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MODELS AND THEORIES OF PURE AND RESOURCE LAMBDA CALCULI

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Direttore della Scuola di dottorato	**Tutore del dottorando**
Prof. Paolo Ugo	**Prof. Antonino Salibra**
	Co-tutore del dottorando
	Prof. Antonio Bucciarelli

Author's web page: `http://www.dsi.unive.it/~acarraro/`

Author's e-mail: acarraro@dsi.unive.it

Author's addresses:

Dipartimento di Informatica
Università Ca' Foscari di Venezia
Stanza 13
Via Torino, 155
30172 Venezia Mestre – Italia

Laboratoire PPS
Université Paris Diderot (Paris 7)
Bureau 6A51
175 Rue du Chevaleret
75013 Paris – France

To Francesca.

Abstract

Part I: A longstanding open problem is whether there exists a model of the untyped lambda calculus in the category **Cpo** of complete partial orderings and Scott continuous functions, whose theory is exactly the least λ-theory $\lambda\beta$ or the least extensional λ-theory $\lambda\beta\eta$: it is Problem 22 in the TLCA list of open problems [56]. In this thesis we analyze the class of reflexive Scott domains, the models of lambda calculus living in the category of Scott domains (a full subcategory of **Cpo**). We isolate, among the reflexive Scott domains, a class of webbed models arising from Scott's information systems, that we call *i-models*. The class of i-models includes, for example, all preordered coherent models, all filter models living in **Cpo** and all extensional reflexive Scott domains. By performing a fine-grained study of an "effective" version of Scott's information systems and i-models we obtain the following main results: there is an important class of i-models which is not complete for the extensional calculus and whose members never have a recursively enumerable order theory.

A closed λ-term M is *easy* if, for any other closed term N, the λ-theory generated by the equation $M = N$ is consistent, while it is *simple easy* if, given an arbitrary intersection type τ, one can find a suitable pre-order on types which allows to derive τ for M. Simple easiness implies easiness. The question whether easiness implies simple easiness constitutes Problem 19 in the TLCA list of open problems [4]. As a byproduct of our work on i-models, we are in the position of solving this problem: we answer negatively, providing a nonempty set of easy, but non simple easy, λ-terms.

Part II: Given a semi-ring with unit which satisfies some conditions, we define an exponential functor on the category of sets and relations which allows to define a denotational model of Differential Linear Logic and of the lambda-calculus with resources. We show that, when the semi-ring has an element which is "infinitary", this model does not validate the Taylor formula and that it is possible to build, in the associated Kleisli cartesian closed category, a model of the pure lambda-calculus which is not sensible. This is a quantitative analogue of the Park's graph model construction in the category of Scott domains.

We initiate a purely algebraic study of Ehrhard and Regnier's resource λ-calculus, by introducing three algebraic varieties: *resource combinatory algebras*, *resource lambda-algebras* and *resource lambda-abstraction algebras*. We establish the relations between them, laying down foundations for a model theory of resource lambda

calculus. We also show that the ideal completion of a resource combinatory algebra (resp. lambda-algebra, lambda-abstraction algebra) induces a "classical" combinatory algebra (resp. lambda-algebra, lambda-abstraction algebra), and that any model of the pure lambda calculus raising from a resource lambda-algebra determines a λ-theory which equates all terms having the same Böhm tree.

Résumé

Partie I: Un problème ouvert depuis longtemps est de savoir s'il existe un modèle du lambda calcul non typé dans la catégorie **Cpo** des ordres partiels complétes et fonctions Scott continues, dont la théorie équationnelle soit exactement la plus petite λ-théorie $\lambda\beta$ ou la plus petite λ-théorie extensionnnelle $\lambda\beta\eta$: c'est le Problème 22 dans la liste de problèmes ouverts TLCA [56]. Dans cette thèse, nous analysons la classe des domaines de Scott réflexifs, les modèles du lambda calcul vivants dans la catégorie des domaines de Scott (une sous-catégorie pleine de **Cpo**). Nous isolons, parmi les domaines de Scott réflexifs, une classe de modèles à trame découlant des systèmes d'information de Scott, que nous appelons *i-modèles*. La classe des i-modèles comprend, par exemple, tous les modèles préordonnées cohérentes, tous les modèles de filtre vivants dans **Cpo** et tous les domaines de Scott réflexifs extensionnels. En réalisant une étude fine d'une version "effective" des systèmes d'information de Scott et des i-modèles, nous obtenons les résultats suivants: il y a une important classe de i-modèles qui n'est pas complète pour le lambda calcul extensionnel et tel que tous ces membres ne ont pas une théorie d'ordre récursivement énumérable.

Un λ-terme clos M est dit *facile* si, pour tout autre terme clos N, la λ-théorie engendrée par l'équation $M = N$ est cohérente, alors qu'il est *simple facile* si, étant donné un type intersection quelconque τ, on peut trouver un pré-ordre sur les types qui permet de dériver le type τ pour M. La facilité simple implique la facilité. La question de savoir si la facilité implique la facilité simple constitue le Problème 19 dans la liste des problèmes ouverts TLCA [4]. Comme sous-produit de notre travail sur les i-modèles, nous sommes en position de résoudre ce problème: nous répondons négativement, en fournissant un ensemble non vide de λ-termes faciles mais non simple faciles.

Partie II: Etant donné un semi-anneau avec unité qui satisfait certaines conditions, nous définissons un foncteur exponentiel sur la catégorie des ensembles et des relations qui permet de définir un modèle dénotationnel de la Logique Linéaire Différentielle et du lambda-calcul avec ressources. Nous montrons que, lorsque le semi-anneau contient un élément qui est "infinitaire", ce modèle ne satisfait pas la formule de Taylor et qu'il est possible de construire, dans la catégorie Cartésienne fermée de Kleisli associée, un modèle du lambda calcul pur qui n'est pas sensible. Il s'agit d'un analogue quantitative de la construction du graphe modèle de Park dans la catégorie des domaines de Scott.

Nous commençons une étude purement algébrique du λ-calcul avec ressources de Ehrhard et Regnier, en introduisant trois variétés algébriques: les *algèbres combinatoires avec ressources*, les *lambda-algèbres avec ressources* et les *algèbres de lambda-abstraction avec ressources*. Nous établissons les relations entre elles, et jetons les bases d'une théorie des modèles du λ-calcul avec ressources. Nous montrons également que la complétion par ideaux d'une algèbre combinatoire (resp. lambda-algèbre, algèbre de lambda-abstraction) avec ressources induit une algèbre combinatoire (resp. lambda-algèbre, algèbre de lambda-abstraction) "classique", et que tout modèle du lambda calcul classique provenant d'une lambda-algèbre avec ressources détermine une λ-théorie qui égalise tous les termes ayant le même arbre de Böhm.

Sommario

Parte I: Una questione aperta da lungo tempo è se esista un modello del lambda calcolo non tipato nella categoria **Cpo** degli ordinamenti parziali completi e funzioni Scott-continue, la cui teoria equazionale sia esattamente la minima λ-teoria $\lambda\beta$ o la minima λ-teoria estensionale $\lambda\beta\eta$: è il Problema 22 nella lista dei problemi aperti TLCA [56]. In questa tesi analizziamo la classe dei dominii di Scott riflessivi, i modelli del lambda calcolo che vivono nella categoria dei dominii di Scott (una sottocategoria piena di **Cpo**). Noi isoliamo, tra i domini riflessivi di Scott, una classe di modelli a trama derivanti dai sistemi informativi di Scott, che noi chiamiamo *i-modelli*. La classe degli i-modelli comprende, per esempio, tutti i modelli preordinati coerenti, tutti i modelli a filtro che vivono in **Cpo** e tutti i dominii di Scott estensionali riflessivi. Effettuando uno studio dettagliato di una versione "effettiva" dei sistemi informativi di Scott e degli i-modelli si ottengono i seguenti risultati principali: c'è un'importante classe di i-modelli tale nessun suo membro ha una teoria dell'ordine ricorsivamente enumerabile e come conseguenza essa non è completa per il lambda calcolo estensionale (ciò implica che $\lambda\beta\eta$ non è la teoria di un modello in tale classe).

Un λ-termine chiuso M è *facile* se, per qualsiasi altro termine chiuso N, la λ-teoria generata dall'equazione $M = N$ è consistente, mentre è *facile semplice* se, dato un tipo intersezione τ arbitrario, si pu trovare un pre-ordine adeguato sui tipi che permette di derivare τ per M. La facilità semplice implica la facilità. La questione se la facilità implichi o meno la facilità semplice costituisce il Problema 19 nella lista dei problemi aperti TLCA [4]. Come sottoprodotto del nostro lavoro sugli i-modelli, siamo in grado di risolvere questo problema: diamo una risposta negativa, fornendo un insieme non vuoto di λ-termini facili ma non facili semplici.

Parte II: Dato un semi-anello con unità che soddisfa alcune condizioni, si definisce un funtore esponenziale della categoria degli insiemi e relazioni che consente di definire un modello denotazionale della Logica Lineare Differenziale e del lambda calcolo con risorse . Si dimostra che, quando il semi-anello possiede un elemento "infinitario", è possibile costruire, nella categoria Cartesiana chiusa di Kleisli associata, un modello del lambda calcolo puro che non valida la formula di Taylor e che non è sensibile. Si tratta di un analogo quantitativo della costruzione del grafo modello di Park nella categoria dei dominii di Scott, che per lungo tempo nessuno era riuscito a replicare nella semantica relazionale.

Abbiamo avviato uno studio puramente algebrico del lambda calcolo con risorse di Ehrhard e Regnier, introducendo tre varietà algebriche: *algebre combinatorie con risorse*, *lambda-algebre con risorse* e *algebre di lambda-astrazione con risorse* . Stabiliamo le relazioni tra di esse, e gettiamo le basi per una teoria dei modelli del lambda calcolo con risorse. Mostriamo anche che il completamento ideale di un'algebra combinatoria (resp. lambda-algebra, algebra di lambda-astrazione) con risorse induce un'algebra combinatoria (resp. lambda-algebra, algebra di lambda-astrazione) "classica", e che qualsiasi modello del lambda calcolo classico proveniente da una lambda-algebra con risorse determina una λ-teoria che identifica tutti i termini che hanno lo stesso albero di Böhm.

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Alberto Carraro
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Preface

The work presented in this thesis is based on some previously published papers, thus much credit for the technical contents and the underlying ideas goes to my co-authors.

In more detail, the results of the first part are fruits of joint research with my supervisor Antonino Salibra. The incompleteness of the class of extensional reflexive Scott domains appeared in the Proceedings of the 24th IEEE Annual Symposium on Logic in Computer Science (LICS'09) [34]. The solution of Problem 19 appeared in the proceedings of the 12th Italian Conference on Theoretical Computer Science (ICTCS'10) [35]. Chapters 2 to 5 contain a completely re-elaborated exposition of these papers, enriched and improved in details. New are the introduction of (effective) Scottian λ-models, in order to give a clearer statement and proof of the main theorem.

The contents of Chapter 8 were obtained as the result of joint research with Antonino Salibra and Thomas Ehrhard. A preliminary version of this work appeared in the proceedings of the 35th International Symposiums on Mathematical Foundations of Computer Science (MFCS 2010) [32].

Also the contents of Chapter 9 were obtained as the result of joint research with Antonino Salibra and Thomas Ehrhard. A preliminary version of this work appeared in the proceedings of the 19th EACSL Annual Conferences on Computer Science Logic (CSL 2010) [33].

Part I and Part II are independent and each one has its own introduction. The first is mainly concerned with the pure lambda calculus whilst the second with the relationship between the pure lambda calculus and the resource lambda calculus. Anyway we claim that the tools developed in the first part can be fruitfully adapted and applied to the framework developed in the second one.

Contents

I

Part one

1
introduction

Lambda theories are congruences on the set of λ-terms, which contain β-conversion; extensional λ-theories are those which contain $\beta\eta$-conversion. Lambda theories arise by syntactical or by semantic considerations. Indeed, a λ-theory may correspond to a possible operational (observational) semantics of λ-calculus, as well as it may be induced by a model of λ-calculus through the kernel congruence relation of the interpretation function.

Although researchers have, till recently, mainly focused their interest on a limited number of them, the lattice of λ-theories ordered by inclusion constitutes a very rich, interesting and complex mathematical structure of cardinality $2^{\aleph_0}$ (see [10, 15, 16]).

After the first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, a large number of mathematical models for λ-calculus, arising from syntax-free constructions, have been introduced in various Cartesian closed categories (ccc, for short) of domains and were classified into semantics according to the nature of their representable functions, see e.g. [10, 15, 95]. Scott continuous semantics [104] is the class of reflexive cpo-models, that are reflexive objects in the category **Cpo** whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [20]) and the strongly stable semantics (Bucciarelli-Ehrhard [26]) are refinements of the continuous semantics, introduced to approximate the notion of "sequential" Scott continuous function; finally "weakly continuous" semantics have been introduced, either for modeling non determinism, or for foundational purposes [15, 41]. In each of these semantics all models come equipped with a partial order, and some of them, called webbed models, are built from lower level structures called "webs". The simplest class of webbed models is the class of graph models, which was isolated in the seventies by Plotkin, Scott and Engeler [48, 95, 107] within the continuous semantics. The class of graph models contains the simplest models of λ-calculus, is itself the easiest describable class, and represents nevertheless $2^{\aleph_0}$ (non-extensional) λ-theories. Another example of a class of webbed models, and the most established one, is the class of filter models, many of which live in Scott continuous semantics. It was isolated at the beginning of the eighties by Barendregt, Dezani and Coppo [11], after the introduction of the intersection type discipline by Coppo and Dezani [37].

Scott continuous semantics and the other mentioned semantics are structurally and equationally rich. Ten years ago, Kerth [71, 72] has proved that in each of the above semantics it is possible to build up $2^{\aleph_0}$ models inducing pairwise distinct λ-theories. Nevertheless, the above denotational semantics do not match all possible operational semantics of λ-calculus. Honsell and Ronchi della Rocca [57] have shown that there exist theories which do not have models in the category **Cpo**. A similar result was obtained by Bastonero and Gouy [12] for the stable semantics. More recently, it has been proved in an uniform way that there are $2^{\aleph_0}$ theories which are omitted by all ordered models of λ-calculus with a bottom element[100] among which $\aleph_0$ are finitely axiomatizable over the least λ-theory $\lambda\beta$.

The question of the existence of a non-syntactical model of $\lambda\beta$ (or $\lambda\beta\eta$, the least extensional λ-theory) has been circulating since at least the beginning of the eighties, but it was only first raised in print in [57]. This problem is still open and constitutes Problem 22 of the TLCA list of open problems [56], which is a list of twenty-two problems that aims at collecting unresolved questions in the subject areas of the TLCA (Typed Lambda Calculi and Applications) series of conferences. Problem 1 and Problem 20 are the only ones that have been solved to date and solutions to Problem 2 and Problem 3 have been announced. A wealth of interesting research and results (partially surveyed in [15] and [16]), have been motivated and inspired by these kind of questions.

In 1995 Di Gianantonio, Honsell and Plotkin succeeded to build an extensional model having theory $\lambda\beta\eta$, living in some weakly continuous semantics [41]. However, the construction of this model as an inverse limit starts from the term model of $\lambda\beta\eta$, and hence involves the syntax of λ-calculus. Furthermore the existence of a model living in Scott semantics itself, or in one of its two refinements, remains completely open. Nevertheless, the authors also proved in [41] that the set of extensional theories representable by models living in Scott continuous semantics has a least element.

In view of the second result of [41], it becomes natural to ask whether, given a (uniformly presented) class of models of λ-calculus, there is a minimum λ-theory represented in it; a question which was raised in [15]. In [30] Bucciarelli and Salibra showed that the answer is also positive for the class of graph models, and that the least λ-theory in this class is different from $\lambda\beta$ and of course $\lambda\beta\eta$.

We notice also that there are only very few theories of non syntactical models which are known to admit an alternative (i.e., non-model-thoretic) description (e.g. via syntactical considerations), and that all happen to coincide either with the theory of Böhm trees [10] or some variations of it, and hence are not recursively enumerable (r.e., for short). This led Berline, Manzonetto and Salibra [17] to raise the following problem, which is a natural extension of the initial problem: can a model living in Scott continuous semantics or in one of its refinements have an r.e. equational theory? This problem was first raised in [16], where it is conjectured that no graph model can have an r.e. theory: in [17] this conjecture is extended to all models living in the continuous semantics, or in its refinements (but of course not in its

weakenings, because of [41]). Based on a notion of an effective model of λ-calculus, in [17] it was shown that the order theory of an effective model cannot be r.e. and its equational theory is different from $\lambda\beta$ ($\lambda\beta\eta$). Effective models are omni-present: in particular, all the models which have been introduced individually in the literature can easily be proved effective. Concerning the above mentioned semantics, it was also proved that no effective model living in the stable or strongly stable semantics has an r.e. equational theory and that no order theory of a graph model can be r.e.

The category of algebraic cpos (a full subcategory of **Cpo**) has many nice properties but unfortunately lacks the essential characteristic of being a ccc: the function space of two algebraic cpos need not be algebraic. This is a serious drawback, as the function-space construction is often used in denotational semantics and the models of λ-calculus are reflexive objects in a ccc. Fortunately there are Cartesian closed (full) subcategories of algebraic cpos. The most important, introduced by Scott [106], is the ccc **Sd** of Scott domains (i.e., bounded complete algebraic cpos). Most of the reflexive cpo-models of λ-calculus introduced in the literature in the last forty years are *reflexive Scott domains*, i.e., reflexive objects in **Sd**.

In this thesis we work with a category equivalent to **Sd** but more "concrete" and easier to work with, namely the ccc **Inf** of information systems introduced by Scott in early eighties [106]. We use **Inf** to isolate a class of structures, that we call *i-webs*, that we use to generate models, then called *i-models* (exactly as graph models arise from total pairs [15]). The class of i-models includes all extensional reflexive Scott domains, all preordered coherent models and all filter models living in **Cpo**. Based on a fine-grained study of an "effective" version of Scott's information systems, in the key technical theorem of this first part we prove that there exists a model of λ-calculus, not living in **Cpo**, whose order theory is contained within the order theory of every i-model and inspired by the work of Berline, Manzonetto and Salibra [17], we show that there is an important class of i-models whose order theories are never r.e., and that there are equations not in $\lambda\beta$ (resp. $\lambda\beta\eta$) which hold in all such i-models.

We work at various levels, that we may roughly divide into the following three.

- At the "lower level" we manipulate i-webs. We consider partial i-webs, extensions and completions. We also consider the effective content of these structures and constructions.

- At the "intermediate level" there are i-models, which include all extensional reflexive Scott domains.

- At the "higher level" we isolate and manipulate a class partially ordered λ-models, that we call *Scottian λ-models*. They include all i-models but have the advantage to be closed under direct product.

For each level we also consider the effective version/content of the structures and constructions.

According to Jacopini [60] a closed λ-term M is *easy* if, for any other closed term N, the λ-theory generated by the equality $M = N$ is consistent. Easy terms can be considered computational processes of a completely non-informative kind. Thus they are suitable candidates for representing inside λ-calculus the undefined value of a partial recursive function. The paradigmatic unsolvable term $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ was shown easy by Jacopini [60] (cf. [10, p. 402]) with a syntactic proof. Other syntactical proofs that certain terms are easy may be found in the literature, e.g., (Jacopini & Venturini Zilli [61, 62]; Intrigila [58]; Berarducci & Intrigila [14]; Kuper [76]; Wang & Zhao [116]).

Baeten and Boerboom gave in [9] the first semantical proof of the easiness of Ω by showing that, for all closed terms M one can build a graph model satisfying the equation $\Omega = M$. Baeten and Boerboom build their graph model by a method of "forcing", which, although much simpler than the forcing techniques used in set theory, is somewhat in the same spirit. Forcing considerations have been extended by Zylberajch [117] to prove the simultaneous easiness of the members of some infinite family of easy terms (see also Berline-Salibra [19] and Berarducci [13]).

However, the semantical methods via graph models have concrete limitations. For example, no semantical proof of the easiness of $\omega_3\omega_3\mathbf{I}$ (where $\omega_3 \equiv \lambda x.xxx$ and $\mathbf{i} \equiv \lambda x.x$) via graph models can exist, in contrast to the case Ω, since Kerth [70] has shown that no graph model satisfies the identity $\omega_3\omega_3\mathbf{I} = \mathbf{I}$. The easiness of the term $\omega_3\omega_3\mathbf{I}$ was proved syntactically in (Jacopini & Venturini Zilli [62]), but was only given a semantic proof in (Alessi et al. [5]), where the authors build, for each closed term M, a filter model of $\omega_3\omega_3\mathbf{I} = M$.

Alessi & Lusin [7] introduced a general technique to prove the easiness of λ-terms through the notion of *simple easiness*. This notion implies easiness and can be handled in a natural way by semantic tools. It allows to prove consistency results via construction of suitable filter models of λ-calculus living in the category **Cpo**: given a simple easy term M and an arbitrary closed term N, it is possible to build (in a canonical way) a non-trivial filter model which equates the interpretation of M and N. In [6] Alessi, Dezani and Lusin prove in such a way the easiness of several terms, like Ω and $\omega_3\omega_3\mathbf{I}$. Besides, simple easiness is interesting in itself, since it has to do with minimal sets of axioms which are needed in order to assign certain types to easy terms.

Problem 19 of the TLCA list was posed by Fabio Alessi and Mariangiola Dezani-Ciancaglini in 2002 (see [4]) and asks whether easiness implies simple easiness.

As a byproduct of our work on i-models, which include all filter models living in **Cpo**, we are in the position of solving this problem: we answer negatively, providing a nonempty set of easy, but non simple easy, λ-terms.

2

Preliminaries

We want to keep this thesis as much self-contained as possible. The purpose of this chapter is to recall some basic notions, terminologies and notations that underly the whole thesis, especially the first part. We generally take Barendregt's classical work [10] for lambda calculus and combinatory logic, and that of Burris and Sankappanavar [31] for universal algebra. Our main references for recursion theory and domain theory are [8, 89, 111, 55].

This chapter is organized as follows: in Section 2.1 and Section 2.2 we review the terminology and notations for the lambda calculus and combinatory logic, respectively. In Section 2.3 we review the definitions of the algebraic models of lambda calculus (combinatory algebras, etc.) and in Section 2.4 we recall how these models arise from reflexive objects in cartesian closed categories. We conclude by surveying in Section 2.5 and Section 2.6 order-theoretic notions, especially regarding models of lambda calculus and representation of Scott domains.

2.1 Lambda-calculus

The *lambda-calculus* is a formalism composed by a set of words, called λ-*terms* (or just *terms*, when no confusion is likely), over an alphabet and by a system of rules specifying how some λ-terms can be *rewritten* or *equated* to others. Each term may be thought of as the definition of a function: the primitive notions of term formation are *application*, the operation of applying a function to an argument, and *lambda abstraction*, the process of forming a function from its "defining expression". The set Λ of λ-terms over a countable set Var of variables is inductively constructed as follows: every variable is a λ-term; if M and N are λ-terms, then so are (MN) and $(\lambda x.M)$ for each variable x.

An occurrence of a variable x in a term is *bound* if it lies in the scope of a lambda abstraction λx, otherwise it is called *free*. The set of free variables of M is denoted by $FV(M)$. A term without free variables is said to be *closed*. The set of closed terms will be denoted by Λ^o.

Letters $M, N, L, \cdots$ usually range over Λ and $x, y, z, \ldots$ range over Var. With regard to the lambda-calculus we follow the notation and terminology of Barendregt (see [10, Ch. 2]). In particular: "$\equiv$" denotes syntactical equality and we adopt his

variable convention. Then $M\{N/x\}$ denotes the result of substituting the term N for all free occurrences of x in M without actually worrying about capture of free the variables of N. In this setting the original Church's axiom scheme

(α) $\lambda x.M = \lambda y.M\{y/x\}$

is built-in the notion of syntactical equivalence and consequently two terms that would only differ in the name of their bound variables, i.e., α-*equivalent*, are considered equal.

There are some distinguished λ-terms that we list here

$$\mathbf{K} \equiv \lambda xy.x \quad \mathbf{S} \equiv \lambda xyz.xz(yz) \quad \mathbf{I} \equiv \lambda x.x \quad \mathbf{1} \equiv \lambda xy.xy \quad \Omega \equiv (\lambda x.xx)(\lambda x.xx)$$

The *equational rules* that determine the lambda-calculus are the followings:

(β) $(\lambda x.M)N = M\{N/x\}$

(app) if $M = M'$ and $N = N'$, then $MN = M'N'$

(ξ) if $M = N$, then $\lambda x.M = \lambda x.N$

(ref) $M = M$

(sym) if $M = N$, then $N = M$

(tran) if $M = N$ and $N = Z$, then $M = Z$

The *extensional* lambda-calculus adds another axiom, which equates all the terms having the same "extensional behavior":

(η) $\lambda x.Mx = M$ if $y \notin \mathrm{FV}(M)$

Two terms which are provably equal using all the rules except (η) are called β-*convertible*, or β-*equivalent*; two λ-terms provably equal using all the rules including (η) are called $\beta\eta$-*convertible*, or $\beta\eta$-*equivalent*.

Definition 2.1.1. *A λ-theory is a set of equations between λ-terms which is closed under the rules (β),(ref),(sym), and (tran). A λ-theory is* extensional *if it is additionally closed under the rule rule (η).*

A λ-theory is equivalently viewed as a set of pairs: it is *consistent* if strictly contained in $\Lambda \times \Lambda$. A a matter of notation for a λ-theory T both $T \vdash M = N$ and $(M, N) \in T$ express the fact that the equation $M = N$ is provable in T. The λ-theory *generated*, or *axiomatized*, by a set of equations is the least λ-theory containing it. The smallest (extensional) λ-theory, containing exactly the pairs of β-($\beta\eta$-)convertible terms, is called $\lambda\beta$ ($\lambda\beta\eta$). Every λ-theory T is uniquely determined by its equations between closed terms, i.e., by the set $T \cap (\Lambda^o \times \Lambda^o)$. If $T' \subseteq \Lambda^o \times \Lambda^o$

we say that T' is a *closed λ-theory* if there exists a λ-theory T such that $T' = T \cap (\Lambda^o \times \Lambda^o)$.

Contexts are terms with some occurrences of algebraic variables (also called "holes"), denoted by ζ_i. A context is inductively defined as follows: ζ_i is a context, x is a context for every variable x, if C_1 and C_2 are contexts then so are C_1C_2 and $\lambda x.C_1$ for each variable x. If $M_1, \ldots, M_k$ are λ-terms we will write $C(\!|M_1, \ldots, M_k|\!)$ for the context $C(\!|\zeta_1, \ldots, \zeta_k|\!)$ where all the occurrences of ζ_i have been simultaneously replaced by M_i, without any renaming of variables. We simply write $C(\!| \ |\!)$ if C occurs has at most one occurrence of a single algebraic variable.

Note that β- and $\beta\eta$-conversion are closed under context formation, i.e., $M = N$ implies $C(\!|M|\!) = C(\!|N|\!)$.

By applying the rules (β) and (η) only from left to right we obtain, respectively, the β- and the η-reduction. In general, given an R-reduction rule, we write $\rightarrow_R$ (resp. $\twoheadrightarrow_R$) to indicate the binary relation over Λ determined by the contextual (resp. contextual, reflexive and transitive) closure of R-reduction.

A λ-term is in *head normal form* (*hnf*, for short) if it is of the form $\lambda x_1 \ldots x_n.yM_1 \cdots M_k$, for some $n, k \geq 0$. A λ-term M is in *normal form* (*nf*, for short) if each of its subterms (including M itself) is in hnf.

A term M is called *solvable* if there exists a context $C(\!|\cdot|\!)$ such that $C(\!|M|\!)$ is β-convertible to $\mathbf{I}$, otherwise M is called *unsolvable*.

The λ-theory $\mathcal{H}$, generated by equating all the unsolvable terms, is consistent by [10, Thm. 16.1.3]. The theory $\mathcal{H}$ admits a unique maximal consistent extension [10, Thm. 16.2.6] $\mathcal{H}^*$, which is an extensional λ-theory, characterized by the following property: $\mathcal{H}^* \vdash M = N$ if and only if $C(\!|M|\!)$ is solvable $\Leftrightarrow C(\!|N|\!)$ is solvable, for all contexts $C(\!| \ |\!)$ (see [10, Thm. 16.2.7]).

A λ-theory T is *semi-sensible* if, and only if, it never equates a solvable and an unsolvable term, while T is *sensible* if, and only if, $\mathcal{H} \subseteq T$ (see [10, Sec. 10.2-16.2]). It turns out that every consistent sensible theory is semi-sensible and that $\mathcal{H}^*$ is the largest semi-sensible λ-theory.

The Böhm tree $BT(M)$ of a λ-term M is a labelled tree. If M is unsolvable, then $BT(M) = \bot$, that is, $BT(M)$ is a tree with a unique node labelled by $\bot$. If M is solvable and $\lambda\bar{x}.yM_1 \cdots M_k$ is its principal hnf, then:

$$BT(M) = \quad \lambda\bar{x}.y$$
$$BT(M_1) \quad \cdots \quad BT(M_k)$$

Following [10], the Böhm tree $BT(M)$ of a λ-term M may also be identified with an ideal (downwards closed and directed subset) of a suitable set of λ-terms. Let $\Lambda^\bot$ be the set normal terms in the ordinary λ-calculus extended with a constant $\bot$ quotiented by the equations $\bot N = \bot$ and $\lambda x.\bot = \bot$ and endow this set with a partial order as follows:

- $\perp \leq M$, for all $M \in \Lambda^\perp$;

- $\lambda x_1 \ldots x_n.yM_1 \cdots M_k \leq N$ iff $N \equiv \lambda x_1 \ldots x_n.yN_1 \cdots N_k$, with $M_j \leq N_j$, for $j = 1, \ldots, k$.

Now define a sequence of functions $BT_n : \Lambda \to \Lambda^\perp$, $n \geq 0$, by induction on n as follows:

- $BT_0(M) = \perp$;

- $BT_{n+1}(\lambda x_1 \ldots x_n.yM_1 \cdots M_k) = \lambda x_1 \ldots x_n.yBT_n(M_1) \cdots BT_n(M_k)$;

- $BT_{n+1}(\lambda x_1 \ldots x_n.(\lambda y.P)QM_1 \cdots M_k) = BT_n(\lambda x_1 \ldots x_n.P\{x := Q\}M_1 \cdots M_k)$.

Now it is clear that $(BT_n(M))_{n \geq 0}$ is a non decreasing sequence of elements of $\Lambda^\perp$. Then finally the Böhm tree of M is the downwards closure of the set $\{BT_n(M) : n \in \mathbb{N}\}$, which is an ideal of $\Lambda^\perp$.

For the definition of (finite and infinite) η-expansion of a Böhm tree see [10, Ch. 10]. We let $\mathcal{BT}$ be the λ-theory which equates exactly the pairs of λ-terms having the same Böhm tree. It turns out that $\mathcal{H} \subseteq \mathcal{BT} \subseteq \mathcal{H}^*$ and that $\mathcal{H}^*$ equates exactly the terms having the same Böhm tree up to possibly infinite η-expansion.

We now recall a fundamental theorem in lambda calculus, that we will use for proving subsequent results.

Theorem 2.1.1 (Böhm's theorem)**.** *If M, N are two closed distinct $\beta\eta$-normal forms, then for all λ-terms P, Q there exists a sequence $\vec{L}$ of λ-terms such that $\lambda\beta \vdash M\vec{L} = P$ and $\lambda\beta \vdash N\vec{L} = Q$.*

As a consequence of Theorem 2.1.1 the $\beta\eta$-normal forms cannot be consistently equated in the lambda calculus. Nonetheless there are some special terms that have a somewhat opposite property. These terms, introduced by Jacopini [60], have been studied by many authors.

Definition 2.1.2 ([60])**.** *A closed λ-term M is* easy *if for every closed term N the λ-theory generated by the equation $M = N$ is consistent.*

The paradigmatic unsolvable term $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ was shown easy by Jacopini [60] (cf. [10, p. 402]). Other proofs of easiness for Ω and other terms may be found in the literature, e.g., (Jacopini & Venturini Zilli [61, 62]; Intrigila [58]; Berarducci & Intrigila [14]; Kuper [76]; Wang & Zhao [116]).

2.2 Combinatory logic

Combinatory logic, as lambda-calculus, is a formalism for writing expressions which define functions, endowed with an equational calculus over them. The terms of combinatory logic, namely *combinatory terms*, are defined by induction as follows:

every variable x is a combinatory term; the constants K and S are combinatory terms; if s, t are combinatory terms, then also (st) is a combinatory term. As for λ-terms, outer parentheses are omitted and $(st)u$ is simply written stu; we list here some special combinatory terms, that will be used in the sequel:

$$K \qquad S \qquad I \equiv SKK \qquad 1 \equiv S(KI)$$

We will denote by $\mathscr{C}$ the set of all combinatory terms and we will let $s, t, u, \ldots$ range over $\mathscr{C}$.

The axioms of combinatory logic are the following:

(C1) $Kxy = x$

(C2) $Sxyz = xz(yz)$

and its rules are those of the traditional Equational Calculus. The equational calculus of combinatory logic is denoted by CL and for $t, u \in \mathscr{C}$, we write $CL \vdash t = u$ if t and s are provably equal in CL.

The fact that combinatory terms can be regarded as function definitions is not evident in combinatory logic, since abstraction and bound variables, unlike in lambda-calculus, are not available. However Schönfinkel [103] and Curry [39] discovered that these two characterizing features of lambda-calculus, can be *simulated* in combinatory logic, thanks to the equational calculus that governs it.

For each variable x one can define inductively a combinatory term $\lambda^* x.t$, where the variable x does not occur, as follows:

- $\lambda^* x.x = I$,

- $\lambda^* x.s = Ks$, if $x \notin \mathrm{FV}(t)$,

- $\lambda^* x.pq = S(\lambda^* x.p)(\lambda^* x.q)$.

Now it is possible to translate every λ-term in Λ into a combinatory term in $\mathscr{C}$ and viceversa via two maps $(\_)_\lambda : \Lambda \to \mathscr{C}$ and $(\_)_{cl} : \mathscr{C} \to \Lambda$ as follows:

- $x_\lambda = x$, for all $x \in \mathrm{Var}$ $\qquad (ts)_\lambda = t_\lambda s_\lambda$

- $x_{cl} = x$, for all $x \in \mathrm{Var}$ $\qquad (MN)_{cl} = M_{cl}N_{cl}$ $\qquad (\lambda x.M)_{cl} = \lambda^* x.M_{cl}$

We remark that $M_{cl,\lambda} \twoheadrightarrow_\beta M$, so that $\lambda\beta \vdash M_{cl,\lambda} = M$ but in general $CL \nvdash t_{\lambda,cl} = t$.

2.3   Algebraic models of lambda-calculus

The very meaning of the expression "algebraic model of lambda-calculus" is not immediate to clarify, since the lambda-calculus, in its commonly accepted formalization, is an higher-order calculus and algebras are naturally associated to languages without variable binding. This apparent mismatch has been for many years subject of interest of many distinguished researchers, which axiomatized a number of mathematical structures, each time giving a way of interpreting λ-terms in order to induce a λ-theory. We highlight those proposals that are characterized by axioms in the language of first-order logic:

- *lambda abstraction algebras* (Pigozzi-Salibra [93]),

- *λ-algebras* (Curry [10, Ch. 5]),

- *λ-models* (Meyer [85] and Scott [10, Ch. 5]),

- *combinatory models* (Meyer [85]).

Lambda abstraction algebras and λ-algebras are even axiomatized by pure equations, thus yielding *varieties*. On the other hand, only lambda abstraction algebras and λ-models allow the use of the standard notion of interpretation, via *environments*, used in universal algebra.

Other proposals are *syntactical λ-algebras* (Hindley-Longo [10, Ch. 5]), *syntactical λ-models* (Hindley-Longo [10, Ch. 5]) and *environment models* (Meyer [85]): their formulation is obtained essentially stipulating the existence of an interpretation function for λ-terms. Luckily the research on this topic produced also many equivalence results and in fact whenever we are given one of the above-mentioned structures, say $\mathbf{A}$, inducing a λ-theory T it is possible to perform a construction yielding a structure $\mathbf{B}$ of another type whose induced λ-theory is again T.

We now recall some basic notions of universal algebra that will be used in the sequel of this thesis.

Let $\mathbf{A}$ be an algebra in a given similarity type and let $\mathscr{T}(A)$ be the set of all *polynomials* over a countable set Var of variables and constants for denoting elements of the universe A.

The terms can be naturally interpreted in $\mathbf{A}$ with the help of "environments", which are particular functions assigning values to variables. More precisely, given a set A, an *A-environment*, is a total function $\rho : \mathrm{Var} \to A$. We let Env_A be the set of all *A*-environments. Given a valuation $\rho \in \mathrm{Env}_A$, the *interpretation* $t_\rho^{\mathbf{A}}$ of a term $t \in \mathscr{T}(A)$ *in* $\mathbf{A}$ *under* ρ is inductively defined as follows:

- $x_\rho^{\mathbf{A}} = \rho(x)$, for all $x \in \mathrm{Var}$,

- $a_\rho^{\mathbf{A}} = a$, for all $a \in A$,

- $(f(t_1, \ldots, t_n))_\rho^{\mathbf{A}} = f^{\mathbf{A}}((t_1)_\rho^{\mathbf{A}}, \ldots, (t_n)_\rho^{\mathbf{A}})$, for any n-ary function symbol f in the type $\mathscr{T}$.

An equation $t = u$ in the type of $\mathbf{A}$ is

(1) *satisfied in* $\mathbf{A}$ *under* ρ, notation $\mathbf{A}, \rho \vDash t = u$, if $t_\rho^{\mathbf{A}} = u_\rho^{\mathbf{A}}$;

(2) *satisfied in* $\mathbf{A}$, notation $\mathbf{A} \vDash t = u$, if $\mathbf{A}, \rho \vDash t = u$ for all $\rho \in \mathrm{Env}_A$.

Moreover if $\mathcal{C}$ is a class of algebras of the same similarity type, then an equation $t = u$ in the type of $\mathcal{C}$ is *satisfied in* $\mathcal{C}$, notation $\mathcal{C} \vDash t = u$, if $\mathbf{A} \vDash t = u$ for all $\mathbf{A} \in \mathcal{C}$.

2.3.1 Combinatory algebras

The first algebraic approach to lambda-calculus was an indirect one, namely via combinatory logic which is a first-order formalism (i.e. without variable binding) and finds a natural algebraic semantics. This way the λ-models the result as a strengthening of the axiomatization of models of combinatory logic.

An *applicative structure* is an algebra with a binary operation $\cdot$ that we call *application*. We may write it infix as $a \cdot b$, or even drop it entirely and write ab. As usual, application associates to the left; hence abc means $(ab)c$. An applicative structure is called *extensional* if the following axiom holds: $\forall xy.(\forall z.(xz = yz) \Rightarrow x = y)$.

Definition 2.3.1. *A* combinatory algebra *(*CA*, for short) is an algebra* $\mathbf{A} = (A, \cdot, K^{\mathbf{A}}, S^{\mathbf{A}})$ *where* $(A, \cdot)$ *is an applicative structure and the following axioms hold in* $\mathbf{A}$:

(CA1) $Kxy = x$

(CA2) $Sxyz = xz(yz)$

A combinatory algebra $\mathbf{A}$ *is called* extensional *if its underlying applicative structure is extensional.*

A *homomorphism of combinatory algebras* $\mathbf{A}$ and $\mathbf{B}$ is a map $f : A \to B$ such that $f(K^{\mathbf{A}}) = K^{\mathbf{B}}$, $f(S^{\mathbf{A}}) = S^{\mathbf{B}}$ and $f(a \cdot^{\mathbf{A}} b) = f(a) \cdot^{\mathbf{B}} f(b)$, for all $a, b \in A$.

