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Put–Call Parities, absence of arbitrage opportunities, and nonlinear pricing rules

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Abstract

When prices of assets traded in a financial market are determined by nonlinear pricing rules, different parities between call and put options have been considered. We show that, under monotonicity, parities between call and put options and discount certificates characterize ambiguity-sensitive (Choquet and/or Šipoš) pricing rules, that is, pricing rules that can be represented via discounted expectations with respect to non-additive probability measures. We analyze how nonadditivity relates to arbitrage opportunities and we give necessary and sufficient conditions for Choquet and Šipoš pricing rules to be arbitrage free. Finally, we identify violations of the Call-Put Parity with the presence of bid–ask spreads.

KEYWORDS

asset pricing, Call–Put Parity, Choquet and/or Šipoš pricing, Discount Certificate–Call Parity, market frictions, no arbitrage, Put–Call Parity

JEL CLASSIFICATION

C71, D81, G12

1 INTRODUCTION

Asset pricing models aim to associate to marketed securities, prices that are consistent with the absence of arbitrage opportunities. Typically, these models assume a reference probability over a state space and frictionless pricing rules, as done, for instance, in the seminal paper of Harrison and Kreps (1979). As a result, assets are valued by a linear pricing rule, or, equivalently, as the discounted expectation with respect to the so-called risk-neutral probability. However, both of these assumptions are questionable. Since at least Ellsberg (1961), the decision theory literature suggests a shift to (nonprobabilistic) uncertainty. A general framework without an objective probability was already studied in Kreps (1981) and was studied more recently by Biagini and Cont (2007), Riedel (2015), Cassese (2008), Cassese (2017), Cassese (2021), and Burzoni et al. (2021). Moreover, substantial evidence of frictions in financial markets (transaction costs, taxes, bid–ask spreads) has prompted the study of nonlinear pricing rules that can account for frictions, as in Garman and Ohlson (1981), Prisman (1986), Ross (1987), Bensaid et al. (1992), and Jouini and Kallal (1995).

This article drops both assumptions of the existence of an objective probability and of the linearity of pricing rules. More precisely, following the ideas first developed in Chateauneuf et al. (1996), we consider Choquet pricing rules, that is, Choquet integrals with respect to a *nonadditive* risk-neutral probability (also called risk-neutral capacity). Choquet pricing rules are not merely a mathematical generalization of linear pricing rule. In fact, as Cerreia-Vioglio et al. (2015) showed in a recent paper, Choquet pricing rules arise if a version of the parity between call and put options is maintained. This result is surprising as it connects the Choquet integral with a parity between call and put options, two apparently unrelated notions.

Our paper makes two main contributions to this stream of literature. First, in Section 3, we study what happens when different parities between call and put options are considered. When prices are nonlinear, as for Choquet pricing rules, there are several ways to define the price parity between call and put options. We show that different parities characterize different pricing rules with respect to nonadditive probability. Second, in Section 4, we impose a general nonarbitrage condition and we analyze its implications on the nonlinear pricing rules that we characterized in the first part of our article. The remainder of the introduction provides further details about our findings.

Section 3 characterizes pricing rules using parities between financial options. As already remarked in the seminal paper of Stoll (1969), there are several ways to replicate call and put options. When prices are nonlinear, different replications strategies lead to different parities between call and put options. One parity, deemed Put–Call Parity (PCP) was considered by Cerreia-Vioglio et al. (2015), while another one, the Call–Put Parity (CPP) was analyzed by Chateauneuf et al. (1996).

Our first contribution in Section 3 provides the formal relationship between the two parities. We show that CPP of Chateauneuf et al. (1996) is a stronger assumption: it corresponds to PCP of Cerreia-Vioglio et al. (2015) plus the absence of bid–ask spreads. Second, we improve the main result of Cerreia-Vioglio et al. (2015). They proved that a pricing rule satisfies PCP, monotonicity, and translation invariance if and only if it is a Choquet pricing rule. We show in Theorem 3.2 that PCP and monotonicity are enough for the characterization. Thus we demonstrate that the connection between PCP and the Choquet pricing rule runs even deeper than suggested by the result of Cerreia-Vioglio et al. (2015). Third, we study what happens when CPP replaces PCP. Theorem 3.3 characterizes CPP by a stronger version of Choquet pricing rules, called Choquet-Šipoš pricing rules. These are pricing rules that are at the same time Choquet and Šipoš integrals. The Šipoš integral is an integral with respect to a nonadditive measure that, in general, can differ from the

Choquet integral, see Šipoš (1979). However, when CPP holds, the two coincide. For sake of completeness, in Appendix A.1, we characterize Šipoš pricing rules that are not also Choquet pricing rules. To do so, we use discount certificates, which are well-known financial options that pay the minimum between the value of an underlying asset and a fixed cap. Theorem A.1 shows that Šipoš pricing rules are characterized by: (i) the parity between discount certificates and call options; and (ii) the absence of bid–ask spreads. To the best of our knowledge, our paper is the first to provide a justification for the use of Šipoš pricing rules for asset pricing.

In Section 4, we assume that pricing rules are either Choquet or Šipoš pricing rules, and we analyze what happens when one imposes the absence of arbitrage opportunities. We say that the market is arbitrage free (AF) if, whenever an agent can build a portfolio that pays a non-negative amount of money at every state, then she has to pay a non-negative price. Parities between put and call options are among the simplest and best understood no-arbitrage relations. However, in general, neither Choquet nor Šipoš pricing rules guarantee that markets are AF. We study which additional conditions one has to impose in order to eliminate arbitrage opportunities. Within this section, Theorem 4.3 contains two distinct results. The first one is that a Choquet pricing rule guarantees no-arbitrage if and only if there exists an additive risk-neutral measure which is smaller than the risk-neutral capacity associated to the pricing rule. On the other hand, the second result shows that a Šipoš pricing rule is AF if and only if it is linear. Therefore, Šipoš pricing rules cannot take into account market frictions.

The implication of Theorem 4.3 is that, whenever markets do not allow arbitrage opportunities but bid–ask spread are observed, the strong parity CPP of Chateauneuf et al. (1996) must be violated. This violation of CPP, in which the marketed price of put options is cheaper than the theoretical one (i.e., the price of building a portfolio that replicates a put), was empirically observed when puts were introduced in financial markets, see Klemkosky and Resnick (1979) and Cremers and Weinbaum (2010). However, it is important to note that this violation of CPP is consistent with the absence of arbitrage opportunities (and with PCP of Cerreia-Vioglio et al. (2015)).

The rest of the paper is organized as follows. Section 2 introduces the framework and our notation. Section 3 studies PCP and CPP and characterizes Choquet and Choquet-Šipoš pricing rules. Section 4 investigates AF Choquet and Šipoš pricing rules. While Section 3 justifies the investigations made in Section 4, we note that the two sections can be read independently. Readers with a specific interest in arbitrage can proceed directly to the latter section. Finally, Section 5 concludes. All proofs are gathered in the appendix.

2 | THE MODEL

This paper considers the simple framework of a stochastic two-date model: $t = 0$ (today) is known, $t = 1$ (tomorrow) is uncertain. Uncertainty is represented by a set Ω (finite or infinite) of *states of nature*, endowed with a σ-algebra $\mathcal{A}$. One, and only one, state of nature will be realized tomorrow and will be known. Note that we do not assume that there is a reference probability defined on $(\Omega, \mathcal{A})$.

A *payoff*, or *contingent claim*, is a bounded, real-valued, $\mathcal{A}$-measurable random variable $x : \Omega \to \mathbb{R}$ (or, if Ω is finite, a vector $x := (x(\omega))_{\omega\in\Omega} \in \mathbb{R}^{\Omega}$) with $x(\omega)$ representing the payoff (money) at $t = 1$ if state ω prevails. We denote $B(\Omega, \mathcal{A})$ the set of all contingent claims. We adopt the convention that if $x(\omega) < 0$, then $|x(\omega)|$ is paid by the agent and if $x(\omega) > 0$ then $|x(\omega)|$ is received. For every $A \in \mathcal{A}$, we denote by $\mathbf{1}_A$ the payoff in $B(\Omega, \mathcal{A})$ defined by $\mathbf{1}_A(\omega) = 1$ if

$\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ otherwise. We set $\mathbf{1}_\emptyset = 0$. Note that $B(\Omega, \mathcal{A})$ comes equipped with the usual pointwise partial order, $\geq$, where $x \geq y$ if and only if $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega$, and we define $x \vee y$ by $(x \vee y)(\omega) := \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$ and $x \wedge y$ by $(x \wedge y)(\omega) := \min\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$. Let $x \in B(\Omega, \mathcal{A})$, we denote $x^+ := x \vee 0$ its *positive part* and $x^- := (-x) \vee 0$ its *negative part*. Finally, we say that two contingent claims $x, y \in B(\Omega, \mathcal{A})$ are *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$.

