepsilon=2\kappa+1$ and $\mu\ge(d+7)/2+\kappa$.*
3. *If $\varepsilon\le -d<0$ and $\mu\ge(d+7)/2+\kappa$, then $K_{\varepsilon;\mu,\kappa,\beta_2,\beta_1}[\mathcal{GW}]\in\Phi_d$.*
4. *$K_{-d;\mu,\kappa,\beta_2,\beta_1}[\mathcal{GW}]\in\Phi_d$ if and only if $\mu\ge(d+1)/2+\kappa$.*

We now assume that for a fixed d, $\lambda>d$. This condition is necessary to ensure integrability of Generalized Cauchy covariance functions, and hence the related spectral density to be bounded. The following result provides a solution to Problem 1.1 when $\phi(\cdot;\boldsymbol{\theta})=\mathcal{C}(\cdot;\delta,\lambda)$ for $0<\delta<2$, so that $\Theta=(0,2)\times(d,\infty)$.

Theorem 3.3. *Let $\mathcal{C}(\cdot;\delta,\lambda)$ for $0<\delta<2$ be the Generalized Cauchy function as defined in Eq. (1.5). Let $K_{\varepsilon;\delta,\lambda;\beta_2,\beta_1}[\mathcal{C}]$ with $0<\beta_1<\beta_2$, be the Zastavnyi operator (1.1) related to the function $\mathcal{C}(\cdot;\delta,\lambda)$.*

1. *Let $\varepsilon>0$. If $\varepsilon\ge\delta>0$ and $\delta<1$, then $K_{\varepsilon;\delta,\lambda;\beta_2,\beta_1}[\mathcal{C}]\in\Phi_\infty$;*
2. *Let $\varepsilon<0$. $K_{\varepsilon;\delta,\lambda;\beta_2,\beta_1}[\mathcal{C}]\in\Phi_\infty$ if and only if $\varepsilon\le-\lambda<0$.*

Two technical lemmas are needed to prove our last result.

Lemma 3.1. *Let $\mathcal{K}_\nu:[0,\infty)\to\mathbb{R}$ be the function defined through (1.4). Let $\nu>0$. Then, for all $z>0$,*

1. $\lim_{z\to+\infty} z\frac{\mathcal{K}_\nu'(z)}{\mathcal{K}_\nu(z)}=-\infty$, *for all $\nu\in(-\infty,+\infty)$;*
2. $\lim_{z\to+0} z\frac{\mathcal{K}_\nu'(z)}{\mathcal{K}_\nu(z)}=-\nu$, *for $\nu>1$.*

Proof. To prove the two assertions, it is enough to use the following result (Baricz et al., 2011),

$$-\sqrt{\frac{\nu}{\nu-1}z^2+\nu^2} < \frac{z\mathcal{K}'_\nu(z)}{\mathcal{K}_\nu(z)} < -\sqrt{z^2+\nu^2}, \tag{3.1}$$

where the left hand side of (3.1) is true for all $\nu > 1$, and the right hand side holds for all $\nu \in \mathbb{R}$. □

Lemma 3.2. *Let $\mathcal{C}(\cdot;2,\lambda)$ be the Cauchy correlation function as defined at (1.5). Let $\beta > 0$. Then, for $d > \lambda/2+2$ and $2\epsilon < -\lambda$ the following assertions are equivalent:*

1. $\beta^\epsilon\widehat{\mathcal{C}}_{d,\beta}(z;2,\lambda)$ *is decreasing with respect to β on $[0,+\infty)$, for every z,λ;*
2. $\beta^{\varepsilon+\frac{2d+\lambda}{4}}\mathcal{K}_{\frac{2d-\lambda}{4}}(\beta)$ *is decreasing with respect to β on $[0,+\infty)$, for every z,λ;*
3. $(\varepsilon+\frac{2d+\lambda}{4})+\beta\frac{\mathcal{K}'_{\frac{2d-\lambda}{4}}(\beta)}{\mathcal{K}_{\frac{2d-\lambda}{4}}(\beta)} < 0,\ \beta\in[0,\infty)$.

Proof. Showing that $\beta^\varepsilon\widehat{\mathcal{C}}_{d,\beta}(z;2,\lambda)$ is decreasing with respect to β is the same as showing that $\beta^{\varepsilon+\lambda/4+d/2}\mathcal{K}_{d/2-\lambda/4}(\beta)$ is decreasing. Point *2* of Lemma 3.2 holds if and only if

$$\beta^{\varepsilon+\frac{2d+\lambda}{4}-1}\left(\left(\varepsilon+\frac{2d+\lambda}{4}\right)+\beta\frac{\mathcal{K}'_{\frac{2d-\lambda}{4}}(\beta)}{\mathcal{K}_{\frac{2d-\lambda}{4}}(\beta)}\right) < 0.$$

Applying point *2* of Lemma 3.1 and the fact that $\beta\mathcal{K}'_{\frac{2d-\lambda}{4}}(\beta)/\mathcal{K}_{\frac{2d-\lambda}{4}}(\beta)$ is decreasing with respect to β, the three assertions of Lemma 3.2 are true if $2d > \lambda+4$ and $2\varepsilon < -\lambda$. The proof is completed. □

We are now able to fix a solution to Problem 1.1 when $\phi(\cdot;\boldsymbol{\theta}) = \mathcal{C}(\cdot;2,\lambda)$, so that $\boldsymbol{\theta}=\nu$ and $\Theta=(0,\infty)$.

Theorem 3.4. *Let $\mathcal{C}(\cdot;2,\lambda)$ be the Cauchy function as defined in Eq. (1.5) and let $K_{\varepsilon;2,\lambda;\beta_2,\beta_1}[\mathcal{C}]$ with $0<\beta_1<\beta_2$ be the Zastavnyi operator (1.1) related to the function $\mathcal{C}(\cdot;2,\lambda)$. Then, for $\lambda < 2d-4$, $K_{\varepsilon;2,\lambda;\beta_2,\beta_1}[\mathcal{C}]\in\Phi_d$ provided $2\varepsilon<-\lambda$.*

Proof. We need to find conditions such that $K_{\varepsilon;2,\lambda;\beta_2,\beta_1}[\mathcal{C}]\in\Phi_d$. This is equivalent to the following condition:

$$\beta_1^\varepsilon\widehat{\mathcal{C}}_{d,\beta_1}(z;2,\lambda)-\beta_2^\varepsilon\widehat{\mathcal{C}}_{d,\beta_2}(z;2,\lambda)\geq 0.$$

Thus, we need to prove that the function $\beta^\varepsilon\widehat{\mathcal{C}}_{d,\beta}(z;2,\lambda)$ is decreasing with respect to β. Using Lemma 3.2, we have that $K_{\varepsilon;2,\lambda;\beta_2,\beta_1}[\mathcal{C}]\in\Phi_d$. □

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2019.108620.

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