Notation: when the combinatory algebra $\mathbf{A}$ is fixed and clear from the context it is common to set $\mathbf{k} = K^{\mathbf{A}}$, $\mathbf{s} = S^{\mathbf{A}}$, $\mathbf{i} = \mathbf{skk}$ and $\boldsymbol{\varepsilon} = \mathbf{s(ki)}$. Committing an abuse of notation we will also use CA to denote the class of all combinatory algebras; we will let the reader distinguish between these two different usages. From Definition 2.3.1 it is clear that CA is indeed a variety (since it is equationally axiomatized).

Combinatory algebras are indeed the natural models of combinatory logic. Given a CA $\mathbf{A}$ we denote by $\mathscr{C}(A)$ the set of *combinatory polynomials over* $\mathbf{A}$, i.e., the elements of $\mathscr{C}$ possibly containing constants from a set A.

The λ-terms can be naturally interpreted in combinatory algebras by means of the translation $(-)_{cl} : \Lambda \to \mathscr{C}$ (see Section 2.2), so that we can associate to each combinatory algebra $\mathbf{A}$ a set of equalities between λ-terms, called its *equational theory*.

Definition 2.3.2 (**Equational theory**). *The* equational theory *of a combinatory algebra* $\mathbf{A}$ *is the set*

$$Eq(\mathbf{A}) = \{M = N : \mathbf{A} \vDash M_{cl} = N_{cl}\}$$

We set the following notation for interpretation of λ-terms in a combinatory algebra: $\llbracket M \rrbracket_\rho^{\mathbf{A}} = (M_{cl})_\rho^{\mathbf{A}}$, *so that*

$$Eq(\mathbf{A}) = \{M = N : \forall \rho \in \mathrm{Env}_A.\ \llbracket M \rrbracket_\rho^{\mathbf{A}} = \llbracket N \rrbracket_\rho^{\mathbf{A}}\}$$

Term models

The simplest way of constructing a combinatory algebra is probably taking the quotient of the set $\mathscr{C}$ of combinatory terms w.r.t. the equivalence determined by CL. Other natural examples of combinatory algebras, coming directly form the lambda calculus, are the so-called *term models*, that we define here.

For a given λ-theory T we define the *open T-equivalence class* of a λ-term M is the set $[M]_T = \{N \in \Lambda : T \vdash M = N\}$; we also define the set $\Lambda/T = \{[M]_T : M \in \Lambda\}$. On Λ/T we define the application $[M]_T \cdot [N]_T = [MN]_T$.

Definition 2.3.3 (**Open term model**). *The* open term model *of T is the structure* $\mathbf{\Lambda/T} = (\Lambda/T, \cdot, [\mathbf{K}]_T, [\mathbf{S}]_T)$.

The *closed T-equivalence class* of a closed λ-term M is the set $[M]_T^o = \{N \in \Lambda^o : T \vdash M = N\}$; we also define the set $\Lambda^o/T = \{[M]_T^o : M \in \Lambda^o\}$. On Λ^o/T we define the application $[M]_T^o \cdot [N]_T^o = [MN]_T^o$.

Definition 2.3.4 (**Closed term model**). *The* closed term model *of T is the structure* $\mathbf{\Lambda^o/T} = (\Lambda^o/T, \cdot, [\mathbf{K}]_T^o, [\mathbf{S}]_T^o)$.

Proposition 2.3.1. *Both* $\mathbf{\Lambda/T}$ *and* $\mathbf{\Lambda^o/T}$ *are combinatory algebras, for any λ-theory T.*

2.3.2 λ-algebras

If $\mathbf{A} \in \mathsf{CA}$, then $Eq(\mathbf{A})$ is not necessarily a λ-theory. For example, if $\mathscr{C}/CL$ is the free combinatory algebra over the denumerable set Var of variables we have that $(\lambda x.(\lambda y.y)x = \lambda x.x) \notin Eq(\mathscr{C}/CL)$.

Such a remark shows that not all combinatory algebras can be viewed as algebraic models of the lambda calculus (via the translation of λ-terms into combinatory

terms), since there are combinatory algebras that do not satisfy all equations of $\lambda\beta$. To cope with this problem another class of algebras was proposed. These algebras, called λ-*algebras*, admit at least four different but equivalent characterizations, by Curry, Hindley-Longo, Barendregt (see [10, Ch. 5]), Selinger (see [108]). We give here Selinger's axiomatization.

Definition 2.3.5. *A λ-algebra (*LA*, for short) is a combinatory algebra satisfying the following five equations:*

(LA1) $K = 1K;$ $\quad\quad Kx = 1(Kx)$

(LA2) $S = 1S;$ $\quad\quad Sx = 1(Sx);$ $\quad\quad Sxy = 1(Sxy)$

(LA3) $S(S(KK)x)y = 1x$

(LA4) $S(S(S(KS)x)y)z = S(Sxz)(Syz)$

(LA5) $K(xy) = S(Kx)(Ky)$

(LA6) $1x = S(Kx)I$

A *homomorphism of λ-algebras* is a homomorphism of the underlying combinatory algebras. Committing an abuse of notation we will also use LA to denote the class of all λ-algebras; we will let the reader distinguish between these two different usages. Indeed LA is a variety (so that in particular LA is closed under homomorphic images).

Again natural examples of λ-algebras come from the lambda calculus.

Proposition 2.3.2. *Both $\Lambda/\mathbf{T}$ and $\Lambda^o/\mathbf{T}$ are λ-algebras, for any λ-theory T.*

Remark 2.3.3. *If $\mathbf{A} \in$ LA, then $Eq(\mathbf{A})$ is not necessarily a λ-theory. An ingenious example of this fact is due to Plotkin [97]. He shows that there exist λ-terms M, N such that $\boldsymbol{\Lambda^o/\lambda\beta\eta} \vDash M = N$ but $\boldsymbol{\Lambda^o/\lambda\beta\eta} \nvDash \lambda x.M = \lambda x.N$.*

Despite the "bug" put in evidence in Remark 2.3.3, the class LA still retains some good properties.

Proposition 2.3.4. *For any λ-algebra $\mathbf{A}$ we have $\lambda\beta \subseteq Eq(\mathbf{A})$ (and $\lambda\beta\eta \subseteq Eq(\mathbf{A})$ if $\mathbf{A}$ is extensional).*

Moreover $Eq(\mathbf{A}) \cap (\Lambda^o \times \Lambda^o)$ is always a closed λ-theory: we shall speak of it as the closed λ-theory *induced* by $\mathbf{A}$. Thus a λ-algebra *indirectly* determines a λ-theory, namely the unique λ-theory of which $Eq(\mathbf{A}) \cap (\Lambda^o \times \Lambda^o)$ is the restriction to closed terms.

2.3.3 λ-models

In order to find the right subclass of CA whose members can be considered as algebraic models of lambda calculus we need to strengthen once again the axiomatization.

Definition 2.3.6. *A λ-model (*LM*, for short) is a λ-algebra satisfying the following axiom, due to Meyer and Scott:*

(LM) $\forall xy.(\forall z.\ xz = yz) \Rightarrow 1x = 1y$

Committing an abuse of notation we will also use LM to denote the class of all λ-models; we will let the reader distinguish between these two different usages. The class LM does not form a variety but it has the following fundamental property.

Proposition 2.3.5. *If $\mathbf{A}$ is a λ-model, then $Eq(\mathbf{A})$ is a λ-theory.*

We say that a λ-theory T *is induced* by a λ-model $\mathbf{A}$ if $Eq(\mathbf{U}) = T$.
Again natural examples of λ-models come from the lambda calculus.

Proposition 2.3.6. *The open term model $\mathbf{\Lambda}/\mathbf{T}$ is a λ-model, for any λ-theory T.*

Remark 2.3.7. *As the reader can notice, closed term models are omitted from Proposition 2.3.6. In fact there are λ-theories T for which $\mathbf{\Lambda^o}/\mathbf{T}$ is not a λ-model. One such example is given by $\lambda\beta\eta$. Another, more exotic, example is due to Jacopini [60].*

Another natural, and maybe more interesting, example of λ-model is given by Böhm trees. We let $\mathfrak{B}$ be the set of all Böhm trees of λ-terms. It is possible to define an application operation on $\mathfrak{B}$ in such a way that choosing as basic combinators the Böhm trees of $\mathbf{K}$ and $\mathbf{S}$, respectively, one obtains a λ-model (see [10, Ch. 19]).

2.4 Models in cartesian closed categories

In the following, $\mathbb{C}$ is a locally small *Cartesian closed category* (*ccc*, for short) and A, B, C are arbitrary objects of $\mathbb{C}$. We denote by $A_1 \& A_2$ the *Cartesian product* (or *direct product*) of A_1 and A_2 and by $\pi_i \in \mathbb{C}(A_1 \& A_2, A_i)$ $(i = 1, 2)$ the associated *projections*. Given a pair of arrows $f_i \in \mathbb{C}(C, A_i)$ $(i = 1, 2)$ $\langle f_1, f_2 \rangle \in \mathbb{C}(C, A_1 \& A_2)$ is the unique arrow such that $\pi_i \circ \langle f_1, f_2 \rangle = f_i$ $(i = 1, 2)$. For $f_i \in \mathbb{C}(A_i, B_i)$ $(i = 1, 2)$ the *product map* $f_1 \& f_2 \in \mathbb{C}(A_1 \& A_2, B_1 \& B_2)$ is defined by $f_1 \& f_2 = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$. By $\top$ is indicated the terminal object characterized by the property that $\mathbb{C}(A, \top)$ has exactly one member for each A.

We will write $A \Rightarrow B$ for the *exponential object* and $\mathrm{ev}_{A,B} \in \mathbb{C}((A \Rightarrow B) \& A, B)$ for the *evaluation morphism* relative to A, B. Whenever A, B are clear from the context we will omit the subscripts.

For all objects A, B, C and arrow $f \in \mathbb{C}(C\&A, B)$ we denote by $\mathrm{cur}_{A,B,C}(f) \in \mathbb{C}(C, A \Rightarrow B)$ the unique morphism such that $\mathrm{ev}_{A,B} \circ (\mathrm{cur}_{A,B,C}(f)\&\mathrm{id}_A) = f$. The family $\{\mathrm{cur}_{A,B,C} : \mathbb{C}(C\&A, B) \to \mathbb{C}(C, A \Rightarrow B)\}_{A,B,C \in \mathbb{C}}$ of maps forms a natural transformation, called *currying*. Whenever A, B, C are clear from the context we will omit the subscripts.

Definition 2.4.1. *A* reflexive object *of a ccc $\mathbb{C}$ is a triple $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ such that U is an object of $\mathbb{C}$ and $\mathrm{Lam} \in \mathbb{C}(U \Rightarrow U, U)$, $\mathrm{Ap} \in \mathbb{C}(U, U \Rightarrow U)$ satisfy $\mathrm{Ap} \circ \mathrm{Lam} = \mathrm{id}_{U \Rightarrow U}$. When moreover $\mathrm{Lam} \circ \mathrm{Ap} = \mathrm{id}_U$, $\mathbf{U}$ is called* extensional.

Whenever there is not risk of confusion, we drop mention to the morphisms Ap and Lam.

For any given object U of $\mathbb{C}$ we set $U^0 = \top$ and $U^{n+1} = U^n\&U$. For a finite set of variables $I = \{x_1, \ldots, x_n\}$ we set $U^I = U^n$ and $\pi_{x_i}^I : U^I \to U$ as the projection morphism that always exists in a ccc.

Given a λ-term M and a subset $I \subseteq_{\mathrm{f}} \mathrm{Var}$, we say that $\bar{x}$ is *adequate for M* if $\mathrm{FV}(M) \subseteq I$. We simply say that I is *adequate* whenever M is clear from the context.

Definition 2.4.2. *Let $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ be a reflexive object. For all $M \in \Lambda$ and all adequate $I \subseteq_{\mathrm{f}} \mathrm{Var}$ the we define a morphism $|M|_I^{\mathbf{U}} \in \mathbb{C}(U^I, U)$ by induction on the structure of M as follows:*

- $|x|_I^{\mathbf{U}} = \pi_x^I$

- $|NZ|_I^{\mathbf{U}} = \mathrm{ev} \circ \langle \mathrm{Ap} \circ |N|_I^{\mathbf{U}}, |Z|_I^{\mathbf{U}} \rangle$

- $|\lambda y.N|_I^{\mathbf{U}} = \mathrm{Lam} \circ \mathrm{cur}(|N|_{I\cup\{y\}}^{\mathbf{U}})$, *where $y \notin I$ (we can always assume this)*

We refer to [10, Ch. 5] for more details about this definition: one useful property is for example that for any $I, J \supseteq \mathrm{FV}(M)\cup\mathrm{FV}(N)$ we have $|M|_I^{\mathbf{U}} = |N|_I^{\mathbf{U}} \Leftrightarrow |M|_J^{\mathbf{U}} = |N|_J^{\mathbf{U}}$. We will usually refer to $|M|_I^{\mathbf{U}}$ as the *interpretation of M in $\mathbf{U}$* even if, strictly speaking, the denotation of the term is an element of $\mathbb{C}(U^I, U)$ and when there will be no danger of confusion, we will just write $|M|_I$, dropping the reference to the object $\mathbf{U}$ and implicitly assuming I adequate and dropping the subscript if the term M is closed.

Via the interpretation function we can associate to $\mathbf{U}$ a set of equalities between λ-terms, called its *equational theory*.

Definition 2.4.3 (**Equational theory**). *The* equational theory *of a reflexive object $\mathbf{U}$ is the set $Th_=(\mathbf{U}) = \{M = N : |M|_I = |N|_I, \ I = \mathrm{FV}(M) \cup \mathrm{FV}(N)\}$.*

Proposition 2.4.1. *If $\mathbf{U}$ is a reflexive object in a ccc, then*

(i) $Th_=(\mathbf{U})$ is a λ-theory,

(ii) $Th_=(\mathbf{U})$ is an extensional λ-theory iff $\mathbf{U}$ is an extensional reflexive object.

We refer to [10, Ch. 5] for the proof of the above proposition.

2.4.1 From reflexive objects to λ-algebras and λ-models

We end this section with a brief review of the connection between reflexive objects in ccc's and λ-algebras/λ-models.

Definition 2.4.4 ([74])**.** *Let* $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ *be a reflexive object in a locally small ccc. It is possible to construct a combinatory algebra*

$$\mathbf{U}^{\bullet} = (\mathbb{C}(\top, U), \cdot, |\mathbf{K}|^{\mathbf{U}}, |\mathbf{S}|^{\mathbf{U}})$$

where $x \cdot y = \mathrm{ev} \circ \langle \mathrm{Ap} \circ x, y \rangle$.

An object A of $\mathbb{C}$ *has enough points* if for all $f, g \in \mathbb{C}(A, A)$ such that $f \neq g$ there exists a morphism $p \in \mathbb{C}(\top, A)$ such that $f \circ p \neq g \circ p$.

Proposition 2.4.2. *Let* $\mathbf{U}$ *be a reflexive object in a locally small ccc. Then*

  (i) $Eq(\mathbf{U}^{\bullet}) = Th_{=}(\mathbf{U})$,

 (ii) $\mathbf{U}^{\bullet}$ *is a* λ*-algebra (extensional if* $\mathbf{U}$ *is),*

(iii) *if* $\mathbf{U}$ *has enough points, then* $\mathbf{U}^{\bullet}$ *is a* λ*-model.*

We refer to [10, Ch. 5] for the proof of the above proposition.

Notation and terminology: in view of Proposition 2.4.2, we will henceforth always consider a reflexive object $\mathbf{U}$ as endowed with its combinatory algebra structure and in fact we will just use the notation $\mathbf{U}$ even when we should use $\mathbf{U}^{\bullet}$; along these lines we will use the notation $Eq(\mathbf{U})$ instead of $Th_{=}(\mathbf{U})$, since this causes no ambiguity.

In the first part of this thesis we will only work with categories in which every object has enough points.

2.5 Partially ordered models of lambda-calculus

In this thesis we are interested in partially ordered λ-models and in the λ-theories that they induce. For this reason we also provide a short summary of the order-theoretic notions we will use.

2.5.1 Scott domains

Let $\mathcal{D} = (D, \leq_D)$ be a *partially ordered set* (*poset*, for short); when there is no ambiguity we write just $\mathcal{D}$ to indicate the poset or just $\leq$ to indicate the order relation. Two elements u and v of D are *compatible* if they have an upper bound in D, i.e., if there exists $z \in D$ such that $z \geq u, v$. A subset $X \subseteq D$ is *directed* if it is non-empty and every pair of elements $u, v \in X$ has an upper bound in X. The set

X is *downward* (resp. *upward*) *closed* if $u \in X$ and $v \leq u$ imply $v \in X$. We use the notation $A \downarrow$ for the least downward closed set containing a subset A of $\mathcal{D}$.

A poset $\mathcal{D}$ is a *complete partial order* (*cpo*, for short) if it has a least element (denoted by $\perp_D$) and every directed set $X \subseteq D$ has a *least upper bound* (or *supremum*) $\sqcup X$. A cpo $\mathcal{D}$ is *bounded complete* if $u \sqcup v$ exists, for all compatible $u, v \in D$.

An element $d \in D$ is called *compact* if for every directed subset $X \subseteq D$ we have that $d \leq \sqcup X$ implies $d \leq v$ for some $v \in X$. We write $\mathcal{K}(\mathcal{D})$ for the collection of all compact elements of $\mathcal{D}$. A cpo $\mathcal{D}$ is *algebraic* if for every $u \in D$ the set $\{d \in \mathcal{K}(\mathcal{D}) : d \leq u\}$ is directed and u is its least upper bound. A bounded complete algebraic cpo is called a *Scott domain*. A *continuous function* from $\mathcal{D}$ to $\mathcal{E}$ is a map $f : D \to E$ such that for any directed $X \subseteq D$ we have $f(\sqcup X) = \sqcup f(X)$.

Example 2.5.1. *The set $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a a Scott domain whose compact elements are exactly the finite subsets of $\mathbb{N}$.*

Example 2.5.2. *Recall from Subsection 2.3.1 the definition of Λ/T, Λ^o/T (where T is a λ-theory) and let $\perp$ be a constant. The* flat domain *$(\Lambda/T)_\perp = \Lambda/T \cup \{\perp\}$ is partially ordered as follows: $\perp \leq [M]_T$ and $[M]_T \leq [N]_T \Leftrightarrow [M]_T = [N]_T$, for all $M, N \in \Lambda$. Then $(\Lambda/T)_\perp$ is a Scott domain in which every element is compact, and the same holds for $(\Lambda^o/T)_\perp$.*

2.5.2 Models in order-enriched ccc's

Throughout this thesis we will work with *order-enriched categories*, i.e. locally small categories in which the hom-sets are partially ordered sets. In particular we will encounter *cpo-enriched* ccc's, that is categories $\mathbb{C}$ such that for every pair of objects A, B, the hom-set $\mathbb{C}(A, B)$ is a cpo and composition, pairing and currying are continuous (plus some other conditions - see [8]).

Notation and terminology: we indicate by **Sd** the category which has all Scott domains as objects and continuous functions between them as arrows. The category **Sd** is a ccc: the exponent of two Scott domains $\mathcal{D}, \mathcal{E}$ is the space $\mathcal{D} \Rightarrow \mathcal{E} = $ **Sd**$(\mathcal{D}, \mathcal{E})$ of continuous maps with pointwise ordering and the Cartesian product $\mathcal{D} \times \mathcal{E}$ is the set-theoretic direct product with pointwise ordering.

Notation and terminology. The reflexive objects of **Sd** are called *reflexive Scott domains*, RSD, for short. Committing an abuse of notation we will also use RSD to denote the class of all reflexive objects of **Sd**; we will let the reader distinguish between these two different usages. An RSD $(\mathcal{D}, \mathrm{Ap}, \mathrm{Lam})$ will be compactly denoted by the corresponding bold letter, **D**.

Interpreting λ-terms in a partially ordered structure one indeed obtains a preorder on Λ. This leads to the concept of *order λ-theory*, that we now define formally in analogy with that of λ-theory.

Definition 2.5.1 (Order λ-theory)**.** *An* order λ-theory *is a set of inequalities between λ-terms which is closed under the rules following axioms and rules:*

(o-β) $(\lambda x.M)N \sqsubseteq M\{N/x\}$, $M\{N/x\} \sqsubseteq (\lambda x.M)N$

(o-app) *if* $M \sqsubseteq M'$ *and* $N \sqsubseteq N'$, *then* $MN \sqsubseteq M'N'$

(o-ref) $M \sqsubseteq M$

(o-tran) *if* $M \sqsubseteq N$ *and* $N \sqsubseteq Z$, *then* $M \sqsubseteq Z$

Terminology: an order λ-theory is *consistent* if it is stricly contained in $\Lambda \times \Lambda$.

Note the lack of the analogue of the ξ-rule from Definition 2.5.1: the reason of this omission is that we want to be able to associate, to every RSD, a set of inequalities between λ-terms, its *order theory*, and we want this set to be an order λ-theory in the sense of Definition 2.5.1. This would not be possible in general, adding a rule like "if $M \sqsubseteq N$ then $\lambda x.M \sqsubseteq \lambda x.N$": the counterexample is due to G. D. Plotkin (see [109, Thm. 2.5]).

Definition 2.5.2 (**Order theory**)**.** *The* order theory *of a* RSD **U** *is the set* $Th_{\le}(\mathbf{U}) = \{M \le N : |M|_I^{\mathbf{U}} \le |N|_I^{\mathbf{U}},\ I = \mathrm{FV}(M) \cup \mathrm{FV}(N)\}.$

Proposition 2.5.1. *If* **U** *is a* RSD*, then* $Th_{\le}(\mathbf{U})$ *is an order λ-theory.*

It is well-known that **Sd** is a cpo-enriched ccc with enough points: thus, in view of the discussion in Section 2.4, any RSD can be considered as a λ-model. Any λ-model arising from an RSD **U** has an additional property, namely that the function space $\mathbf{Sd}(U, U)$ coincides with the space of *representable functions*, that is $f : U \to U$ is continuous iff there exists an element $u \in U$ such that $f = \lambda\!\!\!\lambda\, x \in U.ux$. Moreover in any RSD the application operation is in harmony (is *compatible*) with the partial order and the interpretation function is continuous.

2.6   Scott domains and information systems

Scott domains have an appealing and suggestive representation as information systems, introduced by D. S. Scott himself [106]. Information systems organize themselves into a ccc in tight connection with **Sd**.

2.6.1   The category of information systems

An information system consists of a set A (with a distinguished element Δ_A) together with an entailment relation $\vdash_A$ and a consistency predicate Con_A. We adopt the following notational conventions: letters $\alpha, \beta, \gamma, \ldots$ are used for elements of A; letters $a, b, c, \ldots$ are used for elements of Con_A, usually called *consistent sets*; letters $x, y, z, \ldots$ are used for arbitrary elements of $\mathcal{P}(A)$.

Definition 2.6.1. *An* information system *is a triple* $\mathcal{A} = (A, \mathrm{Con}_A, \vdash_A)$, *where* $\mathrm{Con}_A \subseteq \mathcal{P}_\mathrm{f}(A)$ *is a downward closed family containing all singleton sets, and* $\vdash_A \subseteq \mathrm{Con}_A \times A$ *satisfies the axioms listed below:*

(I1) if $a \in \mathrm{Con}_A$ and $a \vdash_A b$, then $a \cup b \in \mathrm{Con}_A$ (where $a \vdash_A b \overset{def}{=} \forall \beta \in b.\ a \vdash_A \beta$)

(I2) if $\alpha \in a$, then $a \vdash_A \alpha$

(I3) if $a \vdash_A b$ and $b \vdash_A \gamma$, then $a \vdash_A \gamma$

Usually when we write $a \vdash_A \alpha$ we implicitly assume that $a \in \mathrm{Con}_A$.

Note that the meta-notation explained in property (I1) allows to view $\vdash_A$ alternatively as a binary relation on Con_A.

We will usually drop the subscripts from Con_A and $\vdash_A$ when there is no danger of confusion.

Definition 2.6.2. *[106, Def. 5.1] An* approximable relation *between two information systems $\mathcal{A}, \mathcal{B}$ is a relation $R \subseteq \mathrm{Con}_A \times B$ satisfying the following properties:*

(AR1) if $a \in \mathrm{Con}_A$ and $a\ R\ b$, then $b \in \mathrm{Con}_B$ (where $a\ R\ b \overset{def}{=} \forall \beta \in b.\ a\ R\ \beta$)

(AR2) if $a' \vdash_A a\ R\ b \vdash_B \beta'$, then $a'\ R\ \beta'$

Again the meta-notation explained in property (AR1) allows to view R as a relation between Con_A and Con_B.

We will call **Inf** the category which has information systems as objects and approximable relations as arrows. The composition of two morphisms $R \in \mathbf{Inf}(\mathcal{A}, \mathcal{B})$ and $S \in \mathbf{Inf}(\mathcal{B}, \mathcal{C})$ is (using the meta-notation) their usual relational composition: $S \circ R = \{(a, \gamma) \in \mathrm{Con}_A \times C : \exists b \in \mathrm{Con}_B.\ (a, b) \in R \text{ and } (b, \gamma) \in S\}$. The identity morphism of an information system $\mathcal{A}$ is $\mathrm{id}_{\mathcal{A}} = \vdash_A$.

The ccc structure of **Inf** was introduced by Scott [106], and is formulated also by Larsen & Winskel [78]. We report here their version.

Definition 2.6.3. *The Cartesian product $\mathcal{A}\&\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is given by $\mathcal{A}\&\mathcal{B} = (A \uplus B, \mathrm{Con}, \vdash)$ where*

- $A \uplus B = \{(1, \alpha) : \alpha \in A\} \cup \{(2, \beta) : \beta \in B\}$

- $a \in \mathrm{Con}$ *iff* $\{\alpha : (1, \alpha) \in a\} \in \mathrm{Con}_A$ *and* $\{\beta : (2, \beta) \in a\} \in \mathrm{Con}_B$

- $a \vdash (i, \gamma)$ *iff* $\begin{cases} \{\alpha : (1, \alpha) \in a\} \vdash_A \gamma & \text{if } i = 1 \\ \{\beta : (2, \beta) \in a\} \vdash_B \gamma & \text{if } i = 2 \end{cases}$

The category **Inf** also has countable products, defined in the obvious way. We remark the existence of a canonical isomorphism $\varphi_{A,B} : \mathrm{Con}_{A\&B} \cong \mathrm{Con}_A \times \mathrm{Con}_B$ given by $\varphi_{A,B}(X) = (\{\alpha \in A : (1, \alpha) \in X\}, \{\beta \in B : (2, \beta) \in X\})$. In the sequel we will consider such canonical bijection as an equality, hence we will still denote by (a, b) the corresponding element of $\mathrm{Con}_{A\&B}$; such a choice makes the exposition easier.

The projections are $\pi_1 = \{((a, b), \alpha) \in \mathrm{Con}_{A\&B} \times A : a \vdash_A \alpha\}$ and $\pi_2 = \{((a, b), \beta) \in \mathrm{Con}_{A\&B} \times B : b \vdash_B \beta\}$. For $R \in \mathbf{Inf}(\mathcal{C}, \mathcal{A}_1)$ and $S \in \mathbf{Inf}(\mathcal{C}, \mathcal{A}_2)$,

the pairing $\langle R, S \rangle \in \mathbf{Inf}(\mathcal{C}, \mathcal{A}_1 \& \mathcal{A}_2)$ is given by
$\langle R, S \rangle = \{(c, (\alpha, \beta)) : (c, \alpha) \in R, \ (c, \beta) \in S\}$. The information system $\top = (\emptyset, \emptyset, \emptyset)$ is the terminal object.

Definition 2.6.4. *The exponential object $\mathcal{A} \Rightarrow \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is given by $\mathcal{A} \Rightarrow \mathcal{B} = (A \Rightarrow B, \Delta, \mathrm{Con}, \vdash)$ where*

- $A \Rightarrow B = \mathrm{Con}_A \times B$

- $\{(a_1, \beta_1), \ldots, (a_k, \beta_k)\} \in \mathrm{Con}$ *iff* $\forall I \subseteq [1, k].\ (\cup_{i \in I} a_i \in \mathrm{Con}_A \Rightarrow \{\beta_i : i \in I\} \in \mathrm{Con}_B$

- $\{(a_1, \beta_1), \ldots, (a_k, \beta_k)\} \vdash (c, \gamma)$ *iff* $\{\beta_i : c \vdash_A a_i, \ i \in [1, k]\} \vdash_B \gamma$

The currying $\mathrm{cur} : \mathbf{Inf}(\mathcal{C} \& \mathcal{A}, \mathcal{B}) \to \mathbf{Inf}(\mathcal{C}, \mathcal{A} \Rightarrow \mathcal{B})$ is given by

$$\mathrm{cur}(R) = \{(c, (a, \beta)) : ((c, a), \beta) \in R\}$$

(it is not difficult to see that $\mathrm{cur}(R)$ is an approximable relation).

The evaluation morphism $\mathrm{ev} : (\mathcal{A} \Rightarrow \mathcal{B}) \& \mathcal{A} \to \mathcal{B}$ is given by

$$\mathrm{ev} = \{((\{(a_1, \beta_1), \ldots, (a_m, \beta_m)\}, c), \gamma) : c \vdash_A \cup_{i=1}^m a_i, \ \{\beta_1, \ldots, \beta_m\} \vdash_B \gamma\}$$

(again it is not difficult to see that ev is an approximable relation).

2.6.2 Equivalence between the two categories

We already recalled the definition of the category **Sd**. After having recalled the definition of **Inf**, it is now time also to recall why, i.e. in what sense, information systems constitute a representation for Scott domains.

The categories **Inf** and **Sd** are in a strong correspondence. The following definition describes two functors establishing this bridge.

Definition 2.6.5. *The functor $(\_)^+ : \mathbf{Inf} \to \mathbf{Sd}$ is defined as follows. Given $\mathcal{A}, \mathcal{B}$ in **Inf**, we have*

- $\mathcal{A}^+ = \mathbf{Inf}(\top, \mathcal{A})$, *and*

- *for $R \in \mathbf{Inf}(\mathcal{A}, \mathcal{B})$ one has $R^+(x) = \{\beta \in B : \exists a \subseteq x.\ a\ R\ \beta\}$.*

Definition 2.6.6. *The functor $(\_)^- : \mathbf{Sd} \to \mathbf{Inf}$ is defined as follows. Given $\mathcal{D}, \mathcal{E}$ in **Sd**, we have*

- $\mathcal{D}^- = (\mathcal{K}(D), \mathrm{Con}, \vdash)$, *with*

- $\{\alpha_1, \ldots, \alpha_k\} \in \mathrm{Con}$ *iff $\{\alpha_1, \ldots, \alpha_k\}$ has upper bound in $\mathcal{D}$,*

- $\{\alpha_1, \ldots, \alpha_k\} \vdash \beta$ *iff $\sqcup\{\alpha_1, \ldots, \alpha_k\} \geq \beta$, and*

- *for $f \in \mathbf{Sd}(\mathcal{D}, \mathcal{E})$ one has $f^- = \{(\{\alpha_1, \ldots, \alpha_k\}, \beta) : f(\sqcup\{\alpha_1, \ldots, \alpha_k\}) \geq \beta\}$.*

The above functors *preserve the ccc structure*. The following theorem describes precisely the strong equivalence between the two categories.

Theorem 2.6.1. *Both* $(\_)^{+-} : \mathbf{Inf} \to \mathbf{Inf}$ *and* $(\_)^{-+} : \mathbf{Sd} \to \mathbf{Sd}$ *are naturally isomorphic to the identity functor and preserve the ccc structure (up to iso). More precisely for all information systems $\mathcal{A}, \mathcal{B}$, and all Scott domains $\mathcal{D}, \mathcal{E}$, we have*

(i) $\mathcal{A}^{+-} \cong \mathcal{A}$ *(in* $\mathbf{Inf}$*) and* $\mathcal{D}^{-+} \cong \mathcal{D}$ *(in* $\mathbf{Sd}$*);*

(ii) $(\mathcal{A} \Rightarrow \mathcal{B})^+ = \mathbf{Inf}(\mathcal{A}, \mathcal{B}) \cong \mathbf{Sd}(\mathcal{A}^+, \mathcal{B}^+) = \mathcal{A}^+ \Rightarrow \mathcal{B}^+$, $\mathrm{ev}^+ = \mathrm{ev}$, $\mathrm{cur}(R)^+ = \mathrm{cur}(R^+)$;

(iii) $(\mathcal{A}\&\mathcal{B})^+ \cong \mathcal{A}^+ \times \mathcal{B}^+$, $\pi_i^+ = \pi_i$;

(iv) $(\mathcal{D} \Rightarrow \mathcal{E})^- = \mathcal{D}^- \Rightarrow \mathcal{E}^-$, $\mathbf{Sd}(\mathcal{D}, \mathcal{E}) \cong \mathbf{Inf}(\mathcal{D}^-, \mathcal{E}^-)$, $\mathrm{ev}^- = \mathrm{ev}$, $\mathrm{cur}(f)^- = \mathrm{cur}(f^-)$;

(v) $(\mathcal{D} \times \mathcal{E})^- \cong \mathcal{D}^-\&\mathcal{E}^-$, $\pi_i^- = \pi_i$.

Note that $\mathbf{Inf}(\mathcal{A}, \mathcal{B})$ and $\mathbf{Sd}(\mathcal{A}^+, \mathcal{B}^+)$ (resp. $(\mathcal{A}\&\mathcal{B})^+$ and $\mathcal{A}^+ \times \mathcal{B}^+$) are isomorphic objects of $\mathbf{Sd}$. Note that we used the same symbols ($\Rightarrow$) to indicate the exponential functor in both categories and the same symbols for evaluation and projection morphisms and for the currying, leaving to the reader to distinguish between the different usages.

Remark 2.6.2. *The universe of $\mathcal{D}^{-+}$ is the set of all ideals of $\mathcal{D}$ (i.e. upward directed and downward closed subsets). The maps*

- $\mu_{\mathcal{D}}(x) = \{d \in \mathcal{K}(\mathcal{D}) : d \leq x\}$, *for all $x \in \mathcal{D}$*

- $\nu_{\mathcal{D}}(I) = \sqcup I$, *for all ideals I of $\mathcal{D}$*

are components of two inverse natural isomorphisms.

Theorem 2.6.3. (i) *If $\mathbf{A} = (\mathcal{A}, \mathrm{Ap}, \mathrm{Lam})$ is reflexive in $\mathbf{Inf}$, then $\mathbf{A}^+ = (\mathcal{A}^+, \mathrm{Ap}^+, \mathrm{Lam}^+)$ is reflexive in $\mathbf{Sd}$ and $Eq(\mathbf{A}) = Eq(\mathbf{A}^+)$.*

(ii) *If $\mathbf{D} = (\mathcal{D}, \mathrm{Ap}, \mathrm{Lam})$ is reflexive in $\mathbf{Sd}$, then $\mathbf{D}^- = (\mathbf{D}^-, \mathrm{Ap}^-, \mathrm{Lam}^-)$ is reflexive in $\mathbf{Inf}$ and $Eq(\mathbf{D}) = Eq(\mathbf{D}^-)$.*

Proof. (i) The fact that $(\ )^+$ is a ccc functor guarantees that $\mathbf{A}^+$ is reflexive and that $(|M|_I^{\mathbf{A}})^+ = |M|_I^{\mathbf{A}^+}$.

(ii) Similar to (i).

$\square$

Remark 2.6.4. *Another nice aspect of the functors $(\_)^+$ and $(\_)^-$ is the following. Call $\mathbf{FInf}$ the full subcategory of $\mathbf{Inf}$ having as objects all the information systems $\mathcal{A} = (A, \mathrm{Con}, \vdash)$ with $\mathrm{Con} = \mathcal{P}_{\mathrm{f}}(A)$. Then, under $(\_)^+$ and $(\_)^-$, $\mathbf{FInf}$ exactly corresponds to the category $\mathbf{ALat}$ of algebraic lattices and continuous functions.*

2.6.3   Information systems as closure operators

One nice feature is the possibility of defining, for each information system $\mathcal{A}$, an algebraic closure operator on a subset of $\mathcal{P}_f(A)$ from which the entire information system can be recovered. We say that an arbitrary subset $x \subseteq A$ is *finitely consistent* if $\mathcal{P}_f(x) \subseteq \text{Con}$; we let FCon be the set of all finitely consistent subsets of A.

If x is finitely consistent, then we define an operation $(\_) \downarrow_A \colon \text{FCon} \to \text{FCon}$ as follows

$$x \downarrow_{\mathcal{A}} = \{\alpha \in A : \exists a \subseteq_f x.\ a \vdash \alpha\}$$

When the information system $\mathcal{A}$ is clear form the context, we will drop the subscript from $(\_) \downarrow_{\mathcal{A}}$. Note that $(\_) \downarrow$ is an *algebraic closure operator* on FCon, that is, a monotone map satisfying the following conditions:

$$x \subseteq x\downarrow; \qquad x\downarrow\downarrow = x\downarrow; \qquad x\downarrow = \cup_{a \subseteq_f x} a\downarrow$$

Definition 2.6.7. *A* point *of $\mathcal{A}$ is a finitely consistent subset $x \subseteq A$ such that $x = x\downarrow$.*

We denote by $\mathcal{A}^+$ the set of all points of $\mathcal{A}$. The terminology chosen is perfectly coherent, since $\mathcal{A}^+ = \mathbf{Inf}(\top, \mathcal{A})$: such set, ordered by inclusion, is a Scott domain whose compact elements are the points of the form $a\downarrow$, for $a \in \text{Con}$. By Theorem 2.6.1(ii) any approximable relation $R \in \mathbf{Inf}(\mathcal{A}, \mathcal{B})$ can be alternatively seen as a point of $\mathcal{A} \Rightarrow \mathcal{B}$. We will exploit this fact, making use of the closure operator $(\_) \downarrow_{A \Rightarrow B}$ in order to define approximable relations as closures of suitable finitely consistent subsets of $A \Rightarrow B$. A concrete description of $(\_) \downarrow_{A \Rightarrow B}$ is given as follows: for $X \in \text{FCon}_{A \Rightarrow B}$, then $(c, \gamma) \in X\downarrow$ (we omit the subscript, to ease notation) iff

- either $\emptyset \vdash_B c$ or

- $\emptyset \nvdash_B c$ and there exist $(a_1, \beta_1), \ldots, (a_m, \beta_m) \in X$ such that $c \vdash_A \cup_{i=1}^m a_i$ and $\cup_{i=1}^m b_i \vdash_B \gamma$.

3

A unifying theory of webbed models

In practice, all the models built for applications are "webbed models", which means, roughly speaking, that their domain is a subdomain of some $(\mathcal{P}(D), \subseteq)$. Scott's first model $\mathcal{D}_\infty$ was first built in 1969 as an inverse limit of a projective system [104]. Actually $\mathcal{D}_\infty$ is an *extensional* model, i.e. it equates all $\beta\eta$-equivalent terms. A second model, connected to ordinary recursion theory soon followed: Plotkin and Scott's $\mathcal{P}_\omega$ (cf. [105, 107]), built via an elementary construction. Then came Engeler and Plotkin's model $\mathcal{E}$ [48, 96], and then other models, often built with practical purposes. Nearly all of thse models belong to Scott's continuous semantics.

It was noticed progressively thereafter that, in fact, all practical models of the continuous semantics admitted elementary constructions, as "reflexive information systems" of as "filter models". This was already a webbed presentation of the models, and already allowed alternatives to the classical inverse limit construction. Moreover individual filter models themselves were systematically presented and studied, in a proof-theoretic style, as "intersection type assignment systems" (a view which goes back to Coppo-Dezani-Honsell-Longo [38]).