A *pricing rule* is a real-valued function $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ that associates to every payoff $x \in B(\Omega, \mathcal{A})$ the price/cost $f(x)$ at $t = 0$, for the delivery of the random payoff x at $t = 1$, with the convention that $|f(x)|$ is paid if $f(x) > 0$ and received if $f(x) < 0$. Hence $f(x)$ is the buying (ask) price, the price one pays to buy x, and the (bid) selling price is then $-f(-x)$, which is the amount received if one sells x. This paper will take the pricing rule $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ as a primitive concept. Throughout the paper, f will be assumed to satisfy *monotonicity*, that is, $f(x) \geq f(x')$ for all $x, x' \in B(\Omega, \mathcal{A})$ such that $x \geq x'$.

A *capacity* v on the measurable space $(\Omega, \mathcal{A})$ is a set function $v : \mathcal{A} \mapsto \mathbb{R}$ such that $v(\emptyset) = 0$ and which is monotone, that is, for all $A, B \in \mathcal{A}$, $A \subseteq B$ implies $v(A) \leq v(B)$. The capacity is said to be normalized, also called *nonadditive probability*, if $v(\Omega) = 1$. A capacity $v : \mathcal{A} \mapsto \mathbb{R}$ is *concave*, also called submodular, if, for all $A, B \in \mathcal{A}$, $v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$. A *probability* $\mu : \mathcal{A} \mapsto \mathbb{R}$ is a normalized capacity which is (finitely) additive, that is, $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$. The conjugate of the capacity $v : \mathcal{A} \mapsto \mathbb{R}$ is the capacity $v^* : \mathcal{A} \mapsto \mathbb{R}$ defined by $v^*(A) := v(\Omega) - v(A^c)$ for all $A \in \mathcal{A}$ and we say that the capacity v is *auto-conjugate* if $v = v^*$. Note that the capacity v is auto-conjugate if and only if $v(A) + v(A^c) = v(\Omega)$ for all $A \in \mathcal{A}$.

Consider a capacity v on $(\Omega, \mathcal{A})$. The function f is said to be a *Choquet pricing rule* with respect to v if

$$f(x) = \int_\Omega^C x\, dv := \int_{-\infty}^0 (v(\{x \geq t\}) - v(\Omega))\, dt + \int_0^{+\infty} v(\{x \geq t\})\, dt \text{ for all } x \in B(\Omega, \mathcal{A}),$$

where $\{x \geq t\} = \{\omega \in \Omega | x(\omega) \geq t\}$. The notation $\int^C$ indicates that the integral is a Choquet integral (we drop the subscript Ω from the integral when no confusion arises). We will use the notation $\int$ for the standard integral with respect to an additive capacity. Note that in this case, the pricing rule f is linear. We note that the capacity v associated with the Choquet pricing rule f is uniquely defined, it satisfies $v(A) := f(\mathbf{1}_A)$ for all $A \in \mathcal{A}$, and v is called the risk-neutral capacity (associated with the Choquet pricing rule f).

We say that f is a *Šipoš pricing rule* with respect to v if

$$f(x) = \int_\Omega^S x\, dv := \int^C x^+\, dv - \int^C x^-\, dv \quad \text{for all } x \in B(\Omega, \mathcal{A}).$$

Note that Choquet and Šipoš integrals coincide if $x \in B(\Omega, \mathcal{A})$ is non-negative. One fundamental difference between the two integrals is that the former satisfies *translation invariance*, that is, $f(x + t\mathbf{1}_\Omega) = f(x) + f(t\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$, all $t \in \mathbb{R}_+$, while the latter has *no bid–ask spread*, that is, $f(x) = -f(-x)$ for all $x \in B(\Omega, \mathcal{A})$.

Finally, we say that f is a *Choquet–Šipoš pricing rule* when the Choquet and Šipoš integrals coincide with f for the same capacity v, that is

$$f(x) = \int_\Omega^S x\, dv = \int_\Omega^C x\, dv \quad \text{for all } x \in B(\Omega, \mathcal{A}).$$

It turns out that f is a *Choquet–Šipoš pricing rule* if and only if f is a *Choquet pricing rule* with respect to an auto-conjugate capacity υ, see Proposition A.10 in Appendix A.5.

We end this section with a remark. Suppose that f is a *nonzero* Choquet pricing rule with respect to the capacity υ, then one checks that $\upsilon(\Omega) = f(\mathbf{1}_\Omega) > 0$. Thus, we can normalize the capacity υ, by defining the nonadditive probability $\bar{\upsilon} : \mathcal{A} \to \mathbb{R}$ by $\bar{\upsilon}(A) := \frac{\upsilon(A)}{\upsilon(\Omega)}$ for all $A \in \mathcal{A}$, and obtain a riskless interest rate $r > -1$ uniquely defined by $f(\mathbf{1}_\Omega) = \frac{1}{1+r}$. Then the nonzero Choquet pricing rule f can be written as

$$f(x) = \frac{1}{1+r} \int^C x \, d\bar{\upsilon} \text{ for all } x \in B(\Omega, \mathcal{A}).$$

This (nonadditive) probabilistic formulation, taken from Cerreia-Vioglio et al. (2015), allows to interpret the price/cost $f(x)$ of every payoff x as the present value of its nonadditive expectation, where the present value is calculated with the riskless interest rate. The same can be done for Šipoš and Choquet–Šipoš pricing rules.

3 | CHARACTERIZATION OF PRICING RULES THROUGH FINANCIAL PARITIES

3.1 | Call–Put Parity(ies)

This section characterizes the pricing rules that satisfy each of the two different parities between prices of call and put options introduced by Chateauneuf et al. (1996) and Cerreia-Vioglio et al. (2015).

A call option with strike $k \geq 0$ and expiration date T is a financial contract that gives the option buyer the right, but not the obligation, to buy a stock, bond, or commodity at price k at time T. The stock, bond, or commodity is called the underlying asset. A call buyer profits when the underlying asset increases in price. A put option gives the right, but not the obligation, to sell the underlying asset at price k at time T. In our two-period financial economy, given an underlying asset $x \in B(\Omega, \mathcal{A})$, a call option with strike $k \geq 0$ and the related put option are defined, respectively, by

$$c_{x,k} = (x - k\mathbf{1}_\Omega)^+, \; p_{x,k} = (k\mathbf{1}_\Omega - x)^+.$$

Clearly, for all $x \in B(\Omega, \mathcal{A})$, the following equation holds

$$x = c_{x,k} - p_{x,k} + k\mathbf{1}_\Omega, \tag{1}$$

and it says that one can replicate every underlying x by buying a call, selling a put, both with the same strike k, and buying k units of the bond. Given a pricing rule f, Cerreia-Vioglio et al. (2015) used equality in Equation (1) to define the *Put-Call Parity* (*PCP*) by

$$f(x) = f(c_{x,k}) + f(-p_{x,k}) + f(k\mathbf{1}_\Omega), \tag{PCP}$$

for all $x \in B(\Omega, \mathcal{A})$. Obviously, the mathematical equality in Equation (1) can be rewritten as

$$p_{x,k} = c_{x,k} - x + k\mathbf{1}_\Omega. \tag{2}$$

for all $x \in B(\Omega, \mathcal{A})$. Hence a put with strike k can be replicated buying a call with the same strike, selling the underlying asset, and buying k units of the bond. Chateauneuf et al. (1996) used Equation (2) to define a different parity that they named *Call–Put Parity (CPP)*

$$f(p_{x,k}) = f(c_{x,k}) + f(-x) + f(k\mathbf{1}_\Omega). \tag{CPP}$$

for all $x \in B(\Omega, \mathcal{A})$. If f is linear, PCP and CPP are clearly equivalent. The following result provides the formal relationship between the two parities. It shows that CPP of Chateauneuf et al. (1996) is stronger than PCP of Cerreia-Vioglio et al. (2015) as CPP is equivalent to PCP and the absence of bid–ask spreads.

Proposition 3.1. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a pricing rule. Then (i) ⇔ (ii).*

(i) f satisfies CPP;
(ii) f satisfies PCP and no bid–ask spreads.

3.2 | Choquet representation of pricing rules

When frictions are taken into account, the linearity of f is no longer guaranteed. Choquet pricing rules serve as examples of nonlinear pricing rules capable of explaining market frictions. They have been considered first by Chateauneuf et al. (1996), see also Wang et al. (1997), Castagnoli et al. (2002), and Chateauneuf and Cornet (2018), Chateauneuf and Cornet (2022). A full characterization using a financial parity has been given by Cerreia-Vioglio et al. (2015). The main theorem of Cerreia-Vioglio et al. (2015) proves that a pricing rule f satisfies PCP, monotonicity, and translation invariance, if and only if f is a Choquet pricing rule. Their idea is extremely interesting as it connects two apparently unrelated concepts: the well-known financial concept of PCP with the Choquet integral. We generalize their result showing that translation invariance is redundant.

Theorem 3.2. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a monotone pricing rule. Then (i) ⇔ (ii).*

(i) f satisfies PCP;
(ii) f is a Choquet pricing rule.

Note that *only* monotonicity and PCP are required to pin down Choquet pricing rules. These are very weak and desirable assumptions. This highlights the central role played by Choquet pricing rules. From a mathematical point of view, our proof differs from that of Cerreia-Vioglio et al. (2015). Their characterization is based on a representation result by Greco (1982). On the other hand, our proof of Theorem 3.2 consists in showing that the parity PCP is equivalent to the *comonotonic additivity* of the pricing rule f, that is $f(x + x') = f(x) + f(x')$ for all $x, x' \in B(\Omega, \mathcal{A})$ comonotonic, a familiar concept in decision theory. This in turns allows us to use the fundamental result of Schmeidler (1986) who characterized the Choquet integral by the comonotonic additivity property.

A natural question arises however. What happens when one replaces PCP with CPP? A first answer is given by Proposition 1 and Theorem 1, which imply that a monotone pricing rule f satisfies CPP if and only if f is a Choquet pricing rule with no bid–ask spread. The following Theorem 3.3 shows that CPP (together with monotonicity) characterizes Choquet–Šipoš pricing rules.[1]

Theorem 3.3. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a monotone pricing rule. Then (i) ⇔ (ii).*

(i) f satisfies CPP;
(ii) f is a Choquet–Šipoš pricing rule.

Note that, if assets are priced through Choquet–Šipoš pricing rules, then there are no bid–ask spreads. This was also noted in a recent paper of Lécuyer and Lefort (2021) who studied particular Choquet pricing rules given by (normalized) generalized neo-additive capacities (GNAC) (see Chateauneuf et al., 2007). They show that if f is a Choquet pricing rule, then there are no bid–ask spreads if and only if $v(A) + v(A^c) = v(\Omega)$ for all $A \in \mathcal{A}$ (i.e., f is Choquet–Šipoš, see Proposition A.10 in Appendix A.5). Moreover, if f is a GNAC pricing rule, there are no bid–ask spreads if and only if $v(A) + v(A^c) = v(\Omega)$ for at least one $A \in \mathcal{A}$. See also Castagnoli et al. (2004).

To summarize, Section 3.2 characterizes Choquet pricing rules through PCP and Choquet–Šipoš pricing rules through CPP. Finally, Theorem A.1 in Appendix A.1 characterizes general Šipoš pricing rules through the parity between discount certificates and call options.

While the parities between call and put options and discount certificate and call options are some of the best known assumptions about the absence of arbitrage opportunities, the pricing rules obtained in Theorems 3.2 and 3.3 (and Theorem A.1 in Appendix A.1), do not guarantee that markets are AF. The following section defines arbitrage opportunities and shows how one can eliminate them.

4 | ABSENCE OF ARBITRAGE OPPORTUNITIES

In the characterizations given in Section 3, no property is stated about the capacity v associated with the pricing rule f. Without additional properties on v, Choquet and Šipoš pricing rules may leave room for arbitrage opportunities.

Intuitively, a payoff $x \in B(\Omega, \mathcal{A})$ is an arbitrage opportunity if "it allows to make money from nothing." We recall that a *subadditive* pricing rule f satisfies the property that $f(x + x') \leq f(x) + f(x')$ for all $x, x' \in B(\Omega, \mathcal{A})$ and it is said to be AF whenever f is *non-negative*, that is, $x \geq 0$ implies that $f(x) \geq 0$, or, in other words, whenever there is no payoff $x \geq 0$ (with no loss at each state tomorrow) such that $f(x) < 0$ (with a gain today).

Without the subadditivity assumption on f, there is a need to eliminate other arbitrage opportunities. Consider for the moment *buy & sell arbitrage opportunities*, that is, payoffs $x \in B(\Omega, \mathcal{A})$ for which $f(x) + f(-x) < 0$, thus leading to a zero stream of money tomorrow ($x - x = 0$) and an aggregate gain today in buying x and selling x (i.e., buying $-x$). Note that the absence of buy and sell arbitrage opportunities is equivalent to the property of f having non-negative bid–ask spreads.[2] The following result provides several characterization properties of a Choquet pricing rule for which there is no *buy and sell arbitrage opportunity*.

Theorem 4.1. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a Choquet pricing rule with respect to a capacity v. Then the following assertions are equivalent.*

(i) f has no buy & sell arbitrage opportunities*, or equivalently*
f has non-negative bid–ask spreads, that is, $f(x) + f(-x) \geq 0$ for all $x \in B(\Omega, \mathcal{A})$;
(ii) $v \geq v^*$, that is, $v(A) + v(A^c) \geq v(\Omega)$ for all $A \in \mathcal{A}$;

(iii) $-f(-x) \leq \int^S x\, dv \leq f(x)$ *for all* $x \in B(\Omega, \mathcal{A})$.

Note that the previous theorem, part of which was proved when Ω is finite by Chateauneuf and Cornet (2022), introduces the new property that the Šipoš pricing rule is an homogeneous selection of the bid–ask spread interval, that is, $\int^S x\, dv \in [-f(-x), f(x)]$ for all $x \in B(\Omega, \mathcal{A})$. Pricing a new security $x \in B(\Omega, \mathcal{A})$ so that the enriched pricing rule remains buy and sell arbitrage free amounts to choose the price of the new security in the interval $[-f(-x), f(x)]$. Thus, having a canonical way to do it via the Šipoš pricing rule is interesting for both theoretical and practical reasons, as the calculation of the Šipoš integral is standard.

The absence of buy and sell arbitrage opportunity for a pricing rule f can be strengthened in the spirit of Chateauneuf and Cornet (2022) as follows.

Definition 4.2. f is *arbitrage free (AF)* if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in B(\Omega, \mathcal{A})$

$$\sum_{i=1}^{n} x_i \geq 0 \;\Rightarrow\; \sum_{i=1}^{n} f(x_i) \geq 0. \tag{AF}$$

We say equivalently that there are no arbitrage opportunities, that the market is AF, or that a pricing rule f satisfies the AF condition. The interpretation of Definition 4.2 is as follows. Suppose that an agent wants to construct an asset $x \geq 0$, splitting x by buying n securities $x_1, \ldots, x_n$ summing up to x. Then the pricing rule is AF if and only if the aggregate cost of buying separately $x_1, \ldots, x_n$ is non-negative. Thus, we can also interpret it as the absence of *multiple* buy and sell arbitrage opportunities.

Note that Definition 4.2 is stronger than the standard definition (when f is subadditive) that $x \geq 0$ implies $f(x) \geq 0$, and also stronger than assuming the absence of *buy and sell arbitrage opportunities*. When f is subadditive (hence when f is linear too), then f is AF if and only if it is non-negative (i.e., precisely when $x \geq 0$ implies $f(x) \geq 0$). The fundamental theorem of asset pricing famously characterizes linear and AF pricing rules as discounted expectation with respect to a (additive) probability, see Harrison and Kreps (1979).

When frictions are taken into account, the linearity of f is no longer guaranteed. In general, nonlinear nonnegative pricing rules do not guarantee the absence of arbitrage opportunities. To solve this issue, usually pricing rules are required to be sublinear, that is, to satisfy positive homogeneity and subadditivity, see Jouini and Kallal (1995), Castagnoli et al. (2002), Araujo et al. (2018), Burzoni et al. (2021), and Chateauneuf and Cornet (2018). Choquet pricing rules satisfy positive homogeneity, but are *not* subadditive in general. Subadditive Choquet pricing rules were considered by Chateauneuf et al. (1996) and are characterized by concave (submodular) capacities, see also Chateauneuf and Cornet (2022), Cerreia-Vioglio et al. (2015) and Cinfrignini et al. (2023), Cinfrignini et al. (2023) for the particular case of belief functions. Concavity of the capacity gives a sufficient condition to guarantee the absence of arbitrage opportunities for Choquet pricing rules. However, the condition is not necessary.