In this chapter we present a unifying theory of webbed models which allows to view all constructions existing in the literature as particular instances. The categorical inverse limit construction in a category of domains is replaced by a completion process of "partial" structures, which can be treated algebraically and uniformly. Historically, this kind of constructions take source in Krivine [75], Longo [79], and Girard [53]. They can also be related, at least at the level of domains, to the event structures of [88]. However we strongly believe in the value of the systematic and uniform treatment that we propose, at least in view of the results that it allows to obtain.

Section 3.1 introduces various kinds of morphisms between information systems and establish their relations with embedding-projection pairs between domains. These morphisms are used in Section 3.2 for isolating a class of models of lambda calculus arising from what we call *i-webs*. Section 3.3 contains a technical study of various constructions involving restrictions and expansions of i-webs: in particular it deals with *partial i-webs*, which are structures of fundamental importance.

Finally in Section 3.5 we introduce the class of *Scottian λ-models*, whose purpose is to serve a a base for their "effective version" developed in §4 and to formulate the

main results of §5 in a modular and uniform way.

3.1 Morphisms of information systems, and ep-pairs

In Larsen & Winskel [78] substructures of information systems are introduced in relation to the search of exact solutions of recursive domain equations. In [42] Droste & Göbel further develop this theme, showing how substructures of information systems correspond to embedding-projection pairs of domains. In this [42] the authors also mention an "algebraic-style" (or "model-theoretic") notion of isomorphisms of information systems, which differs from the categorical definition of isomorphism in the category **Inf**.

We develop this idea and distinguish a basic kind of model-theoretic morphism and two refinements: *b-morphisms* and *f-morphisms*, which play different roles in our theory. In particular it turns out that b-morphisms are useful for the construction of models, while f-morphisms are useful for relating order theories of these models.

Notation. Let $\psi : A \to B$ be a function. Then for $x \subseteq A$ and $y \subseteq B$ we set

$$\psi[x] = \{\psi(\alpha) : \alpha \in x\} \quad \text{and} \quad \psi^{-1}[y] = \{\alpha : \psi(\alpha) \in y\}$$

Definition 3.1.1. *Let $\mathcal{A}, \mathcal{B}$ be ISs. A morphism from $\mathcal{A}$ to $\mathcal{B}$ is a map $\psi : A \to B$ satisfying the following property:*

(Mo) $a \in \mathrm{Con}_A$ iff $\psi[a] \in \mathrm{Con}_B$

Warning. The morphisms described in Definition 3.1.1 are in general not arrows neither in the category **Inf**, nor in **epInf**.

Definition 3.1.2. *A morphism from $\psi : \mathcal{A} \to \mathcal{B}$ is*

(1) a b-morphism ("b" for backward) if it satisfies the following property

(bMo) if $\psi[a] \vdash_B \psi(\alpha)$, then $a \vdash_A \alpha$

(2) a f-morphism ("f" for forward) if it satisfies the following property

(fMo) if $a \vdash_A \alpha$, then $\psi[a] \vdash_B \psi(\alpha)$

(3) a bf-morphism if it is both a b-morphism and a f-morphism.

Example 3.1.1. *Any set A "is" an information system $\mathcal{A} = (A, \mathcal{P}_{\mathrm{f}}(A), \ni)$. For two sets A, B a bf-morphism $\psi : \mathcal{A} \to \mathcal{B}$ between the two corresponding information systems is just an injective function.*

Example 3.1.2. *A* preordered set with coherence *(pc-set, for short) is a triple* $(A, \leq, \between)$, *where A is a non-empty set, $\leq$ is a preorder on A and $\between$ is a coherence (i.e., a reflexive, symmetric relation on A) compatible with the preorder (see [15, Def. 120]). A pc-set "is" an information system* $\mathcal{A} = (A, \mathcal{P}_{\mathrm{f}}^{\mathrm{coh}}(A), \vdash)$, *where $\mathcal{P}_{\mathrm{f}}^{\mathrm{coh}}(A)$ is the set of finite coherent subsets of A and $a \vdash \alpha$ iff $\exists \beta \in a. \beta \geq \alpha$.*

For two pc-sets $(A, \leq_A, \between_A)$ and $(B, \leq_B, \between_B)$ a bf-morphism $\psi : \mathcal{A} \to \mathcal{B}$ between the corresponding information systems is a function satisfying $\alpha \leq_A \alpha'$ iff $\psi(\alpha) \leq_B \psi(\alpha')$ and $\alpha \between_A \alpha'$ iff $\psi(\alpha) \between_B \psi(\alpha')$.

Example 3.1.3. *An* extended abstract type structure *(*EATS*, for short, [38, Def. 1.1]) is a partially ordered algebra $(A, \wedge, \to, \omega)$, where "$\wedge$" and "$\to$" are binary operations and "ω" is nullary one. Then the structure $\mathcal{A} = (A, \mathcal{P}_{\mathrm{f}}(A), \vdash)$, where $a \vdash \alpha$ iff $(\bigwedge a) \leq \alpha$, is an information system.*

For two EATS*s $(A, \wedge_A, \to_A, \omega_A)$ and $(B, \wedge_B, \to_B, \omega_B)$ a bf-morphism $\psi : \mathcal{A} \to \mathcal{B}$ between the corresponding information systems is a function satisfying $\alpha \leq_A \alpha'$ iff $\psi(\alpha) \leq_B \psi(\alpha')$.*

Notation. Let $\psi : A \to B$ be a morphism. Then we define a derived map $\vec{\psi} : A \Rightarrow A \to B \Rightarrow B$ setting $\vec{\psi}(a, \alpha) = (\psi[a], \psi(\alpha))$.

Proposition 3.1.1. *Let $\psi : \mathcal{A} \to \mathcal{B}$ be a morphism. Then $\vec{\psi}$ is a morphism too. Moreover*

(i) if ψ is a b-morphism, then $\vec{\psi}$ is a b-morphism too,

(ii) if ψ is a f-morphism, then $\vec{\psi}$ is a f-morphism too.

Proof. Easy. $\square$

It is not difficult to check that if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are ISs and $\psi : \mathcal{A} \to \mathcal{B}$, $\chi : \mathcal{B} \to \mathcal{C}$ are b-morphisms, then their composition $\chi \circ \psi : A \to C$ is a b-morphism too and that the identity function id is always a b-morphism. Hence we can define a new category.

Definition 3.1.3. *We let* **bInf** *(resp.* **bfInf**) *be the category which has all* IS*s as objects and b-morphisms (resp. bf-morphisms) as arrows.*

We will soon see that b(f)-morphisms are related to particular pairs of arrows which are often of interest in cpo-enriched categories.

Definition 3.1.4. *Let $\varepsilon : B \to A$ and $\pi : A \to B$ be morphisms in a cpo-enriched category $\mathbb{C}$. The couple (ε, π) is an* embedding-projection pair from B to A *(ep-pair, for short) if $\pi \circ \varepsilon = \mathrm{id}_B$ and $\varepsilon \circ \pi \leq \mathrm{id}_A$.*

Notation. We use the notation $(\varepsilon, \pi) : B \to A$ to indicate that (ε, π) is an ep-pair from B to A.

It is well-known that $(\mathrm{id}_B, \mathrm{id}_B) : B \to B$ is an ep-pair and that if $(\varepsilon', \pi') : C \to B$ and $(\varepsilon, \pi) : B \to A$ are ep-pairs, then their *composition* $(\varepsilon, \pi) \circ (\varepsilon', \pi') = (\varepsilon \circ \varepsilon', \pi' \circ \pi)$ is an ep-pair from C to A. Therefore the collection of all objects of $\mathbb{C}$ together with ep-pairs as arrows define a category, that we will call $\mathsf{ep}\mathbb{C}$.

The following lemma is folklore.

Lemma 3.1.2. *If $(\varepsilon, \pi) : \mathcal{E} \to \mathcal{D}$ is an ep-pair in **Sd**, then*

(i) $\forall x \in D. \forall y \in E. \; \varepsilon(y) \leq x \Leftrightarrow y \leq \pi(x)$,

(ii) $\pi(x) = \sqcup \{ y \in E : \varepsilon(y) \leq x \}$,

(iii) if $e \in \mathcal{K}(\mathcal{E})$, then $\varepsilon(e) \in \mathcal{K}(\mathcal{D})$.

Lemma 3.1.2 says, among other things, that all information regarding an ep-pair of continuous maps is encoded in ε, since the projection is completely determined by the embedding. We are now about to see that this feature of ep-pairs in **Sd** allows to put them in a bijective correspondence with bf-morphisms.

Recall Definition 2.6.5 and Definition 2.6.6, where the two functors $(\_)^+$ and $(\_)^-$ are described.

Definition 3.1.5. *Let $\psi : \mathcal{B} \to \mathcal{A}$ be a b-morphism. Define two functions $\psi_1 : \mathcal{B}^+ \to \mathcal{A}^+$ and $\psi_2 : \mathcal{A}^+ \to \mathcal{B}^+$ as follows:*

- *$\psi_1(y) = \psi[y] \downarrow_A$*

- *$\psi_2(x) = \psi^{-1}[x] \downarrow_B$*

Call $\mathbf{F}$ the mapping of information systems and b-morphisms given by $\mathcal{A} \mapsto \mathcal{A}^+$ and $\psi \mapsto (\psi_1, \psi_2)$.

Lemma 3.1.3. *The mapping $\mathbf{F} : \mathsf{b}\mathbf{Inf} \to \mathsf{ep}\mathbf{Sd}$ is a functor.*

Proof. Note that both ψ_1 and ψ_2 are well-given by property (Mo) and send points to points and their continuity is easy to prove. Let $y \in \mathcal{B}^+$. Then $\psi_2(\psi_1(y)) = \psi^{-1}[\psi[y] \downarrow_A] \downarrow_B = y$, using property (bMo). Let $x \in \mathcal{A}^+$. Then $\psi_1(\psi_2(x)) = \psi[\psi^{-1}[x] \downarrow_B] \downarrow_A \subseteq x$, since x is a point.

Finally $\mathbf{F}(\mathrm{id}_{\mathcal{B}}) = (\mathrm{id}_{\mathcal{B}^+}, \mathrm{id}_{\mathcal{B}^+})$ and $\mathbf{F}(\psi \circ \psi') = \mathbf{F}(\psi) \circ \mathbf{F}(\psi')$. $\qquad\square$

Therefore a b-morphism $\psi : \mathcal{B} \to \mathcal{A}$ yields and ep-pair from $\mathcal{B}^+$ to $\mathcal{A}^+$ in **Sd**.

Definition 3.1.6. *Call $\mathbf{G}$ the mapping of Scott domains and ep-pairs given by $\mathcal{D} \mapsto \mathcal{D}^-$ and $(\varepsilon, \pi) \mapsto \varepsilon\restriction_{\mathcal{K}(\mathcal{E})}$, where $(\varepsilon, \pi) : \mathcal{E} \to \mathcal{D}$.*

Lemma 3.1.4. *The mapping $\mathbf{G} : \mathsf{ep}\mathbf{Sd} \to \mathsf{bf}\mathbf{Inf}$ is a functor.*

Proof. Let $(\varepsilon, \pi) : \mathcal{E} \to \mathcal{D}$ be an ep-pair in **Sd**. Note that by Lemma 3.1.2, $\varepsilon{\upharpoonright}_{\mathcal{K}(\mathcal{E})}$: $\mathcal{E}^- \to \mathcal{D}^-$ is a well-given total function.

Let $x \subseteq_{\mathrm{f}} \mathcal{K}(\mathcal{E})$ and let e and upper bound of x in $\mathcal{E}$. Then clearly $\varepsilon(e)$ is an upper bound of $\varepsilon[x]$ in $\mathcal{D}$. Conversely if d is an upper bound of $\varepsilon[x]$ in $\mathcal{D}$ then $\pi(d)$ is an upper bound of $x = \pi[\varepsilon[x]]$ in $\mathcal{E}$. This proves property (Mo).

Suppose $\sqcup \varepsilon[x] \geq_{\mathcal{D}} \varepsilon(e)$. Then $\sqcup \pi[x] = \pi(\sqcup \varepsilon[x]) \geq_{\mathcal{E}} \pi(\varepsilon(e)) = e$. This proves property (bMo).

Assume $\sqcup x \geq_{\mathcal{E}} e$. Then $\sqcup \varepsilon[x] = \varepsilon(\sqcup x) \geq_{\mathcal{D}} \varepsilon(e)$. This proves property (fMo).

Finally $\mathbf{G}(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}) = \mathrm{id}_{\mathcal{D}^-}$ and $\mathbf{G}((\varepsilon', \pi') \circ (\varepsilon, \pi)) = \mathbf{G}(\varepsilon', \pi') \circ \mathbf{G}(\varepsilon, \pi)$. $\qquad\square$

Therefore an ep-pair $(\varepsilon, \pi) : \mathcal{E} \to \mathcal{D}$ yields a bf-morphism from $\mathcal{E}^-$ to $\mathcal{D}^-$.

Theorem 3.1.5. *For any ep-pair (ε, π) in **Sd** we have $(\mathbf{F} \circ \mathbf{G})(\varepsilon, \pi) = (\varepsilon^{-+}, \pi^{-+})$.*

Proof. Let $(\varepsilon, \pi) : \mathcal{E} \to \mathcal{D}$ be an ep-pair and let $\psi = \varepsilon{\upharpoonright}_{\mathcal{K}(\mathcal{E})}$. Then it is not difficult to check that $\psi_1 = \varepsilon^{-+}$ and $\psi_2 = \pi^{-+}$. $\qquad\square$

3.2 i-webs and i-models

In this section we apply our dovetailed study of morphisms of information systems: in particular we will use the fundamental fact that b-morphisms generate ep-pairs. We will obtain an axiomatization of a class of structures which is easily recognizable as a common theory for various existing definitions of *webbed models* like Krivine models, filter models and graph models (see [15] for a survey).

After having introduced these structures and having described how λ-terms are interpreted into them, we will show that this class is general enough to include, up to isomorphism, all coadditive reflexive Scott domains (see Definition 3.2.3).

The following definitions give the axiomatization of the classes of structures we are principally interested in: *i-webs* and *i-models*.

Definition 3.2.1. *An* i-web *is a pair* $\mathbf{A} = (\mathcal{A}, \phi_A)$ *where*

- $\mathcal{A}$ *is an information system,*

- $\phi_A : \mathcal{A} \Rightarrow \mathcal{A} \to \mathcal{A}$ *is a b-morphism.*

We denote by iW the class of all i-webs.

Definition 3.2.2. *Let* $\mathbf{A}$ *be an i-web. The* i-model *generated by* $\mathbf{A}$ *is the triple* $\mathbf{A}^+ = (\mathcal{A}^+, (\phi_A)_1, (\phi_A)_2)$. *We denote by* iM *the class of all i-models.*

When the i-web $\mathbf{A}$ is clear from the context, we may drop the subscript from ϕ_A. The next theorem justifies the name "i-model".

Theorem 3.2.1. *If* $\mathbf{A}$ *is an i-web, then* $\mathbf{A}^+$ *is a* RSD, *and hence a model of lambda calculus.*

Proof. Immediate, since by Theorem 3.1.5 $((\phi_A)_1, (\phi_A)_2) : \mathcal{A}^+ \Rightarrow \mathcal{A}^+ \to \mathcal{A}^+$ is an ep-pair in **Sd**, and hence $\mathbf{A}^+$ is a reflexive Scott domain. $\square$

From the proof of Theorem 3.2.1 it is evident that $\mathbf{A}^+$ is something more than a RSD: it has an additional property, called *coadditivity* (see [38]).

Definition 3.2.3. *A* RSD $\mathbf{D} = (\mathcal{D}, \mathrm{Ap}, \mathrm{Lam})$ *is* coadditive *if* $(\mathrm{Lam}, \mathrm{Ap}) : \mathcal{D} \Rightarrow \mathcal{D} \to \mathcal{D}$ *is an ep-pair.*

Theorem 3.2.1 says that i-model is a coadditive RSD. The next completes the picture: from the point of view of the equational and order theories, the class iM is equivalent to the class of all coadditive RSDs.

Theorem 3.2.2. *For every coadditive* RSD $\mathbf{D}$*, there is an i-model with the same equational and order theories of* $\mathbf{D}$*.*

Proof. Let $\mathbf{D}$ be a coadditive RSD. Since $\mathbf{D}$ is isomorphic to $\mathbf{D}^{-+}$ and $(\ )^{-+}$ is a ccc functor, by Theorem 3.2.2(i),(ii) we have that $Eq(\mathbf{D}) = Eq(\mathbf{D}^{-+})$ and obviously also $Or(\mathbf{D}) = Or(\mathbf{D}^{-+})$. Finally since $(\ )^{-+}$ is continuous on hom-sets we have that $\mathbf{D}^{-+}$ is a coadditive RSD and by Theorem 3.1.5 $\mathbf{D}^{-+}$ is an i-model. $\square$

Comment. This theorem is fundamental for our subsequent results: it says that in order to study the equational incompleteness of the class of coadditive RSDs, we can use the class iM.

Notation. Throughout this section when speaking of an i-web $\mathbf{A}$ we will make lowercase greek letters $\alpha, \beta, \gamma, \ldots$ range over A, roman letters $a, b, c, \ldots$ range over Con_A, and capital roman letters $X, Y, Z, \ldots$ range over $\mathrm{Con}_{A \Rightarrow A}$. A finite sequence of consistent sets is denoted $\bar{a} = (a_1, \ldots, a_n)$.

We now give some examples of i-webs and i-models.

Example 3.2.1 (**Graph Models**). *A* total pair *([15]) is a set A together with an injection $i_A : \mathcal{P}_\mathrm{f}(A) \times A \to A$ and a* graph model *generated by the pair (A, i_A) is then obtained by taking the powerset of A (see [15, Def. 120]). Recall from Example 3.1.1 that a set A yields an information system $\mathcal{A}$. The total pair (A, i_A) then yields an i-web $\mathbf{A} = (\mathcal{A}, i_A)$. The graph model generated by (A, i_A) corresponds exactly to the i-model $\mathbf{A}^+$. The injectivity of i_A corresponds to the requirement of being a b-morphism.*

Example 3.2.2 (**Preordered Coherent Models**). *Recall from Example 3.1.2 how a pc-set A determines an information system $\mathcal{A}$. A* pc-web *(see [15, Def. 153]) is determined by a pc-set together with a map $\phi : \mathcal{P}_\mathrm{f}^\mathrm{coh}(A) \times A \to A$ satisfying:*

(1) $\phi(a, \alpha) \eqcirc \phi(b, \beta)$ iff $(a \cup b \in \mathcal{P}_\mathrm{f}^\mathrm{coh}(A) \Rightarrow \alpha \eqcirc \beta)$

(2) if $\phi(a, \alpha) \leq \phi(b, \beta)$, then $\alpha \leq \beta$ and $(\forall \gamma \in b \; \exists \delta \in a. \gamma \leq \delta)$.

A pc-web is a particular instance of i-web and properties (1),(2) say exactly that ϕ is a b-morphism. Krivine models [15, Sec. 5.6.2] are pc-webs in which $\asymp = A \times A$, while graph models [15, Sec. 5.5] are pc-webs in which $\asymp = A \times A$ and $\le$ is the equality.

Example 3.2.3 (**Filter Models**). *Recall from Example 3.1.3 how a pc-set A determines an information system $\mathcal{A}$. Recall from [38, Def. 2.12,Thm. 2.13] that the Filter models living in the Scott semantics are obtained by taking the set of filters of* EATS*s satisfying the following condition:*

(*) $\bigwedge_{i=1}^{n}(\alpha_i \to \beta_i) \le \gamma \to \delta$*, then* $(\bigwedge_{i \in \{i : \gamma \le \alpha_i\}} \beta_i) \le \delta$

In such a case, defining $\phi : \mathcal{P}_{\mathrm{f}}(A) \times A \to A$ by $\phi(a, \alpha) = (\bigwedge a) \to \alpha$ one obtains a b-morphism and hence an i-web $\mathbf{A} = (\mathcal{A}, \phi_A)$ and the corresponding filter model is exactly the i-model $\mathbf{A}^+$.

In the forthcoming Theorem 3.2.3 we describe the interpretation function associated to an i-model. The reader can directly check how such description generalizes the various interpretations in preordered coherent models and filter models.

Notation. We recall a notation, commonly used in the literature, to denote the update of environments. If $\rho : \mathrm{Var} \to \mathcal{A}^+$ and $u \in \mathcal{A}^+$ we define a new environment $\rho[y := u]$ by setting

$$\rho[y := u](x) = \begin{cases} u & \text{if } x = y \\ \rho(x) & \text{otherwise} \end{cases}$$

Theorem 3.2.3. *Let $\mathbf{A} = (\mathcal{A}, \phi_A)$ be an i-web. Then the interpretation function $[\![\_]\!]_{\_}^{\mathbf{A}} : \Lambda \times \mathrm{Env}_{\mathcal{A}^+} \to \mathcal{A}^+$ associated to the λ-model $\mathbf{A}^+$ is as follows:*

- $[\![x_i]\!]_{\rho}^{\mathbf{A}} = \rho(x)$

- $[\![\lambda y.M]\!]_{\rho}^{\mathbf{A}} = \{\phi_A(a, \alpha) : \alpha \in [\![M]\!]_{\rho[y := a\downarrow]}^{\mathbf{A}}\} \downarrow_A$

- $[\![MN]\!]_{\rho}^{\mathbf{A}} = \{\beta : \exists a \subseteq_{\mathrm{f}} [\![N]\!]_{\rho}^{\mathbf{A}}. \ (a, \beta) \in \{(a', \beta') : \phi_A(a', \beta') \in [\![M]\!]_{\rho}^{\mathbf{A}}\} \downarrow_{A \Rightarrow A}\}$

Proof. This interpretation is obtained instantiating the interpretation function associated to a RSD, as described for example in Barendregt's book [10, Ch. 5] to the case in which the retraction is given by the functions $(\phi_A)_1(y) = \phi_A[y] \downarrow_A$ and $(\phi_A)_2(x) = \phi_A^{-1}[x] \downarrow_{A \Rightarrow A}$. $\qquad\square$

Interpretation in an i-web $\mathbf{A}$ can also be expressed by means of a "typing system" with judgements of the form $\Gamma \rhd^{\mathbf{A}} M : \alpha$, where

- α is an element of A,

- Γ is a *context* of the form $x_1 : a_1, \ldots, x_n : a_n$ with $\bar{x}$ adequate and $a_i \in \mathrm{Con}_A$, for each $i \in [1, n]$. We shall also abbreviate $x_1 : a_1, \ldots, x_n : a_n$ with the more compact notation $\bar{x} : \bar{a}$.

Notation. Whenever there is no danger of confusion, we drop the reference to the i-web $\mathbf{A}$, writing simply $\Gamma \triangleright M : \alpha$. Moreover for $a \subseteq_{\mathrm{f}} A$ we will write $\Gamma \triangleright M : a$ as a shorthand for $\forall \alpha \in a.\ \Gamma \triangleright M : \alpha$.

$$\frac{a_i \vdash_A \alpha}{\bar{x} : \bar{a} \triangleright x_i : \alpha}\ [\mathsf{var}]$$

$$\frac{X \in \mathrm{Con}_{A \Rightarrow A} \qquad \phi_A[X] \vdash_A \beta \qquad \overset{(\forall (a, \alpha) \in X)}{\Gamma, y : a \triangleright M : \alpha}}{\Gamma \triangleright \lambda y.M : \beta}\ [\mathsf{abs}]$$

$$\frac{X \in \mathrm{Con}_{A \Rightarrow A} \quad X \vdash_{A \Rightarrow A} (a, \beta) \qquad \Gamma \triangleright M : \phi_A[X] \qquad \Gamma \triangleright N : a}{\Gamma \triangleright MN : \beta}\ [\mathsf{app}]$$

Recall now also the categorical interpretation $|\_|_{\bar{x}}^{\mathbf{A}}$ given in Definition 2.4.2, §2. Recall also that the elements (if any) $(\bar{a}, \alpha) \in |M|_{\bar{x}}^{\mathbf{A}}$ are pairs in $\mathrm{Con}_A^n \times A$, where n is the length of $\bar{x}$ (in view of the isomorphism $\mathrm{Con}_{A \& A} \cong \mathrm{Con}_A \times \mathrm{Con}_A$ that we consider in §2). The connection between the different descriptions of the interpretation map is the following.

Proposition 3.2.4. *For any λ-term M, any adequate $\bar{x} \in \mathrm{Var}^n$ and any $\bar{a} \in \mathrm{Con}^n$*

$$(\bar{a}, \alpha) \in |M|_{\bar{x}}^{\mathbf{A}} \quad \Leftrightarrow \quad \bar{x} : \bar{a} \triangleright^{\mathbf{A}} M : \alpha \quad \Leftrightarrow \quad \alpha \in [\![M]\!]_{\rho[\bar{x} := \bar{a} \downarrow]}^{\mathbf{A}}$$

where $\bar{a} \downarrow$ is a shorthand for $a_1 \downarrow_A, \ldots, a_n \downarrow_A$.

We now explicitly describe the λ-model structure induced on $\mathbf{A}^+$ by an i-web $\mathbf{A}$ (see Definition 2.4.4). The universe of the λ-model is the set of points of $\mathcal{A}$, ordered by inclusion, and the basic combinators are $|\mathbf{K}|^{\mathbf{A}}$ and $|\mathbf{S}|^{\mathbf{A}}$, respectively. Finally the application operation is given by

$$u \cdot v = \{\beta \in A : \exists a \subseteq_{\mathrm{f}} v.\ (a, \beta) \in \{(a', \beta') : \phi_A(a', \beta') \in u\} \downarrow_{A \Rightarrow A}\}$$

for all points u, v.

3.3 Partial i-webs

It will be of crucial importance for us to consider information systems which are "almost" i-webs. The first step is to generalize Definition 3.1.1 in order to take into account partial functions.

Definition 3.3.1. *Let $\mathcal{A}, \mathcal{B}$ be $\mathsf{IS}s$. A* partial morphism *(resp.* partial b-morphism, partial f-morphism) *from $\mathcal{A}$ to $\mathcal{B}$ is a partial function $\psi : A \hookrightarrow B$ that, on its domain, behaves like a morphism (resp. b-morphism, f-morphism).*

Definition 3.3.2. *A* partial i-web *is a pair* $\mathbf{S} = (\mathcal{S}, \phi_S)$ *where*

- $\mathcal{S}$ *is an information system,*

- $\phi_S : \mathcal{S} \Rightarrow \mathcal{S} \hookrightarrow \mathcal{S}$ *is a partial b-morphism.*

Terminology. It is evident that an i-web is also a partial i-web. We will say that $\mathbf{S}$ is a *proper* partial i-web if ϕ_S is not a total function.

Remark 3.3.1. *Any finite partial i-web is a proper partial i-web.*

Example 3.3.1. *A* partial pair *([15]) is a set A together with a partial injection* $i_A : \mathcal{P}_{\mathrm{f}}(A) \times A \to A$. *Any partial pair is a partial i-web.*

Given a partial i-web $\mathbf{S}$ it still makes sense to consider a "typing system" with judgements of the form $\Gamma \rhd^{\mathbf{S}} M : \alpha$, exactly as we did for i-webs. Clearly such typing system does not correspond to the interpretation function of a model, but it will still be useful in the subsequent proofs.

In our case a partial i-web $\mathbf{S}$ is obtained by "taking a piece" of a i-web $\mathbf{A}$. In what follows we describe this operation, that we call *restriction*.

Definition 3.3.3. *Let $\mathcal{A}$ be an* IS *and let $S \subseteq A$. Then we define an information system* $\mathcal{A}{\upharpoonright}_S = (S, \mathrm{Con}_S, \vdash_S)$, *where*

- $\mathrm{Con}_S = \{a \in \mathrm{Con}_A : a \subseteq S\}$

- $\vdash_S = \{(a, \alpha) \in S \Rightarrow S : a \vdash_A \alpha\}$

Definition 3.3.4. *Let $\mathcal{A}, \mathcal{S}$ be information systems. We say that $\mathcal{S}$ is a* restriction *of $\mathcal{A}$ (and $\mathcal{A}$ is an* extension *of $\mathcal{S}$), notation $\mathcal{S} \preceq \mathcal{A}$, if $\mathcal{S} = \mathcal{A}{\upharpoonright}_S$.*

Notation. We will write $\mathcal{S} \prec \mathcal{A}$ if $\mathcal{S} \preceq \mathcal{A}$ and $\mathcal{S} \neq \mathcal{A}$. Note that since consistency and entailment in $\mathcal{S}$ are univocally determined by the set S, then we have that $\mathcal{S} \prec \mathcal{A} \Leftrightarrow \mathcal{S} \preceq \mathcal{A}, \ S \subset A$.

It is useful to note that the relation $\preceq$ behave nicely w.r.t. formation of exponential objects.

Lemma 3.3.2. *If $\mathcal{S} \preceq \mathcal{A}$, then $\mathcal{S} \Rightarrow \mathcal{S} \preceq \mathcal{A} \Rightarrow \mathcal{A}$.*

In Definition 3.3.3 we saw how to create smaller information systems from existing ones. A fundamental step for our results will be to take pieces of i-webs: however in general this operation does not yield a smaller i-web, but rather a smaller partial i-web. We define in general the operation of taking a piece of a partial i-web (and hence, in particular, of an i-web) relying on Definition 3.3.3.

Definition 3.3.5. *Let $\mathbf{S} = (\mathcal{S}, \phi_S)$ be a partial i-web. and let $B \subseteq S$. Then we define the partial i-web $\mathbf{S}{\upharpoonright}_B = (\mathcal{S}{\upharpoonright}_B, \phi_S{\upharpoonright}_B)$, where $\phi_S{\upharpoonright}_B = \phi_S \cap ((B \Rightarrow B) \times B)$.*

We also extend the relations given in Definition 3.3.4 to the case of partial i-webs.

Definition 3.3.6. *Let* $\mathbf{S}, \mathbf{B}$ *be partial i-webs. We say that* $\mathbf{B}$ *is a* restriction *of* $\mathbf{S}$ *(and* $\mathbf{S}$ *is an* extension *of* $\mathbf{B}$*), notation* $\mathbf{B} \preceq \mathbf{S}$*, if* $\mathbf{B} = \mathbf{S}{\upharpoonright}_B$.

Notation. We will write $\mathbf{B} \prec \mathbf{S}$ if $\mathbf{B} \preceq \mathbf{S}$ and $\mathbf{B} \neq \mathbf{S}$. Once again all data about $\mathbf{B}$ are univocally determined by the set B, and hence we have that $\mathbf{B} \prec \mathbf{S} \Leftrightarrow \mathbf{B} \preceq \mathbf{S}$, $B \subset S$.

3.3.1 Extensions of partial i-webs

We saw in Theorem 3.2.1 that i-webs give rise to models of the lambda calculus. This is not the case for partial i-webs: however it is possible to construct an i-web starting from a partial one by *completing* it. This idea has its origins with graph models: the completion method for building graph models from partial pairs dates back to Longo's presentation [79] of Plotkin's and Scott's graph model $\mathcal{P}_\omega$. Later on completion methods were developed and used on a wide scale by Kerth [71, 70].

In this subsection $\mathbf{S} = (\mathcal{S}, \phi)$ is a partial i-web, where $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ (we omit subscripts to ease readability).

Notation. We indicate by $\mathsf{do}(\phi)$ the domain of ϕ and we define $\overline{\mathsf{do}}(\phi) = (S \Rightarrow S) - \mathsf{do}(\phi)$.

Our goal is to show the existence of a specific kind of extensions of $\mathbf{S}$, obtained by adding to S all elements of $\overline{\mathsf{do}}(\phi)$ and extending in a particular way the function ϕ and the entailment $\vdash$. The only parameter that we let vary among these extensions will be the consistency predicate.

Definition 3.3.7. *A partial i-web* $\mathbf{S}^* = (\mathcal{S}^*, \phi^*)$ *is a* free extension *of* $\mathbf{S}$*, (where* $\mathcal{S}^* = (S^*, \mathrm{Con}^*, \vdash^*)$*) if the following conditions are satisfied:*

(E1) $S^* = S \cup \overline{\mathsf{do}}(\phi)$

(E2) $\mathrm{Con} \subseteq \mathrm{Con}^* \subseteq \{x \in \mathcal{P}_{\mathrm{f}}(S^*) : x \cap S \in \mathrm{Con}, \ x - S \in \mathrm{Con}_{S \Rightarrow S}\}$

(E3) $a \vdash^* \alpha$ *iff either* $a \cap S \vdash \alpha$ *or* $\alpha \in a$

(E4) $\phi^*(a, \alpha) = \begin{cases} \phi(a, \alpha) & \text{if } (a, \alpha) \in \mathsf{do}(\phi) \\ (a, \alpha) & \text{if } (a, \alpha) \in \overline{\mathsf{do}}(\phi) \end{cases}$

So far we have not yet given any reason to believe that any partial i-web $\mathbf{S}$ admits a free extension. We will do that in the forthcoming Theorem 3.3.5, ensuring that Definition 3.3.7 is never "vacuous". We stress that free extensions, despite the connotation of the term *free* in some fields of mathematics, need not to be unique. However there is still a motivation behind this terminology: a free extension, if it exists, is univocally determined by the predicate Con^* and the rest of the structure is constructed "algorithmically" in function of Con^* (this point will be important for the matters of §4).

Remark 3.3.3. *Note that if* $\mathbf{S}^*$ *is a free extension of* $\mathbf{S}$ *according to Definition 3.3.7, then* $\mathbf{S} \preceq \mathbf{S}^*$ *(see Definition 3.3.6) and* $\boldsymbol{do}(\phi^*) = \mathcal{S} \Rightarrow \mathcal{S}$ *(this makes sense since* $\mathcal{S} \Rightarrow \mathcal{S} \preceq \mathcal{S}^* \Rightarrow \mathcal{S}^*$ *by Lemma 3.3.2). Moreover if* $\mathbf{S}$ *is an i-web, then* $\mathbf{S}$ *is a free extension of itself, whilst if* $\mathbf{S}$ *is a proper partial i-web, then all its free extensions* $\mathbf{S}^*$ *are such that* $\mathbf{S} \prec \mathbf{S}^*$.

If $\mathbf{S}$ is a proper partial i-web, then the consistency predicate Con^* has to contain new sets, consisting of old and new tokens (i.e. pairs) mixed together, in order for ϕ^* to be a b-morphism. There is a certain degree of freedom in adding new consistent sets, but there are constraints imposed by the fact that we want $\mathcal{S}^*$ to be an information system. The following definition characterizes the predicates that can be taken as defining consistency in a free extension of $\mathbf{S}$.

Definition 3.3.8. *Let* $\mathbf{S}$ *be a partial i-web. A family* $\mathcal{X} \subseteq \mathcal{P}_{\mathrm{f}}(S \cup \overline{\boldsymbol{do}}(\phi))$ *is called* $\mathbf{S}$-*compatible if for all* $y \subseteq_{\mathrm{f}} S \cup \overline{\boldsymbol{do}}(\phi)$, *for all* $X \subseteq_{\mathrm{f}} S \Rightarrow S$, *and for all* $x \in \mathcal{X}$ *we have*

(C1) if $y \in \mathrm{Con}$, *then* $y \in \mathcal{X}$

(C2) $x \cap S \in \mathrm{Con}$ *and* $x - S \in \mathrm{Con}_{S \Rightarrow S}$

(C3) if $y \subseteq_{\mathrm{f}} x \cup (x \cap S){\downarrow}_S$, *then* $y \in \mathcal{X}$

(C4) if $X \in \mathrm{Con}_{S \Rightarrow S}$, *then* $(X \cap \overline{\boldsymbol{do}}(\phi)) \cup (\phi[X \cap \boldsymbol{do}(\phi)]) \in \mathcal{X}$

(C5) if $(X \cap \overline{\boldsymbol{do}}(\phi)) \cup \phi[X \cap \boldsymbol{do}(\phi)] \in \mathcal{X}$, *then* $X \in \mathrm{Con}_{S \Rightarrow S}$

Comment. We briefly discuss Definition 3.3.8 in order to motivate it. An $\mathbf{S}$-compatible family is a candidate to be the Con^* of Definition 3.3.7. Property (C3) says that the family $\mathcal{X}$ must be closed w.r.t. entailment in $\mathcal{S}^*$. Property (C4),(C5) ensure that ϕ^* is a morphism. All images of sets that are consistent in $\mathcal{S} \Rightarrow \mathcal{S}$ must be consistent in $\mathcal{S}^*$. All anti-images of sets that are consistent in $\mathcal{S}^*$ must be consistent in $\mathcal{S} \Rightarrow \mathcal{S}$: one has to be careful not add to many consistent sets, since the predicate $\mathrm{Con}_{S \Rightarrow S}$ can't be modified in the construction of $\mathcal{S}^*$.

Theorem 3.3.4. *A partial i-web* $\mathbf{S}$ *has a free extension iff there exists an* $\mathbf{S}$-*compatible family.*

Proof. Let $\mathcal{X}$ be an $\mathbf{S}$-compatible family. Set $\mathrm{Con}^* := \mathcal{X}$ and set accordingly the structure $\mathbf{S}^* = (\mathcal{S}^*, \phi^*)$ as specified in Definition 3.3.7. We start by proving that $\mathcal{S}^*$ is an IS, by considering the four properties of Definition 2.6.1, §2.

(I1) Suppose $a \in \mathrm{Con}^*$ and $a \vdash^* b$. Then by property (C3) it follows immediately that $b \in \mathrm{Con}^*$.

(I2) If $\alpha \in a$, then $a \vdash^* \alpha$ by definition of $\vdash^*$.

(I3) Suppose $a \vdash^* \{\alpha_1, \ldots, \alpha_k\}$ and $\{\alpha_1, \ldots, \alpha_k\} \vdash^* \gamma$. If $\gamma \in \{\alpha_1, \ldots, \alpha_k\}$ then clearly $a \vdash^* \gamma$. Otherwise $\{\alpha_1, \ldots, \alpha_k\} \cap S \vdash \gamma$ and since $a \cap S \vdash \{\alpha_1, \ldots, \alpha_k\} \cap S$ we can conclude using the property (I3) of $\mathcal{S}$.

(I4) Immediate, by definition.

By construction $\mathsf{do}(\phi^*) = S \Rightarrow S$, so it suffices to show that ϕ^* is a b-morphism.

(Mo) Immediate by properties (4) and (5).

(bMo) Assume that $\phi^*[X] \vdash^* \phi^*(\alpha)$. There are two cases to be dealt with. If $\alpha \in \mathsf{do}(\phi)$, then $\phi^*[X] \cap S \vdash \phi(\alpha)$ and we derive $\phi[X \cap \mathsf{do}(\phi)] \vdash \phi(\alpha)$ so that by (bMo) for ϕ we have that $X \cap \mathsf{do}(\phi) \vdash_{S \Rightarrow S} \alpha$ and hence $X \vdash_{S \Rightarrow S} \alpha$.

If $\alpha \notin \mathsf{do}(\phi)$, then $\alpha = \phi^*(\alpha) \in \phi^*[X]$, so that $\alpha \in X$ and this concludes the proof.

$\square$

Theorem 3.3.5. *Every partial i-web admits a minimal free extension.*

Proof. The minimal extension is obtained by defining, for x ranging over $\mathcal{P}_{\mathrm{f}}(S^*)$,

$$x \in \mathrm{Con}^* \text{ iff } \exists Y \in \mathrm{Con}_{S \Rightarrow S}. \ x \subseteq_{\mathrm{f}} (Y \cap \overline{\mathsf{do}}(\phi)) \cup \phi[Y \cap \mathsf{do}(\phi)] \!\downarrow_S$$

It is not difficult to check that Con^* is the minimal **S**-compatible family. $\square$

Example 3.3.2. *Let* $\mathbf{S} = (S, i_S)$ *be a partial pair. Then one step of the completion of* $\mathbf{S}$ *in the sense of [15, Def. 104] is the maximal free extension of* $\mathbf{S}$.

Remark 3.3.6. *If* $\mathbf{S}$ *is a finite partial i-web with* $\mathrm{Con} = \mathcal{P}_{\mathrm{f}}(S)$, *then* $\mathcal{P}_{\mathrm{f}}(S^*)$ *is* $\mathbf{S}$-*compatible.*

3.3.2   Completions of partial i-webs

The goal of this subsection is to give a method that for any given partial i-web $\mathbf{S}$ allows to construct an i-web $\mathbf{S}_\omega$ such that $\mathbf{S} \preceq \mathbf{S}_\omega$. The i-web $\mathbf{S}_\omega$ will be the union of a family $\{\mathbf{S}_n\}_{n \geq 0}$ of partial i-webs obtained by constructing each time a free extension (see Definition 3.3.7, Subsection 3.3.1).

Definition 3.3.9. *Let* $\mathbf{S}$ *be a partial i-web and let* $\{\mathbf{S}_n\}_{n \geq 0}$ *be a sequence of partial i-webs such that* $\mathbf{S}_0 = \mathbf{S}$ *and* $\mathbf{S}_{m+1}$ *is a free extension of* $\mathbf{S}_m$, *for each* $m \in \mathbb{N}$. *In such a case the family* $\{\mathbf{S}_n\}_{n \geq 0}$ *is called a* development *of* $\mathbf{S}$.

Note that for each $m \in \mathbb{N}$ we have $\mathbf{S}_m \preceq \mathbf{S}_{m+1}$.

Definition 3.3.10. *Let* **S** *be a partial i-web. The* completion *of* **S** *with respect to a development* $\{\mathbf{S}_n\}_{n \geq 0}$ *of* **S** *is the structure* $\mathbf{S}_\omega = (\mathcal{S}_\omega, \phi_\omega)$, *where*

$$\mathcal{S}_\omega := (\cup_{m<\omega} S_m, \cup_{m<\omega}\mathrm{Con}_m, \cup_{m<\omega} \vdash_m) \ and \ \phi_\omega := \bigcup_{m<\omega} \phi_m$$

Theorem 3.3.7. $\mathbf{S}_\omega$ *is an i-web and* $\mathbf{S} \preceq \mathbf{S}_\omega$.

Proof. The union of the family $\{\mathbf{S}_n\}_{n \geq 0}$ gives an information system, since such family is a $\preceq$-chain. Similarly ϕ_ω, being the union of a sequence of b-morphisms extending one another, is a b-morphism and moreover its domain is $\mathcal{S}_\omega \Rightarrow \mathcal{S}_\omega$. Last we observe that trivially $\mathbf{S} \preceq \mathbf{S}_\omega$. $\qquad\square$

A development $\mathfrak{S} = \{\mathbf{S}_n\}_{n \geq 0}$ of a partial i-web **S** is a $\preceq$-chain. Moreover if **S** is a proper partial i-web, then $\mathfrak{S}$ is a $\prec$-chain.

Terminology. Since $\mathfrak{S}$ by definition contains **S**, we may sometimes speak of *the completion of* $\mathfrak{S}$, rather then the completion of **S** w.r.t. $\mathfrak{S}$.

Example 3.3.3. *Let* $\mathbf{S} = (S, j_S)$ *be a partial pair. Then the completion of* **S** *in the sense of [15] is the smallest solution of the set-theoretic equation* $D = S \cup ((\mathcal{P}_f(D) \times D) - \mathsf{do}(j_S))$, *equipped with the total injection*

$$i(a, \alpha) = \begin{cases} j_S(a, \alpha) & if\ (a, \alpha) \in \mathsf{do}(j_S) \\ (a, \alpha) & if\ (a, \alpha) \in \overline{\mathsf{do}}(j_S) \end{cases}$$

This is an instance of our completion with respect to the development obtained by taking each time the maximal free extension.

Terminology. We say that a (partial) i-web **S** *has full consistency* if $\mathrm{Con}_S = \mathcal{P}_f(S)$.

Remark 3.3.8. *If* **S** *is a finite partial i-web with full consistency, then it admits a development* $\{\mathbf{S}_n\}_{n \geq 0}$ *such that* $\mathbf{S}_n$ *has full consistency, for each* $n \geq 0$, *and a completion* $\mathbf{S}_\omega$ *with full consistency.*

Proof. Apply Remark 3.3.6. $\qquad\square$

3.3.3 Morphisms of partial i-webs

In Section 3.1 we defined algebraic-style morphisms of information systems. We proceed by defining algebraic-style morphisms between partial i-webs, i.e. functions which not only preserve the information system structure, but also commute with the partial b-morphisms involved. A particular instance of our definition are the morphisms of graph models considered by Kerth [71] and Schellinx [102].

The fundamental application of morphisms between i-webs is to establish relations between order and equational theories of the corresponding i-models; in the case of partial i-webs it is possible to infer interesting relations between the i-models generated by the completion process.

Definition 3.3.11. *Let* $\mathbf{B}, \mathbf{C}$ *be partial i-webs. A* morphism *from* $\mathbf{B}$ *to* $\mathbf{C}$ *(resp. f-morphism)is a morphism (resp. f-morphism) ψ of the underlying* IS*s satisfying the following additional property:*

(lMo) if $(a, \beta) \in do(\phi_B)$, then $\vec{\psi}(a, \beta) \in do(\phi_C)$ and $\psi(\phi_B(a, \beta)) = \phi_C(\vec{\psi}(a, \beta))$

We may sometimes say "morphism of partial i-webs", instead of just "morphism from $\mathbf{B}$ to $\mathbf{C}$", to stress its additional property of commuting with the b-morphisms of the involved partial i-webs.

Proposition 3.3.9. *Let* $\mathbf{B}, \mathbf{C}$ *be partial i-webs and let $\psi : \mathbf{B} \to \mathbf{C}$ be a f-morphism of partial i-webs. Then*

$$x_1 : a_1, \ldots, x_n : a_n \rhd^{\mathbf{B}} M : \alpha \quad implies \quad x_1 : \psi[a_1], \ldots, x_n : \psi[a_n] \rhd^{\mathbf{C}} M : \psi(\alpha)$$

Proof. As usual we abbreviate $x_1 : a_1, \ldots, x_n : a_n$ by $\bar{x} : \bar{a}$. Moreover we abbreviate $x_1 : \psi[a_1], \ldots, x_n : \psi[a_n]$ by $\bar{x} : \psi[\bar{a}]$. The proof is by induction on type judgements. If the last rule used is

$$\frac{a_i \vdash_B \alpha}{\bar{x} : \bar{a} \rhd^{\mathbf{B}} x_i : \alpha} \text{ [var]}$$

then by properties (Mo) and (fMo) we have

$$\frac{\psi[a_i] \vdash_C \psi(\alpha)}{\bar{x} : \psi[\bar{a}] \rhd^{\mathbf{C}} x_i : \psi(\alpha)} \text{ [var]}$$

If the last rule used is

$$\frac{X \in \text{Con}_{B \Rightarrow B} \qquad \phi_B[X] \vdash_B \gamma \qquad \overset{(\forall (b, \beta) \in X)}{\bar{x} : \bar{a}, y : b \rhd^{\mathbf{B}} M : \beta}}{\bar{x} : \bar{a} \rhd^{\mathbf{B}} \lambda y.M : \gamma} \text{ [abs]}$$

then by induction hypothesis and property (lMo) we have

$$\frac{\vec{\psi}[X] \in \text{Con}_{C \Rightarrow C} \quad \phi_C[\vec{\psi}[X]] = \vec{\psi}[\phi_B[X]] \vdash_C \psi(\gamma) \quad \overset{(\forall \vec{\psi}(b, \beta) \in \vec{\psi}[X])}{\bar{x} : \psi[\bar{a}], y : \psi[b] \rhd^{\mathbf{C}} M : \psi(\beta)}}{\bar{x} : \psi[\bar{a}] \rhd^{\mathbf{C}} \lambda y.M : \psi(\gamma)} \text{ [abs]}$$

If the last rule used is

$$\frac{X \in \text{Con}_{B \Rightarrow B} \quad X \vdash_{B \Rightarrow B} (b, \beta) \quad \bar{x} : \bar{a} \rhd^{\mathbf{B}} M : \phi_B[X] \quad \bar{x} : \bar{a} \rhd^{\mathbf{B}} N : b}{\bar{x} : \bar{a} \rhd^{\mathbf{B}} MN : \beta} \text{ [app]}$$

Then by induction hypothesis and property (lMo) we have

$$\frac{\begin{array}{c} \vec{\psi}[X] \in \text{Con}_{C \Rightarrow C} \\ \vec{\psi}[X] \vdash_{C \Rightarrow C} \vec{\psi}(b, \beta) \quad \bar{x} : \psi[\bar{a}] \rhd^{\mathbf{C}} M : \psi[\phi_B[X]] = \phi_C[\vec{\psi}[X]] \quad \bar{x} : \psi[\bar{a}] \rhd^{\mathbf{C}} N : \psi[b] \end{array}}{\bar{x} : \psi[\bar{a}] \rhd^{\mathbf{C}} MN : \psi(\beta)} \text{ [app]}$$

$\square$

3.4 Finiteness properties of i-webs

In this section we prove an important "finiteness" property of i-webs. Whenever we are given two closed λ-terms M, N and an element α belonging to an i-web $\mathbf{A}$ such that $\alpha \in |M|^{\mathbf{A}^+} - |N|^{\mathbf{A}^+}$, then there exists a finite partial i-web $\mathbf{S}$ such that $\mathbf{S} \prec \mathbf{A}$ and $\alpha \in |M|^{\mathbf{S}_\omega^+} - |N|^{\mathbf{S}_\omega^+}$, where $\mathbf{S}_\omega$ is a suitable completion of $\mathbf{S}$.

   Notation. We write $\bar{x} : \bar{a} \not\rhd^{\mathbf{A}} M : \alpha$ to mean that the judgement $\bar{x} : \bar{a} \rhd^{\mathbf{A}} M : \alpha$ is not obtainable.

   The precise statement of the main theorem of this section is the following.

Theorem 3.4.1. *Let $\mathbf{A}$ be a i-web and suppose $\bar{x} : \bar{a} \rhd^{\mathbf{A}} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{A}} N : \alpha$. Then there exist a finite partial i-web $\mathbf{S} \prec \mathbf{A}$ and a completion $\mathbf{S}_\omega$ of $\mathbf{S}$ such that*

(i) $\bar{x} : \bar{a} \rhd^{\mathbf{S}} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{S}} N : \alpha$ and

(ii) $\bar{x} : \bar{a} \rhd^{\mathbf{S}_\omega} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{S}_\omega} N : \alpha$

   The proof of Theorem 3.4.1 is divided in three parts:

   (1) construction of $\mathbf{S}$ and proof of item (i) (Lemma 3.4.2),

   (2) construction of $\mathbf{S}_\omega$,

   (3) proof of item (ii) for the $\mathbf{S}_\omega$ constructed at step (2) (Lemma 3.4.7).

Lemma 3.4.2. *Let $\mathbf{A}$ be a i-web and suppose $\bar{x} : \bar{a} \rhd^{\mathbf{A}} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{A}} N : \alpha$. Then there exists a finite partial i-web $\mathbf{S} \prec \mathbf{A}$ such that $\bar{x} : \bar{a} \rhd^{\mathbf{S}} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{S}} N : \alpha$.*

Proof. Suppose $\bar{x} : \bar{a} \rhd^{\mathbf{A}} M : \alpha$. We show how to construct, by induction on this judgement, a finite set $S_{(\bar{a},\alpha)}$ such that $\bar{x} : \bar{a} \rhd^{\mathbf{A} \restriction S_{(\bar{a},\alpha)}} M : \alpha$.

   If the last rule used is

$$\frac{a_i \vdash_B \alpha}{\bar{x} : \bar{a} \rhd^{\mathbf{A}} x_i : \alpha} \; [\mathsf{var}]$$

then we set $S_{(\bar{a},\alpha)} = a_i \cup \{\alpha\}$. This indeed gives $\bar{x} : \bar{a} \rhd^{\mathbf{A} \restriction S_{(\bar{a},\alpha)}} x_i : \alpha$.

   If the last rule used is

$$\frac{X \in \mathrm{Con}_{A \Rightarrow A} \qquad \phi_A[X] \vdash_A \alpha \qquad \overset{(\forall (b,\beta) \in X)}{\bar{x} : \bar{a}, y : b \rhd^{\mathbf{A}} P : \beta}}{\bar{x} : \bar{a} \rhd^{\mathbf{A}} \lambda y.P : \alpha} \; [\mathsf{abs}]$$

then by induction hypothesis we have already constructed $S_{((\bar{a},b),\beta)}$, for all pairs $(b,\beta) \in X$. We finally set $S_{(\bar{a},\alpha)} = \{\alpha\} \cup (\cup_{(b,\beta \in X)} S_{((\bar{a},b),\beta)})$ and we obtain $\bar{x} : \bar{a} \rhd^{\mathbf{A} \restriction S_{(\bar{a},\alpha)}} \lambda y.P : \alpha$.

If the last rule used is

$$\frac{X \in \mathrm{Con}_{A \Rightarrow A} \quad X \vdash_{A \Rightarrow A} (b, \alpha) \qquad \bar{x} : \bar{a} \rhd^{\mathbf{A}} P : \phi_A[X] \qquad \bar{x} : \bar{a} \rhd^{\mathbf{A}} Q : b}{\bar{x} : \bar{a} \rhd^{\mathbf{A}} PQ : \alpha} \, [\mathsf{app}]$$

then by induction hypothesis we have already constructed $S_{(\bar{a}, \gamma)}$, for all $\gamma \in b$ and $S_{(\bar{a}, \phi_B(d, \delta))}$, for all $(d, \delta) \in X$. We finally set $S_{(\bar{a}, \alpha)} = \{\alpha\} \cup (\cup_{\gamma \in b} S_{(\bar{a}, \gamma)}) \cup (\cup_{(d, \delta) \in X} S_{(\bar{a}, \phi_B(d, \delta))})$ and we obtain $\bar{x} : \bar{a} \rhd^{\mathbf{A} \upharpoonright S_{(\bar{a}, \alpha)}} PQ : \alpha$.

Finally let $\mathbf{S} = \mathbf{A} \upharpoonright_{S_{(\bar{a}, \alpha)}}$. Since $\mathbf{S}$ is a restriction of $\mathbf{A}$, (see Definition 3.3.5) we have $\mathbf{S} \preceq \mathbf{A}$ and since $\mathbf{S}$ is partial and finite we have $\mathbf{S} \prec \mathbf{A}$. $\qquad \square$

Construction of the development

From now on $\mathbf{A}$ is a given i-web and $\alpha \in A$, $\bar{a} \in \mathrm{Con}_A^n$, $M, N \in \Lambda$ are such that $\bar{x} : \bar{a} \rhd^{\mathbf{A}} M : \alpha$ and $\bar{x} : \bar{a} \not\rhd^{\mathbf{A}} N : \alpha$ and $\mathbf{S}$ is the partial i-web constructed in Lemma 3.4.2.

We construct by induction on $n \in \mathbb{N}$ a development $\{\mathbf{S}_n\}_{n \geq 0}$ of $\mathbf{S}$ together with a family $\{\psi_n : \mathbf{S}_n \to \mathbf{A}\}_{n \geq 0}$ of f-morphisms of i-webs.

Stage 0

We set $\mathbf{S}_0 := \mathbf{S}$ and $\psi_0(\alpha) = \alpha$, for all $\alpha \in S_0$.

It suffices to observe that the inclusion of S_0 in A is trivially an f-morphisms of partial i-webs from $\mathbf{S}_0$ to $\mathbf{A}$.

Stage $n + 1$

Now suppose that the partial i-web $\mathbf{S}_n$ has been defined, together with the f-morphism $\psi_n : \mathbf{S}_n \to \mathbf{A}$.

The forthcoming definition gives the family of subsets of S_{n+1} that will be used to construct a free extension of $\mathbf{S}_n$.

Definition 3.4.1. *We set $\mathcal{X}_n$ as the family of all sets $x \subseteq_f S_n \cup \overline{do}(\phi_n)$ such that either*

(1) there exists $a \in \mathrm{Con}_n$ and $X \in \mathrm{Con}_{S_n \Rightarrow S_n}$ such that $X \subseteq \overline{do}(\phi_n)$ and $\psi_n[a] \cup \phi_A[\vec{\psi}_n[X]] \in \mathrm{Con}_A$ and $x = a \cup X$ or

(2) there exists $X \in \mathrm{Con}_{S_n \Rightarrow S_n}$ such that $x \subseteq_f (X \cap \overline{do}(\phi_n)) \cup \phi_n[X \cap do(\phi_n)]) \downarrow_{S_n}$.

Lemma 3.4.3. *$\mathcal{X}_n$ is $\mathbf{S}_n$-compatible.*

Proof. We prove the properties defining a $\mathbf{S}_n$-compatible family (see Definition 3.3.8).

(C1) All elements of Con_n can be obtained by choosing $X = \emptyset$ in clause (1).

(C2) Immediate for the sets added by clause (1). Regarding those added by clause (2) it suffices to observe that $\phi[Y \cap do(\phi_n)]) \downarrow_{S_n}$ is finitely consistent in S_n.

(C3) Let $x \in \mathcal{X}_n$ and let $y \subseteq_f x \cup (x \cap S_n)\!\downarrow_{S_n}$.

If x is added by clause (1), then $y = b \cup X$, for some $b \in \text{Con}_n$ such that $a \vdash_n b$. Since ψ_n is a morphism, we have $\psi_n[b] \cup \phi_A[\vec{\psi_n}[X]] \in \text{Con}_A$ and hence $y \in \mathcal{X}_n$ again by clause (1).

If x is added by clause (2), then evidently also y is added to $\mathcal{X}_n$ by clause (2).

(C4) Let $X \in \text{Con}_{S_n \Rightarrow S_n}$. Then $(X \cap \overline{\text{do}}(\phi_n)) \cup (\phi_n[X \cap \text{do}(\phi_n)])$ is in $\mathcal{X}_n$ by clause (2).

(C5) Let $x = (X \cap \overline{\text{do}}(\phi_n)) \cup \phi_n[X \cap \text{do}(\phi_n)] \in \mathcal{X}_n$.

If x is added by clause (1), then $\psi_n[\phi_n[X \cap \text{do}(\phi_n)]] \cup \phi_A[\vec{\psi_n}[X \cap \overline{\text{do}}(\phi_n)]] \in \text{Con}_A$. Since ψ_n is a morphism of partial i-webs, we have $\phi_A[\vec{\psi_n}[X \cap \text{do}(\phi_n)]] \cup \phi_A[\vec{\psi_n}[X \cap \overline{\text{do}}(\phi_n)]] = \phi_A[\vec{\psi_n}[X]] \in \text{Con}_A$. Since $\phi_A \circ \vec{\psi_n}$ is a morphism, we can conclude that $X \in \text{Con}_{S_n \Rightarrow S_n}$.

If x is added by clause (2), then evidently $X \in \text{Con}_{S_n \Rightarrow S_n}$.

$\square$

Definition 3.4.2. *We define $\mathbf{S}_{n+1}$ as the free extension of $\mathbf{S}_n$ obtained by setting $\text{Con}_{n+1} := \mathcal{X}_n$ (recall from Definition 3.3.7 that it is the only parameter of the construction).*

Definition 3.4.3. *We define the function $\psi_{n+1} : S_{n+1} \to A$ as follows:*

$$\psi_{n+1}(\alpha) = \begin{cases} \psi_n(\alpha) & \text{if } \alpha \in S_n \\ \phi_A(\vec{\psi_n}(b, \beta)) & \text{if } \alpha = (b, \beta) \in S_{n+1} - S_n \end{cases}$$

Note that ψ_{n+1} is a well-given total function extending ψ_n.

Lemma 3.4.4. *The function $\psi_{n+1} : \mathbf{S}_{n+1} \to \mathbf{A}$ is an f-morphism of partial i-webs.*

Proof. We first prove the properties defining a f-morphism.

(Mo) ($\Rightarrow$) Suppose $x \in \text{Con}_{n+1}$. We consider the two cases of Definition 3.4.1.

If x is added by clause (1), i.e. $x = a \cup X$ for suitable a and X, then $\psi_{n+1}[x] = \psi_n[a] \cup \phi_A[\vec{\psi_n}[X]] \in \text{Con}_A$, by clause (1) itself.

If x is added by clause (2), then $x \subseteq_f (X \cap \overline{\text{do}}(\phi_n)) \cup (\phi_n[X \cap \text{do}(\phi_n)])\!\downarrow_{S_n}$, for some $X \in \text{Con}_{S_n \Rightarrow S_n}$. Now let $y = (X \cap \overline{\text{do}}(\phi_n)) \cup \phi_n[X \cap \text{do}(\phi_n)]$. We first observe that

$$\begin{aligned} \psi_{n+1}[y] &= \phi_A[\vec{\psi_n}[X \cap \overline{\text{do}}(\phi_n)]] \cup \psi_n[\phi_n[X \cap \text{do}(\phi_n)]] \\ &= \phi_A[\vec{\psi_n}[X \cap \overline{\text{do}}(\phi_n)]] \cup \phi_A[\vec{\psi_n}[X \cap \text{do}(\phi_n)]] \\ &\quad \text{since } \psi_n \text{ is a morphism of partial i-webs} \\ &= \phi_A[\vec{\psi_n}[X]] \end{aligned}$$

This proves that $\psi_{n+1}[y] \in \mathrm{Con}_A$. Now using property (fMo) ψ_n we obtain that $\psi_{n+1}[y] \vdash_A \psi_{n+1}[x]$, and hence $\psi_{n+1}[x] \in \mathrm{Con}_A$.

($\Leftarrow$) By the very definition of Con_{n+1}, in particular by the clause (1) of the definition of $\mathcal{X}_n$.

(fMo) Trivial.

We conclude proving property (lMo) for ψ_{n+1}. Let $(a, \alpha) \in S_n \Rightarrow S_n$. Then $\psi_{n+1}(\phi_{n+1}(a, \alpha)) = \phi_A(\vec{\psi}_n(a, \alpha)) = \phi_A(\vec{\psi}_{n+1}(a, \alpha))$, by definition of ψ_{n+1} and the fact that it extends ψ_n. $\qquad\square$

Remark 3.4.5. *The sequence $\{\mathbf{S}_n\}_{n \geq 0}$ whose i-th element is given by Definition 3.4.2 is a development of $\mathbf{S}$.*

Definition 3.4.4. *We let $\mathbf{S}_\omega$ be the completion of $\mathbf{S}$ with respect to the development $\{\mathbf{S}_n\}_{n \geq 0}$ (recall Definition 3.3.10). We set the function $\psi_\omega : S_\omega \to A$ as $\psi_\omega = \cup_{n < \omega} \psi_n$.*

Note that the definition of ψ_ω is well-given, since each f-morphism ψ_{n+1} of partial i-webs extends ψ_n.

Lemma 3.4.6. *The map $\psi_\omega : \mathbf{S}_\omega \to \mathbf{A}$ is an f-morphism of i-webs.*

Proof. Easy. $\qquad\square$

In the following theorem $\mathbf{S}_\omega$ and ψ_ω are the i-web and the f-morphism of Definition 3.4.4. It is important to recall that $\bar{a} \in \mathrm{Con}_{S_0}^n$ and $\alpha \in S_0$ (recall that S_0 is produced by Lemma 3.4.2).

Lemma 3.4.7. $\bar{x} : \bar{a} \rhd^{\mathbf{S}_\omega} M : \alpha$ *and* $\bar{x} : \bar{a} \not\rhd^{\mathbf{S}_\omega} N : \alpha$.

Proof. The fact that $\bar{x} : \bar{a} \rhd^{\mathbf{S}_\omega} M : \alpha$ is an immediate consequence of Lemma 3.4.2. Now to conclude we need to show that $\bar{x} : \bar{a} \not\rhd^{\mathbf{S}_\omega} N : \alpha$. Suppose by the way of contradiction, that $\bar{x} : \bar{a} \rhd^{\mathbf{S}_\omega} N : \alpha$. Now by Lemma 3.4.6 and Proposition 3.3.9 we can infer $\bar{x} : \psi_\omega[\bar{a}] \rhd^{\mathbf{A}} N : \psi_\omega(\alpha)$ and hence in fact $\bar{x} : \bar{a} \rhd^{\mathbf{A}} N : \alpha$, since $\psi_\omega[\bar{a}] = \psi_0[\bar{a}] = \bar{a}$ and $\psi_\omega(\alpha) = \psi_0(\alpha) = \alpha$. Contradiction. $\qquad\square$

Comment. Lemma 3.4.2 and 3.4.7 together prove Theorem 3.4.1. This latter theorem is extremely important, since it says that every inequality that fails in some i-model also fails in an i-model generated by the completion of a finite partial i-web.

Remark 3.4.8. *If $\mathbf{A}$ has full-consistency, then $\mathbf{S}_\omega$ has full-consistency. Hence every inequality that fails in some i-model $\mathbf{A}^+$ with full-consistency also fails in an i-model generated by the completion of a development containing only partial i-webs with full-consistency.*

3.5 Scottian λ-models

A crucial point for us is that the classes iM and, in general, RSD are not closed under the formation of direct (Cartesian) products. For the results that we will develop it is useful to define a class of λ-models which is

- general enough to contain all i-models generated by completions of finite partial i-webs, but

- well-behaved enough to be closed under direct products and to retain all good properties of i-models w.r.t. order theories and continuity of the interpretation function.

As a result of our analysis, we find convenient to use the following class of mathematical objects.

Definition 3.5.1. *A* Scottian λ-model *(*SLM*, for short) is a λ-model* $\mathbf{D} = (\mathcal{D}, \cdot, \mathbf{k}, \mathbf{s})$ *such that*

(1) $\mathcal{D}$ *is a Scott domain,*

(2) *application is continuous in both arguments w.r.t.* $\leq_{\mathcal{D}}$ *and* $\bot \cdot \bot = \bot$,

(3) $(\mathcal{K}(\mathcal{D}), \cdot)$ *is an applicative substructure of* $\mathbf{D}$.

Committing an abuse of notation we will also use SLM to denote the class of all Scottian λ-models; we will let the reader distinguish between these two different usages.

Warning. Contrarily to λ-models arising from reflexive Scott domains, the representable functions in a SLM are just a subset of the continuous ones, in general. Moreover an arbitrary RSD need not to be an SLM, because of requirement (3). However certain RSDs are SLMs: we will prove that this is the case for the i-models generated by completions of finite partial i-webs.

We now examine arbitrary direct products of Scottian λ-models. Since such constructions rely inevitably on the products of the underlying Scott domains, we take the occasion here to detail the structure of such a direct product.

Definition 3.5.2. *Let* $\{\mathcal{D}_i\}_{i \in I}$ *be a family of Scott domains and let* $P = \prod_{i \in I} D_i$. *We define the* support *of a sequence* $x = \langle d_i \rangle_{i \in I}$ *such that* $d_i \in D_i$, *for each* $i \in I$, *as the set* $\mathsf{su}(x) = \{i \in I : d_i >_i \bot_i\}$.

Proposition 3.5.1. *Let* $\{\mathcal{D}_i\}_{i \in I}$ *be a family of Scott domains and let* $E = \prod_{i \in I} D_i$. *Then* $\mathcal{E} = (E, \sqsubseteq)$ *is a Scott domain (* $\sqsubseteq$ *is defined coordinate-wise) and the set* $\mathcal{K}(\mathcal{E})$ *of compact elements of* $\mathcal{E}$ *is the set of all sequences* $x = \langle d_i \rangle_{i \in I}$ *such that* $d_i \in \mathcal{K}(\mathcal{D}_i)$, *for each* $i \in I$, *and* $\mathsf{su}(x)$ *is finite.*

Proof. Standard. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In particular the set $\text{Env}_{\mathcal{D}}$ of valuations from Var to $\mathcal{D}$, which "is" the power $\mathcal{D}^{\text{Var}}$, is a Scott domain and $\mathcal{K}(\text{Env}_{\mathcal{D}})$ is the set of all valuations ρ such that the set $\mathsf{su}(\rho) = \{x \in \text{Var} : \rho(x) > \bot\}$ is finite.

Proposition 3.5.2. *For any set I, if $\{\mathbf{D}_i\}_{i \in I}$ is a family of SLMs, then $\prod_{i \in I} \mathbf{D}_i$ is a SLM, where the partial order, the application operation and the basic combinators are defined pointwise.*

Proof. Let $\mathbf{P} = \prod_{i \in I} \mathbf{D}_i$. We just observe that $(\mathcal{K}(\mathcal{P}), \cdot)$ is a combinatory subalgebra of $\mathbf{P}$: since the bottom element $\bot_{\mathcal{D}_i}$ is always compact and $\bot_{\mathcal{D}_i} \cdot \bot_{\mathcal{D}_i} = \bot_{\mathcal{D}_i}$, the application of two sequences of compact elements with finite support gives a sequence of compact elements with finite support. $\qquad\square$

In Section 2.3 we defined the notion of satisfaction of equations in algebras. Since λ-terms can be interpreted in a SLM **D** and such interpretations belong to a partial order, we now examine the notion of inequalities in partially ordered algebras.

An inequality $t \sqsubseteq u$ in the type of a partially ordered algebra **A** is

(1) *satisfied in* **A** *under ρ*, notation $\mathbf{A}, \rho \vDash t \sqsubseteq u$, if $t_\rho^{\mathbf{A}} \le u_\rho^{\mathbf{A}}$;

(2) *satisfied in* **A**, notation $\mathbf{A} \vDash t \sqsubseteq u$, if $\mathbf{A}, \rho \vDash t \sqsubseteq u$ for all $\rho \in \text{Env}_A$.

To a given Scottian λ-model **D** we can associate a set of inequalities between λ-terms.

Definition 3.5.3 (**Order theory**). *The* order theory *of an SLM* **D** *is the set* $Or(\mathbf{D}) = \{M \sqsubseteq N : \mathbf{D} \vDash M_{cl} \sqsubseteq N_{cl}\}$.

Recall that with our notation we have $M \sqsubseteq N \in Or(\mathbf{D})$ iff $\forall \rho \in \text{Env}_D.\ [\![M]\!]_\rho^{\mathbf{D}} \le [\![N]\!]_\rho^{\mathbf{D}}$.

Proposition 3.5.3. *If* **D** *is a Scottian λ-model, then $Or(\mathbf{D})$ is an order λ-theory.*

Notation and terminology: recalling the discussion of Section 2.4, as for equational theories, also for order theories there is no possible ambiguity: if an RSD **U** can be endowed with an SLM structure, then the order theory in the sense of Definition 2.5.2 §2 coincides with its order theory in the sense of Definiton 3.5.3. We will just use the notation **U** also when we should use $\mathbf{U}^\bullet$ and we will use the notation $Or(\mathbf{U})$ instead of $Th_\le(\mathbf{U})$, since this causes no ambiguity. Finally in the sequel we will just say "order theory" instead of "order λ-theory", since every time we consider the order theory of a SLM, this will be an order λ-theory.

We now give some propositions stating important properties of Scottian λ-models w.r.t the continuity of the interpretation function and the orderability of the interpretations of $\beta\eta$-normal forms.

Let $\Lambda_\bot$ be the flat domain obtained adjoining a constant $\bot$ to the set of λ-terms partially ordered as follows: $\bot \le M$ and $M \le N \Leftrightarrow M \equiv N$, for all $M, N \in \Lambda$.

Proposition 3.5.4. *Let $\mathbf{D}$ be an SLM. Then the map $[\![-]\!]_{-}^{\mathbf{D}} : \Lambda_{\perp} \times \mathrm{Env}_{\mathbf{D}} \to \mathbf{D}$ is continuous.*

Proof. Standard. $\qquad\square$

Proposition 3.5.5. *Let $\mathbf{D}$ be a non-trivial Scottian λ-model. Then for any two closed distinct $\beta\eta$-normal forms M, N we have $[\![M]\!]^{\mathbf{D}} \not\leq [\![N]\!]^{\mathbf{D}}$.*

Proof. Suppose by contradiction that $[\![M]\!]^{\mathbf{D}} \leq [\![N]\!]^{\mathbf{D}}$. Then by Böhm's Theorem 2.1.1 there exists a sequence $\vec{L}$ of λ-terms such that $\lambda\beta \vdash M\vec{L} = \lambda xy.y$ and $\lambda\beta \vdash N\vec{L} = \lambda xy.x$. But then using the monotonicity of application in $\mathbf{D}$ we obtain that for arbitrary $a, b \in D$

$$a = [\![\lambda xy.x]\!]^{\mathbf{D}}ab = [\![M\vec{L}]\!]^{\mathbf{D}}ab \leq [\![N\vec{L}]\!]^{\mathbf{D}}ab = [\![\lambda xy.y]\!]^{\mathbf{D}}ab = b$$

and this contradicts the non-triviality of $\mathbf{D}$. $\qquad\square$

Proposition 3.5.6. *If $\mathbf{S}_{\omega}$ is the completion of a finite partial i-web, then $\mathbf{S}_{\omega}^{+}$ is an SLM.*

Proof. By Theorem 3.2.1 $\mathbf{S}_{\omega}^{+}$ is an RSD and hence the application is continuous in both arguments.

We now prove that the compact elements of $\mathbf{S}_{\omega}^{+}$ form an applicative substructure. Our first observation is that if for any point u of $\mathcal{S}_{\omega}$ we have

$$u \text{ compact in } \mathcal{S}_{\omega}^{+} \Leftrightarrow u \text{ finite point of } \mathcal{S}_{\omega}$$

Let u, v be finite points of $\mathcal{S}_{\omega}$. Then their application

$$u \cdot v = \{\beta \in S_{\omega} : \exists a \subseteq_{\mathrm{f}} v. \ (a, \beta) \in \{(a', \beta') : \phi_{\omega}(a', \beta') \in u\}\!\downarrow_{S_{\omega} \Rightarrow S_{\omega}}\}$$

is clearly a finite point of $\mathcal{S}_{\omega}$. Moreover $\emptyset\!\downarrow_{S_{\omega}} \cdot \emptyset\!\downarrow_{S_{\omega}} = \{\beta \in S_{\omega} : (\emptyset, \beta) \in \emptyset\!\downarrow_{S_{\omega} \Rightarrow S_{\omega}}\} = \emptyset\!\downarrow_{S_{\omega}}$, and this concludes the proof. $\qquad\square$

4

Numbered structures and effectivity

There is a field across logic and mathematics that aims at establishing the scope and limits of finite computation by means of algorithms on *any* set of data: it could be called *Effective Algebra*. A set A of data, together with some basic functions and/or relations, form an algebraic structure $\mathbf{A}$. In Effective Algebra the approach to analyzing computation in the algebra A is to apply the theory of computable functions on $\mathbb{N}$, using a surjection $\nu : \mathbb{N} \to A$ called a *numbering*. Algorithmic properties of $\mathbf{A}$ are measured by the algorithmic properties of the number-theoretic representation of $\mathbf{A}$ via ν. In particular, the concept of *effective* (or *computable*, for many authors) algebraic structure can be defined. Thus Effective Algebra studies what data can be represented algorithmically, and what sets and functions can be defined by algorithms, using the same concepts as those that underpin the Church-Turing Thesis for algorithms on $\mathbb{N}$. It also studies algebraic structures that can be algorithmically approximated. Algebraic structures can be found throughout mathematics and computer science, and their applications. Effective Algebra encompasses a wide range of subjects, some of which are well developed mathematical theories, while others are awaiting systematic investigation.

The move to the numbering or enumeration of arbitrary sets using natural numbers was made in Rabin [99, 98] and Malcev [81]. The idea is a generalization of the Gödel numbering of logical syntax. M.O. Rabin defined encodings of the form $\iota : A \to \mathbb{N}$, and proved several interesting results about computable groups, rings and fields. In particular, he established that the algebraic closure of any computable field is computable, using Artin's construction of the algebraic closure.

A.I. Malcev studied effective algebraic structures using numberings of the form $\nu : \mathbb{N} \to A$. Most of the notions that we use are adaptations of those of Malcev, who began the theory of numberings, numbered sets and structures in a series of papers (see his selected works [82]). Thorough mathematical accounts of computable sets in Universal Algebra have been developed by Yu. Ershov and others (see [49, 50, 51]).

With this background, a number of authors have considered domains and information systems effectively, soon afterwards their birth. Fundamental results for effectively given domains are due to Scott [106], Smyth [110], Kanda [63, 68, 67, 66, 64, 65], and Kanda-Park [69]; "effective" results on information systems can be found in Scott [106], Coppo-Dezani-Longo [36]. Larsen-Winskel [78] and Droste-

Göbel [42], establish an effective version of the duality between information systems and Scott domains and further develop the study of *computable elements*, already begun in [106, 78, 69, 110].

In this chapter we recall the basics notions regarding Gödel numberings and numerations (Section 4.1) and recursivity over these structures. Section 4.2 reviews effective domains, effective information systems and the relations between them.

Finally in Section 4.4 we study the effective version of Scottian λ-models (see §3), whose purpose is to allow a high-level and modular formulation of the main results of §5.

4.1   Encodings and numbered structures

We denote by $\mathbb{N}$ the set of natural numbers. A set $X \subseteq \mathbb{N}$ is r.e. if it is the domain of a partial recursive function. The complement of a r.e. set is called a *co-r.e. set*. If both X and its complement are r.e., X is called *decidable* (or recursive).

Definition 4.1.1. *An* encoding *of a set A is an injection $\sharp(\_) : A \to \mathbb{N}$. The number $\sharp a$ is called the* Gödel number *of $a \in A$. An encoding is* decidable *if the set $\sharp A = \{\sharp a \in \mathbb{N} : a \in A\}$ is decidable.*

We now recall some well-known examples of encodings that we will use later on.

Example 4.1.1. *The bijection $\langle \_, \_ \rangle : \mathbb{N}^2 \to \mathbb{N}$ given by $\langle n, m \rangle = (n + m)^2 + n + 1$ is a decidable encoding.*

Clearly for any given $k > 2$ it is possible to define a bijective decidable encoding $\langle \_, \dots, \_ \rangle : \mathbb{N}^k \to \mathbb{N}$ by induction setting $\langle n_1, \dots, n_{k-1}, n_k \rangle = \langle \langle n_1, \dots, n_{k-1} \rangle, n_k \rangle$.

Example 4.1.2. *The bijection $e : \mathcal{P}_{\mathrm{f}}(\mathbb{N}) \to \mathbb{N}$ given by $e(\emptyset) = 0$ and $e(\{n_1, \dots, n_k\}) = \sum_{i=1}^{k} 2^{n_i}$ (where $n_1 < \cdots < n_k$) is a decidable encoding.*

Example 4.1.3. *Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers. Recall Gödel's famous recursive β-function $\beta : \mathbb{N}^2 \to \mathbb{N}$ having the property that for any sequence $n_1, \dots, n_k$ of natural numbers there exists a number m such that $\beta(m, i) = n_i$, for all $i = 1, \dots, k$.*

Then one defines the injection $(\_)^* : \mathbb{N}^* \to \mathbb{N}$ as follows: $(n_0, \dots, n_k)^*$ codes the sequence $(n_1, \dots, n_k)$ in such a way that $\beta((n_0, \dots, n_k)^*, i) = n_i$, for all $i = 1, \dots, k$ and $lh((n_0, \dots, n_k)^*) = \beta((n_0, \dots, n_k)^*, 0) = k$.