Theorem 4.3 provides a full characterization of Condition AF for both Choquet and Šipoš pricing rules. For Choquet pricing rules, there are no arbitrage opportunities if and only if there exists an additive non-negative set function μ below the capacity v. Formally, the *anticore* of a capacity v (already considered in Gilboa and Lehrer and used in the context of pricing rules by Araujo et al. (2012) is defined as

$$AC(v) = \{\mu : \mathcal{A} \to \mathbb{R}_+,\ \mu \text{ is additive},\ \mu \leq v,\ \text{and } \mu(\Omega) = v(\Omega)\}. \tag{AC}$$

It is well known that if a capacity v is concave, then the associated Choquet pricing rule f is subadditive and $AC(v) \neq \emptyset$. Theorem 4.3 shows that if f is a Choquet pricing rule with respect to v, then it is AF if and only if $AC(v) \neq \emptyset$. A similar result was proved in Chateauneuf and Cornet (2022) when Ω is finite. However, the proof for the case of Ω infinite is different.

Finally, Theorem 4.3 provides also a characterization of AF Šipoš pricing rules that turns out to be a negative result since a Šipoš pricing rule is AF if and only if it is linear. Therefore Šipoš pricing rules that are also AF cannot take into account *any* friction.

Theorem 4.3. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a pricing rule. Then,*

(i) If f is a Choquet pricing rule, then f satisfies AF if and only if $AC(v) \neq \emptyset$.
(ii) If f is a Šipoš pricing rule, then f satisfies AF if and only if f is linear.

We briefly discuss the fact that *stronger* arbitrage opportunities such as $x := \sum_{i=1}^n x_i \geq 0, x \neq 0$, and $\sum_{i=1}^n f(x_i) = 0$, are not ruled out by our definition of AF. Since in the primitives of our model, there is no notion of null sets, it is not straightforward to deal with this case when the state space Ω is infinite.[3] Some authors, such as Burzoni et al. (2021) and Cassese (2021), replace null sets with a strengthening of the pointwise order, which they include in the definition of the financial market. Since in this section, we assumed that f is Choquet (or Šipoš), we can define the partial order $>^v$:

$$x >^v 0 \Leftrightarrow x \geq 0 \text{ and } \exists \varepsilon > 0 \text{ such that } v^*(\omega | x(\omega) \geq \varepsilon) > 0,$$

denoting v^* the conjugate capacity of v. Note that, because of Theorem 4.1, we have $v \geq v^*$ and therefore also $v(\omega | x(\omega) \geq \varepsilon) > 0$. We require the existence of $\varepsilon > 0$ to avoid consideration of σ-additivity. Equipped with $>^v$, we can provide the following definition.

Definition 4.4. f is *strongly arbitrage free (AF*)* if it satisfies *AF* and moreover, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in B(\Omega, \mathcal{A})$ one has

$$\sum_{i=1}^n x_i >^v 0 \ \Rightarrow \ \sum_{i=1}^n f(x_i) > 0. \tag{AF*}$$

Clearly AF* implies AF. Corollary 4.5 shows that the two notions are equivalent. Therefore, when the payoff $x := \sum_{i=1}^n x_i$ is strictly positive on a set of strictly positive "measure," the agent will pay a strictly positive price.

Corollary 4.5. *Let f be a Choquet or Šipoš pricing rule. Then AF* is equivalent to AF.*

To summarize, this section has proved that, under the mild assumptions of monotonicity and no-arbitrage, the unique parity, which would lead to frictions in accordance with observed bid–ask spreads, is PCP considered in Cerreia-Vioglio et al. (2015). We remark that under PCP, monotone pricing rules are of the Choquet type. If, moreover, non-negative bid–ask spreads are present in the market, then it turns out that $f(p_{x,k}) \leq f(c_{x,k}) + f(-x) + f(k\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$, $k \geq 0$, that is, a violation of CPP considered by Chateauneuf et al. (1996). This price violation says that it is cheaper to buy a put option with strike k in the market, rather than "constructing" one by buying a call, selling the underlying x, and buying k units of the bond. This miss-pricing was actually observed when put options were introduced in the markets, see Gould and Galai (1974), Klemkosky and Resnick (1979), and Sternberg (1994).

5 CONCLUSION

The first part of our paper studies different formulations of the famous parity between call and put options in a framework where pricing rules are nonlinear and not subadditive. Cerreia-Vioglio et al. (2015) study a parity called PCP. They prove that PCP, together with translation invariance and monotonicity, characterizes Choquet pricing rules. Chateauneuf et al. (1996) study a different parity, called CPP. Our first result studies the relationship between the two parities and shows that CPP is equivalent to PCP and no bid–ask spreads. Our second result improves the characterization of Cerreia-Vioglio et al. (2015) as it shows that translation invariance is redundant. Third, replacing PCP by CPP, we obtain Choquet-Šipoš pricing rules, which are pricing rules that are at the same time Choquet and Šipoš pricing rules. For the sake of completeness, in Appendix A.1, we characterize Šipoš pricing rules using the Discount Certificate–Call Parity and no bid–ask spreads.

The second part of our paper studies the implications of an AF condition on pricing rules. We show that, in general, Choquet and Šipoš pricing rules are not AF. If f is a Choquet pricing rule with respect to a capacity v, then the market is AF if and only if the anticore of v is nonempty. If f is a Šipoš pricing rule, then f is AF if and only if it is linear. Therefore, Šipoš pricing rules that are also AF cannot take into account *any* friction. Finally, we characterize bid–ask spreads as violations of CPP. This shows that, for pricing rules *à la* Choquet, one can observe at the same time PCP, absence of arbitrage opportunities, and a violation of CPP (i.e., non-negative bid–ask spreads).

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ENDNOTES

[1] A further characterization of Choquet–Šipoš pricing rules is given in Proposition A.10 in Appendix A.5 using the risk-neutral capacity v associated to f.

[2] Note that buy and sell arbitrage opportunities cannot arise for Šipoš and Choquet-Šipoš pricing rules since Šipoš pricing rules satisfy no bid–ask spread.

[3] For Ω finite, the counting measure is (implicitly) used, see Chateauneuf and Cornet (2022) and a characterization of the absence of strong arbitrage opportunities is provided in terms of the anticore.

[4] See also Theorem 5 in Marinacci and Montrucchio (2004).

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APPENDIX

The appendix contains the proofs of the statements in the main text. We begin, however, with Appendix A.1 in which we characterize Šipoš pricing rules through the use of the financial parity between discount certificates and call options.

A.1 | Discount Certificate–Call Parity and Šipoš representation of pricing rules

In order to pin down Šipoš pricing rules we first introduce discount certificates. A *discount certificate* on an asset x with cap $k \geq 0$, denoted $d_{x,k}$, is a contingent claim that in state ω pays x if $x(\omega) \leq k$ and pays k if $x(\omega) > k$, or equivalently,

$$d_{x,k} = x \wedge k\mathbf{1}_\Omega.$$

Noting that $d_{x,k} = (x - k\mathbf{1}_\Omega) \wedge 0 + k\mathbf{1}_\Omega$ and recalling that $c_{x,k} := [x - k\mathbf{1}_\Omega]^+ = (x - k\mathbf{1}_\Omega) \vee 0$ one can conclude

$$x = c_{x,k} + d_{x,k}. \tag{A.1}$$

Therefore, asset x can be replicated by buying the call $c_{x,k}$ and the discount certificate $d_{x,k}$. Given a pricing rule f, Cerreia-Vioglio et al. (2015) used Equation (A.1) to define the *Discount Certificate–Call Parity (DCP)* as follows:

$$f(x) = f(c_{x,k}) + f(d_{x,k}). \tag{DCP}$$

for all $x \in B(\Omega, \mathcal{A})$. In their paper, they used DCP, monotonicity and translation invariance in order to derive another characterization of Choquet pricing rules. The following Theorem A.1 shows that DCP and no bid–ask spreads pin down Šipoš pricing rules.

Theorem A.1. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a monotone pricing rule. Then (i) ⇔ (ii).*

(i) f satisfies no bid–ask spreads and DCP;
(ii) f is a Šipoš pricing rule.

Proof. The proof of Theorem A.1 is given in Appendix A.5. □

As we noted in Section 3.1, when pricing rules are nonlinear it is important to pay attention to the replication strategy, as different strategies imply different parities. The same is true with the Discount Certificate–Call Parity. In fact, Equation (A.1) can be rewritten as $c_{x,k} = x - d_{x,k}$. Therefore, one can replicate a call by buying the underlying x and selling a discount certificate. This replication strategy would suggest defining the following parity:

$$f(c_{x,k}) = f(x) + f(-d_{x,k}). \tag{DCP*}$$

for all $x \in B(\Omega, \mathcal{A})$, which may differ from DCP if f is not linear. The following Proposition A.2 relates DCP and DCP* in the same way Proposition 3.1 was relating PCP and CPP. Thus, in view of Theorem A.1, a monotone pricing rule is a Šipoš pricing rule if and only if it satisfies DCP*.