Example 4.1.4. *A useful example of encoding is the Gödel encoding $\sharp(\_) : \Lambda \to \mathbb{N}$ of the set of λ-terms. This encoding has the property that the sets $\{\sharp x \in \mathbb{N} : x \in \mathrm{Var}\}$, $\{\sharp M \in \mathbb{N} : M \in \Lambda\}$, $\{(\sharp M, \sharp x) \in \mathbb{N}^2 : x \in \mathrm{FV}(M)\}$ and $\{\sharp M \in \mathbb{N} : M \in \Lambda^o\}$ are all decidable. It is well-known that $\sharp(\_)$ can be assumed to be a bijection. Moreover there exists a bijective decidable encoding $\flat(\_) : \Lambda^o \to \mathbb{N}$.*

Definition 4.1.2 (**Numeration**,[49]). *A numeration is a pair (A, ν), where A is a set and $\nu : \mathbb{N} \to A$ is a surjective total map; the function ν is called a numbering of A. We usually write ν_n in place of $\nu(n)$.*

In general we will use letters $\nu, \xi, \zeta, \dots$ to indicate numberings.

The encodings presented so far are also useful to create new numerations from existing ones in an effective way. Natural examples of numerations come from the lambda-calculus and from classical recursion theory.

Example 4.1.5. *Recall from Subsection 2.3.1 the definition of Λ/T, Λ^o/T (where T is a λ-theory). One can define a numbering $\nu : \mathbb{N} \to \Lambda/T$ by setting $\nu(\sharp M) = [M]_T$ and a numbering $\zeta : \mathbb{N} \to \Lambda^o/T$ by setting $\nu(\natural M) = [M]_T^o$.*

Example 4.1.6. *Also flat term models admit simple numberings. Recall from Example 2.5.2 the definitions of $(\Lambda/T)_\perp$ and $(\Lambda^o/T)_\perp$. One can define a numbering $\nu : \mathbb{N} \to (\Lambda/T)_\perp$ by setting*

$$\nu(n) = \begin{cases} \perp & \text{if } n = 0 \\ [M]_T & \text{if } n = \sharp M + 1 > 0 \end{cases}$$

and a numbering $\zeta : \mathbb{N} \to (\Lambda^o/T)_\perp$ in similar way.

Example 4.1.7. *Let $\mathcal{PR}$ be the set of all partial recursive functions and let $\mathcal{RE}$ be a the set of all r.e. subsets of $\mathbb{N}$. Kleene has shown how to encode partial recursive functions and how to obtain a numbering $n \mapsto \varphi_n$ of $\mathcal{PR}$. Then a numeration of $\mathcal{RE}$ is obtained via the numbering $n \mapsto \mathcal{W}_n = dom(\varphi_n)$.*

Numerations and set-theoretic constructions

Finite sets. Given a numeration (A, ν), we define a numeration $(\mathcal{P}_f(A), \mathcal{P}_f(\nu))$, by setting $\mathcal{P}_f(\nu)(e(\{n_1, \dots, n_k\})) = \{\nu_{n_1}, \dots, \nu_{n_k}\}$, where e is the encoding of Example 4.1.2.

Tuples. Given numerations $(A_i, \nu^{(i)})$ $(1 \leq i \leq k)$ we define a numeration $(\prod_{i=1}^k A_i, \prod_{i=1}^k \nu^{(i)})$, by setting $(\prod_{i=1}^k \nu^{(i)})(\langle n_1, \dots, n_k \rangle) = (\nu_{n_1}, \dots, \nu_{n_k})$, where $\langle \dots \rangle$ is the encoding of Example 4.1.3.

4.1.1 Recursivity on numbered structures

We now recall how a number of notions from recursion theory carry over in the context of numbered structures.

Definition 4.1.3. *A numeration (A, ν) is decidable (resp. r.e.) iff $\{(n, m) : \nu_n = \nu_m\}$ is a decidable (resp. r.e.) relation.*

We remark that decidable numerations are called *positive* by Visser [115].

Remark 4.1.1. *The constructions defined at the end of Section 4.1 have an effective character in the sense that if we start from decidable (resp. r.e.) numerations, then we end up with decidable (resp. r.e.) numerations.*

Definition 4.1.4. *Let (A, ν) be a numeration. A subset $X \subseteq A$ is ν-r.e. (resp. ν-co-r.e., ν-decidable) if $\nu^{-1}(X) = \{n : \nu_n \in X\}$ is an r.e. (resp. co-r.e., decidable) subset of $\mathbb{N}$.*

Remark 4.1.2. *Consider the numberings of flat term models given in Example 4.1.6. Then a subset $X \subseteq (\Lambda/T)_\perp$ is r.e. (resp. decidable) iff $X - \{\perp\}$ is r.e. (resp. decidable) w.r.t. the numbering of Λ/T. The same relation exists between $(\Lambda^o/T)_\perp$ and Λ^o/T.*

Definition 4.1.5. *A binary relation $R \subseteq A \times B$ between two numerations $(A, \xi), (B, \nu)$ is (ξ, ν)-r.e. (resp. (ξ, ν)-co-r.e., (ξ, ν)-decidable) if $\{(m, n) : R(\xi_m, \nu_n)\}$ is an r.e. (resp. co-r.e., decidable)relation*

Note that a numeration is r.e. (resp. decidable) iff the equality relation is r.e. (resp. decidable).

Definition 4.1.5 extends straightforwardly to arbitrary k-ary relations between numerations. For k-ary relation over a power of (A, ν), we shall simply say ν-r.e. (resp. ν-co-r.e., ν-decidable) if the suitable conditions are satisfied. We shall even drop the references to the numerations when this will be either clear from the context or irrelevant for our statement.

Note also that if $(A, \leq, \xi)$ is a numbered poset and $\leq$ is ξ-decidable then ξ is a decidable numeration.

Definition 4.1.6. *A λ-theory T is r.e. if the set $\{(\sharp M, \sharp N) : (M, N) \in T\}$ is r.e. An order theory O is r.e. if the set $\{(\sharp M, \sharp N) : (M, N) \in O\}$ is r.e.*

Remark 4.1.3. *Definition 4.1.6 can be equivalently reformulated using the theory of numbered structures. In this context a λ-theory T is r.e. iff the equality relation on Λ/T is r.e..*

An order theory O uniquely determines a λ-theory T_O, which is the closure of the set of all pairs (M, N) such that both $(M, N) \in O$ and $(N, M) \in O$. Then an order theory O is is r.e. if the partial order that it induces on Λ/T_O is r.e.

Definition 4.1.7. *Let $(A, \nu), (B, \zeta)$ be numerations. A total function $f : A \to B$ is (ν, ζ)-computable there exists a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $f(\nu_n) = \zeta_{\varphi(n)}$, for all $n \in \mathbb{N}$. In such a case, we say that φ tracks f.*

As for relations, Definition 4.1.7, easily extends to an arbitrary k-ary partial function $f : \prod_{i=1}^k A_i \hookrightarrow B$. As for the terminology, we shall speak of $(\prod_{i=1}^k \nu^{(i)}, \zeta)$-computability (assuming these are the source and the targe numberings) and of ν-computability if both the source and the target of f are powers of a single numeration (A, ν) and the suitable conditions are satisfied.

We remark that the notion of computable function appearing in Definition 4.1.7 corresponds to the notion of *morphism* of numerations in the terminology of Visser [115]. The following Proposition is an elementary generalization of a well-known fact in computability theory.

Proposition 4.1.4. *Let $(A, \nu), (B, \zeta)$ be r.e. numerations, and let $f : A \to B$ be a (ν, ζ)-computable function. Then*

   (i) if Y is ζ-r.e., then $f[Y]^{-1}$ is ν-r.e.

   (ii) if Y is ζ-co-r.e., then $f[Y]^{-1}$ is ν-co-r.e.

We conclude this section recalling an important theorem of Visser, which will be fundamental to prove our subsequent results.

Terminology: Let A be a set. For a subset $X \subseteq A$ we will say that X is *non-trivial* if $X \neq \emptyset$ and $X \neq A$.

Theorem 4.1.5 (Visser's theorem,[115]**).** *Let T be a λ-theory and consider the numeration (Λ, ν). Then any pair of non-trivial ν-co-r.e. and T-closed subsets $X, Y \subseteq \Lambda$ has nonempty intersection.*

Example 4.1.8. *Recall the concept of easy (closed) λ-term from Definition 2.1.2. The set E of all easy λ-terms is non-trivial, co-r.e. and $\lambda\beta$-closed.*

4.2 Effective Scott domains and effective information systems

4.2.1 Effective Scott domains

Definition 4.2.1 (Effective Scott domain,[111]**).** *A Scott domain $\mathcal{D}$ is* effective *if there exists a numeration $(\mathcal{K}(\mathcal{D}), \nu)$ of its compact elements such that the sets $\{(x, y) \in \mathcal{K}(\mathcal{D})^2 : \exists z \in \mathcal{K}(\mathcal{D}). \ z \geq x, y\}$, $\{(x, y, z) \in \mathcal{K}(\mathcal{D})^3 : z = x \sqcup y\}$ are both ν-decidable.*

When the context requires it, an effective Scott domain will be indicated by a pair $(\mathcal{D}, \nu)$, where ν is the numeration of its compact elements. It follows, by Definition 4.2.1, that the order and the equality relations on $\mathcal{K}(\mathcal{D})$ are ν-decidable, i.e., ν is a decidable numbering of $\mathcal{K}(\mathcal{D})$.

Definition 4.2.2. *An element $x \in \mathcal{D}$ is called* r.e. *(resp.* decidable*) if the set $\{y \in \mathcal{K}(\mathcal{D}) : y \leq x\}$ is ν-r.e. (resp. ν-decidable). We write $\mathcal{D}^{r.e.}$ (resp. $\mathcal{D}^{dec}$) to indicate the set of all r.e. (resp. decidable) elements of $\mathcal{D}$. Clearly $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{D}^{dec}$.*

We are now interested in defining a numeration of $\mathcal{D}^{r.e.}$ amenable to the effective numeration ν of $\mathcal{K}(\mathcal{D})$. The natural surjection $\zeta : \mathbb{N} \to \mathcal{D}^{r.e.}$ defined by $\zeta(n) = x$ iff $\mathcal{W}_n = \nu^{-1}(\{y \in \mathcal{K}(\mathcal{D}) : y \leq x\})$ is not a good choice. Using standard techniques of recursion theory it is possible to overcome the difficulties and construct a well-behaved numeration of $\mathcal{D}^{r.e.}$. The next definition states precisely what properties are required to the numeration of the r.e. elements of $\mathcal{D}$.

Definition 4.2.3. *A numeration $(\mathcal{D}^{r.e.}, \zeta)$ is* adequate *w.r.t. $(\mathcal{D}, \xi)$ if it has the following two properties:*

(1) the inclusion mapping $\iota : \mathcal{K}(\mathcal{D}) \to \mathcal{D}^{r.e.}$ is (ξ, ζ)-computable,

(2) the relation $\{(x, y) \in \mathcal{K}(\mathcal{D}) \times \mathcal{D}^{r.e.} : x \leq y\}$ is (ξ, ζ)-r.e.

Proposition 4.2.1. *For any effective Scott domain $(\mathcal{D}, \xi)$ it is possible to get in a uniform way a standard adequate numeration $(\mathcal{D}^{r.e.}, \widehat{\xi})$.*

Proof. See [111, Ch. 10, Thm. 4.4]. □

The particular construction of $\widehat{\xi}$ from ξ turns out to be inessential, since for any two adequate numerations ν, ζ there exists a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\nu = \zeta \circ \varphi$. Henceforth we will always implicitly consider effective Scott domains $(\mathcal{D}, \xi)$ equipped with an adequate numeration $(\mathcal{D}^{r.e.}, \widehat{\xi})$.

Definition 4.2.4. *An subset $A \subseteq \mathcal{D}^{r.e.}$ is called* completely r.e. *(resp.* completely co-r.e., *resp.* completely decidable*) if A is $\widehat{\xi}$-r.e. (resp. $\widehat{\xi}$-co-r.e., resp. $\widehat{\xi}$-decidable).*

Example 4.2.1. *The singleton set $\{\bot_{\mathcal{D}}\}$ is completely co-r.e.*

Theorem 4.2.2. *Let $(\mathcal{D}, \xi)$ be an effective Scott domain and let $A \subseteq \mathcal{D}^{r.e.}$. Then A is completely r.e. if and only if there exists an r.e. set $E \subseteq \mathbb{N}$ such that $A = \{v \in \mathcal{D}^{r.e.} : \exists n \in E. \ \widehat{\xi}_n \in \mathcal{K}(\mathcal{D}) \text{ and } \widehat{\xi}_n \leq v\}$.*

Proof. See [111, Thm. 5.2]. □

Lemma 4.2.3. *Let $(\mathcal{D}, \xi)$ be an effective Scott domain. If $x \in \mathcal{D}^{dec}$, then $\{y \in \mathcal{D}^{r.e.} : y \leq x\}$ is completely co-r.e.*

Proof. It suffices to apply Theorem 4.2.2 to show that $\{y \in \mathcal{D}^{r.e.} : y \not\leq x\}$ is completely r.e. □

Example 4.2.2. *The set $(\mathcal{P}(\mathbb{N}), \subseteq)$, equipped with the coding of the finite subsets of $\mathbb{N}$ is an effective Scott domain whose r.e. (resp. decidable) elements are exactly the r.e. (resp. decidable) sets. Hence $\mathcal{P}(\mathbb{N})^{r.e.} = \mathcal{RE}$, with the adequate numeration $e \mapsto \mathcal{W}_e$.*

Example 4.2.3. *The flat domains $(\Lambda/T)_\bot$ and $(\Lambda^o/T)_\bot$ (see Example 2.5.2) are effective Scott domains in which every element is decidable.*

Proposition 4.2.4. *Let* $(\mathcal{D}, \nu), (\mathcal{E}, \zeta)$ *be effective Scott domains. Then it is possible to obtain, in a canonical way, two numerations* $(\mathcal{K}(\mathcal{D} \Rightarrow \mathcal{E}), \nu \rightsquigarrow \zeta)$ *and* $(\mathcal{K}(\mathcal{D} \times \mathcal{E}), \nu \times \zeta)$ *giving* $\mathcal{D} \Rightarrow \mathcal{E}$ *and* $\mathcal{D} \times \mathcal{E}$, *respectively, a structure of effective Scott domain.*

For the proof of Proposition 4.2.4 we refer to [111, Ch. 10].

Thus it is possible in particular to talk about r.e. continuous functions, i.e., elements of $(\mathcal{D} \Rightarrow \mathcal{E})^{r.e.}$, characterized by the property that the set of all compact functions below them is $(\nu \rightsquigarrow \zeta)$-r.e. The next propositions give alternative characterizations of the elements of $(\mathcal{D} \Rightarrow \mathcal{E})^{r.e.}$.

Proposition 4.2.5. *Let* $(\mathcal{D}, \nu), (\mathcal{E}, \zeta)$ *be effective Scott domains. For all functions* $f \in \mathbf{Sd}(\mathcal{D}, \mathcal{E})$ *the following are equivalent:*

(i) $f \in (\mathcal{D} \Rightarrow \mathcal{E})^{r.e.}$

(ii) the trace $\{(d, e) \in \mathcal{K}(\mathcal{D}) \times \mathcal{K}(\mathcal{E}) : e \leq_E f(d)\}$ *of* f *is* (ν, ζ)-r.e.

(iii) f *maps* ν-r.e. *elements of* $\mathcal{D}$ *to* ζ-r.e. *elements of* $\mathcal{E}$ *and the restriction* $f{\restriction}_{\mathcal{D}^{r.e.}}$ *is* $(\widehat{\nu}, \widehat{\zeta})$-computable.

Proposition 4.2.5 (i) $\Leftrightarrow$ (ii) with its proof can be found in [111, Ch. 10,Prop. 3.7], while Proposition 4.2.5 (i) $\Leftrightarrow$ (iii) with its proof can be found in [111, Ch. 10,Prop. 4.14].

Definition 4.2.5. *We indicate by* $\mathbf{ESd}$ *the category which has effective Scott domains as objects; the hom-sets are given by* $\mathbf{ESd}(\mathcal{D}, \mathcal{E}) = \mathbf{Sd}(\mathcal{D}, \mathcal{E})^{r.e.}$.

Proposition 4.2.6. *The category* $\mathbf{ESd}$ *is a cartesian closed subcategory of* $\mathbf{Sd}$.

4.2.2   Effective information systems

We have seen that the category $\mathbf{Sd}$ has an "effective counterpart", namely $\mathbf{ESd}$. Analogously, it is possible to define an "effective counterpart" of $\mathbf{Inf}$, developing the notion of *effective information system*.

Definition 4.2.6 ([42])**.** *An* effective information system *is pair* $(\mathcal{A}, \xi)$ *where* $\mathcal{A}$ *is an information system and* (A, ξ) *is a numeration such that*

- *the predicate* Con *is* $\mathcal{P}_f(\xi)$-decidable,
- *the relation* $\vdash$ *is* $(\mathcal{P}_f(\xi), \xi)$-decidable.

Notation and terminology. The *trivial numeration* of $\mathbb{N}$ is the numeration $(\mathbb{N}, \iota)$, where ι is the identity function.

Example 4.2.4. *Consider the set* $\mathbb{N}$, *equipped with its trivial numeration. The consistency predicate* $\mathrm{Con}(a) \Leftrightarrow a \in \mathcal{P}_f(\mathbb{N})$ *is* $\mathcal{P}_f(\iota)$-decidable *and the entailment relation* $a \vdash n \Leftrightarrow n \in a$ *is* $(\mathcal{P}_f(\iota), \iota)$-decidable, *so that* $(\mathbb{N}, \mathrm{Con}, \vdash)$ *is an effective information system.*

The class of effective information systems is closed under the construction of products and function spaces, admitting an analogue of Proposition 4.2.4.

These constructions are make use of the following . Let $(\mathcal{A}, \nu), (\mathcal{B}, \zeta)$ be effective information systems.

We define a numbering $\nu \& \zeta : \mathbb{N} \to A \& B$ (recall Definition 2.6.3 §2) as follows

$$(\nu \& \zeta)(\langle n, m \rangle) = \begin{cases} (1, \nu_m) & \text{if } n \text{ odd} \\ (2, \zeta_m) & \text{if } n \text{ even} \end{cases}$$

and a numbering $\nu \Rightarrow \zeta : \mathbb{N} \to A \Rightarrow B$ (recall Definition 2.6.4 §2) as follows

$$(\nu \Rightarrow \zeta)(\langle e(\{n_1, \ldots, n_k\}), m \rangle) = \begin{cases} (\{\nu_{n_1}, \ldots, \nu_{n_k}\}, \zeta_m) & \text{if } \{\nu_{n_1}, \ldots, \nu_{n_k}\} \in \text{Con}_A \\ (\emptyset, \zeta_m) & \text{otherwise} \end{cases}$$

where we used the numerations of Examples 4.1.2 and 4.1.3.

Note in particular that even if in general $\nu \Rightarrow \zeta \neq \mathcal{P}_{\mathrm{f}}(\nu) \times \zeta$, we have that in an effective information system the entailment "$\vdash$" is $(\mathcal{P}_{\mathrm{f}}(\xi), \xi)$-decidable iff it is $\xi \Rightarrow \xi$-decidable.

Proposition 4.2.7 ([42]). $(\mathcal{A}, \nu), (\mathcal{B}, \zeta)$ *be effective information systems. Then* $(\mathcal{A} \& \mathcal{B}, \nu \& \zeta)$ *and* $(\mathcal{A} \Rightarrow \mathcal{E}, \nu \Rightarrow \zeta)$ *are effective information systems.*

Definition 4.2.7. *We indicate by* **EInf** *the category which has effective information systems as objects; the hom-sets are given by* $\mathbf{EInf}(\mathcal{A}, \mathcal{B}) = \{R \in \mathbf{Inf}(\mathcal{A}, \mathcal{B}) : R \text{ is } (\nu \Rightarrow \zeta)\text{-r.e.}\}$, *where* ν *and* ζ *are the numberings of* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively.*

Proposition 4.2.8. *The category* **EInf** *is a cartesian closed subcategory of* **Inf**.

4.2.3   Equivalence between the two categories

Now it makes sense to wonder whether or not the functors $(\ )^+$ and $(\ )^-$ also establish an exact correspondence between **EInf** and **ESd**: this is indeed the case (see [42]).

If $(\mathcal{A}, \xi)$ is an effective information system, then a point x of $\mathcal{A}$ is ξ-r.e. (resp. ξ-decidable) iff $\{n \in \mathbb{N} : \xi_n \in x\}$ is a r.e. (resp. decidable) set. Thus the elements of $(\mathcal{A}^+)^{r.e.}$ (resp. $(\mathcal{A}^+)^{dec}$) are exactly those r.e. (resp. decidable) subsets of A which are also points of $\mathcal{A}$. Note that for any r.e. finitely consistent subset x of A, $x \downarrow \in (\mathcal{A}^+)^{r.e.}$, since $\alpha \in x \downarrow \Leftrightarrow \exists b \subseteq x. \ b \vdash \alpha$.

Theorem 4.2.9. *For all information systems* $\mathcal{A}, \mathcal{B}$ *we have*

  (i) *if* $\mathcal{A}$ *is effective, then* $\mathcal{A}^+$ *is an effective Scott domain,*

  (ii) *if* $R \in \mathbf{EInf}(\mathcal{A}, \mathcal{B})$, *then* $R^+ \in \mathbf{ESd}(\mathcal{A}^+, \mathcal{B}^+)$ *and* $\mathbf{EInf}(\mathcal{A}, \mathcal{B}) \cong \mathbf{ESd}(\mathcal{A}^+, \mathcal{B}^+)$.

For all Scott domains $\mathcal{D}, \mathcal{E}$ *we have*

  (i) *if* $\mathcal{D}$ *is effective, then* $\mathcal{D}^-$ *is an effective information system,*

  (ii) *if* $f \in \mathbf{ESd}(\mathcal{D}, \mathcal{E})$, *then* $f^- \in \mathbf{EInf}(\mathcal{D}^-, \mathcal{E}^-)$ *and* $\mathbf{ESd}(\mathcal{D}, \mathcal{E}) \cong \mathbf{EInf}(\mathcal{D}^-, \mathcal{E}^-)$.

Proof. See [42]. ☐

4.3 Effective (partial) i-webs and effective completions

An important breakthrough in the study of models of lambda calculus is made in [17], where the authors consider reflexive objects in the category **ESd** and are able to make very interesting connections with recursivity properties of order theories of graph models. We extend their results and develop the effective counterparts of the main structures and constructions of §3.

We start by generalizing Definition 4.1.7 to the case of partial functions.

Definition 4.3.1. *Let $(A, \nu), (B, \zeta)$ be numerations. A partial function $f : A \hookrightarrow B$ is (ν, ζ)-computable if $dom(f)$ is a ν-r.e. set and there exists a partial recursive function $\varphi : \mathbb{N} \hookrightarrow \mathbb{N}$ satisfying the following conditions:*

(1) $\nu_n \in dom(f)$ iff $n \in dom(\varphi)$,

(2) $f(\nu_n) = \zeta_{\varphi(n)}$.

The function f is strongly (ν, ζ)-computable *if it is (ν, ζ)-computable and $dom(f)$ is ν-decidable. In any a case, we still say that φ tracks f.*

Clearly any total (ν, ζ)-computable function is also strongly (ν, ζ)-computable.

Example 4.3.1. *If $(\mathcal{D}, \nu)$ is an effective Scott domain, then the supremum function on $\mathcal{K}(\mathcal{D})$, which is partial, is strongly ν-computable.*

Notation. The range of a function f is indicated by $\mathsf{ra}(f)$.

Definition 4.3.2. *A partial function $f : (A, \nu) \hookrightarrow (B, \zeta)$ between two numerations is* completely (ν, ζ)-computable *if it is strongly (ν, ζ)-computable and $\mathsf{ra}(f)$ is ζ-decidable.*

Definition 4.3.3. *A partial i-web $\mathbf{S}$ is* effective *if there exists a numbering σ of S such that $(\mathcal{S}, \sigma)$ is an effective IS and $\phi_S : \mathcal{S} \Rightarrow \mathcal{S} \hookrightarrow \mathcal{S}$ is a completely $(\sigma \Rightarrow \sigma, \sigma)$-computable function.*

Note that, in particular, an i-web $\mathbf{A}$ is effective if there exists a numbering σ of A such that $(\mathcal{A}, \sigma)$ is an effective IS and $\phi_A : \mathcal{A} \Rightarrow \mathcal{A} \to \mathcal{A}$ is a strongly $(\sigma \Rightarrow \sigma, \sigma)$-computable function.

We now want to investigate the effectivity properties of the completions of effective partial i-webs. All free extensions of an effective partial i-web $\mathbf{S}$ have the same set of tokens, namely $S^* = S \cup \overline{\mathsf{do}}(\phi_S)$ (see Definition 3.3.7 §3). The following definition gives a standard way of constructing a numbering of S^* from a numbering of S.

Definition 4.3.4. *Let σ be the numbering of* **S**. *We define a numbering σ^* for the set $S^* = S \cup \overline{do}(\phi_S)$ as follows*

$$
\sigma_n^* = \begin{cases} \sigma_{\frac{n}{2}} \textit{ if } n \textit{ is even} \\ (\sigma \Rightarrow \sigma)_{\frac{n-1}{2}} \textit{ if } n \textit{ is odd and } (\sigma \Rightarrow \sigma)_{\frac{n-1}{2}} \in \overline{do}(\phi_S) \\ \sigma_0 \textit{ otherwise} \end{cases}
$$

Theorem 4.3.1. *Let* **S** *be an effective partial i-web and let* **S*** *be a fixed free extension of* **S**, *together with the numbering σ^* of Definition 4.3.4. If* Con^* *is* $\mathcal{P}_{\mathrm{f}}(\sigma^*)$-*decidable, then* **S*** *is an effective partial i-web.*

Proof. Using the hypothesis and the fact that ϕ_S is completely computable (for **S**) it is routine to check that the predicate $\vdash^*$ is decidable and that ϕ^* is completely computable w.r.t. the new numerations involving σ^*. $\qquad\qquad\square$

Let $\{\mathbf{S}_n\}_{n \geq 0}$ be a development of an effective partial i-web **S**. Even if each member of the development if effective, there is no general reason for which the completion of **S** should be an effective i-web.

Following this observation we define *effective families* of effective ISs. Such notion will be the base for defining *effective completions*, proceeding in analogy to Definition 3.3.9 and 3.3.10 §3.

Terminology. There is a specific terminology used in the literature of recursion theory for designing families of recursive functions/predicates which are "algorithmically" produced: such families are called *uniform* [73].

Definition 4.3.5. *A countable family $\{(\mathbf{S}_n, \nu^{(n)})\}_{n \in \mathbb{N}}$ of effective partial i-webs is* effective *if*

- *the predicate* Con_m *is* $\mathcal{P}_{\mathrm{f}}(\nu^{(m)})$-*decidable uniformly in* m,

- *the predicate* $\vdash_m$ *is* $(\nu^{(m)} \Rightarrow \nu^{(m)})$-*decidable uniformly in* m,

- *the function* ϕ_{S_m} *is* $(\nu^{(m)} \Rightarrow \nu^{(m)}, \nu^{(m)})$-*completely computable uniformly in* m.

For example in the case of Definition 4.3.5 it means that there is a certain recursive function that, given an integer m, returns an index of the recursive predicate that decides consistency in the partial i-web $\mathbf{S}_m$.

All completions of an effective partial i-web **S** have the same set of tokens, namely S_ω (see Definition 3.3.10, §3). The following definition gives a standard way of constructing a numbering of S_ω from a numbering of S.

Definition 4.3.6. *Let* **S** *be an effective partial i-web and let $\{\mathbf{S}_n\}_{n \geq 0}$ be a development of* **S**. *Let $\sigma^{(n)}$ be the numbering of $\mathbf{S}_n$ obtained by iterating n times the construction of Definition 4.3.4. We define a numeration (S_ω, ζ) as $\zeta_{\langle m,k \rangle} = \sigma_k^{(m)}$.*

Definition 4.3.7. *Let* $\mathbf{S}$ *be an effective partial i-web and let* $(\mathbb{N}, \iota)$ *be the trivial numeration of* $\mathbb{N}$*. A development* $\{\mathbf{S}_n\}_{n \geq 0}$ *of* $\mathbf{S}$ *is* effective *if it is an effective family of effective information systems.*

Theorem 4.3.2. *Let* $\mathbf{S}$ *be an effective partial i-web and let* $\{\mathbf{S}_n\}_{n \geq 0}$ *be an effective development of* $\mathbf{S}$*. Then the completion of* $\mathbf{S}$ *w.r.t. this development is an effective i-web.*

Proof. Let $\mathbf{S}$ be an effective partial i-web and let $\mathbf{S}_\omega$ be its completion w.r.t. an effective development $\{\mathbf{S}_n\}_{n \geq 0}$. Moreover take ζ to be the numbering of Definition 4.3.6.

By using the corresponding properties of ϕ_n, $\vdash_n$ and Con_n, it is not difficult then to see that $\vdash_\omega$ and Con_ω are all ζ-decidable and ϕ_ω is strongly $(\zeta \Rightarrow \zeta, \zeta)$-computable. Now since $S_\omega - \mathsf{ra}(\phi_\omega) = S_0 - \mathsf{ra}(\phi_0)$, it is clear that $\mathsf{ra}(\phi_\omega)$ is ζ-decidable and hence ϕ_ω is completely $(\zeta \Rightarrow \zeta, \zeta)$-computable. This proves that $\mathcal{S}_\omega$ is an effective i-web. $\square$

Remark 4.3.3. *A development* $\{\mathbf{S}_n\}_{n \geq 0}$ *of* $\mathbf{S}$ *is effective iff the predicate* Con_n *is* $\mathcal{P}_{\mathrm{f}}(\nu^{(n)})$*-decidable uniformly in* n*.*

Proof. All data of $\mathbf{S}_{n+1}$ are determined algorithmically in function of Con_{n+1} and $\mathbf{S}_n$. $\square$

We conclude the section with a very important definition, which is about adding a "further effective dimension".

Definition 4.3.8. *A countable family of* $\{\mathfrak{S}_m\}_{m \in \mathbb{N}}$ *effective developments of finite partial i-webs is* effective *if any member* $\mathfrak{S}_m = \{(\mathbf{S}_{m,n}, \nu^{(m,n)})\}_{n \in \mathbb{N}}$ *is such that*

- *the predicate* $\mathrm{Con}_{m,n}$ *is* $\mathcal{P}_{\mathrm{f}}(\nu^{(m,n)})$*-decidable uniformly in* m *and* n*,*

- *the predicate* $\vdash_{m,n}$ *is* $(\nu^{(m,n)} \Rightarrow \nu^{(m,n)})$*-decidable uniformly in* m *and* n*,*

- *the function* $\phi_{S_{m,n}}$ *is* $(\nu^{(m,n)} \Rightarrow \nu^{(m,n)}, \nu^{(m,n)})$*-completely computable uniformly in* m *and* n*.*

4.4 Effective Scottian lambda-models

In Section Section 3.5 of §3 we defined the class SLM and discussed its main features. In the present section we will propose an "effective" version of Scottian λ-models and prove that it enjoys the "effective" version of the properties enjoyed by SLM. The main purpose of the additional hypotheses is having a numbered structure with a computable interpretation of λ-terms: then clearly the interpretation establishes a bridge that can be "crossed backwards" from numerations of Scott domains to numerations of term models in order to infer results about the recursivity of order and equational theories of those "effective" versions of Scottian λ-models.

We start by recalling Definition 4.2.1, which describes effective Scott domains.

Definition 4.4.1 (**Effective Scott domain**). *An Scott domain $\mathcal{D}$ is effective if there exists a numeration $(\mathcal{K}(\mathcal{D}), \nu)$ of its compact elements such that the sets $\{(x, y) \in \mathcal{K}(\mathcal{D})^2 : \exists z \in \mathcal{K}(\mathcal{D}). \; z \geq x, y\}$, $\{(x, y, z) \in \mathcal{K}(\mathcal{D})^3 : z = x \sqcup y\}$ are both ν-decidable.*

Then we give the definition of *effective combinatory algebra*, which is an instance of a more general definition of effective algebra (see [55, Ch. 12] and [111, Ch. 10]).

Definition 4.4.2 (**Effective applicative structure**). *An applicative structure $\mathbf{A} = (A, \cdot)$ is effective if there exists a decidable numeration (A, ν) of its universe such that there exists a binary total recursive function which tracks its application operation (see Definition 4.1.7).*

When the context requires it, effective applicative structures will be indicated by pairs $(\mathbf{A}, \nu)$ to put in evidence the involved numerations.

We are about to define the effective counterpart of the class SLM: it will be a class of λ-models with some effectivity properties. Evidently we won't instantiate the abstract definition of effective algebra (e.g. [55, Ch. 12], [111, Ch. 10]) to the case of λ-models, since it is well-known that effective combinatory algebras do not exist (see [10, Ch. 5]).

We want this class to be

- general enough to contain all i-models generated by effective completions of finite partial i-webs, but

- well-behaved enough to be closed under certain direct products and to have a computable interpretation function.

As a result of our analysis, we find convenient to use the following class of mathematical objects.

Definition 4.4.3. *A SLM $\mathbf{D} = (\mathcal{D}, \cdot, \mathbf{k}, \mathbf{s})$ is effective if there exists a numbering ν of $\mathcal{K}(\mathcal{D})$ such that*

- *$(\mathcal{D}, \nu)$ is an effective Scott domain,*

- *$(\mathcal{K}(\mathcal{D}), \cdot)$ is an effective applicative substructure of $\mathbf{D}$ (w.r.t. ν),*

- *$\mathbf{k}, \mathbf{s} \in \mathcal{D}^{r.e.}$,*

- *$\mathbf{i} \in \mathcal{D}^{dec}$.*

The class of effective Scottian λ-models will be noted ESLM; committing an abuse of notation we will use the same acronym to speak of a single effective Scottian λ-model and we will let the reader distinguish between these two different usages.

We remark that if $\mathbf{D}$ is an ESLM, then $\{y \in \mathcal{D}^{r.e.} : y \leq \mathbf{i}\}$ is a completely co-r.e. set, by Lemma 4.2.3.

Indeed the intuition suggests that ESLM cannot be closed under arbitrary direct products: such class is instead closed under the formation of *effective products*, which are direct products of *effective families* (see the forthcoming Defintion 4.4.4). Intuitively a family of ESLMs is effective if there is a uniform way of generating the effective predicates and functions of its members.

We recall that the union of a countable family of countable numerations can be numbered in a standard way. We will assume this fact in the forthcoming definition, which is an adaptation of the definition of effective family of effective rings (see for example [55, Ch. 12]).

Definition 4.4.4. *A countable family* $\{(\mathbf{D}_n, \nu^{(n)})\}_{n\in\mathbb{N}}$ *of* ESLM*s is* effective *if*

- *the relation* $\{(x,y) \in \mathcal{K}(\mathcal{D}_m)^2 : \exists z \in \mathcal{K}(\mathcal{D}_m). \ z \geq_m x, y\}$ *is* $\nu^{(m)}$*-decidable uniformly in* m,

- *the relation* $\{(x,y,z) \in \mathcal{K}(\mathcal{D}_m)^3 : z = x \sqcup_m y\}$ *is* $\nu^{(m)}$*-decidable uniformly in* m,

- *the function* $\cdot_m : \mathcal{K}(\mathcal{D}_m)^2 \to \mathcal{K}(\mathcal{D}_m)$ *is* $(\nu^{(m)} \times \nu^{(m)}, \nu^{(m)})$*-computable uniformly in* m,

- *the set* $\{x \in \mathcal{K}(\mathcal{D}_m) : x \leq_m \mathbf{k}_m\}$ *is* $\nu^{(m)}$*-r.e. uniformly in* m,

- *the set* $\{x \in \mathcal{K}(\mathcal{D}_m) : x \leq_m \mathbf{s}_m\}$ *is* $\nu^{(m)}$*-r.e. uniformly in* m,

- *the set* $\{x \in \mathcal{K}(\mathcal{D}_m) : x \leq_m \mathbf{i}_m\}$ *is* $\nu^{(m)}$*-decidable uniformly in* m.

Proposition 4.4.1. *The direct product of an effective family of* ESLM*s is an* ESLM*.*

Proof. Let $\{(\mathbf{D}_n, \nu^{(n)})\}_{n\in\mathbb{N}}$ be an effective family of ESLMs: we will prove each point of Definition 4.4.3. First of all recall that by Proposition 3.5.2 the product is still an SLM.

Let $P = \prod_{n\in\mathbb{N}} D_n$ and $\mathcal{P} = (P, \sqsubseteq)$, where $\sqsubseteq$ is the pointwise ordering. For simplicity we indicate an element $\langle d_n \rangle_{n\in\mathbb{N}} \in \mathcal{P}$ as a function $\lambda\!\!\!\lambda\, n \in \mathbb{N}. \ d_n$. We now devise a numbering ξ of $\mathcal{K}(\mathcal{P})$ as follows:

$$\xi(n) = \begin{cases} \lambda\!\!\!\lambda\, m \in \mathbb{N}. \ (\text{if } (m \in \{s_1, \ldots, s_k\}) \text{ then } \nu^{(m)}(t_j) \text{ else } \perp_m) \\ \quad \text{if } n = (\langle s_1, t_1 \rangle, \ldots, \langle s_k, t_k \rangle)^* \text{ and } \forall j, h \in [1,k]. \ s_j \neq s_h \\ \lambda\!\!\!\lambda\, m \in \mathbb{N}. \ \perp_m \\ \quad \text{otherwise} \end{cases}$$

Recall the idea of support of a sequence given in Definition 3.5.2, §3. Given a compact element ξ_m we can indeed effectively compute $\mathsf{su}(\xi_m) = \{s_1, \ldots, s_k\}$ and the sequence $(t_1, \ldots, t_k)$ of integers such that for all $i = 1, \ldots, k$ the s_i-th coordinate of ξ_m is the element $\xi_m(s_i) = \nu^{(s_i)}(t_i) \in \mathcal{K}(\mathcal{D}_{s_i})$.

Now let for compact elements ξ_m, ξ_n, ξ_h of $\mathcal{P}$ we have that

- ξ_m and ξ_n have upper-bound iff for all $s \in \mathsf{su}(\xi_m) \cup \mathsf{su}(\xi_n)$, $\xi_m(s)$ and $\xi_n(s)$ have upper-bound in $\mathcal{D}_s$

- $\xi_h = \xi_m \sqcup \xi_n$ iff for all $h \in \mathsf{su}(\xi_m) \cup \mathsf{su}(\xi_n)$, $\xi_h(s) = \xi_m(s) \sqcup \xi_n(s)$

and the above predicates are ξ-decidable, by the uniformity of the predicates of de components of the product. This proves that $(\mathcal{P}, \xi)$ is an effective Scott domain. Similarly one can prove that the application function on $\mathcal{K}(\mathcal{P})$ is ξ-computable, that $\mathbf{k}, \mathbf{s} \in \mathcal{P}^{r.e.}$ and $\mathbf{i} \in \mathcal{P}^{dec}$. $\square$

Corollary 4.4.2. *For any* ESLM $\mathbf{D}$*, the set* Env_D *of valuations from* Var *to* $\mathbf{D}$ *(which is the power* $\mathbf{D}^{\mathrm{Var}}$*) is an* ESLM*.*

Proof. Since Var can be equipped with a bijective numbering. Uniformity is immediate since the effective predicates and functions are the same for each coordinate. $\square$

We now give some theorems stating important properties of ESLMs w.r.t the computability of the interpretation function.