Proposition A.2. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a monotone pricing rule. Then (i) ⇔ (iii).*

(i) f satisfies no bid–ask spreads and DCP;
(iii) f satisfies DCP.*

Proof. The proof is similar to the one of Proposition 3.1 and it is, therefore, omitted. □

A.2 | Summary of the properties of pricing rules

We gather here the properties of pricing rules used in the proofs and some abbreviations. Also, recall that $c_{x,k} = (x - k\mathbf{1}_\Omega)^+$, $p_{x,k} = (k\mathbf{1}_\Omega - x)^+$ and $d_{x,k} = x \wedge k\mathbf{1}_\Omega$. We will need the following basic equalities:

$$(x - k\mathbf{1}_\Omega)^+ = x \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega,\ -(k\mathbf{1}_\Omega - x)^+ = x \wedge k\mathbf{1}_\Omega - k\mathbf{1}_\Omega, \text{ and } x \wedge k\mathbf{1}_\Omega = (x - k\mathbf{1}_\Omega) \wedge 0 + k\mathbf{1}_\Omega. \quad \text{(A.2)}$$

1. Monotonicity: $f(x) \geq f(x')$ for all $x \geq x'$.
2. Translation invariance (TI): $f(x + k\mathbf{1}_\Omega) = f(x) + f(k\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$ and all $k \in \mathbb{R}_+$.
3. Put–Call Parity (PCP): $f(x) = f((x - k\mathbf{1}_\Omega)^+) + f(-(k\mathbf{1}_\Omega - x)^+) + f(k\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$ and all $k \in \mathbb{R}_+$.
4. Call–Put Parity (CPP): $f((k\mathbf{1}_\Omega - x)^+) = f((x - k\mathbf{1}_\Omega)^+) + f(-x) + f(k\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$ and all $k \in \mathbb{R}_+$.
5. Discount Certificate–Call Parity (DCP): $f(x) = f((x - k\mathbf{1}_\Omega)^+) + f(x \wedge k\mathbf{1}_\Omega)$ for all $x \in B(\Omega, \mathcal{A})$, all $k \geq 0$.
6. No bid–ask spread: $f(-x) = -f(x)$ for all $x \in B(\Omega, \mathcal{A})$.

A.3 | Proof of Proposition 3.1

Proof. $[(i) \Rightarrow (ii)]$ We first prove that f has no bid–ask spread. Note that $f(0) = 0$ is a consequence of CPP (taking $x = 0$ and $k = 0$). Take now $k = 0$ in CPP, we get

$$f((-x)^+) = f(x^+) + f(-x) + f(0) \quad (*)$$

Replacing x by $-x$ in the above equation, we get

$$f((-x)^+) = f(x^+) + f(-x) + f(0) \quad (**)$$

Consequently, from (*) and (**), using the fact that $f(0) = 0$, we get $f(x) = f(x^+) - f((-x)^+) = -f(-x)$. Thus, f has no bid–ask spread. We then deduce that PCP holds since

$$\begin{aligned} f((x - k\mathbf{1}_\Omega)^+) + f(k\mathbf{1}_\Omega) &= -f(-x) + f((k\mathbf{1}_\Omega - x)^+) && \text{[from CPP]} \\ &= f(x) - f(-(k\mathbf{1}_\Omega - x)^+) && \text{[since } f \text{ has no bid–ask spread]} \end{aligned}$$

$[(ii) \Rightarrow (i)]$ Clearly CPP holds since

$$\begin{aligned} f((x - k\mathbf{1}_\Omega)^+) + f(k\mathbf{1}_\Omega) &= f(x) - f(-(k\mathbf{1}_\Omega - x)^+) && \text{[from PCP]} \\ &= -f(-x) + f((k\mathbf{1}_\Omega - x)^+) && \text{[since } f \text{ has no bid–ask spread]} \end{aligned}$$

□

A.4 | Proof of Theorem 3.2

The following lemmata will prove useful in the proof of Theorem 3.2.

Lemma A.3. *A pricing rule* $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ *satisfies translation invariance (TI) if and only if*

$$f(x + k\mathbf{1}_\Omega) = f(x) + kf(\mathbf{1}_\Omega) \text{ for all } x \in B(\Omega, \mathcal{A}) \text{ and all } k \in \mathbb{R}.$$

Proof. [⇒] We first prove that, for all $x \in B(\Omega, \mathcal{A})$ and all $k \in \mathbb{R}$, $f(x + k\mathbf{1}_\Omega) = f(x) + f(k\mathbf{1}_\Omega)$. Indeed, for $k \in \mathbb{R}_+$, this follows from the TI of f. Moreover, by TI of f, we have $f(0) = 0$ (taking $x = 0$ and $k = 0$) and $0 = f(0) = f(-k\mathbf{1}_\Omega + k\mathbf{1}_\Omega) = f(-k\mathbf{1}_\Omega) + f(k\mathbf{1}_\Omega)$ (taking $x = -k\mathbf{1}_\Omega$). Thus $f(-k\mathbf{1}_\Omega) = -f(k\mathbf{1}_\Omega)$ for all $k \in \mathbb{R}_+$. Finally, $f(x) = f(x - k\mathbf{1}_\Omega + k\mathbf{1}_\Omega) = f(x - k\mathbf{1}_\Omega) + f(k\mathbf{1}_\Omega)$, hence for all $k \in \mathbb{R}_+$

$$f(x - k\mathbf{1}_\Omega) = f(x) - f(k\mathbf{1}_\Omega) = f(x) + f(-k\mathbf{1}_\Omega).$$

We complete the proof by showing that $f(k\mathbf{1}_\Omega) = kf(\mathbf{1}_\Omega)$ for all $k \in \mathbb{R}$. First, let $n \in \mathbb{N}$ and $t \in \mathbb{R}$, then TI implies $f(nt\mathbf{1}_\Omega) = f((n-1)t\mathbf{1}_\Omega + t\mathbf{1}_\Omega) = f((n-1)t\mathbf{1}_\Omega) + f(t\mathbf{1}_\Omega)$ and by induction $f(nt\mathbf{1}_\Omega) = nf(t\mathbf{1}_\Omega)$. Consequently, from the first part of the proof, $f(-n\mathbf{1}_\Omega) = -nf(\mathbf{1}_\Omega)$ for all $n \in \mathbb{N}$. Second, let $q = \frac{n}{m} \in \mathbb{Q}$ (with $n \in \mathbb{Z}, m \in \mathbb{N} \setminus \{0\}$). Then by what we just proved $nf(\mathbf{1}_\Omega) = f(n\mathbf{1}_\Omega) = f(mq\mathbf{1}_\Omega) = mf(q\mathbf{1}_\Omega)$. Thus $f(q\mathbf{1}_\Omega) = \frac{n}{m}f(\mathbf{1}_\Omega) = qf(\mathbf{1}_\Omega)$. Finally, let $k \in \mathbb{R}$, then there are two sequences $(q_n^1)_n \subseteq \mathbb{Q}$, $(q_n^2)_n \subseteq \mathbb{Q}$ such that $q_n^1 \uparrow k$ and $q_n^2 \downarrow k$. By the first part of the proof and by monotonicity of f, for all n, $q_n^1 f(\mathbf{1}_\Omega) = f(q_n^1\mathbf{1}_\Omega) \le f(k\mathbf{1}_\Omega) \le f(q_n^2\mathbf{1}_\Omega) = q_n^2 f(\mathbf{1}_\Omega)$. Letting $n \to \infty$ shows that $f(k\mathbf{1}_\Omega) = kf(\mathbf{1}_\Omega)$.

[⇐] The proof is immediate noticing that $f(k\mathbf{1}_\Omega) = kf(\mathbf{1}_\Omega)$ for all $k \in \mathbb{R}$. □

Lemma A.4. *Let* $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ *be monotone and translation invariant, let* $k := f(\mathbf{1}_\Omega)$*, then*

$$|f(x) - f(y)| \le k\|x - y\|_\infty \text{ for all } x, y \in B(\Omega, \mathcal{A}).$$

Proof. For all $x, y \in B(\Omega, \mathcal{A})$, one has $x \le y + \|x - y\|_\infty \mathbf{1}_\Omega$. Since f is monotone and translation invariant, we deduce that $f(x) \le f(y + \|x - y\|_\infty \mathbf{1}_\Omega) = f(y) + k\|x - y\|_\infty$. Exchanging the role of x and y, we get $f(y) \le f(x) + k\|y - x\|_\infty$. Thus, $|f(x) - f(y)| \le k\|x - y\|_\infty$. □

Lemma A.5. *A pricing rule* $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ *satisfies PCP if and only if it satisfies TI and the following Buy and Sell Additivity Property:*

$$f(x) = f(x \wedge 0) + f(x \vee 0) = f(x^+) + f(-(-x)^+) \quad \text{for all } x \in B(\Omega, \mathcal{A}). \tag{A.3}$$

Proof. [⇒] First, we have $f(0) = 0$ by PCP, taking $x = 0$ and $k = 0$.