Proposition 4.4.3. *If* $\mathbf{D}$ *is an effective* SLM*, then* $(\mathcal{D}^{r.e.}, \cdot, \mathbf{k}, \mathbf{s})$ *is a combinatory subalgebra of* $\mathbf{D}$*.*

Proof. It is not difficult to see, that the continuity of application and its computability on the compact elements imply that the application is an r.e. element of $\mathbf{Sd}(\mathcal{D} \times \mathcal{D}, \mathcal{D})^{r.e.}$, by analyzing its trace (see Proposition 4.2.5). Finally again by Proposition 4.2.5 we conclude that $\mathcal{D}^{r.e.}$ is closed under application. $\square$

In the following theorem we let $\Lambda_\perp^o$ (resp. $\mathscr{C}_\perp^o$) be the flat domain of closed λ-terms (resp. closed combinatory terms) obtained by adjoining a constant $\perp$. We also let $\sharp(-) : \mathbb{N} \to \mathscr{C}_\perp^o$ be a bijective numbering obtained by a Gödel encoding like that of Example 4.1.4. Note that $(\mathscr{C}_\perp^o)^{r.e.} = \mathscr{C}_\perp^o$ (and $(\Lambda_\perp^o)^{r.e.} = \Lambda_\perp^o$ as well).

Theorem 4.4.4. *Let* $(\mathbf{D}, \nu)$ *be an* ESLM*. Then*

(i) $[\![M]\!]^{\mathbf{D}} \in \mathbf{D}^{r.e.}$*, for each* $M \in \Lambda_\perp^o$*,*

(ii) the continuous function $[\![-]\!]^{\mathbf{D}} : \Lambda_\perp^o \to \mathbf{D}^{r.e.}$ *is* $(\sharp, \widehat{\nu})$*-computable.*

Proof. Recall that $[\![M]\!]^{\mathbf{D}} := [\![M_{cl}]\!]^{\mathbf{D}}$ by definition and the translation $M \mapsto M_{cl}$ is evidently computable (w.r.t. the involved numerations). In view of this fact, we can obtain items (i) and (ii) by proving that the trace of the interpretation function $[\![-]\!]^{\mathbf{D}} : \mathscr{C}_\perp^o \to \mathbf{D}$ is $(\sharp, \nu)$-r.e. and applying Proposition 4.2.5 (in a rather trivialized way, since $(\mathscr{C}_\perp^o)^{r.e.} = \mathscr{C}_\perp^o$ and $\widehat{\sharp} = \sharp$). By Proposition 4.4.3 we have that $\{d \in \mathcal{K}(\mathbf{D}) : d \leq_{\mathcal{D}} [\![t]\!]^{\mathbf{D}}\}$ is ν-r.e., for each $t \in \mathscr{C}_\perp^o$ and the trace $\{(t, d) \in \mathscr{C}_\perp^o \times \mathcal{K}(\mathbf{D}) : d \leq_{\mathcal{D}} [\![t]\!]^{\mathbf{D}}\}$ is the union of a uniform family of ν-r.e. sets. $\square$

The following theorems show that effective families of effective developments of finite partial i-webs give rise to effective Scottian λ-models.

Theorem 4.4.5. *If* $\mathbf{S}_\omega$ *is an effective completion of a finite partial i-web, then* $\mathbf{S}_\omega^+$ *is an* ESLM.

Proof. By Proposition 3.5.6 §3 $\mathbf{S}_\omega^+$ is an SLM and by Theorem 4.2.9 its universe is an effective Scott domain. We prove the remaining defining properties of ESLM:

(1) its compact elements form an effective applicative substructure of $\mathbf{S}_\omega^+$,

(2) $[\![\lambda xy.x]\!] \in (\mathcal{S}_\omega^+)^{r.e.}$ and $[\![\lambda xyz.xz(yz)]\!] \in (\mathcal{S}_\omega^+)^{r.e.}$,

(3) $[\![\lambda x.x]\!] \in (\mathcal{S}_\omega^+)^{dec}$.

As for (1), we already know by Proposition 3.5.6 that the compact elements are closed under composition. The computability of application it is easily derivable looking at its definition

$$u \cdot v = \{\beta \in S_\omega : \exists a \subseteq_{\mathrm{f}} v. \ (a,\beta) \in \{(a',\beta') : \phi_\omega(a',\beta') \in u\}\!\downarrow_{S_\omega \Rightarrow S_\omega}\}$$

since ϕ_ω is a strongly computable function.

Concerning points (2),(3) we use the description of interpretation described in Theorem 3.2.3 §3.

As for (2) it is easy to prove, by induction on the structure of λ-terms, that $[\![M]\!]_\rho \in (\mathcal{S}_\omega^+)^{r.e.}$, for each $M \in \Lambda$, and each $\rho \in (\mathrm{Env}_{\mathcal{S}_\omega^+})^{r.e.}$. Hence in particular $[\![\lambda xy.x]\!], [\![\lambda xyz.xz(yz)]\!] \in (\mathcal{S}_\omega^+)^{r.e.}$.

As for (3) by definition of $\mathcal{S}_\omega$ we have that $x\!\downarrow_{\mathcal{S}_\omega} = x \ \cup (x \cap S)\!\downarrow_S$ and hence $\alpha \in x\!\downarrow_{\mathcal{S}_\omega}$ iff $\alpha \in x \vee (\alpha \in S \wedge \exists a \in (\mathrm{Con}_S \cap x). \ a \vdash_S \alpha)$. The above predicate is clearly decidable, since Con_S is a finite set and the predicate $\alpha \in S$ is ζ-decidable (recall the definition of ζ from the proof of Theorem 4.3.2). Finally we have

$$
\begin{aligned}
[\![\lambda x.x]\!]^{\mathcal{S}_\omega} \ &= \{\phi_\omega(a,\alpha) : a \vdash_\omega \alpha\}\!\downarrow_{S_\omega} \\
&= \{\phi_0(a,\alpha) : a \vdash_0 \alpha, \ (a,\alpha) \in \mathsf{do}(\phi_0)\}\!\downarrow_{S_0} \\
&\quad \cup \{(b,\beta) : \beta \in b, \ (b,\beta) \notin \mathsf{do}(\phi_0)\} \\
&\quad \cup \{(c,\gamma) : c \cap S_0 \vdash_0 \gamma \in b, \ (c,\gamma) \notin \mathsf{do}(\phi_0)\}
\end{aligned}
$$

This set is decidable since S_0 is finite and ϕ_n is completely computable, for each $n \in \mathbb{N}$. $\square$

Theorem 4.4.6. *Let* $\{\mathfrak{S}_m\}_{m \in \mathbb{N}}$ *be an effective family of effective developments of finite partial i-webs and let* $\{\mathbf{A}_m\}_{m \in \mathbb{N}}$ *be the family of all completions of such developments. Then the product* $\prod_{m \in \mathbb{N}} \mathbf{A}_m^+$ *is an* ESLM.

Proof. Each i-web $\mathbf{A}_m$ is an effective i-web by Theorem 4.3.2, and each $\mathbf{A}_m^+$ is an ESLM by Theorem 4.4.5. Finally the conditions of Definition 4.3.8 ensure that $\{\mathbf{A}_m^+\}_{m \in \mathbb{N}}$ is an effective family of ESLMs and hence we conclude by Proposition 4.4.1. $\square$

Incompleteness results for Scott semantics

This chapter contains the core results of the paper [34], which constitute the main theorems of the first part of this thesis. These theorems establish properties of order and equational theories of i-models (and hence also of coadditive RSDs) whose order theory extend that of a particular ESLM, and are organized into a general and systematic *method* that yield equational incompleteness and limitations of semantical proof methods.

Outline of the method. The main idea is to apply computability theory in the context of lambda models, as done in [18]. The key step for the proof is the construction of an effective Scottian λ-model $\mathbf{P}$ with the following properties:

(i) $Or(\mathbf{P}) \subseteq Or(\mathbf{A}^+)$, for every i-model $\mathbf{A}^+$ (here we use the finiteness property of i-models);

(ii) $Or(\mathbf{P})$ is not r.e. (use Visser's theorem 4.1.5 ([115]));

(iii) $Eq(\mathbf{P}) \neq \lambda\beta, \lambda\beta\eta$ (use Selinger's theorem [109, Cor. 4]).

Finally from item (i) one can extend results (ii),(iii) to all i-models and moreover applying Visser's theorem and Proposition 3.5.5 (which follows from Böhm's Theorem) one can infer (non-constructively) the existence of an easy term which cannot be proved easy in the class iM.

In particular this last point makes use of the following formal definition, of "to be proved easy in a class of models".

Definition 5.0.5. *Let $\mathfrak{B}$ be a class of λ-models. A closed λ-term M is $\mathfrak{B}$-easy if for each closed λ-term N there exists a member $\mathbf{B}$ of $\mathfrak{B}$ such that $[\![M]\!]^{\mathbf{B}} = [\![N]\!]^{\mathbf{B}}$.*

The property of $\mathfrak{B}$-easiness allows to somehow measure the power of the semantical proof methods that can be developed in $\mathfrak{B}$. For example on may wonder whether or not a given class of models can be used to prove the easiness of all easy terms. The following theorem gives a sufficient conditions for proving that a class $\mathfrak{B}$ cannot be used to give semantical proofs of easiness for all easy terms.

A particular instance of Definition 5.0.5 is Alessi-Lusin's *simple easiness* [7], but we postpone the discussion of this point to Section 5.2.

5.1 The main theorems

Theorem 5.1.1. *Let* **P** *be an* ESLM*. For any non-trivial order λ-theory O if $Or(\mathbf{P}) \subseteq O$, then O is not r.e.*

Proof. Lemma 4.2.3 §4 the set $X = \{a \in \mathbf{P}^{r.e.} : a \leq [\![\mathbf{I}]\!]^{\mathbf{P}}\}$ is completely co-r.e. Now let $Y = \{N \in \Lambda^o : [\![N]\!]^{\mathbf{P}} \leq [\![\mathbf{I}]\!]^{\mathbf{P}}\}$. We now observe the following things:

- Y is a $\lambda\beta$-closed subset of Λ^o, since it is the anti-image of X via the interpretation function and for the same reason Y is co-r.e., by applying Theorem 4.4.4(ii) and Proposition 4.1.4 §4;

- Y is non-trivial, since $Y \subset \Lambda^o$ (otherwise by Proposition 3.5.5 **P** would be trivial) and $\mathbf{I} \in Y$;

Now let $Z = \{N \in \Lambda^o : O \vdash N \leq \mathbf{I}\}$. We observe the following things:

- Z is $\lambda\beta$-closed;

- $Y \subseteq Z$, since $Or(\mathbf{P}) \subseteq O$;

- Z is non-trivial, since $\mathbf{I} \in Z$ and $Z \subset \Lambda^o$ (otherwise O would be trivial).

Now suppose, by the way of contradiction, that O is r.e. Then Z is an r.e. subset of Λ^o. Hence Y and $\Lambda^o - Z$ are two disjoint non-trivial $\lambda\beta$-closed and co-r.e. subsets of Λ^o, contradicting Theorem 4.1.5 §4. □

In the following theorem we want to speak about a very general class of λ-models. An *ordered λ-model* (OLM, for short) is a λ-model whose application is monotone in both arguments. Committing an abuse of notation we will also use OLM to denote the class of all ordered λ-models; we will let the reader distinguish between these two different usages. Clearly any OLM induces an order theory (as in Definition 3.5.3 §3) which is an order λ-theory (see Definition 2.5.1 and Proposition 3.5.3 §3). Finally OLM encompasses all classes of domain models encountered so-far, like SLM, iM and RSD.

Theorem 5.1.2. *Let* **P** *be an* ESLM*. Then for any subclass $\mathfrak{U}$ of* OLM *not containing trivial models, if $Or(\mathbf{P}) \subseteq \bigcap_{\mathbf{U} \in \mathfrak{U}} Or(\mathbf{U})$ then:*

(i) $Or(\mathbf{U})$ is not r.e., for each member $\mathbf{U}$ of $\mathfrak{U}$,

(ii) $Eq(\mathbf{U}) \neq \lambda\beta, \lambda\beta\eta$, for each member $\mathbf{U}$ of $\mathfrak{U}$,

(iii) $\bigcap_{\mathbf{U} \in \mathfrak{U}} Eq(\mathbf{U}) \neq \lambda\beta, \lambda\beta\eta$,

(iv) there exists an easy λ-term that is not $\mathfrak{U}$-easy.

Proof. (i) First observe that $Or(\mathbf{U})$ is a non-trivial order λ-theory. Applying Theorem 5.1.1 we conclude that it is not r.e.

(ii) If $Eq(\mathbf{U})$ is not r.e., then clearly $Eq(\mathbf{U}) \neq \lambda\beta$, $Eq(\mathbf{U}) \neq \lambda\beta\eta$. If $Eq(\mathbf{U})$ is r.e. then $Or(\mathbf{U})$ strictly contains $Eq(\mathbf{U})$ (since by (i) $Or(\mathbf{U})$ is not r.e.) and therefore $\leq_{\mathbf{U}}$ is non-trivial on the interpretation of λ-terms. The conclusion follows from Selinger's result stating that in any partially ordered λ-model, whose theory is $\lambda\beta$ ($\lambda\beta\eta$), the interpretations of distinct closed terms are incomparable [109, Cor. 4].

(iii) An easy re-adaptation of (ii).

(iv) Let E be the set of all easy terms. By Theorem 5.1.2(i) the set $Y = \{N \in \Lambda^o : [\![N]\!]^{\mathbf{P}} \leq [\![\mathbf{I}]\!]^{\mathbf{P}}\}$ is a nonempty co-r.e. set of λ-terms. By Theorem 4.1.5 the intersection of the nonempty co-r.e. set X and of the nonempty co-r.e. set E of all easy terms is co-r.e. and nonempty and hence there is a term $M \in X \cap E$. Clearly M is easy. Now suppose by the way of contradiction that M is $\mathfrak{B}$-easy. Then by definition there exists a member $\mathbf{U}$ of $\mathfrak{U}$ such that $[\![M]\!]^{\mathbf{U}} = [\![\lambda xy.x]\!]^{\mathbf{U}}$. From $M \in X$ it follows that $[\![M]\!]^{\mathbf{P}} \leq [\![\mathbf{I}]\!]^{\mathbf{P}}$ and by hypothesis this implies $[\![M]\!]^{\mathbf{U}} \leq [\![\mathbf{I}]\!]^{\mathbf{U}}$ so that in turn $[\![\lambda xy.x]\!]^{\mathbf{U}} \leq [\![\mathbf{I}]\!]^{\mathbf{U}}$. This contradicts Corollary 3.5.5.

$\square$

Comment. Clearly Theorem 5.1.2(ii) states the equational incompleteness of the class $\mathfrak{U}$. But even if no member of the class $\mathfrak{U}$ induces the theories $\lambda\beta$ and $\lambda\beta\eta$, one could still hope that the intersection of the equational theories of all members of the class were $\lambda\beta$ (or $\lambda\beta\eta$, if the models in $\mathfrak{U}$ are extensional). Theorem 5.1.2(iii) says that this cannot be the case: there are equations not belonging to $\lambda\beta$ (or $\lambda\beta\eta$) which hold in all members of $\mathfrak{U}$. Finally Theorem 5.1.2(iv) says that not all semantical proof of easiness can be carried out in the class $\mathfrak{U}$: however the proof does not give a concrete example of an easy term which is not $\mathfrak{U}$-easy.

The rest of this chapter is dedicated to the concrete construction of an effective Scottian λ-model with remarkable properties.

Definition 5.1.1. *We define an enumeration $\{\mathbf{E}_m\}_{m \in \mathbb{N}}$ of all finite partial i-webs whose tokens are natural numbers. First regard at m as the encoding of a quadruple $m = \langle n_1, n_2, n_3, n_4 \rangle$ where n_1 codes a finite subset X_1 of $\mathbb{N}$, n_2 codes a finite subset X_2 of $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$, n_3 codes a finite subset X_3 of $\mathcal{P}_{\mathrm{f}}(\mathbb{N}) \times \mathbb{N}$, and n_4 codes a finite subset X_4 of $\mathcal{P}_{\mathrm{f}}(\mathbb{N}) \times \mathbb{N} \times \mathbb{N}$.*

- *If X_2 is a consistency predicate for X_1, X_3 is an entailment for X_1, and X_4 is the graph of a partial b-morphism for the information system given by X_1, X_2, X_3, then $\mathbf{E}_m$ is the partial i-web defined by these data.*

- *Otherwise $\mathbf{E}_m$ is the partial i-web with empty web and completely undefined b-morphism.*

Now if $\mathbf{E}$ is a finite partial i-web of natural numbers then it has potentially infinitely many developments: we can effectively generate any finite subfamily of any development of $\mathbf{E}$ according to the following algorithm, which is supposed to run on a multitasking system (recall the notion of compatible family and of extension from definitions 3.3.8 and 3.3.7, §3):

```
1 Completion(E)
2 begin
3     foreach X ⊆ 𝒫_f(E ∪ do(φ_E)) do
4         if (X is E-compatible) then
5             E' := Extension(E, X);
6             run Completion(E') for 2 seconds;

7 end
```

The objective is to stress that $\texttt{Extension}(\mathbf{E}, \mathcal{X})$ can be considered implementable by a real algorithm, since once an $\mathbf{E}$-compatible family $\mathcal{X}$ is given, all data of the extension of $\mathbf{E}$ by $\mathcal{X}$ are effectively determined.

Comment. Let $\mathbf{E}_n$ be the n-th member of $\mathfrak{E}$. The effective partial i-webs produced by $\texttt{Completion}(\mathbf{E}_n)$ form a **finitely branching** $\prec$-tree with root $\mathbf{E}_n$, since every finite partial i-web admits a finite number of extensions (corresponding to the finite number of compatible families for it).
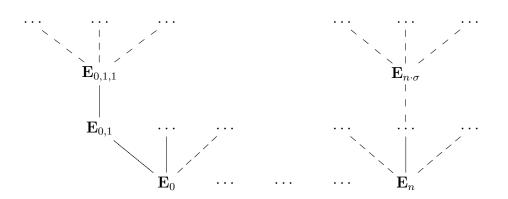
```
1 EnumerateCompletions
2 begin
3     foreach n ∈ ℕ do
4         run Completion(E_n) for 2 seconds;
5 end
```

Let us call $\mathfrak{F}$ the set of all finite partial i-webs produced by "running forever" the program $\texttt{EnumerateCompletions}$. We can index these partial i-webs by sequences of natural numbers as follows:

- the index of a root $\mathbf{E}_n$ is the sequence containing just n;

- given a finite partial i-web $\mathbf{E}_\sigma$, the indices of its extensions are $\sigma \cdot 1, \ldots, \sigma \cdot k$, where k is the number of $\mathbf{E}_\sigma$-compatible families and $\sigma \cdot i$ is the concatenation of σ and the sequence containing just the number i.

For each sequence σ, there is a moment of time at which $\mathbf{E}_\sigma$ is produced by the program $\texttt{EnumerateCompletions}$.

Remark 5.1.3. $\mathfrak{F}$ *is a $\prec$-forest in which the developments of the finite partial i-webs $\mathbf{E}_n$ (for $n \in \mathbb{N}$) are exactly the $\prec$-chains which are both upwards and downwards closed in $\mathfrak{F}$.*

Definition 5.1.2. *We let $\mathfrak{S}_m$ be the unique infinite branch of the tree with root $\mathbf{E}_m$ all whose members are finite partial i-webs with full consistency (this exists because of Remark 3.3.8).*

Theorem 5.1.4. *$\{\mathfrak{S}_m\}_{m \in \mathbb{N}}$ is an effective family of effective developments of finite partial i-webs.*

Definition 5.1.3. *We let* fiW *be the class of all i-webs with full consistency and we let* fiM *be the class of all i-models $\mathbf{A}^+$ with $\mathbf{A}$ in* fiW.

Theorem 5.1.5. *Let $\mathbf{C}_m$ be the completion of the development $\mathfrak{S}_m$ and let $\mathbf{P} = \prod_{m \in \mathbb{N}} \mathbf{C}_m^+$. Then $\mathbf{P}$ is an* ESLM.

Proof. Each $\mathfrak{S}_m$ is an effective development of some finite partial i-web of natural numbers with full consistency and $\{\mathfrak{S}_m\}_{m \in \mathbb{N}}$ is an effective family of effective developments of finite partial i-webs with full consistency. Therefore by Theorem 4.4.6 $\mathbf{P}$ is an ESLM. $\qquad\square$

Let $\mathbf{P}$ be the ESLM constructed at the end of §4.

Theorem 5.1.6. *$Or(\mathbf{P}) \subseteq \bigcap_{\mathbf{A} \in\ \mathsf{fiW}} Or(\mathbf{A}^+)$.*

Proof. By Theorem 3.4.1 §3 for any i-web $\mathbf{A}$ with full consistency and any inequality $M \sqsubseteq N$ which fails in $\mathbf{A}^+$ there exists a finite partial i-web $\mathbf{S} \prec \mathbf{A}$ and a suitable completion $\mathbf{S}_\omega$ with full consistency such that $M \sqsubseteq N$ fails in $\mathbf{S}_\omega^+$. Now recall that $\mathbf{P} = \prod_{m \in \mathbb{N}} \mathbf{C}_m^+$, where each $\mathbf{C}_m$ is an i-web with full consistency. We conclude observing that the i-web $\mathbf{S}_\omega$ is, up to bf-isomorphism, one of the $\mathbf{C}_m$'s. $\qquad\square$

Corollary 5.1.7. *Let $\mathbf{A}$ be an i-web with full-consistency. Then $Or(\mathbf{A}^+)$ is not r.e., $Eq(\mathbf{A}^+) \neq \lambda\beta$ and $Eq(\mathbf{A}^+) \neq \lambda\beta\eta$.*

Corollary 5.1.8. *There are equations not in $\lambda\beta, \lambda\beta\eta$ holding in all i-models in* fiM.

Corollary 5.1.9. *There exists a closed λ-term that is not* fiM-*easy.*

From Corollary 5.1.9 it follows simply the solution of Problem 19 of TLCA list [4].

5.2 The solution of Problem 19 of TLCA list

The purpose of this section is to explain why Corollary 5.1.9 solves Problem 19 of TLCA list.

We review the class of filter models of λ-calculus that arise from easy intersection type systems. Alessi and Lusin [7] have shown the easiness of the simple easy λ-terms through this class of models. More precisely, given a simple easy λ-term M and an arbitrary closed λ-term N there exists an easy intersection type system which generates a filter model satisfying the identity $M = N$.

The point that we want to make clear is the following: if a term is simple easy, then it is fiM-easy.

5.2.1 Easy intersection type theories and filter models

An *intersection type language* $\mathbb{T}$ is a set of formulas, called *types*, built on a given set of constants by means of the type constructors "$\wedge$" and "$\rightarrow$". The constant ω belongs to any intersection type language. The letter $\alpha, \beta, \gamma, \delta$ will range over constants (different from ω), while $\sigma, \tau, \ldots$ over types.

The concept of an easy intersection type theory over an intersection type language was defined for the first time in Alessi et al. [5, Def. 2].

Definition 5.2.1. *[5, Def. 2],[7, Def. 1.2] An* easy intersection type theory *(eitt, for short) over an intersection type language $\mathbb{T}$ is the set of inequalities of the form $\sigma \leq \tau$ ($\sigma, \tau \in \mathbb{T}$) derivable from a collection $\mathcal{T}$ of axioms and rules such that:*

(1) $\mathcal{T}$ contains the following axioms and rules

$$\overline{\sigma \leq \omega} \qquad \overline{\omega \leq \omega \rightarrow \omega} \qquad \overline{\sigma \leq \sigma \wedge \sigma}$$

$$\overline{\sigma \wedge \tau \leq \sigma} \qquad \overline{\sigma \wedge \tau \leq \tau} \qquad \overline{(\sigma \rightarrow \tau) \wedge (\sigma \rightarrow \tau') \leq \sigma \rightarrow (\tau \wedge \tau')}$$

$$\frac{\sigma \leq \sigma' \quad \tau \leq \tau'}{\sigma \wedge \tau \leq \sigma' \wedge \tau'} \qquad \frac{\sigma' \leq \sigma \quad \tau \leq \tau'}{\sigma \rightarrow \tau \leq \sigma' \rightarrow \tau'}$$

It is customary to define an equivalence relation $\sim$ on types as follows: $\sigma \sim \tau$ iff $\sigma \leq \tau \leq \sigma$.

(2) *Besides the axioms and rules of item (1), $\mathcal{T}$ does not contain further rules and it only contains axioms of the following two shapes: $\alpha \leq \beta$ or $\alpha \sim \bigwedge_{i \in I}(\gamma_i \to \tau_i)$, where α, β, γ_i are constants with $\alpha, \beta \not\equiv \omega$, and $\tau_i \in \mathbb{T}$;*

(3) *for each constant $\alpha \not\equiv \omega$ there exists exactly one axiom of the shape $\alpha \sim \bigwedge_{i \in I}(\gamma_i \to \tau_i)$;*

(4) *if $\mathcal{T}$ contains $\alpha \sim \bigwedge_{i \in I}(\gamma_i \to \tau_i)$ and $\beta \sim \bigwedge_{j \in J}(\delta_j \to \sigma_j)$, then $\mathcal{T}$ contains also $\alpha \leq \beta$ iff for each $j \in J$ there exists $i_j \in I$ such that $\delta_j \leq \gamma_{i_j}$ and $\tau_{i_j} \leq \sigma_j$.*

We ambiguously denote by $\mathcal{T}$ the eitt generated by the set $\mathcal{T}$ of rules and axioms.

We write $\sigma \leq_{\mathcal{T}} \tau$ to indicate that $\sigma \leq \tau$ is derivable from $\mathcal{T}$. A *filter* of a eitt $\mathcal{T}$ is a nonempty subset $X \subseteq \mathbb{T}$ which is upward closed w.r.t. $\leq_{\mathcal{T}}$ and closed under $\wedge$; the filter generated by a subset Y of $\mathbb{T}$ will be denoted by $\uparrow Y$. $\mathcal{F}_{\mathcal{T}}$ denotes the set of all filters of $\mathcal{T}$.

For any eitt $\mathcal{T}$ it is possible to define a filter model in the ccc **ALat** of algebraic lattices and Scott-continuous functions. We report such construction from [38].

Theorem 5.2.1. *([38, 7]) The triple $(\mathcal{F}_{\mathcal{T}}, F, G)$ is a reflexive object in the category* **ALat** *via the maps $F : \mathcal{F}_{\mathcal{T}} \to [\mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}]$ and $G : [\mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}] \to \mathcal{F}_{\mathcal{T}}$ defined by*

$$F(X)(Y) = \{\tau \in \mathbb{T} : \exists \sigma \in Y. \, \sigma \to \tau \in X\}; \quad G(f) = \uparrow \{\sigma \to \tau \in \mathbb{T} : \tau \in f(\uparrow \sigma)\}.$$

The interpretation of a closed λ-term M in the filter model generated by the eitt $\mathcal{T}$ is denoted by $[\![M]\!]^{\mathcal{T}}$.

Let $\mathcal{T}, \mathcal{S}$ be eitt over the type languages $\mathbb{T}$ and $\mathbb{S}$ respectively. We say that $\mathcal{S}$ is a *conservative extension* of $\mathcal{T}$, written $\mathcal{T} \sqsubseteq \mathcal{S}$, if $\mathbb{T} \subseteq \mathbb{S}$ and, for all $\tau, \sigma \in \mathbb{T}$, $\tau \leq_{\mathcal{T}} \sigma$ iff $\tau \leq_{\mathcal{S}} \sigma$.

Definition 5.2.2. *An unsolvable term M is* simple easy *if for every eitt $\mathcal{T}$ over the type language $\mathbb{T}$ and every type $\tau \in \mathbb{T}$ there exists a conservative extension $\mathcal{S}$ of $\mathcal{T}$ such that*

$$\sigma \in [\![M]\!]^{\mathcal{S}} \text{ iff } (\exists \sigma' \in \mathbb{T}) \, \sigma' \wedge \tau \leq_{\mathcal{S}} \sigma \text{ and } \sigma' \in [\![M]\!]^{\mathcal{T}},$$

for all types σ in the type language of $\mathcal{S}$.

Theorem 5.2.2. *[7, Thm. 3.5] Let M be a simple easy term. Then M is easy, because, for every closed term N, there exists an eitt $\mathcal{T}$ such that $[\![M]\!]^{\mathcal{T}} = [\![N]\!]^{\mathcal{T}}$.*

5.2.2 Filter models as i-models

In this subsection we show that every eitt is an i-web and that every filter model built over an eitt is an i-model; this is of course an explanation of the reason why our results concerning i-models does apply to the problem posed by Alessi and Dezani-Ciancaglini.

Let $\mathbb{T}$ be a type language. As a matter of notation, if $a = \{\sigma_1, \ldots, \sigma_n\} \subseteq_{\mathrm{f}} \mathbb{T}$ we write $\wedge a$ as a shorthand for $\sigma_1 \wedge \ldots \wedge \sigma_n$.

We define the structure $\mathcal{A}_{\mathcal{T}} = (\mathbb{T}, \vdash_{\mathcal{T}})$ by setting $a \vdash_{\mathcal{T}} b$ ($a, b \subseteq_{\mathrm{f}} \mathbb{T}$) iff $\wedge a \leq \wedge b$, and this way obviously we get $\emptyset \vdash_{\mathcal{T}} \omega$. Indeed $\mathcal{A}_{\mathcal{T}}$ is an information system (a similar observation appears already in [38]). As a consequence, in the exponential $\vec{\mathcal{A}}_{\mathcal{T}}$ of $\mathcal{A}_{\mathcal{T}}$ we have $\{(a_1, b_1), \ldots, (a_n, b_n)\} \vdash_{\vec{\mathcal{A}}_{\mathcal{T}}} (c, d)$ iff either $\omega \leq \wedge d$ or there exists $J \subseteq \{1, \ldots, n\}$, $J \neq \emptyset$ such that $\wedge c \leq \wedge(\cup_{j \in J} a_j)$ and $\wedge(\cup_{j \in J} b_j) \leq \wedge d$.

In order to define an i-web we now set

$$\phi_{\mathcal{T}} : \mathcal{P}_{\mathrm{f}}(\mathbb{T}) \times \mathcal{P}_{\mathrm{f}}(\mathbb{T}) \to \mathbb{T} \quad \text{with} \quad \phi_{\mathcal{T}}(a, b) = \wedge a \to \wedge b.$$

Now it is clear that $\phi_{\mathcal{T}}$ is a b-morphism if, and only if, the following implication holds:

$$\bigwedge_{i=1}^{n} (\wedge a_i \to \wedge b_i) \leq \wedge c \to \wedge d \Rightarrow$$

either $\omega \leq \wedge d$ or $\exists J \subseteq \{1, \ldots, n\}$, $J \neq \emptyset$. $\wedge c \leq \wedge(\cup_{j \in J} a_j)$ and $\wedge(\cup_{j \in J} b_j) \leq \wedge d$.

We can conclude by observing that the above implication holds in any eitt (see [5, Thm. 2.3]). Hence $\mathbf{A}_{\mathcal{T}} = (\mathcal{A}_{\mathcal{T}}, \phi_{\mathcal{T}})$ is an i-web.

It is an easy matter to show that the i-model $\mathbf{A}_{\mathcal{T}}^{+} = (\mathcal{A}_{\mathcal{T}}^{+}, (\phi_{\mathcal{T}})_1, (\phi_{\mathcal{T}})_2)$ (see Definition 3.2.2, §3) coincides with the filter λ-model $\mathcal{F}_{\mathcal{T}}$ of Theorem 5.2.1.

II

Part two

6

Introduction

In the '90s Boudol [24] introduced the *λ-calculus with multiplicities*, an extension of λ-calculus where arguments may come in limited availability and mixed together. After one decade Ehrhard and Regnier [44] introduced the *differential λ-calculus*, a conservative (see [44, Prop. 19]) extension of the λ-calculus with differential constructions, in which the linear application of a term M to an argument roughly corresponds to applying the derivative of M in 0 (which is a linear function) to that argument. The presence of linear application, and linear substitution force the enrichment of the calculus with an operation of sum with a neutral element. In [45, 47] Ehrhard and Regnier introduce a simple subsystem of the differential λ-calculus, that they call *resource λ-calculus*, and establish a correspondence between *differential nets*, a variation of Girard's [59] linear logic proof-nets (without promotion rule), and resource λ-calculus. Very recently, Tranquilli [112] enriched the resource λ-calculus with a promotion operator (bearing strong similarities to Boudol's λ-calculus with multiplicities), establishing a correspondence with differential interaction nets extended with promotion. Tranquilli's resource calculus has been recently studied from the syntactical point of view by Pagani and Tranquilli [91], for confluence results, and by Pagani and Ronchi Della Rocca [90] for results about solvability. Regarding the semantics of these calculi, the first studies were conducted by Boudol et al. [25] for the λ-calculus with multiplicities. In [29] Bucciarelli et al. define categorical models for the differential λ-calculus.

There have been several attempts to reformulate the λ-calculus as a purely algebraic theory. The earliest and best known algebraic models are the *combinatory algebras* of Schönfinkel and Curry [39]. Combinatory algebras, as well as their remarkable subclass of λ-*algebras*, have a purely equational characterization but yield somewhat weak notions of models of the λ-calculus. In fact, the combinatory interpretation of λ-calculus does not satisfy the so-called ξ-rule: under the interpretation, $M = N$ does not necessarily imply $\lambda x.M = \lambda x.N$. Thus, the class of λ-algebras is not sound for λ-theories, and one is forced to consider the non-equational class of λ-models (see [10]). There are many advantages in using algebraic languages rather than languages with binders, particularly in connection with equational reasoning. The former have well-understood model theory, and the models are closed under standard constructions such as cartesian products, subalgebras, quotients and free

algebras. The above-mentioned problem with the ξ-rule seems to suggest that the λ-calculus is not quite equivalent to an algebraic theory. The lattice of λ-theories is isomorphic to the congruence lattice of the term algebra of the least λ-theory $\lambda\beta$. This remark is the starting point for studying λ-calculus by universal algebraic methods, through the variety (i.e. equational class of algebras) generated by the term algebra of $\lambda\beta$, which Salibra [101] has shown to be axiomatized by the finite scheme of identities characterizing λ-abstraction algebras. These algebras, introduced by Pigozzi and Salibra [93], are intended as an alternative to combinatory algebras, which keeps the lambda notation and hence all the functional intuitions. In [94] the connections between the variety of λ-abstraction algebras and the other algebraic models of λ-calculus are explained; it is also shown that the free extension of a λ-algebra can be turned into a λ-abstraction algebra, thus validating all rules of the λ-calculus, including the ξ-rule. The algebraic approach to λ-calculus has been fruitful in studying the structure of the lattice of λ-theories and in generalizing the Stone representation theorem for Boolean algebras to combinatory and λ-abstraction algebras (see [80, 84, 83]). The Stone theorem has been also applied to provide an algebraic incompleteness theorem that encompasses incompleteness results for all known semantics of λ-calculus.

Chapter 9 contains an extended survey of the results obtained in our paper [33], in which we initiate a purely algebraic study of Ehrhard and Regnier's resource λ-calculus, by introducing three equational classes of algebras: *resource combinatory algebras*, *resource lambda-algebras* and *resource lambda-abstraction algebras*. We establish the relations between them, laying down foundations for a model theory of resource λ-calculus. We also show that the ideal completion of a resource combinatory (resp. lambda-, lambda-abstraction) algebra induces a "classical" combinatory (resp. lambda-, lambda-abstraction) algebra, and that any model of the classical λ-calculus raising from a resource lambda-algebra determines a λ-theory which equates all terms having the same Böhm tree.

7
Preliminaries

This chapter is organized as follows: Section 7.1 recalls some basic notions regarding semi-rings and modules, which are used in the presentation of resource lambda calculus and will be the base for formulating our generalization of the ordinary category of sets and relations. In Section 7.2 we review the preliminaries regarding the resource lambda calculus. In Section 7.3 we survey a presentation of the category of sets and relations.

7.1 Semirings and modules

The theory developed in the second part of this thesis relies on basic concepts of linear algebra. A classical and complete reference on the subject is Serge Lang's book [77]. The reader familiar with the concepts regarding semirings, modules and matrices may entirely skip this section, which is included just for the purpose of self-containment of the exposition.

Definition 7.1.1. *A* semiring *is an algebra* $\mathbf{A} = (A, +, \cdot, 0, 1)$ *such that*

(S1) $(A, +, 0)$ *is a commutative monoid,*

(S2) $(A, \cdot, 1)$ *is a monoid,*

(S3) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ $\qquad (x + y)$ *and* $z = (x \cdot z) + (y \cdot z)$

(S4) $0 \cdot x = x \cdot 0 = 0$

A semiring $\mathbf{A}$ is *commutative* if $(A, \cdot, 1)$ is a commutative monoid and it is *idempotent* if $(A, +, 0)$ is a join-semilattice with bottom.

Note that the difference between rings and semirings is that addition yields only a commutative monoid, not necessarily a commutative group. Specifically, elements in semirings do not necessarily have an inverse for the addition. The last axiom is omitted from the definition of a ring because it follows automatically from the the other ring axioms using the fact that $(A, +, 0)$ is a group; for semirings instead it has to be stated explicitly. As usual the symbol "$\cdot$" will be afterwards omitted from the notation, i.e., $x \cdot y$ is simply written xy.

Definition 7.1.2. *A* left **A**-module **M** *over a semiring* $\mathbf{A} = (A, +, \cdot, 0, 1)$ *is a monoid* $\mathbf{M} = (M, \oplus, \overline{0})$ *together with an operation* $(\_ \odot \_) : A \times M \to M$ *(called* scalar multiplication*) such that for all* $p, q \in A$ *and all* $x, y \in M$ *we have*

(M1) $p \odot (x + y) = (p \odot x) \oplus (p \odot y)$

(M2) $(p + q) \odot x = (p \odot x) \oplus (p \odot x)$

(M3) $(p \cdot q) \odot x = p \odot (q \odot x)$

(M4) $1 \odot x = x$

In an expression of the form $p \odot x$, the element $p \in A$ is called a *scalar*, or *coefficient*. *Right* **A**-*modules* are defined analogously to left **A**-modules, but with a scalar multiplication $(\_ \odot \_) : M \times A \to M$.

Of course every semiring **A** is itself an **A**-module by defining the scalar multiplication as the multiplication of the semiring.

There is another less obvious standard example of **A**-module, given by a construction that we will use heavily in the rest of this chapter.

Definition 7.1.3. *Given a set X and a semi-ring **A**,* free **A**-module over X, *notation,* $\mathbf{A}\langle X \rangle$ *is the* **A**-*module whose universe is the set $A\langle X \rangle$ of all functions $\mu : X \to A$ such that* $\mathsf{su}(\mu) = \{x \in X : \mu(x) \neq 0\}$ *(the* support *of μ). The sum of two elements $\mu, \nu \in A\langle X \rangle$ is defined pointwise, i.e. $(\mu + \nu)(x) = \mu(x) + \nu(x)$ for all $x \in X$. Note that we use the same symbol for the addition in **A** and $\mathbf{A}\langle X \rangle$. We also keep the notation 0 for the function given by $0(x) = 0$ for all $x \in X$: this is of course the neutral element of the monoid $\mathbf{A}\langle X \rangle$ with respect to sum. The scalar multiplication of an element $\mu \in A\langle X \rangle$ by a coefficient $p \in \mathbf{A}$ is again given pointwise, i.e., $(p\mu)(x) = p\mu(x)$, for all $x \in X$. Note that once again do use the same notation for multiplication of **A** and scalar multiplication of $\mathbf{A}\langle X \rangle$.*

Note that since **A** has a multiplicative unit, X can be naturally seen as a subset of $\mathbf{A}\langle X \rangle$.