Second, Equation (A.3) follows from PCP, taking $k = 0$, since $f(0) = 0$.

We now prove that f is translation invariant. Let $x \in B(\Omega, \mathcal{A})$ and let $k \ge 0$. Then one has

$$\begin{aligned} f(x) &= f(x^+) + f(-(-x)^+) && \text{by (A.3)} \\ &= f((x + k\mathbf{1}_\Omega - k\mathbf{1}_\Omega)^+) + f(-(k\mathbf{1}_\Omega - x - k\mathbf{1}_\Omega)^+) \\ &= f(x + k\mathbf{1}_\Omega) - f(k\mathbf{1}_\Omega) && \text{by PCP.} \end{aligned}$$

Consequently, f is translation invariant.

[⇐] Fix $x \in B(\Omega, \mathcal{A})$ and $k \geq 0$. Then PCP holds since we have

$$\begin{aligned} f((x-k\mathbf{1}_\Omega)^+) + f(-(k\mathbf{1}_\Omega - x)^+) &= f(x-k\mathbf{1}_\Omega) && \text{from (A.3)} \\ &= f(x) - f(k\mathbf{1}_\Omega) && \text{from TI and Lemma A.3.} \end{aligned}$$

□

Lemma A.6. *A monotone pricing rule $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ satisfies PCP if and only if it is comonotonic additive.*

Proof. $[(i) \Rightarrow (ii)]$ The following steps prove that f is comonotonic additive.

Step A.7. $f(x+y) = f(x) + f(y)$ for all comonotonic and positive step functions $x, y \in B(\Omega, \mathcal{A})$.

Proof. Let $x, y \in B(\Omega, \mathcal{A})$ be comonotonic and positive step functions. By comonotonicity, there is a partition $A_1, \dots, A_n$ of Ω such that

$$x = x_1\mathbf{1}_{A_1} + \cdots + x_n\mathbf{1}_{A_n}, \text{ with } 0 \leq x_1 \leq \cdots \leq x_n,$$

$$y = y_1\mathbf{1}_{A_1} + \cdots + y_n\mathbf{1}_{A_n}, \text{ with } 0 \leq y_1 \leq \cdots \leq y_n.$$

Equivalently, one can write

$$x = \sum_{i=1}^n X_i, \text{ where } X_i = (x_i - x_{i-1})\mathbf{1}_{A_i \cup \cdots \cup A_n}, \ i = 1, \dots, n \text{ and } x_0 = 0,$$

$$y = \sum_{i=1}^n Y_i, \text{ where } Y_i = (y_i - y_{i-1})\mathbf{1}_{A_i \cup \cdots \cup A_n}, \ i = 1, \dots, n \text{ and } y_0 = 0.$$

We first show that $f(x) = \sum_{i=1}^n f(X_i)$, $f(y) = \sum_{i=1}^n f(Y_i)$, and $f(x+y) = \sum_{i=1}^n f(X_i + Y_i)$. It is enough to prove that for $i = 1, \dots, n-1$,

$$f(X_i + X_{i+1} + \cdots + X_n) = f(X_i) + f(X_{i+1} + \cdots + X_n).$$

For i fixed $1 \leq i \leq n$, set $\Sigma_{i,n} := X_i + \cdots + X_n$. Then we have

$$\Sigma_{i+1,n} = (x_{i+1} - x_i)\mathbf{1}_{A_{i+1}} + (x_{i+2} - x_i)\mathbf{1}_{A_{i+2}} + \cdots + (x_n - x_i)\mathbf{1}_{A_n},$$

$$X_i = \Sigma_{i,n} \wedge (x_i - x_{i-1})\mathbf{1}_\Omega = [\Sigma_{i,n} - (x_i - x_{i-1})\mathbf{1}_\Omega] \wedge 0 + (x_i - x_{i-1})\mathbf{1}_\Omega \text{ and}$$

$$\Sigma_{i+1,n} = \Sigma_{i,n} \vee (x_i - x_{i-1})\mathbf{1}_\Omega - (x_i - x_{i-1})\mathbf{1}_\Omega = [\Sigma_{i,n} - (x_i - x_{i-1})\mathbf{1}_\Omega] \vee 0.$$

Let us show that $f(\Sigma_{i,n}) = f(X_i + \Sigma_{i+1,n}) = f(X_i) + f(\Sigma_{i+1,n})$. Using Buy and Sell Additivity (A.3) of f and Lemma A.3, we get

$$\begin{aligned} f(X_i) + f(\Sigma_{i+1,n}) &= f([\Sigma_{i,n} - (x_i - x_{i-1})\mathbf{1}_\Omega] \wedge 0 + (x_i - x_{i-1})\mathbf{1}_\Omega) + f([\Sigma_{i,n} - (x_i - x_{i-1})\mathbf{1}_\Omega] \vee 0) \\ &= f(\Sigma_{i,n} - (x_i - x_{i-1})\mathbf{1}_\Omega) + f((x_i - x_{i-1})\mathbf{1}_\Omega) \\ &= f(\Sigma_{i,n}) = f(X_i + \Sigma_{i+1,n}). \end{aligned}$$

By induction, it is easy to see that $f(x) = \sum_{i=1}^n f(X_i)$. Similarly, we prove that $f(y) = \sum_{i=1}^n f(Y_i)$, and $f(x+y) = \sum_{i=1}^n f(X_i + Y_i)$.

To conclude the proof, we show that $f(X_i + Y_i) = f(X_i) + f(Y_i)$ for all i. Note that $X_i = a\mathbf{1}_A$, $Y_i = b\mathbf{1}_A$ with $A := A_i \cup \cdots \cup A_n$, $a := x_i - x_{i-1} \geq 0$, $b := y_i - y_{i-1} \geq 0$. Thus we only need to prove that $f(a\mathbf{1}_A + b\mathbf{1}_A) = f(a\mathbf{1}_A) + f(b\mathbf{1}_A)$. Indeed, if $x := (a+b)\mathbf{1}_A - a\mathbf{1}_\Omega = -a\mathbf{1}_{A^c} + b\mathbf{1}_A$, we have $x \vee 0 = b\mathbf{1}_A$, $x \wedge 0 = -a\mathbf{1}_{A^c} = a\mathbf{1}_A - a\mathbf{1}_\Omega$. Thus, using TI and Buy & Sell Additivity (A.3) of f, we obtain

$$\begin{aligned} f(a\mathbf{1}_A + b\mathbf{1}_A) + f(-a\mathbf{1}_\Omega) &= f(x) = f(x \vee 0) + f(x \wedge 0) \\ &= f(b\mathbf{1}_A) + f(a\mathbf{1}_A - a\mathbf{1}_\Omega) = f(b\mathbf{1}_A) + f(a\mathbf{1}_A) + f(-a\mathbf{1}_\Omega). \end{aligned}$$

Thus, $f(X_i + Y_i) = f(a\mathbf{1}_A + b\mathbf{1}_A) = f(a\mathbf{1}_A) + f(b\mathbf{1}_A) = f(X_i) + f(Y_i)$. □

Step A.8. $f(x+y) = f(x) + f(y)$ for all positive and comonotonic $x, y \in B(\Omega, \mathcal{A})$.

Proof. Consider $x, y \in B(\Omega, \mathcal{A})$ such that $x \geq 0$ and $y \geq 0$. Define for all $n \in \mathbb{N}$

$$x_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \mathbf{1}_{\left\{\frac{i}{2^n} < x \leq \frac{i+1}{2^n}\right\}} \quad \text{and} \quad y_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \mathbf{1}_{\left\{\frac{i}{2^n} < y \leq \frac{i+1}{2^n}\right\}}.$$

Since x and y are bounded above, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$x_n \leq x \leq x_n + \frac{1}{2^n}\mathbf{1}_\Omega \quad \text{and} \quad y_n \leq y \leq y_n + \frac{1}{2^n}\mathbf{1}_\Omega.$$

It is straightforward to check that x_n and y_n are comonotonic since x and y are comonotonic. Therefore, Step A.7 implies $f(x_n + y_n) = f(x_n) + f(y_n)$. Since f is continuous for the sup norm by Lemma A.4, passing to the limit when $n \to \infty$, one gets $f(x+y) = f(x) + f(y)$ □

Step A.9. For all $x, y \in B(\Omega, \mathcal{A})$ comonotonic, $f(x+y) = f(x) + f(y)$.