Notation and terminology: the elements of $\mathbf{A}\langle X \rangle$ will be called **A**-*multisets over X*, or simply **A**-multisets, when X is clear from the context. Given $x \in X$, we denote as $[x] \in \mathbf{A}\langle X \rangle$ the function given by $[x](y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ (i.e., $[x](y) = \delta_{x,y}$, the *Kronecker operator*. If $\mu \in \mathbf{A}\langle X \rangle$ we define the *cardinality of μ* by $\sharp\mu = \Sigma_{x \in \mathsf{su}(\mu)}\mu(x) \in A$. In general then an element of $\mathbf{A}\langle X \rangle$ is a finite sum of the form $\Sigma_{i=1}^{n} p_i[x_i]$.

Notation and terminology: For an **A**-module **B** we denote by $\mathbf{B}^{n,m}$ the **A**-module of all $n \times m$ matrices over B with usual addition and scalar multiplication. If $M \in \mathbf{B}^{n,m}$ is a matrix we write $M^{\perp}$ in order to indicate its *transpose*, i.e. the matrix in $\mathbf{B}^{m,n}$ obtained from M by exchanging rows and columns. We will also

make use of the multiplication of $n \times m$ matrices over $\mathbf{A}$ by $m \times k$ matrices over $\mathbf{B}$ given by

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix} \begin{pmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mk} \end{pmatrix} = \begin{pmatrix} \Sigma_{j=1}^{m} p_{1j}s_{j1} & \cdots & \Sigma_{j=1}^{m} p_{1j}s_{jk} \\ \vdots & \ddots & \vdots \\ \Sigma_{j=1}^{m} p_{nj}s_{j1} & \cdots & \Sigma_{j=1}^{m} p_{nj}s_{jk} \end{pmatrix}$$

If 1 is the multiplicative unit of $\mathbf{A}$, we indicate by $\mathbf{1}_n \in \mathbf{A}^{n,1}$ the vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Note that

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix} \mathbf{1}_m = \begin{pmatrix} \sum_{i=1}^{m} p_{1i} \\ \vdots \\ \sum_{i=1}^{m} p_{ni} \end{pmatrix} \quad \text{and} \quad \mathbf{1}_n^{\perp} \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} = \sum_{i=1}^{n} p_i$$

7.2 The resource lambda calculus

In this section we present the *resource lambda calculus* ($r\lambda$-calculus, for short), which belongs to a family of resource-sensible calculi [91, 90, 44, 114, 47]. In this thesis we will work with one particular version of these calculi, namely the one obtained by choosing coefficients in the boolean semiring $\mathbf{2}$. We will state and prove results about this calculus without worrying about making any further reference to this fact, just saying "resource lambda-calculus" (or just "$r\lambda$-calculus") instead of "resource lambda-calculus with truth-values coefficients".

The $r\lambda$-calculus has three syntactic categories: *terms* that are in functional position, *bags* that are in argument position and represent multisets of linear resources, and *finite sums of terms* representing all possible results of a computation.

The following grammar form the terms:

$$(\Lambda^r) \; t, s, p ::= x \mid \lambda x.t \mid tP \qquad \text{terms}$$
$$(\Lambda^b) \; P ::= [t_1, \ldots, t_k] \qquad (k \geq 0) \qquad \text{bags}$$

Terms are the real protagonists of the $r\lambda$-calculus. The term $\lambda x.t$ represents the *λ-abstraction* and tP the *application* of a term t to a *bag* P, which is a multiset of terms. An application of the form $(\lambda x.t)P$ should be thought of as a linear function fed with collection of linear resources P, each of which is available exactly once for the function $\lambda x.t$.

Sums. There are many ways of "feeding" a function $(\lambda x.t)$ with a bag P of arguments, but instead of choosing one of them non-deterministically, the reduction keeps track of them all in form of a sum. These sums live in free module $\mathbf{2}\langle \Lambda^r \rangle$, where $\mathbf{2}$ is the semiring $\{0, 1\}$ with addition and multiplication given by join and meet, respectively. We use metavariables $\mathbb{T}, \mathbb{S}, \ldots$ to range over $\mathbf{2}\langle \Lambda^r \rangle$.

The α-*equivalence* relation and the set $\mathrm{FV}(t)$ of *free variables of t* are defined as usual, like in ordinary λ-calculus [10]. We write $\deg_x(t)$ for the number of free occurrences of x in t. Hereafter, sums of terms are considered up to α-equivalence, associativity, commutativity and idempotence of the sum.

Notice that the grammar for terms does not include any sums, so they may arise only on the "surface". However, as syntactic sugar – and *not* as actual syntax – we consider sums of bags ($\mathbb{P} \in \mathbf{2}\langle \Lambda^b \rangle$) and extend all the constructors to sums by multilinearity, setting for instance $(\Sigma_i t_i)(\Sigma_j P_j) := \Sigma_{i,j} t_i P_j$, in such a way that the following equations hold:

$$\lambda x.(\Sigma_i t_i) = \Sigma_i \lambda x.t_i$$
$$\mathbb{T}(\Sigma_i P) = \Sigma_i \mathbb{T} P_i \quad (\Sigma_i t_i)\mathbb{P} = \Sigma_i t_i \mathbb{P}$$
$$[\Sigma_i t_i] = \Sigma_i [t_i] \qquad (\Sigma_i P_i) \uplus \mathbb{P} = \Sigma_i P_i \uplus \mathbb{P}$$

where we recall that $\uplus$ is the union of multisets. As an example of this *extended syntax*, we can write $(x_1 + x_2)[y_1 + y_2]$ instead of $x_1[y_1] + x_1[y_2] + x_2[y_1] + x_2[y_2]$.

Observe that in the particular case of empty sums, we get $\lambda x.0 := 0$, $\mathbb{T}0 := 0$, $0\mathbb{P} := 0$, $[0] := 0$ and $0 \uplus \mathbb{P} := 0$. Thus 0 annihilates anything (note that, formally speaking, the symbol "0" occurring in these definitions is not always *the same* 0, but each time the additive neutral of the suitable module).

We now introduce two kinds of substitutions: the usual λ-calculus substitution and a linear one, which is proper to differential and resource calculi (see [47, 90, 91, 29, 114]).

Expressions $A, B \in \Lambda^r \cup \Lambda^b$ are either terms or bags and sums of expressions are ranged over by $\mathbb{A}, \mathbb{B} \in \mathbf{2}\langle \Lambda^r \rangle \cup \mathbf{2}\langle \Lambda^b \rangle$.

Let A be an expression and let $s \in \Lambda^r$. The *(capture-free) substitution of s for x in A*, denoted by $A\{s/x\}$, is defined as usual. Accordingly, $A\{\mathbb{S}/x\}$ denotes a term of the extended syntax. Last, we define the application of a substitution to a sum as in $\mathbb{A}\{\mathbb{S}/x\}$ by linearity in $\mathbb{A}$.

The *linear (capture-free) substitution of s for x in A*, denoted by $A\langle s/x \rangle$, is defined as follows[1]:

- $y\langle s/x \rangle = \begin{cases} s & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$

- $(\lambda y.t)\langle s/x \rangle = \lambda y.t\langle s/x \rangle$ with $y \notin \mathrm{FV}(s)$, $x \not\equiv y$,

- $(tP)\langle s/x \rangle = t\langle s/x \rangle P + t(P\langle s/x \rangle)$,

- $[t_1, \ldots, t_k]\langle s/x \rangle = \Sigma_{i=1}^k [t_1, \ldots, t_i\langle s/x \rangle, \ldots, t_k]$.

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[1]In this definition we strongly use the extended syntax.

Roughly speaking, linear substitution replaces the resource to *exactly one* linear free occurrence of the variable. In presence of multiple occurrences, all possible choices are made and the result is the sum of them.

For example $(y[x][x])\langle \lambda z.z/x \rangle = y[\lambda z.z][x] + y[x][\lambda z.z]$.

Turning to the extension of linear substitution to sums: the term $A\langle \mathbb{S}/x \rangle$ belongs to the extended syntax, and we define $\mathbb{A}\langle \mathbb{S}/x \rangle$ by linearity in $\mathbb{A}$, as we did for usual substitution.

Observe that $\mathbb{A}\langle \mathbb{S}/x \rangle$ is linear in $\mathbb{A}$ and in $\mathbb{S}$, whereas $\mathbb{A}\{\mathbb{S}/x\}$ is linear in $\mathbb{A}$ but not in $\mathbb{S}$.

Linear substitutions commute in the sense expressed by the next lemma, whose proof is rather classic and is omitted.

Lemma 7.2.1 (Schwarz Lemma, cf. [44, 46])**.** *For any sum of expressions* $\mathbb{A}$, *and any* $\mathbb{T}, \mathbb{S} \in \mathbf{2}\langle \Lambda^r \rangle$ *and* $y \notin \mathrm{FV}(\mathbb{T}) \cup \mathrm{FV}(\mathbb{S})$ *we have:*

$$\mathbb{A}\langle \mathbb{T}/y \rangle \langle \mathbb{S}/x \rangle = \mathbb{A}\langle \mathbb{S}/x \rangle \langle \mathbb{T}/y \rangle + \mathbb{A}\langle \mathbb{T}\langle \mathbb{S}/x \rangle /y \rangle$$

In particular, if $x \notin \mathrm{FV}(\mathbb{T})$ *the two substitutions commute.*

Given a bag $P = [L_1, \ldots, L_k]$ where x does not occur free, it makes sense now to set $\mathbb{A}\langle P/x \rangle := \mathbb{A}\langle L_1/x \rangle \cdots \langle L_k/x \rangle$, because this expression does not depend on the enumeration $L_1, \ldots, L_k$. In particular, $\mathbb{A}\langle []/x \rangle = \mathbb{A}$.

We are going to introduce the reduction rules defining the operational semantics of the $r\lambda$-calculus and show that it enjoys Church-Rosser and strong normalization, even in the untyped version of the calculus.

Definition 7.2.1. *The reduction of the* $r\lambda$-calculus *is generated by the following rule:*

- $(\lambda x.M)P \to_\beta M\langle P/x \rangle \{0/x\}$,

We remark that $M\langle L_1/x \rangle \cdots \langle L_k/x \rangle \{0/x\}$ is equal to $\sum_{\sigma \in \mathfrak{S}_k} M\{L_{\sigma(1)}/x^1, \ldots, L_{\sigma(k)}/x^k\}$ if $\deg_x(M) = k$ and is equal to 0 otherwise, where $x^1, \ldots, x^k$ are the k free occurrences of x in M.

Theorem 7.2.2 ([91, 44, 113])**.** *The* $r\lambda$-calculus *is strongly normalizing and Church-Rosser.*

7.2.1 Taylor expansion of ordinary lambda terms

Originally the resource λ calculus arose as a fragment of the differential λ-calculus, more precisely as a target language for writing the *Taylor expansion* of ordinary λ-terms. The Taylor expansion is a mapping of ordinary λ-terms to possibly infinite sums of differential λ-terms and bears strong similarities with the idea of Taylor expansion in analysis. Usually, when f is a sufficiently regular function from a

vector space E to a vector space F (finite dimensional spaces, or Banach spaces, typically), at all points $x \in E$, f has nth derivatives for all $n \in \mathbb{N}$ and these derivatives are maps $f^{(n)} : E \times E^n \to F$ with the same regularity as f and such that $f^{(n)}(x, u_1, \ldots, u_n) = f^{(n)}(x) \cdot (u_1, \ldots, u_n)$ is n-linear and symmetric in $u_1, \ldots, u_n$. When one is lucky, and usually locally only, the Taylor formula holds. Around 0 it reads

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \cdot \underbrace{(u, \ldots, u)}_{n \text{ times}}$$

If we want to Taylor expand completely λ-terms, which after all are functions, we need to extend the language with explicit differentials, or more precisely a construction of *differential application* of a term M to n terms $N_1, \ldots, N_n$, as done in [44]. The idea is that if M represents a function f from E to F and if $N_1, \ldots, N_n$ represent n vectors $u_1, \ldots, u_n \in E$, then this new construction $D^n M \cdot (N_1, \ldots, N_n)$ will represent the function from E to F which maps x to $f^{(n)}(x) \cdot (u_1, \ldots, u_n)$, and therefore this construction is linear and symmetric in the N_i's. The Taylor expansion of a single lambda-calculus application MN would then read

$$\sum_{n=0}^{\infty} \frac{1}{n!} (D^n M \cdot \underbrace{(N, \ldots, N)}_{n \text{ times}}) 0$$

If we want now to Taylor expand *all* the applications occurring in a λ-term, we see that the usual lambda-calculus application in its generality will become useless; only application to 0 is needed. This is exactly the purpose of the construction $t[s_1, \ldots, s_n]$ of the $r\lambda$-calculus; with the notations of the differential lambda-calculus, the expression becomes $(D^n M \cdot (N_1, \ldots, N_n)) 0$.

So in this sense the resource lambda-calculus is a "target language" for completely Taylor expanding ordinary λ-terms. The expansion of an ordinary term M will be an infinite linear combination of resource terms, with rational coefficients (actually, inverses of positive integers). Before going further we settle some technical details needed to correctly formalize the Taylor expansion formula in the resource lambda calculus.

First of all we need to have infinite sums. To this purpose we define $\mathbf{2}\langle \Lambda^r \rangle_\infty$ as the set of all $\mathbf{2}$-valued functions with domain Λ^r with pointwise defined addition and scalar multiplication. Note that we don't require for elements of $\mathbf{2}\langle \Lambda^r \rangle_\infty$ to vanish for almost all arguments. Note also that $\mathbf{2}\langle \Lambda^r \rangle \cong \mathcal{P}_{\mathrm{f}}(\Lambda^r)$ and $\mathbf{2}\langle \Lambda^r \rangle_\infty \cong \mathcal{P}(\Lambda^r)$: in the sequel we will make use of these facts identifying sets and linear combinations with coefficients in $\mathbf{2}$.

Let us use M^* for the (complete) Taylor expansion of M, which is an element of $\mathbf{2}\langle \Lambda^r \rangle_\infty$. By what we said, this operation should obey $(MN)^* = \sum_{n=0}^{\infty} M^*[\underbrace{N^*, \ldots, N^*}_{n \text{ times}}]$

as well as $x^* = x$ and $(\lambda x.M)^* = \lambda x.M^*$ (recalling that coefficients in our case are

only 0/1) From these equations we obtain that

$$M^* = \sum_{s \in \mathcal{T}(M)} s$$

where $\mathcal{T}(M) \subseteq \Lambda^r$ is defined inductively by the following clauses

- $\mathcal{T}(x) = \{x\}$,

- $\mathcal{T}(\lambda x.M) = \{\lambda x.s : s \in \mathcal{T}(M)\}$,

- $\mathcal{T}(MN) = \{s[t_1, \ldots, t_m] : s \in \mathcal{T}(M), \ [t_1, \ldots, t_m] \in \mathcal{M}_f(\mathcal{T}(N))\}$.

By Theorem 7.2.2, each $r\lambda$-term t possesses a unique normal form, indicated by $\mathrm{NF}(t)$.

Given an ordinary λ-term M, it makes sense therefore to apply NF to each of the simple terms occurring in its Taylor expansion, defining

$$\mathrm{NF}(M^*) = \Sigma_{s \in \mathcal{T}(M)}\mathrm{NF}(s)$$

In [46] the authors go further, proving that this sum is equal to $BT(M)^*$, the Taylor expansion of the Böhm tree of M. To give a meaning to this notion, we need first to define $\mathcal{T}(B)$, when B is a Böhm tree. The easiest way to do this is looking at a Böhm tree as an ideal of $\Lambda^\perp$ (see §1). Now the definition of $\mathcal{T}(B)$ comes in two steps: first adding the clause $\mathcal{T}(\perp) = \emptyset$ we can translate all terms in $\Lambda^\perp$, which are the same thing as the finite Böhm trees. Then for an arbitrary Böhm tree B we set $\mathcal{T}(B) = \cup\{\mathcal{T}(M) : M \in B\}$ (this is a directed union since B is an ideal). Of course, all these resource terms are normal. Given a Böhm tree B it makes sense finally to define its Taylor expansion, as it has been done for ordinary λ-terms: $B^* = \Sigma_{s \in \mathcal{T}(B)}s$.

Theorem 7.2.3 ([46])**.** *Let M be an ordinary λ-term and let u be a normal simple $r\lambda$-term. Then $u \in \mathcal{T}(BT(M))$ if and only if there exists $s \in \mathcal{T}(M)$ such that $u \in$* ***su***$(\mathrm{NF}(s))$*. Moreover, when this simple term exists, it is unique, so that $BT(M)^* = \mathrm{NF}(M^*)$.*

7.3 The category of sets and relations

We denote by **Rel** the category whose objects are all the sets and whose arrows are the relations between them, so that $\mathbf{Rel}(A, B) = \mathcal{P}(A \times B)$ for all sets A, B. It is well-known that **Rel** is a symmetric monoidal closed category. We briefly review the definitions involved here. As already done before we adopt the conventions of using letters $\alpha, \beta, \gamma, \ldots$ for elements of a set A, letters $a, b, c, \ldots$ for elements of $\mathcal{P}_f(A)$ and letters $x, y, z, \ldots$ for elements of $\mathcal{P}(A)$ in general.

The composition of two morphisms $R \in \mathbf{Rel}(A, B)$ and $S \in \mathbf{Rel}(B, C)$ is their usual relational composition: $S \cdot R = \{(\alpha, \gamma) \in A \times C : \exists \beta \in B. \, (\alpha, \beta) \in R \text{ and } (\beta, \gamma) \in S\}$. The identity morphism of a set A is $\mathrm{id}_A = \{(\alpha, \alpha) : \alpha \in A\}$.

The tensor product of two sets A, B is $A \otimes B = A \times B$. The unit of the tensor product is the singleton set $\mathbf{1} = \{*\}$. The linear exponent if given by $A \multimap B = A \times B$, the linear currying cur $: \mathbf{Rel}(C \otimes A, B) \to \mathbf{Rel}(C, A \multimap B)$ is $\mathrm{cur}(R) = \{(\gamma, (\alpha, \beta)) : ((\alpha, \gamma), \beta) \in R\}$ and the linear evaluation morphism ev $: (A \multimap B) \otimes A \to B$ is
ev $= \{(((\alpha, \beta), \alpha), \beta) : \alpha \in A, \, \beta \in B\}$. The definition of tensor product can be extended to arbitrary countable families of sets in the obvious way.

The category $\mathbf{Rel}$ is also cartesian. The Cartesian product of two sets A_1, A_2 is $A_1 \& A_2 = A_1 \uplus A_2$ and the projections $\pi_i \in \mathbf{Rel}(A_1 \& A_2, A_i)$ $(i = 1, 2)$ are given by $\pi_i = \{((i, \gamma), \gamma) : \gamma \in A_i\}$, $i = 1, 2$. For $R \in \mathbf{Rel}(C, A_1)$ and $S \in \mathbf{Rel}(C, A_2)$, the pairing $\langle R, S \rangle \in \mathbf{Rel}(C, A_1 \& A_2)$ is given by $\langle R, S \rangle = \{(\gamma, (1, \alpha)) : (\gamma, \alpha) \in R\} \cup \{(\gamma, (2, \beta)) : (\gamma, \beta) \in S\}$. The terminal object is $\top = \emptyset$. More generally $\mathbf{Rel}$ also has countable products.

It is also well-known that there exists a(t least one) comonad over $\mathbf{Rel}$ which is a symmetric strong monoidal endofunctor on $\mathbf{Rel}$. The best-known such comonad is functor $\mathcal{M}_{\mathrm{f}}(\_)$ which takes a set A to the set $\mathcal{M}_{\mathrm{f}}(A)$ of all finite multisets over A and a relation $R \in \mathbf{Rel}(A, B)$ to the relation $\mathcal{M}_{\mathrm{f}}(R) \in \mathbf{Rel}(\mathcal{M}_{\mathrm{f}}(A), \mathcal{M}_{\mathrm{f}}(B))$ given by $\mathcal{M}_{\mathrm{f}}(R) = \{([\alpha_1, \ldots, \alpha_k], [\beta_1, \ldots, \beta_k]) \in \mathcal{M}_{\mathrm{f}}(A) \times \mathcal{M}_{\mathrm{f}}(B) : \forall i = 1, \ldots, k. \, (\alpha_i, \beta_i) \in R\}$. The *digging* for this comonad is the natural transformation whose component at set A is the relation $\mathrm{dig}_A \in \mathbf{Rel}(\mathcal{M}_{\mathrm{f}}(A), \mathcal{M}_{\mathrm{f}}(\mathcal{M}_{\mathrm{f}}(A)))$ given by $\mathrm{dig}_A = \{(X, [Y_1, \ldots, Y_k]) : X = \Sigma_{i=1}^{k} Y_i\}$ and the *dereliction* is the natural transformation whose component at set A is the relation $\mathrm{der}_A \in \mathbf{Rel}(\mathcal{M}_{\mathrm{f}}(A), A)$ given by $\mathrm{der}_A = \{([\alpha], \alpha) : \alpha \in A\}$.

The fundamental (Seely) natural isomorphism $\varphi_{A,B} : \mathcal{M}_{\mathrm{f}}(A \& B) \cong \mathcal{M}_{\mathrm{f}}(A) \otimes \mathcal{M}_{\mathrm{f}}(B)$ then arises as the map $\varphi_{A,B}([(1, \alpha_1), \ldots, (1, \alpha_k), (2, \beta_1), \ldots, (2, \beta_h)]) = ([\alpha_1, \ldots, \alpha_k], [\beta_1, \ldots, \beta_h])$.

Then $\mathbf{Rel}$ a new-Seely category whose co-Kleisli category $\mathbf{MRel}$ is a ccc. We describe explicitly the ccc-structure of $\mathbf{MRel}$. In the sequel we will consider the canonical bijection between $\mathcal{M}_{\mathrm{f}}(A \& B)$ and $\mathcal{M}_{\mathrm{f}}(A) \otimes \mathcal{M}_{\mathrm{f}}(B)$ as an equality, hence we will still denote by (a, b) the corresponding element of $\mathcal{M}_{\mathrm{f}}(A \& B)$. Such a choice makes the exposition easier to read.

The objects are again all the sets and the arrows from A to B are the relations between $\mathcal{M}_{\mathrm{f}}(A)$ and B so that $\mathbf{MRel}(A, B) = \mathcal{P}(\mathcal{M}_{\mathrm{f}}(A) \times B)$ for all sets A, B. The composition of two morphisms $R \in \mathbf{MRel}(A, B)$ and $S \in \mathbf{MRel}(B, C)$ is defined as $S \circ R = \mathrm{dig}_A \cdot !R \cdot S = \{(\Sigma_{i=1}^{k} a_i, \gamma) \in \mathcal{M}_{\mathrm{f}}(A) \times C : \exists k \geq 0. \exists \beta_1, \ldots, \beta_k \in B. \, (a_i, \beta_i) \in R \text{ and } ([\beta_1, \ldots, \beta_k], \gamma) \in S\}$. The identity morphism of a set A is $\mathrm{id}_A = \{([\alpha], \alpha) : \alpha \in A\}$.

The Cartesian product "$\&$" is the same as in $\mathbf{Rel}$. The exponential object $A \Rightarrow B$ of A and B is given by $A \Rightarrow B = \mathcal{M}_{\mathrm{f}}(A) \times B$, the currying cur $: \mathbf{MRel}(C \& A, B) \to \mathbf{MRel}(C, A \Rightarrow B)$ is $\mathrm{cur}(R) = \{(c, (a, \beta)) : ([(a, c)], \beta) \in R\}$ and the evaluation

morphism $\mathrm{ev} : (A \Rightarrow B)\&A \to B$ is $\mathrm{ev} = \{(([(a,\beta)], a), \beta) : a \in \mathcal{M}_{\mathrm{f}}(A),\ \beta \in B\}$.

Here the points of a set A are the relations between $\mathcal{M}_{\mathrm{f}}(\emptyset)$ and A and hence, up to isomorphism, are the subsets of A. Observe that no object A of **MRel**, excluding the terminal $\top$, has enough points, so that the category **MRel** is a category without enough points.

For a set S, we denote by $\mathcal{M}_{\mathrm{f}}(S)^{(\omega)}$ the set of all $\mathbb{N}$-indexed sequences σ of multisets over S with the property that $\sigma_i = [\,]$ for all but a finite number of indices i, where of course σ_i indicates the i-th element of σ.

8

The relational semantics of ordinary and resource lambda calculus

The category **Rel** of sets and relations is a quite standard denotational model of linear logic which underlies most denotational models of this system (coherence spaces, hypercoherence spaces, totality spaces, finiteness spaces ...). In this completely elementary setting, a formula is interpreted as a set, and a proof of that formula is interpreted as a subset of the set interpreting the formula.

Logical connectives are interpreted very simply: tensor product, par and linear implication are interpreted as cartesian products direct product (with) and direct sums (plus) are interpreted as disjoint union. The linear negation of a set is the same set: it is a remarkable feature of linear logic that it admits such a "degenerate" semantics of types, which is nonetheless non trivial in the sense that not all proofs are identified.

Exponentials are traditionally interpreted by the operation which maps a set X to the set of all finite multisets of elements of X (the origin of this idea can be found in [54]). One might be tempted to use finite sets instead of finite multisets since, in the coherence space semantics, the exponential can be interpreted by an operation which maps a coherence space to the set of its finite cliques (with a suitable coherence). In the relational model however, such an interpretation of the exponentials based on finite sets is not possible as it leads to a dereliction which is not natural (in the categorical sense).

With this standard multiset-based interpretation of exponentials, the relational model interprets also the differential extensions of Linear Logic and of the lambda calculus presented in [44, 45, 43] and the resource lambda calculus surveyed in §7, Section 7.2; in the same chapter, Section 7.2.1, it is recalled that Taylor expanding completely a λ-term M one obtains a (generally infinite) linear combination of resource terms and that, if one normalizes each resource term occurring in that formal sum, one obtains the Taylor expansion of the Böhm tree of M.

This results implies that, in a denotational model which validates the Taylor expansion formula in the sense that the interpretation of a λ-term M is equal to the interpretation of its Taylor expansion, the interpretation of an unsolvable λ-term is necessarily equal to 0 (i.e. the empty set). Since the multiset-based exponential of

Rel validates the Taylor expansion formula, any model of the pure lambda calculus in the corresponding Cartesian closed category, such as the model presented in [27, 28], is bound to be sensible (at least if differential operations are interpreted in the standard way). This seems to be a serious limitation in the equational expressive power of this kind of semantics.

This problem arose during a general investigation undertaken in collaboration with T. Ehrhard and A. Salibra, whise scope is to develop an algebraic setting for differential extensions of the lambda calculus, in the spirit of [92].

In the present chapter (which surveys [32]) we discuss the issue of the equational expressive power of the standard relational semantics (originated in [54]) and we propose a generalization of it, by changing the usual multiset-based exponential. We then prove that our solution overcomes the limitation imposed by the Taylor expansion formula.

8.1   A denotational model of the Taylor formula

It is well-known that in the category **MRel** there exist a(t least one) reflexive object. One such object is constructed as follows:

- $U_0 = \emptyset$,

- $U_{n+1} = \mathcal{M}_{\mathrm{f}}(U_n)^{(\omega)}$,

- $U = \cup_{n \geq 0} U_n$.

Every element $\sigma \in U$ is a sequence of finite multisets of U. We write $a :: \sigma$ for the element $\sigma' \in U$ such that $\sigma'_1 = a$ and $\sigma'_{i+1} = \sigma'_i$. We can now define the reflexive object $\mathbf{U} = (U, \mathrm{Ap}_U, \mathrm{Lam}_U)$ where

- $\mathrm{Ap}_U = \{([(a, \sigma)], a :: \sigma) : a \in \mathcal{M}_{\mathrm{f}}(U), \ \sigma \in U\} \in \mathbf{MRel}(U \Rightarrow U, U)$,

- $\mathrm{Lam}_U = \{([a :: \sigma], (a, \sigma)) : a \in \mathcal{M}_{\mathrm{f}}(U), \ \sigma \in U\} \in \mathbf{MRel}(U, U \Rightarrow U)$.

It is not difficult to check that $\mathrm{Lam}_U \circ \mathrm{Ap}_U = \mathrm{id}_U$ and $\mathrm{Ap}_U \circ \mathrm{Lam}_U = \mathrm{id}_{U \Rightarrow U}$ so that in fact **U** is a model of the extensional lambda calculus.

It is also well-known that $Eq(\mathbf{U}) = \mathcal{H}^*$ (recall these notions from §2 and 1, respectively), so that the object **U** has a sensible equational theory. One may now ask a very general question about the relational semantics, i.e., the class of all reflexive objects in the category **MRel**: whether or not there exists one such object with a non-sensible equational theory. We will see that this is not the case: not only *every* reflexive object **U** equates all unsolvable λ-terms, but is also equates the terms having the same Böhm tree.

In order to prove such a claim, we need first to report some basic result on what is probably the most interesting feature of the relational semantics: the fact that

it models the resource lambda calculus and that it *validates the Taylor expansion formula*. We will soon give meaning to such sentence.

The category **MRel** is not just a ccc. It also possesses the *differential structure* needed to interpret the differential lambda calculus, as well as the resource lambda calculus. We don't explain here the details about differential structures in monoidal and/or cartesian categories: we just point the interested reader to existing references ([29, 22, 23, 52]) for that matter.

Let now U be a set: in the rest of this chapter we will make lowercase greek letters $\alpha, \beta, \gamma, \ldots$ range over elements U and roman letters $a, b, c, \ldots$ range over $\mathcal{M}_{\mathrm{f}}(U)$.

Recall from §2 the definition of interpretation $|M|_{\bar{x}}^{\mathbf{U}} \in \mathbb{C}(U^{\bar{x}}, U)$ of an ordinary λ-term M with free variables $\mathrm{FV}(M) \subseteq \bar{x}$ in a reflexive object $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ of a ccc $\mathbb{C}$. In the case of the category **MRel**, in particular, this interpretation is the following:

$$|x_i|_{\bar{x}}^{\mathbf{U}} = \{(([\,]\ldots,[\alpha],\ldots,[\,]),\alpha) : \alpha \in U\}$$

$$|\lambda y.M|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{j=1}^k \bar{a}_j, \alpha) : \exists (b_1, \beta_1), \ldots, (b_k, \beta_k) \in U \Rightarrow U.\ ((\bar{a}_j, b_j), \beta_j) \in |M|_{\bar{x},y}^{\mathbf{U}} \\ (j = 1, \ldots, k),\ ([(b_1, \beta_1), \ldots, (b_k, \beta_k)], \alpha) \in \mathrm{Lam}\}$$

$$|MN|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{i=1}^h \bar{c}_i + \Sigma_{j=1}^k \bar{a}_j, \alpha) : \exists \beta_1, \ldots, \beta_k \in U.\exists \gamma_1, \ldots, \gamma_h \in U. \\ (\bar{a}_j, \beta_j) \in |N|_{\bar{x}}^{\mathbf{U}},\ (j = 1, \ldots, k),\ (\bar{c}_i, \gamma_i) \in |M|_{\bar{x}}^{\mathbf{U}}\ (i = 1, \ldots, h), \\ ([\gamma_1, \ldots, \gamma_h], ([\beta_1, \ldots, \beta_k], \alpha)) \in \mathrm{Ap}\}$$

As already anticipated the resource lambda calculus (with truth-values coefficients) can be soundly interpreted in a reflexive object $\mathbf{U}$ of **MRel**. A simple resource λ-term t with $\mathrm{FV}(t) \subseteq \bar{x}$ is mapped to a relation $|t|_{\bar{x}}^{\mathbf{U}} \in \mathbf{MRel}(U^{\bar{x}}, U)$ in the following way:

$$|x_i|_{\bar{x}}^{\mathbf{U}} = \{(([\,]\ldots,[\alpha],\ldots,[\,]),\alpha) : \alpha \in U\}$$

$$|\lambda y.s|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{j=1}^k \bar{a}_j, \alpha) : \exists (b_1, \beta_1), \ldots, (b_k, \beta_k) \in U \Rightarrow U.\ ((\bar{a}_j, b_j), \beta_j) \in |s|_{\bar{x},y}^{\mathbf{U}} \\ (j = 1, \ldots, k),\ ([(b_1, \beta_1), \ldots, (b_k, \beta_k)], \alpha) \in \mathrm{Lam}\}$$

$$|t[s_1, \ldots, s_k]|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{i=1}^h \bar{c}_i + \Sigma_{j=1}^k \bar{a}_j, \alpha) : \exists \beta_1, \ldots, \beta_k \in U.\exists \gamma_1, \ldots, \gamma_h \in U. \\ (\bar{a}_j, \beta_j) \in |s_j|_{\bar{x}}^{\mathbf{U}},\ (j = 1, \ldots, k),\ (\bar{c}_i, \gamma_i) \in |t|_{\bar{x}}^{\mathbf{U}} \\ (i = 1, \ldots, h),\ ([\gamma_1, \ldots, \gamma_h], ([\beta_1, \ldots, \beta_k], \alpha)) \in \mathrm{Ap}\}$$

Finally the interpretation is extended to arbitrary resource terms as follows:

- $|0|_{\bar{x}}^{\mathbf{U}} = \emptyset$,

- $|\Sigma_{i=1}^h t_i|_{\bar{x}}^{\mathbf{U}} = \cup_{i=1}^h |t_i|_{\bar{x}}^{\mathbf{U}}$.

An equation $t = s$ between resource λ-terms is *(absolutely) satisfied in* $\mathbf{U}$, notation $\mathbf{U} \models^{\mathrm{abs}} t = s$, iff $|t|_{\bar{x}}^{\mathbf{U}} = |s|_{\bar{x}}^{\mathbf{U}}$, where $\bar{x} = \mathrm{FV}(M) \cup \mathrm{FV}(N)$. The *(resource) equational theory* of $\mathbf{U}$ is defined as $Eq^r(\mathbf{U}) = \{t = s : \mathbf{U} \models^{\mathrm{abs}} t = s\}$.

The word "soundly" that we used, referring to the interpretation of terms, is motivated by the following theorem.

Theorem 8.1.1. *The set $Eq^r(\mathbf{U})$ is a resource λ-theory.*

Recall now from the definition of Taylor expansion that if M is an ordinary λ-term with $FV(M) = \bar{x}$, then for each $t \in \mathcal{T}(M)$ we have $FV(t) = \bar{x}$. Then the Taylor expansion itself can be interpreted in $\mathbf{U}$ by simply setting $|\Sigma_{t \in \mathcal{T}(M)} t|_{\bar{x}}^{\mathbf{U}} = \cup_{t \in \mathcal{T}(M)} |t|_{\bar{x}}^{\mathbf{U}}$.

Fnally the following theorem precisely explains what we mean by saying that the Taylor expansion formula holds in the category $\mathbf{MRel}$.

Theorem 8.1.2. *For any reflexive object $\mathbf{U}$ in $\mathbf{MRel}$ the following equation holds:*

$$|M|_{\bar{x}}^{\mathbf{U}} = |\Sigma_{t \in \mathcal{T}(M)} t|_{\bar{x}}^{\mathbf{U}}$$

The *relational semantics* is the collection of lambda algebras arising from reflexive objects in $\mathbf{MRel}$. One may now ask a very general question about the relational semantics, i.e., the class of all reflexive objects in the category $\mathbf{MRel}$: whether or not there exists one such object with a non-sensible equational theory. We will see that this is not the case: not only *every* reflexive object $\mathbf{U}$ equates all unsolvable λ-terms, but it also equates the terms having the same Böhm tree.

The next theorem is a direct consequence of Theorem 8.1.2 and Theorem 7.2.3 (from §7). Recall from §1 that $\mathcal{BT}$ is the λ-theory equating all ordinary λ-terms having the same Böhm tree.

Theorem 8.1.3. *For any reflexive object $\mathbf{U}$ in $\mathbf{MRel}$ we have $\mathcal{BT} \subseteq Eq(\mathbf{U})$.*

Proof. Let M, N be ordinary λ-terms and suppose $BT(M) = BT(N)$. Then of course $BT(M)^* = BT(N)^*$ and in turn $\Sigma_{t \in \mathcal{T}(M)} NF(t) = \Sigma_{s \in \mathcal{T}(N)} NF(s)$, by Theorem 7.2.3. Now we have

$$\begin{aligned}
|M|_{\bar{x}}^{\mathbf{U}} &= |\Sigma_{t \in \mathcal{T}(M)} t|_{\bar{x}}^{\mathbf{U}}, \text{ by Theorem 8.1.2,} \\
&= |\Sigma_{t \in \mathcal{T}(M)} NF(t)|_{\bar{x}}^{\mathbf{U}} \\
&= |\Sigma_{s \in \mathcal{T}(N)} NF(s)|_{\bar{x}}^{\mathbf{U}} \\
&= |\Sigma_{t \in \mathcal{T}(N)} t|_{\bar{x}}^{\mathbf{U}} \\
&= |N|_{\bar{x}}^{\mathbf{U}}
\end{aligned}$$

$\square$

Theorem 8.1.3 states the (large) incompleteness of the standard relational semantics: more precisely this semantics omits all λ-theories strictly below $\mathcal{BT}$. In the next section we will se how the relational semantics can be generalized in order to overcome the limitation stated in Theorem 8.1.3.

We conclude this section by presenting interpretation of terms in a reflexive object $\mathbf{U} = (U, Ap, Lam)$ as a typing system by simply generalizing De Carvalho's system R [40]. A type is an element of U. A typing context is a finite partial function

from variables to $\mathcal{M}_{\mathrm{f}}(U)$; they are indicated as sequences $x_1 : a_1, \ldots, x_n : a_n$, where $\{x_1, \ldots, x_n\}$ is the *domain* of the context, and ranged over by capital greek letters like Γ, Δ; if $\Gamma = x_1 : a_1, \ldots, x_n : a_n$, then clearly $\Gamma(x_i) = a_i$. If $\Gamma_1, \ldots, \Gamma_k$ are contexts with the same domain $x_1, \ldots, x_n$ then $\Sigma_{i=1}^k \Gamma_i$ is the context with domain $x_1, \ldots, x_n$ assigning to x_i the multiset sum $\Sigma_{i=1}^k \Gamma_i(x_i)$.

The typing judgements are of the form $\Gamma \rhd^{\mathbf{U}} M : \alpha$, to be read as the "term M can be assigned type α in the context Γ according to the typing determined by $\mathbf{U}$". We omit the superscript $\mathbf{U}$ when the underlying reflexive object will be clear from the context. The typing rules are the followings:

$$\frac{}{x_1 : [\,], \ldots, x_i : [\alpha], \ldots, x_n : [\,] \rhd x_i : \alpha} \; [\mathsf{var}]$$

$$\frac{\Gamma_j, y : b_j \rhd^{\mathbf{U}} M : \beta_j \; (j = 1, \ldots, k) \quad ([(b_1, \beta_1), \ldots, (b_k, \beta_k)], \alpha) \in \mathrm{Lam}}{\Sigma_{j=1}^k \Gamma_j \rhd \lambda y.M : \alpha} \; [\mathsf{abs}]$$

$$\frac{\Gamma_i \rhd M : \gamma_i \; (i = 1, \ldots, h) \quad \Delta_j \rhd N : \beta_j \; (j = 1, \ldots, k) \quad ([\gamma_1, \ldots, \gamma_h], ([\beta_1, \ldots, \beta_k], \alpha)) \in \mathrm{Ap}}{\Sigma_{i=1}^h \Delta_i + \Sigma_{j=1}^k \Gamma_j \rhd MN : \alpha} \; [\mathsf{app}]$$

The precise sense in which interpretation can be seen as typing is stated in the following proposition.