Proof. Let $x, y \in B(\Omega, \mathcal{A})$ be comonotonic. We can choose $k \geq 0$ such that $x' = x + k\mathbf{1}_\Omega \geq 0$ and $y' = y + k\mathbf{1}_\Omega \geq 0$. Since f satisfies TI, we have

$$f(x' + y') = f(x + y + 2k\mathbf{1}_\Omega) = f(x+y) + f(2k\mathbf{1}_\Omega) = f(x+y) + 2f(k\mathbf{1}_\Omega).$$

By Step A.8, noticing that x' and y' are comonotonic, and using again TI, we have

$$f(x' + y') = f(x') + f(y') = f(x) + f(k\mathbf{1}_\Omega) + f(y) + f(k\mathbf{1}_\Omega).$$

Hence $f(x + y) = f(x) + f(y)$. □

[$(ii) \Rightarrow (i)$] Since f is comonotonic additive, it satisfies Buy & Sell Additivity (A.3) and TI, that is,

$$f(x) = f(x^+) + f(-(-x)^+) \quad \text{for all } x \in B(\Omega, \mathcal{A}),$$

$$f(x + k\mathbf{1}_\Omega) = f(x) + f(k\mathbf{1}_\Omega) \ \ \text{for all } x \in B(\Omega, \mathcal{A}) \ \ \text{and all } k \in \mathbb{R}.$$

This follows from the facts that x^+ and $-(-x)^+$ are comonotonic and x and $k\mathbf{1}_\Omega$ are also comonotonic. Consequently f satisfies PCP by Lemma A.5. □

Proof of Theorem 3.2. From Lemma A.6, f satisfies PCP if and only if f is comonotonic additive. From Schmeidler (1986), the comonotonic additivity of f is equivalent to the fact that f is a Choquet pricing rule. □

A.5 | Proofs of Proposition A.10, Theorem 3.3, Theorem 4.1, and Theorem A.1

We prove first Theorem 4.1 and we use it to prove Proposition A.10 and Theorem 3.3. Then we prove Theorem A.1 (the statement is in Appendix A.1).

Proof of Theorem 4.1. [$(i) \Rightarrow (iii)$] Note that Choquet pricing rules are Buy and Sell Additive, that is, for all $x \in B(\Omega, \mathcal{A})$, $\int^C x\, dv = \int^C x^+\, dv + \int^C -x^-\, dv$; indeed, from Schmeidler (1986), every Choquet integral is comonotonic additive and, for all $x \in B(\Omega, \mathcal{A})$, x^+ and $-x^-$ are comonotonic. Then we have

$$\int^C x\, dv - \int^S x\, dv = \left[\int^C x^+\, dv + \int^C -x^-\, dv\right] - \left[\int^C x^+\, dv - \int^C x^-\, dv\right]$$

$$= \int^C x^-\, dv + \int^C -x^-\, dv = f(x^-) + f(-x^-) \geq 0 \qquad [\text{by } (i)]$$

$$\int^S x\, dv + \int^C -x\, dv = -\int^S -x\, dv + \int^C -x\, dv \geq 0 \qquad [\text{from above}]$$

[$(iii) \Rightarrow (i)$] Immediate.

[$(i) \Rightarrow (ii)$] Fix $A \in \mathcal{A}$ and consider $x := \mathbf{1}_A$. Then (ii) holds since

$$\begin{aligned} 0 \leq f(-x) + f(x) &= f(\mathbf{1}_{A^c} - \mathbf{1}_\Omega) + f(\mathbf{1}_A) && [\text{by } (i)] \\ &= f(\mathbf{1}_{A^c}) - f(\mathbf{1}_\Omega) + f(\mathbf{1}_A) && [\text{from TI}] \\ &= v(A^c) - v(\Omega) + v(A) && . \end{aligned}$$

[$(ii) \Rightarrow (i)$] Note that we are done as soon as we prove (i) for $x \geq 0$ since f is a Choquet integral and therefore it satisfies TI. But, from (ii), we have $\upsilon^* \leq \upsilon$, and for $x \geq 0$, we get from standard properties of the Choquet integral

$$-f(-x) = -\int^C (-x)\,d\upsilon = \int^C x\,d\upsilon^* \leq \int^C x\,d\upsilon = f(x).$$

□

Proposition A.10. *Let $f : B(\Omega, \mathcal{A}) \to \mathbb{R}$ be a Choquet pricing rule with respect to υ. Then the following are equivalent:*

(i) $f(x) + f(-x) = 0$ *for all* $x \in B(\Omega, \mathcal{A})$.
(ii) $\upsilon = \upsilon^*$, *that is,* $\upsilon(A) + \upsilon(A^c) = \upsilon(\Omega)$ *for all* $A \in \mathcal{A}$.
(iii) f *is a Choquet–Šipoš pricing rule.*

Proof. [$(i) \Leftrightarrow (ii)$] Define the pricing rule $f^* : B(\Omega, \mathcal{A}) \to \mathbb{R}$ by $f^*(x) := -f(-x)$. If f is a Choquet pricing rule w.r.t. υ, then f^* is a Choquet pricing rule w. r. t. υ^*. Then Assertion (i) holds if and only if $f(x) + f(-x) \geq 0$ and $f^*(x) + f^*(-x) \geq 0$ for all x, hence if and only if $\upsilon \geq \upsilon^*$ and $\upsilon = (\upsilon^*)^* \geq \upsilon^*$ by Theorem 4.1. That is, $\upsilon = \upsilon^*$.

[$(i) \Leftrightarrow (iii)$] If f is a Choquet–Šipoš pricing rule, then $-f(-x) = -\int^S -x\,d\upsilon = \int^S x\,d\upsilon = f(x)$ for all x. Hence, (i) is satisfied. Conversely, if f is a Choquet pricing rule such that, for all x, $f(x) + f(-x) = 0$, then in particular, $f(x) + f(-x) \geq 0$. By Theorem 4.1, $\int^S x\,d\upsilon \in [-f(-x), f(x)] = \{f(x)\}$ for all x. Thus f is a Šipoš pricing rule. □

Proof of Theorem 3.3. By Proposition 3.1, f satisfies CPP if and only if f satisfies PCP and $f(x) + f(-x) = 0$ for all x. By Theorem 3.2, f satisfies PCP, if and only if f is a Choquet pricing rule. By Proposition A.10, f is a Choquet pricing rule and satisfies $f(x) + f(-x) = 0$ for all x if and only if f is a Choquet–Šipoš pricing rule. Thus, (i) is equivalent to (ii). □

Notation: Let us denote $B_+(\Omega, \mathcal{A}) = \{x \in B(\Omega, \mathcal{A}) \mid x \geq 0\}$ and let TI_+ (respectively, DCP_+) denote TI (respectively, DCP_+) restricted to $B_+(\Omega, \mathcal{A})$.

Proof of Theorem A.1. [$(i) \Rightarrow (ii)$] By taking $k = 0$, DCP implies property A.3, that is, that f is Buy and Sell Additive. Moreover DCP implies TI_+. For all $x \in B_+(\Omega, \mathcal{A})$, $k \geq 0$, apply DCP to $x + k\mathbf{1}_\Omega$ and k to get

$$f(x + k\mathbf{1}_\Omega) = f((x + k\mathbf{1}_\Omega) \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega) + f((x + k\mathbf{1}_\Omega) \wedge k\mathbf{1}_\Omega) = f(x) + f(k\mathbf{1}_\Omega).$$

Doing the same proof as in Step A.7, Step A.8 of Lemma A.6, we can show that f satisfies Comonotonic Additivity on $B_+(\Omega, \mathcal{A})$. Then by Schmeidler (1986), f is a Choquet pricing rule on $B_+(\Omega, \mathcal{A})$. Taking $k = 0$, from DCP (first equality), and no bid–ask spread (third equality), we deduce that for all $x \in B(\Omega, \mathcal{A})$

$$f(x) = f(x \wedge 0) + f(x \vee 0) = f(x^+) + f(-x^-) = f(x^+) - f(x^-).$$

Since $x^+, x^- \in B_+(\Omega, \mathcal{A})$, and since f is a Choquet integral on $B_+(\Omega, \mathcal{A})$

$$f(x) = \int^C x^+ dv - \int^C x^- dv = \int^S x^- dv,$$

that is, f is a Šipoš pricing rule.

$[(ii) \Rightarrow (i)]$ By Theorem 5(ii), in Šipoš (1979), the Šipoš integral is monotone when v is a capacity. It is easy to see that a Šipoš pricing rule satisfies and no bid–ask spread. Only DCP is left to be shown. Using and no bid–ask spread, we get for all $x \in B(\Omega, \mathcal{A})$

$$f(x) = f(x^+) - f(x^-) = f(x^+) + f(-x^-) = f(x \vee 0) + f(x \wedge 0),$$

that is, f is Buy and Sell Additive.

We show that f satisfies DCP_+. Note that $(x - k\mathbf{1}_\Omega)^+$ and $x \wedge k\mathbf{1}_\Omega$ are comonotonic. Since f is a Šipoš integral, it is a Choquet integral on $B_+(\Omega, \mathcal{A})$ and therefore if satisfies comonotonic additivity on $B_+(\Omega, \mathcal{A})$. Using the fact that $x = (x - k\mathbf{1}_\Omega)^+ + x \wedge k\mathbf{1}_\Omega$, we get $f(x) = f((x - k\mathbf{1}_\Omega)^+) + f(x \wedge k\mathbf{1}_\Omega)$. Using Equation (A.2), we obtain

$$f(x \vee 0) = f((x \vee 0) \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega) + f((x \vee 0) \wedge k\mathbf{1}_\Omega) = f(x \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega) + f((x \wedge k\mathbf{1}_\Omega) \vee 0)$$

since $k \geq 0$ implies $(x \vee 0) \vee k\mathbf{1}_\Omega = x \vee k\mathbf{1}_\Omega$ and $(x \vee 0) \wedge k\mathbf{1}_\Omega(\omega) = 0 = (x \wedge k\mathbf{1}_\Omega) \vee 0(\omega)$ if $x(\omega) \leq 0$ and $(x \vee 0) \wedge k\mathbf{1}_\Omega(\omega) = x \wedge k\mathbf{1}_\Omega(\omega) = (x \wedge k\mathbf{1}_\Omega) \vee 0(\omega)$ if $x(\omega) \geq 0$. Also,

$$f(x \wedge 0) = f((x \wedge k\mathbf{1}_\Omega) \wedge 0).$$

Replacing $f(x \vee 0)$ and $f(x \wedge 0)$ in Equation (A.3) and applying Equation (A.3) once again, one obtains

$$f(x) = f(x \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega) + f((x \wedge k\mathbf{1}_\Omega) \vee 0) + f((x \wedge k\mathbf{1}_\Omega) \wedge 0) = f(x \vee k\mathbf{1}_\Omega - k\mathbf{1}_\Omega) + f(x \wedge k\mathbf{1}_\Omega),$$

that is, DCP holds. □

A.6 | Proofs of Theorem 4.3 and Corollary 4.5

Proof of Theorem 4.3. [Proof of the Choquet Part] [⇒] Assume that f satisfies AF and is a Choquet pricing rule (note that the proof of ⇒ works also if f is a Šipoš pricing and will be needed in the second part). Let v^* be the conjugate of v. We prove that (i) v^* is balanced and (ii) $AC(v) = \text{core}(v^*)$, hence $\text{core}(v^*) \neq \emptyset$ by Schmeidler (1968) that extends the result by Bondareva (1963) and Shapley (1967) to infinite spaces Ω.[4]

We first prove that v^* is balanced. Fix $n \in \mathbb{N}$ and let $a_1, \dots, a_n \geq 0$ and $A_1, \dots, A_n \in \mathcal{A}$ such that $\sum_{i=1}^n a_i \mathbf{1}_{A_i} - \mathbf{1}_\Omega = 0$. Then by AF, $\sum_{i=1}^n f(a_i \mathbf{1}_{A_i}) + f(-\mathbf{1}_\Omega) \geq 0$. Since f is a Choquet or a Šipoš pricing rule, $f(a_i \mathbf{1}_{A_i}) = a_i f(\mathbf{1}_{A_i}) = a_i v(A_i)$ and $f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega) = -v(\Omega)$. Therefore, $\sum_{i=1}^n a_i v(A_i) \geq v(\Omega)$ and clearly $\sum_{i=1}^n a_i v^*(A_i) \leq v^*(\Omega)$. Then by Schmeidler (1968), one gets

$$\text{core}(v^*) := \left\{ \mu \in \text{ba}(\mathcal{A}) : \mu(A) \geq v^*(A) \text{ for all } A \in \mathcal{A} \text{ and } \mu(\Omega) = v^*(\Omega) \right\} \neq \emptyset.$$

But v^* is a capacity since v is a capacity. Hence all $\mu \in \text{core}(v^*)$ are non-negative. But the set $\{\mu \in \text{ba}(\mathcal{A}) : \mu \geq 0, \text{ and } \mu(\Omega) = v(\Omega)\}$ is the set of positive, additive set functions on Ω such that $\mu(\Omega) = v(\Omega)$. Therefore,

$$AC(v) := \{\mu : \mathcal{A} \mapsto \mathbb{R} \ : \ \mu \text{ is positive, additive, } \mu \leq v, \text{ and } \mu(\Omega) = v(\Omega)\} = \text{core}(v^*) \neq \emptyset.$$

[$\Leftarrow$] Assume $AC(v) \neq \emptyset$ and let $\mu \in AC(v)$. Let $x_1, \dots, x_n \in B(\Omega, \mathcal{A})$ such that $x := \sum_{i=1}^n x_i \geq 0$. Then $\int^C x_i \, dv \geq \int x_i d\mu$ for all i (since $\mu \leq v$ and $\mu(\Omega) = v(\Omega)$). Since $x \geq 0$, one gets

$$\sum_{i=1}^n f(x_i) := \sum_{i=1}^n \int^C x_i \, dv \geq \sum_{i=1}^n \int x_i d\mu = \int x d\mu \geq 0.$$

This proves that f is arbitrage free. □

[Proof of the Šipoš Part] If a Šipoš pricing rule f is linear, then it is clearly AF.

We now prove the converse implication. Suppose that f is a Sipoš pricing rule that satisfies AF. Then, there exists a capacity v such that $f(x) = \int^S x \, dv$ for all $x \in B(\Omega, \mathcal{A})$. For f to be linear, it is sufficient to prove that v is additive since for v additive one has $f(x) = \int^C x^+ \, dv - \int^C x^- \, dv$ (since f is a Šipoš pricing rule) hence $f(x) = \int [x^+ - x^-] \, dv = \int x \, dv$ (since v is additive). We end the proof by showing that v is additive. First note that $\mathbf{1}_\Omega - \mathbf{1}_A - \mathbf{1}_{A^c} = 0 = \mathbf{1}_A + \mathbf{1}_{A^c} - \mathbf{1}_\Omega$. Since f is AF and $f(-x) = -f(x)$ for all $x \in B(\Omega, \mathcal{A})$ (as f is a Šipoš pricing rule), one has

$$v(\Omega) - v(A) - v(A^c) = f(\mathbf{1}_\Omega) + f(-\mathbf{1}_A) + f(-\mathbf{1}_{A^c}) \geq 0,$$

$$v(A) + v(A^c) - v(\Omega) = f(\mathbf{1}_A) + f(\mathbf{1}_{A^c}) + f(-\mathbf{1}_\Omega) \geq 0.$$

Therefore, $v(A) + v(A^c) = v(\Omega)$, that is, v is auto-conjugate.

Finally, from the first part of the theorem, $AC(v) \neq \emptyset$. Let $\mu \in AC(v)$, then $\mu(A) \leq v(A)$ and $\mu(A^c) \leq v(A^c)$ for all $A \in \mathcal{A}$. Since v is auto-conjugate, neither of the previous inequalities can be strict, otherwise, summing up, we would get $v(\Omega) = \mu(\Omega) = \mu(A) + \mu(A^c) < v(A) + v(A^c) = v(\Omega)$, a contradiction. Therefore, $\mu(A) = v(A)$ for all $A \in \mathcal{A}$. This proves that v is additive and the Sipoš pricing rule f is linear. □

Proof of Corollary 4.5. We only give the proof that if f is a Choquet pricing rule, then AF implies AF*. Since AF holds, by Theorem 4.3, $AC(v) \neq \emptyset$. Let $\mu \in AC(v)$ and consider $x_1, \dots, x_n \in B(\Omega, \mathcal{A})$ such that $x := \sum_{i=1}^n x_i >^v 0$. Then $\int^C x_i \, dv \geq \int x_i d\mu$ for all i (since $\mu \leq v$ and $\mu(\Omega) = v(\Omega)$). Since $x >^v 0$, $\exists \varepsilon > 0$ s.t. $v^*(\omega | x(\omega) \geq \varepsilon) > 0$. Let $A = \{\omega | x(\omega) \geq \varepsilon\}$, one gets

$$\sum_{i=1}^n f(x_i) := \sum_{i=1}^n \int^C x_i \, dv \geq \sum_{i=1}^n \int x_i d\mu = \int x d\mu \geq \int x dv^* \geq \int \varepsilon 1_A dv^* = \varepsilon v^*(A) > 0.$$

The second inequality is true because $\mu \in \mathit{core}(v^*)$. This proves that f satisfies AF*. □