Proposition 8.1.4. *For all λ-terms M with $\mathrm{FV}(M) \subseteq \bar{x} = x_1, \ldots, x_n$ and all $\bar{a} = a_1, \ldots, a_n \in \mathcal{M}_{\mathrm{f}}(U)^n$, and all $\alpha \in U$ we have $(\bar{a}, \alpha) \in |M|_{\bar{x}}^{\mathbf{U}}$ iff $x_1 : a_1, \ldots, x_n : a_n \rhd^{\mathbf{U}} M : \alpha$.*

Proof. By induction on the structure of M. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

8.2 Exponentials with infinite multiplicities

We saw in Section 8.1 that every reflexive object in the category **MRel** validates the Taylor expansion formula in the sense that the interpretation of a λ-term M is equal to the interpretation of its Taylor expansion, causing every model in **MRel** to be sensible.

The present section proposes an exit way to this limitation, by introducing new exponential operations on the category **Rel**. The idea is quite simple: we replace the set $\mathbb{N}$ of natural numbers (which are used for counting multiplicities of elements in multisets) with more general semi-rings which typically contain "infinite elements" ω such that $\omega + 1 = \omega$. *Mutatis mutandis*, the various structures of the exponentials (functorial action, dereliction, etc.) are interpreted as with the ordinary multiset-based exponentials. For these structures to satisfy the required equations, some rather restrictive conditions have to be satisfied by the considered semi-ring: the semi-rings which satisfy these conditions are called "*multiplicity semi-rings*". We show that such a semi-ring must contain $\mathbb{N}$ and we exhibit multiplicity semi-rings with infinite elements.

In these models with infinite multiplicities, the differential constructions are available but the Taylor formula does not hold. It is possible to find morphisms $f : A \to B$ (in the associated cartesian closed category) which are $\neq 0$ but are such that, for all $n \in \mathbb{N}$, the n-th derivative $f^{(n)}(0) : A^n \to B$ is equal to 0. The Taylor expansion of such a function is the 0 map, and hence the function is different from its Taylor expansion. This is analogous to the well-known smooth (C^∞) map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = e^{-\frac{1}{|x|}}$ for $x \neq 0$: all the derivatives of f at 0 are equal to 0 and hence there is no neighborhood of 0 where f coincides with its Taylor expansion at 0. In some sense, f is infinitely flat at 0, and we obtain a similar effect with our infinite multiplicities. For any multiplicity semi-ring which contains an infinite element, we build a model of the pure lambda calculus which is not sensible.

Definition 8.2.1. *A* multiplicity semi-ring $\mathbf{A} = (A, \cdot, +, 0, 1)$ *is a commutative semi-ring which satisfies the following properties:*

(MS1) *if* $n_1 + n_2 = 0$, *then* $n_1 = n_2 = 0$

(MS2) *if* $n_1 + n_2 = 1$, *then either* $n_1 = 0$ *or* $n_2 = 0$

(MS3) *if* $n_1 + n_2 = m_1 + m_2$, *then there exists a matrix* $M \in \mathbf{A}^{2,2}$ *such that*

$$M\mathbf{1}_2 = \binom{n_1}{n_2} \quad and \quad M^\perp\mathbf{1}_2 = \binom{m_1}{m_2}$$

(MS4) *if* $nm = p_1 + p_2$, *then there exists a vector* $V \in \mathbf{A}^{2,1}$ *and a matrix* $M \in \mathbf{A}^{2,2}$ *such that*

$$\mathbf{1}_2^\perp V = n \quad and \quad MV = \binom{p_1}{p_2} \quad and \quad M^\perp\mathbf{1}_2 = \binom{m}{m}$$

As a matter of terminology we say that a semiring:

- is *positive* if it has property (MS1),

- is *discrete* if it has property (MS2),

- has the *additive splitting property* if it has property (MS3),

- has the *multiplicative splitting property* if it has property (MS4).

The motivations for Condition (MS4) is mainly technical: it is essential in the proof of Lemma 8.3.5. It has also an intuitive content, describing what happens when an element of A can be written both as a sum and as a product. The proof that this property holds in $\mathbb{N}$ is based on Euclidean division. We conjecture that this property is independent from Conditions (MS1), (MS2) and (MS3).

The splitting conditions are expressed in a sort of binary way but they can be generalized to arbitrary arities.

Notational convention for indices. We shall use quite often multiple indices, written as subscript as in "a_{ijk}" which has three indices i, j and k. When there are no ambiguities, these indices will not be separated by commas. We insert commas when we use multiplication on these indices, as in "$a_{i,2j,k}$" for instance.

We first generalize condition (MS3).

Lemma 8.2.1. *Let* $\mathbf{A}$ *be a semi-ring satisfying (MS3). Then the following generalized version of (MS3) holds in* $\mathbf{A}$

(MS3)' *if* $\Sigma_{i=1}^{l} n_i = \Sigma_{j=1}^{r} p_j$, *then there exists a matrix* $S \in \mathbf{A}^{l,r}$ *such that*

$$S\mathbf{1}_r = \begin{pmatrix} n_1 \\ \vdots \\ n_l \end{pmatrix} \quad and \quad S^{\perp}\mathbf{1}_l = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$$

Proof. Assume first that $l = 2$ and let us prove the result by induction on r. For $r = 1$ one takes $S = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Assume that the property holds for r and let us prove it for $r + 1$. If $n_1 + n_2 = \Sigma_{j=1}^{r} p_j + p_{r+1}$ by applying condition (MS3) we can find a matrix $T = (t_{ij}) \in \mathbf{A}^{2,2}$ such that

$$T\mathbf{1}_2 = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad and \quad T^{\perp}\mathbf{1}_2 = \begin{pmatrix} \Sigma_{j=1}^{r} p_j \\ p_{r+1} \end{pmatrix}$$

In particular we have $t_{11} + t_{12} = \Sigma_{j=1}^{r} p_j$. By inductive hypothesis we can a matrix $U = (u_{ij}) \in \mathbf{A}^{2,r}$ such that

$$U\mathbf{1}_r = \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix} \quad and \quad U^{\perp}\mathbf{1}_2 = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$$

Then we define a matrix $V \in \mathbf{A}^{2,r+1}$ by setting

$$V = \begin{pmatrix} u_{11} & \cdots & u_{1r} & t_{12} \\ u_{21} & \cdots & u_{2r} & t_{22} \end{pmatrix}$$

It is easy to check that V is the desired matrix.

Now we prove the result for an arbitrary value of l by induction on this parameter. For $l = 1$ we set $S = \begin{pmatrix} p_1 & \cdots & p_r \end{pmatrix}$. Assume that the result holds for l and let us prove it for $l+1$. By assumption we have $\Sigma_{i=1}^{l} n_i + n_{l+1} = \Sigma_{j=1}^{r} p_j$ so we can apply the

property that we just proved (where $l = 2$ and r is arbitrary). Let $T = (t_{ij}) \in \mathbf{A}^{2,r}$ be a matrix such that

$$T\mathbf{1}_r = \begin{pmatrix} \Sigma^l_{i=1}n_i \\ n_{l+1} \end{pmatrix} \quad \text{and} \quad T^\perp\mathbf{1}_2 = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$$

In particular we have $\Sigma^l_{i=1}n_i = \Sigma^r_{j=1}t_{1j}$ and $n_{l+1} = \Sigma^r_{j=1}t_{2j}$. By inductive hypothesis we can find a matrix $U = (u_{ij}) \in \mathbf{A}^{l,r}$ such that

$$U\mathbf{1}_r = \begin{pmatrix} n_1 \\ \vdots \\ n_l \end{pmatrix} \quad \text{and} \quad U^\perp\mathbf{1}_l = \begin{pmatrix} t_{11} \\ \vdots \\ t_{r1} \end{pmatrix}$$

Then we define a matrix $V \in \mathbf{A}^{l+1,r}$ by setting

$$V = \begin{pmatrix} u_{11} & \cdots & u_{1r} \\ \vdots & \ddots & \vdots \\ u_{l1} & \cdots & u_{lr} \\ t_{21} & \cdots & t_{2r} \end{pmatrix}$$

It is easy to check that V is the desired matrix. □

Along the same lines we generalize condition (MS4).

Lemma 8.2.2. *Let $\mathbf{A}$ be a semi-ring satisfying (MS3) and (MS4). Then the following generalized version of (MS4) holds in $\mathbf{A}$*

(MS4)' if $mp = \Sigma^k_{i=1}n_i$, then there exists a vector $P \in \mathbf{A}^{l,1}$ and a matrix $M \in \mathbf{A}^{k,l}$ with $l = 2^{k-1}$ such that

$$\mathbf{1}^\perp_l P = p \quad \text{and} \quad MP = \begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix} \quad \text{and} \quad M^\perp\mathbf{1}_k = \begin{pmatrix} m \\ \vdots \\ m \end{pmatrix}$$

Proof. We proceed by induction on k. For $k = 1$ one has $l = 1$ and takes $P = (p)$ and $M = (m)$.

Assume that the result holds for k, let $l = 2^{k-1}$, and let us prove the statement for $k + 1$. Suppose $mp = \Sigma^{k-1}_{i=1}n_i + (n_k + n_{k+1})$. By inductive hypothesis we can find a vector $P = (p_i) \in \mathbf{A}^{l,1}$ and a matrix $M = (m_{ij}) \in \mathbf{A}^{k,l}$ such that

$$\mathbf{1}^\perp_l P = p \quad \text{and} \quad MP = \begin{pmatrix} n_1 \\ \vdots \\ n_{k-1} \\ n_k + n_{k+1} \end{pmatrix} \quad \text{and} \quad M^\perp\mathbf{1}_k = \begin{pmatrix} m \\ \vdots \\ m \end{pmatrix}$$

In particular we have $\Sigma_{j=1}^{l} m_{kj} p_j = n_k + n_{k+1}$ and hence by Lemma 8.2.1 we can find a matrix $R = (r_{ij}) \in \mathbf{A}^{l,2}$ such that

$$R\mathbf{1}_2 = \begin{pmatrix} m_{k1}p_1 \\ \vdots \\ m_{kl}p_l \end{pmatrix} \quad \text{and} \quad R^{\perp}\mathbf{1}_l = \begin{pmatrix} n_k \\ n_{k+1} \end{pmatrix}$$

In particular we have $p_j m_{kj} = r_{j1} + r_{j2}$ for each $j = 1, \ldots, l$ and thus by condition (MS4) for each $j = 1, \ldots, l$ we can find a vector $V^j = \begin{pmatrix} v_1^j \\ v_2^j \end{pmatrix} \in \mathbf{A}^{2,1}$ and a matrix $S^j = (s_{it}^j) \in \mathbf{A}^{2,2}$ such that

$$\mathbf{1}_2^{\perp} V^j = p_j \quad \text{and} \quad S^j V^j = \begin{pmatrix} r_{j1} \\ r_{j2} \end{pmatrix} \quad \text{and} \quad (S^j)^{\perp}\mathbf{1}_2 = \begin{pmatrix} m_{kj} \\ m_{kj} \end{pmatrix}$$

Hence in particular we have $n_i = \Sigma_{j=1}^{l} m_{ij}(v_{11}^j + v_{21}^j)$ for each $i = 1, \ldots, k-1$ and $n_k = \Sigma_{j=1}^{l} r_{j1} = \Sigma_{j=1}^{l} s_{11}^j v_1^j + s_{21}^j v_2^j$ and $n_{k+1} = \Sigma_{j=1}^{l} r_{j1} = \Sigma_{j=1}^{l} s_{12}^j v_1^j + s_{22}^j v_2^j$.

Finally let us define a vector $U \in \mathbf{A}^{2l,1}$ by setting

$$U = \begin{pmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_1^l \\ v_2^l \end{pmatrix}$$

and let us define a matrix $T \in \mathbf{A}^{k+1,l}$

$$T = \begin{pmatrix} m_{11} & m_{11} & \cdots & m_{1l} & m_{1l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{k-1,1} & m_{k-1,1} & \cdots & m_{k-1,l} & m_{k-1,l} \\ s_{11}^1 & s_{21}^1 & \cdots & s_{11}^l & m_{21}^l \\ s_{21}^1 & s_{22}^1 & \cdots & s_{21}^l & m_{22}^l \end{pmatrix}$$

Then U and T are the vector and the matrix we were looking for since

$$\mathbf{1}_l^{\perp} U = p \quad \text{and} \quad TU = \begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix} \quad \text{and} \quad T^{\perp}\mathbf{1}_k = \begin{pmatrix} m \\ \vdots \\ m \end{pmatrix}$$

$\square$

Proposition 8.2.3. *Any multiplicity semi-ring* $\mathbf{A}$ *contains an isomorphic copy of* $\mathbb{N}$.

Proof. One defines a map $f : \mathbb{N} \to \mathbf{A}$ by induction on natural numbers by setting $f(0) = 0$ and $f(n + 1) = f(n) + 1$, that is $f(n) = \Sigma_{i=1}^{n} 1$. This map is a semi-ring morphism as easily checked by induction on natural numbers again. We prove that f is injective, so let $p \in \mathbb{N}$ and let us prove that $f(n) = f(n + p)$ implies $p = 0$ by induction on n. For $n = 0$ assume that $\Sigma_{i=1}^{p} 1 = 0$. Applying condition (MS1) we get easily $p = 0$ (by induction on p actually). Assume now that $\Sigma_{i=1}^{n+1+p} 1 = \Sigma_{i=1}^{n+1} 1$, that is $\Sigma_{i=1}^{n+1+p} 1 = \Sigma_{i=1}^{n} 1 + 1$. By condition (MS3) one can find $r_{11}, r_{12}, r_{21}, r_{22} \in \mathbf{A}$ such that $n + p = r_{11} + r_{12}$, $1 = r_{21} + r_{22}$, $n = r_{11} + r_{21}$, and $1 = r_{12} + r_{22}$. By condition (MS2) there are two cases to consider:

- either $r_{22} = 1$ and $r_{21} = r_{12} = 0$

- or $r_{22} = 0$ and $r_{21} = r_{12} = 1$

In both cases we have $n + p = n$ and hence $p = 0$ by inductive hypothesis. $\qquad \square$

We shall simply say that $\mathbf{A}$ contains $\mathbb{N}$, that is $\mathbb{N} \subseteq A$. In particular, a multiplicity semi-ring cannot be finite. An element $m \in A$ will be said to be *infinite* if $m = m + 1$.

The elements of a multiplicity semi-ring should be considered as generalized natural numbers. We give here examples of such semi-rings.

The most canonical example of multiplicity semi-ring is the set $\mathbb{N}$ of natural numbers, with the ordinary addition and multiplication. Of course $\mathbb{N}$ has no infinite element.

Proposition 8.2.4. $\mathbb{N}$ *is a multiplicity semi-ring.*

Proof. Let us check condition (MS3), so let $n_1, n_2, p_1, p_2 \in \mathbb{N}$ be such that $n_1 + n_2 = p_1 + p_2$ and let q be this common value. Pick arbitrarily sets $I_1, I_2, J_1, J_2 \subseteq \{1, \ldots, q\}$ of respective cardinality n_1, n_2, p_1, p_2. It suffices to take $r_{ij} = \sharp(I_i \cap J_j)$.

We now prove condition (MS4). We apply Euclidean division by p and we get $n_1 = q_1 p + r_1$ and $n_2 = q_2 p + r_2$ where $r_1, r_2 < p$. We have $r_1 + r_2 = p(m - q_1 - q_2)$, and since $r_1, r_2 < p$, we must have either $m - q_1 - q_2 = 0$ or $m - q_1 - q_2 = 1$.

In the first case we have $r_1 = r_2 = 0$. Pick $p_1, p_2 \in \mathbb{N}$ such that $p_1 + p_2 = p$. Set $m_{11} = m_{12} = q_1$ and $m_{21} = m_{22} = q_2$. Then we have $m_{11} + m_{21} = m_{12} + m_{22} = m$, $p_1 m_{11} + p_2 m_{12} = p_1 q_1 + p_2 q_1 = p q_1 = n_1$ and $p_1 m_{21} + p_2 m_{22} = p_1 q_2 + p_2 q_2 = p q_2 = n_2$ as required.

Assume now that $m - q_1 - q_2 = 1$. We set $p_1 = r_1$, $p_2 = r_2$, $m_{11} = q_1 + 1$, $m_{12} = q_1$, $m_{21} = q_2$ and $m_{22} = q_2 + 1$. We have $m_{11} + m_{21} = m_{12} + m_{22} = q_1 + q_2 + 1 = m$. Next we have $p_1 m_{11} + p_2 m_{12} = r_1(q_1 + 1) + r_2 q_1 = (r_1 + r_2) q_1 + r_1 = p q_1 + r_1 = n_1$. Similarly we have $p_1 m_{21} + p_2 m_{22} = n_2$, as required. $\qquad \square$

Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ be the *completed set of natural numbers*. We extend addition to this set by $n + \omega = \omega + n = \omega$, and multiplication by $0\omega = \omega 0 = 0$ and $n\omega = \omega n = \omega$ for $n \neq 0$, so that $\overline{\mathbb{N}}$ has exactly one infinite element, namely ω.

Proposition 8.2.5. $\overline{\mathbb{N}}$ *is a multiplicity semi-ring.*

Proof. We check condition (MS3), so assume that $n_1 + n_2 = p_1 + p_2 = q$. If $q \neq \omega$, then we have $n_i, p_j \in \mathbb{N}$ for each i, j and we use condition (MS3) for $\mathbb{N}$. Assume that $q = \omega$. Without loss of generality we can assume that $n_1 = p_1 = \omega$. We can take $r_{11} = \omega$, $r_{22} = 0$, $r_{12} = p_2$ and $r_{21} = n_2$. Last we check condition (MS3), so assume that $pm = n_1 + n_2 = q$. Assume first that $q \in \mathbb{N}$. If $q\mathbb{N}0$ we know that $p, m, n_1, n_2 \in \mathbb{N}$ and we can use condition (MS4) in $\mathbb{N}$. If $q = 0$, then $n_1 = n_2 = 0$ and we must have $m = 0$ or $p = 0$. If $p = \omega$ and $m = 0$ then we can take $p_1 = \omega$, $p_2 = 0$, $m_{11} = m_{12} = m_{21} = m_{22} = 0$. If $p = 0$ and $m = \omega$, we take $p_1 = p_2 = 0$, $m_{11} = m_{21} = \omega$ and $m_{12} = m_{22} = 0$. We are left with the case were $q = \omega$. Without loss of generality we can assume that $n_1 = \omega$, and of course we must have $m \neq 0$ and $p \neq 0$. Assume first that $p = \omega$. Then we can take $p_1 = \omega$, $p_2 = n_2$, $m_{11} = m$, $m_{21} = 0$, $m_{12} = m'$ such that $m' + 1 = m$ and $m_{22} = 1$. Assume last that $m = \omega$. Then we can take $p_1 = p'$ with $p' + 1 = p$, $p_2 = 1$, $m_{11} = \omega$, $m_{21} = \omega$, $m_{12} = \omega$ and $m_{22} = n_2$. $\qquad\square$

A more interesting example is $\mathbb{N}_2 = (\mathbb{N}^+ \times \mathbb{N}) \cup \{0\}$. The element (n, d) of this set (with $n \neq 0$) will be denoted as $n\omega^d$. We extend this notation to the case where $n = 0$, identifying $0\omega^d$ with 0, which is quite natural with these notations. Addition is defined as follows (0 being of course neutral for this operation)

$$n\omega^d + m\omega^e = \begin{cases} (n + m)\omega^d & \text{if } d = e \\ n\omega^d & \text{if } n \neq 0 \text{ and } e < d \\ m\omega^e & \text{if } m \neq 0 \text{ and } d < e \end{cases}$$

and multiplication is defined by $(n\omega^d)(m\omega^e) = nm\omega^{d+e}$. This semi-ring has infinitely many infinite elements: all the elements $n\omega^d$ of $\mathbb{N}_2$ with $n \neq 0$ and $d \neq 0$ are infinite.

Proposition 8.2.6. $\mathbb{N}_2$ *is a multiplicity semi-ring.*

Proof. A simple case analysis shows that this addition is associative and, obviously, commutative. Distributivity is easily checked as well, so that we have defined a semiring. Observe that $\omega + 1 = \omega$, but $\omega + \omega \neq \omega$ and actually, unlike in $\overline{\mathbb{N}}$, the only element $n \in \mathbb{N}_2$ such that $n + n = n$ is 0.

Let us check property (MS3), so assume that $n_1\omega^{d_1} + n_2\omega^{d_2} = p_1\omega^{e_1} + p_2\omega^{e_2}$. If $d_1 = d_2$ and $e_1 = e_2$, we are reduced to the splitting property of $\mathbb{N}$. If $d_1 = d_2$ and $e_1 > e_2$, then we have $(n_1 + n_2)\omega^{d_1} = p_1\omega^{e_1}$. Then a matrix works for our purpose is

$$\begin{pmatrix} n_1\omega^{d_1} & p_2\omega^{e_2} \\ n_2\omega^{d_1} & 0 \end{pmatrix}$$

The last case to consider (up to commutativity of addition) is $d_1 > d_2$ and $e_1 > e_2$. Then we know that $n_1\omega^{d_1} = p_1\omega^{e_1}$. Then a matrix works for our purpose is

$$\begin{pmatrix} n_1\omega^{d_1} & p_2\omega^{e_2} \\ n_2\omega^{d_2} & 0 \end{pmatrix}$$

Let us check condition (MS4), so assume that $m\omega^c p\omega^e = n_1\omega^{d_1} + n_2\omega^{d_2}$. If $d_1 = d_2$ then we are reduced to the property (MS4) of $\mathbb{N}$. Assume $d_2 < d_1$ (and of course $n_1 \neq 0$ and $n_2 \neq 0$). So we have $pm\omega^{e+c} = n_1\omega^{d_1}$. Our goal is to find a vector V and a matrix M with

$$\mathbf{1}_2^{\perp} V = p\omega^e \;\; \text{and} \;\; MV = \begin{pmatrix} n_1\omega^{d_1} \\ n_2\omega^{d_2} \end{pmatrix} \;\; \text{and} \;\; M^{\perp}\mathbf{1}_2 = \begin{pmatrix} m\omega^c \\ m\omega^c \end{pmatrix}$$

We consider several cases.

$(c \leq d_2)$ Then we set $V = \begin{pmatrix} p\omega^e \\ n_2\omega^{d_2-c} \end{pmatrix}$ and $M = \begin{pmatrix} m\omega^c & (m-1)\omega^c \\ 0 & \omega^c \end{pmatrix}$

$(c > d_2,\, e > 0)$ Then we set $V = \begin{pmatrix} p\omega^e \\ n_2 \end{pmatrix}$ and $M = \begin{pmatrix} m\omega^c & m\omega^c \\ 0 & \omega^{d_2} \end{pmatrix}$

$(c > d_2,\, e = 0)$ Then we set $V = \begin{pmatrix} 1 \\ p-1 \end{pmatrix}$ and $M = \begin{pmatrix} m\omega^c & m\omega^c \\ n_2\omega^{d_2} & 0 \end{pmatrix}$

By direct calculation it is possible to check that in each case the vector and the matrix chosen work for the purpose. $\square$

We also give a non-example: the semi-ring $\mathbf{2}$ of truth values (see §7) is not a multiplicity semi-ring, since it does not satisfy property (MS1).

8.3   An exponential functor based on multiplicity semi-rings

The following is the definition of what we will prove to be an endofunctor on **Rel** parameterized by the choice of a multiplicity semi-ring, one of whose instances is the bang sending a set to the set of its finite multisets, already defined on **Rel**. From now on $\mathbf{A}$ denotes a multiplicity semi-ring.

Definition 8.3.1. *For a set X we define $!_{\mathbf{A}}X = \mathbf{A}\langle X \rangle$ and for a relation $R \in$ **Rel**(X, Y), we define $!_{\mathbf{A}}R \in$ **Rel**$(!_{\mathbf{A}}X, !_{\mathbf{A}}Y)$ as the set of all pairs (μ, ν) such that one can find $\sigma \in \mathbf{A}\langle X \times Y \rangle$ with $\mathsf{su}(\sigma) \subseteq R$ and*

$$\forall x \in X.\; \mu(x) = \sum_{y \in Y} \sigma(x, y) \;\; \text{and} \;\; \nu(y) = \sum_{x \in X} \sigma(x, y) \qquad (*)$$

We say a σ satisfying property $(*)$ *is a witness of (μ, ν) for R.* Observe that all these sums are finite because $\sigma \in \mathbf{A}\langle X \times Y \rangle$.

Warning: in the rest of the chapter we will drop the subscript from the notation $!_{\mathbf{A}}(-)$. In fact the particular choice of the multiplicity semi-ring with which

implement the exponential functor will not matter: all subsequent results do hold whatever multiplicity semi-ring **A** is chosen.

Recall that for $R \in \mathbf{Rel}(X, Y)$ and $S \in \mathbf{Rel}(Y, Z)$ we denote as $R; S = S \cdot R = \mathbf{Rel}(X, Z) = \{(x, z) \in X \times Z : \exists y \in Y.\ x\ R\ y\ S\ z\}$ the relational composition of R and S.

Lemma 8.3.1. $!(-) : \mathbf{Rel} \to \mathbf{Rel}$ *is an endofunctor on* **Rel**.

Proof. It is clear from this definition that $!\mathrm{id}_X = \mathrm{id}_{!X}$. We now prove that $!(S \cdot R) = !S \cdot !R$. First let $(\mu, \pi) \in !(S \cdot R)$. Let φ be a witness of (μ, π) for $S \cdot R$. For each $(x, z) \in S \cdot R$, let us choose $f(x, z) \in Y$ such that $(x, f(x, z)) \in R$ and $(f(x, z), z) \in S$. Let $\nu \in \mathbf{A}\langle Y \rangle$ be given by

$$\nu(y) = \sum_{f(x,z)=y} \varphi(x, z)$$

This sum is finite because φ has a finite support. Moreover if $y \in \mathsf{su}(\nu)$ then we must have $y = f(x, z)$ for some $(x, z) \in \mathsf{su}(\varphi)$ and there are only finitely many such pairs (x, z) so ν has finite support: $\nu \in !Y$. We check that $(\mu, \nu) \in !R$, and for this we exhibit a witness, namely $\sigma \in \mathbf{A}\langle X \times Y \rangle$, given by

$$\sigma(x, y) = \sum_{f(x,z)=y} \varphi(x, z)$$

Indeed we have

$$\forall x \in X.\ \sum_{y \in Y} \sigma(x, y) = \sum_{y \in Y} \sum_{f(x,z)=y} \varphi(x, z) = \sum_{(x,z) \in R} \varphi(x, z) = \mu(x)$$

$$\forall y \in Y.\ \sum_{x \in X} \sigma(x, y) = \sum_{x \in X} \sum_{f(x,z)=y} \varphi(x, z) = \sum_{f(x,z)=y} \varphi(x, z) = \nu(y)$$

One checks similarly that $(\nu, \pi) \in !S$ and hence $(\mu, \pi) \in !S \cdot !R$.

Conversely, let $(\mu, \pi) \in !S \cdot !R$. Let $\nu \in !Y$ be such that $(\mu, \nu) \in !R$ and $(\nu, \pi) \in !S$ and let $\sigma \in \mathbf{A}\langle X \times Y \rangle$ and $\tau \in \mathbf{A}\langle X \times Y \rangle$ be corresponding witnesses. Let $y \in Y$; we have

$$\sum_{x \in X} \sigma(x, y) = \sum_{z \in Z} \tau(y, z) = \nu(y)$$

By Lemma 8.2.1, we can find $\varphi^y \in \mathbf{A}\langle X \times Y \rangle$ such that

$$\forall x \in X.\ \sigma(x, y) = \sum_{z \in Z} \varphi^y(x, z) \ \text{ and } \ \forall x \in X.\ \tau(y, z) = \sum_{x \in X} \varphi^y(x, z)$$

Let $\varphi = \sum_{y \in \mathsf{su}(\nu)} \varphi^y$. Let $x \in X$, we have

$$\mu(x) = \sum_{y \in Y} \sigma(x, y) = \sum_{y \in Y} \sum_{z \in Z} \varphi^y(x, z) = \sum_{z \in Z} \sum_{y \in Y} \varphi^y(x, z) = \sum_{z \in Z} \varphi(x, z)$$

Similarly one show that $\pi(z) = \Sigma_{x \in X} \varphi(x, z)$. Last observe that if $(x, z) \in \mathsf{su}(\varphi)$, one has $(x, z) \in \mathsf{su}(\varphi^y)$ for some y. For such a y we have $(x, y) \in \mathsf{su}(\sigma) \subseteq R$ and $(y, z) \in \mathsf{su}(\tau) \subseteq S$. This shows that $\mathsf{su}(\varphi) \subseteq S \cdot R$, so that φ is a witness of (μ, π) for $S \cdot R$, and hence $(\mu, \pi) \in !(S \cdot R)$. $\qquad\square$

Lemma 8.3.2. *Let $R \subseteq X \times Y$ and let $(\mu_i, \nu_i) \in !R$ and $p_i \in A$, for $i = 1, \ldots, n$. Then $(\Sigma_{i=1}^n p_i \mu_i, \Sigma_{i=1}^n p_i \nu_i) \in !R$.*

Proof. For each i, choose a witness σ_i of (μ_i, ν_i) for R. Then $\Sigma_{i=1}^n p_i \sigma_i$ is a witness of $(\Sigma_{i=1}^n p_i \mu_i, \Sigma_{i=1}^n p_i \nu_i)$ for R. $\qquad\square$

8.3.1   Comonad structure of the new exponential modality

We introduce the fundamental comonadic structure of the exponential functor, which consists of two natural transformations usually called *dereliction* (the counit of the comonad) and *digging* (the comultiplication of the comonad). This entire subsection is devote to the proofs of the comonad axioms and such proof is articulated in several lemmas: three of them (8.3.3, 8.3.6, 8.3.8) contain the principal results and the others (8.3.4, 8.3.5, 8.3.7) are strictly technical, so the reader may rapidly skim through them.

Dereliction. We set $\mathrm{der}_X = \{([\alpha], \alpha) : \alpha \in X\} \in \mathbf{Rel}(!X, X)$.

Lemma 8.3.3. $\mathrm{der} : ! \Rightarrow \mathrm{id}$ *is a natural transformation.*

Proof. Let $R \in \mathbf{Rel}(X, Y)$. We must show that $R \cdot \mathrm{der}_X = \mathrm{der}_Y \cdot !R$. Let $\mu \in !X$ and $y \in Y$. Assume first that $(\mu, y) \in R \cdot \mathrm{der}_X$; this means that there exists $x \in X$ such that $(\mu, x) \in \mathrm{der}_X$ and $(x, y) \in R$. Hence we have $\mu = [x]$. We have $([x], [y]) \in !R$ and hence also $(\mu, y) \in \mathrm{der}_Y \cdot !R$.

Conversely assume that $(\mu, y) \in \mathrm{der}_Y \cdot !R$, so that $(\mu, [y]) \in !R$, and let $\sigma \in \mathbf{A}\langle X \times Y \rangle$ be a witness. We have $\Sigma_{x \in X} \sigma(x, y') = [y](y')$ for each $y' \in Y$. By conditions (MS1) and (MS2) one has $\sigma(x, y') = 0$, for all $x \in X$ and all $y' \neq y$, and there exists $x \in X$ such that $\sigma(x, y) = 1$ and $\sigma(x', y) = 0$ for all $x' \neq x$. We have therefore $\mu = [x]$. Since $(x, y) \in R$, this shows that $(\mu, y) \in R \cdot \mathrm{der}_X$ because $([x], x) \in \mathrm{der}_X$. $\qquad\square$

Let's take $\mathbf{2}$ as semi-ring of coefficients. Then $!_{\mathbf{2}}X \cong \mathcal{P}_{\mathrm{f}}(X)$ is indeed an endofunctor on $\mathbf{Rel}$ but $\mathbf{2}$ is not a multiplicity semi-ring since it does not satisfy condition (MS2). As a consequence of this fact if we define a dereliction morphism as $\mathrm{der}_X = \{(\{\alpha\}, \alpha) : \alpha \in X\}$, then we don't get a natural transformation, as pointed out by T. Ehrhard.

Example 8.3.1. *The naturality diagram does not commute for $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta\}$ and the relation $R = \{(\alpha_1, \beta), (\alpha_2, \beta)\}$.*

$$
\begin{CD}
!A @>!R>> !B \\
@V\mathrm{der}_A VV @VV\mathrm{der}_B V \\
A @>>R> B
\end{CD}
$$

In fact $(\{\alpha_1, \alpha_2\}, \beta) \in \mathrm{der}_B \cdot !R - R \cdot \mathrm{der}_A$.

The digging operation is more problematic and some preliminaries are required.

Lemma 8.3.4. *Let X, Y be sets and let $R \subseteq X \times Y$. Let $\nu_1, \nu_2 \in !Y$ and $\mu \in !X$. If $(\mu, \nu_1 + \nu_2) \in !R$, then one can find $\mu_1, \mu_2 \in !X$ such that $\mu_1 + \mu_2 = \mu$ and $(\mu_i, \nu_i) \in !R$ for $i = 1, 2$.*

Proof. Let $\sigma \in \mathbf{A}\langle X \times Y \rangle$ be a witness of (μ, ν) for R and let $y \in Y$. We have $\nu_1(y) + \nu_2(y) = \Sigma_{x \in X} \sigma(x, y)$. By Lemma 8.2.1 we can find $\varphi_i^y \in \mathbf{A}\langle X \rangle$ (for $i = 1, 2$) such that $\nu_i(y) = \Sigma_{x \in X} \varphi_i^y(x)$ (for $i = 1, 2$) and $\sigma(x, y) = \varphi_1^y(x) + \varphi_2^y(x)$. Let $\sigma_i(x, y) = \varphi_i^y(x)$. Then $\sigma_1(x, y) + \sigma_2(x, y) = \sigma(x, y)$ and this shows that $\mathsf{su}(\sigma_i) \subseteq R$ (for $i = 1, 2$) using condition (MS1). We have $\Sigma_{x \in X} \sigma_i(x, y) = \nu_i(y)$ for all $y \in Y$ (for $i = 1, 2$). We set $\mu_i(x) = \Sigma_{y \in Y} \sigma_i(x, y)$. Then $\mu_i \in !X$ (for $i = 1, 2$) since σ_i has finite support. Moreover $(\mu_i, \nu_i) \in !R$ with witness σ_i (for $i = 1, 2$). We conclude because $\mu_1(x) + \mu_2(x) = \Sigma_{y \in Y}(\sigma_1(x, y) + \sigma_2(x, y)) = \Sigma_{y \in Y} \sigma(x, y) = \mu(x)$. $\square$

Given $M \in !!X$, we define $\Sigma(M) \in !X$ as follows:

$$
\Sigma(M) = \sum_{\mu \in !X} M(\mu)\mu
$$

Since M has finite support, this sum is actually a finite sum (the linear combination with coefficients $M(\mu) \in \mathbf{A}$ is taken in the module $!X$).

Digging. We set $\mathrm{dig}_X = \{(\Sigma(M), M) : M \in !!X\} \in \mathbf{Rel}(!X, !!X)$.

The next lemma is the main tool for proving the naturality of digging. It combines the two generalized splitting properties (MS3)' and (MS4)' of $\mathbf{A}$.

Lemma 8.3.5. *Let X, Y be sets and let $R \subseteq X \times Y$ be finite. There exists $q(R) \in \mathbb{N}$ with the following property: for any $\mu \in !X$, $\pi \in !Y$ and $p \in \mathbf{A}$ if $(\mu, p\pi) \in !R$ then one can find $p_1, \ldots, p_{q(R)} \in \mathbf{A}$ and $\mu_1, \ldots, \mu_{q(R)} \in !X$ such that $\Sigma_{j=1}^{q(R)} p_j = p$, $\Sigma_{j=1}^{q(R)} p_j \mu_j = \mu$ and $(\mu_j, \pi) \in !R$ for each $j = 1, \ldots, q(R)$.*

Proof. Let $I = \{x \in X : \exists y \in Y. \ (x,y) \in R\}$ and $J = \{y \in Y : \exists x \in X. \ (x,y) \in R\}$. Given $y \in J$, let $\deg_y(R) = \sharp\{x \in X : (x,y) \in R\} - 1 \in \mathbb{N}$ and $\deg_y(R) = \Sigma_{y \in J} \deg_y(R)$. We prove the result by induction on $\deg(R)$.

Assume first that $\deg(R) = 0$, so that, for any $y \in J$, there is exactly one $x \in I$ such that $(x,y) \in R$: let us define $g : J \to I$ to be the surjective function associating each $y \in J$ to that one $x \in I$, so that $R = \{(g(y),y) : y \in J\}$. Let σ be a witness of $(\mu, p\pi)$ for R. For all $y \in J$ we have $p\pi(y) = \Sigma_{x \in X}\sigma(x,y) = \sigma(g(y),y)$ and for all $x \in I$ we have $\mu(x) = \Sigma_{g(y)=x}\sigma(x,y) = p\Sigma_{g(y)=x}\pi(y)$. Let $\tau \in \mathbf{A}\langle X \times Y \rangle$ be defined by

$$\tau(x,y) = \begin{cases} \pi(y) & \text{if } g(y) = x \\ 0 & \text{otherwise} \end{cases}$$

then clearly $\mathsf{su}(\tau) \subseteq R$ and τ is a witness of (μ', π) for R, where $\mu' \in\ !X$ is given by $\mu'(x) = \Sigma_{g(y)=x}\pi(x)$. Since $p\mu' = \mu$, we obtained the required property (with $q(R) = 1$, $p_1 = p$ and $\mu_1 = \mu'$).

Assume now that $\deg(R) > 0$ and let us pick some $y \in J$ such that $k = \deg(R)_y + 1 > 1$. Let $x_1, \ldots, x_k$ be a repetition-free enumeration of the elements x of I such that $(x,y) \in R$. We have

$$p\pi(y) = \sum_{i=1}^{k} \sigma(x_i, y)$$

Let $l = 2^{k-1}$. By Lemma 8.2.2 there exists a vector $V \in \mathbf{A}^{l,1}$ and a matrix $M \in \mathbf{A}^{k,l}$ such that

$$\mathbf{1}_l^T V = p \ \text{ and } \ M^T \mathbf{1}_k = \begin{pmatrix} \pi(y) \\ \vdots \\ \pi(y) \end{pmatrix} \ \text{ and } \ MV = \begin{pmatrix} \sigma(x_1, y) \\ \vdots \\ \sigma(x_k, y) \end{pmatrix}$$

Let $y_1, \ldots, y_k$ be pairwise distinct new elements which do not belong neither to X nor to Y and let $Y' = (Y - \{y\}) \cup \{y_1, \ldots, y_k\}$. We define a new relation to which we'll be able to apply the inductive hypothesis as follows:

$$S = \{(x,y') \in R : y' \neq y\} \cup \{(x_i, y_i) : i = 1, \ldots, k\}$$

Then we have $\deg(S) = \deg(R) - k + 1 < \deg(R)$. Let $\tau \in \mathbf{A}\langle X \times Y' \rangle$ be given by

$$\tau(x,z) = \begin{cases} \sigma(x,z) & \text{if } z \notin \{y_1, \ldots, y_k\} \\ \sigma(x_i, y) & \text{if } z = y_i \text{ and } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

Let

$$V = \begin{pmatrix} p_1 \\ \vdots \\ p_l \end{pmatrix} \ \text{ and } \ M = \begin{pmatrix} m_{11} & \ldots & m_{1l} \\ \vdots & \ddots & \vdots \\ m_{k1} & \ldots & m_{kl} \end{pmatrix}$$

It is clear that $\mathsf{su}(\tau) \subseteq S$. Moreover τ is a witness of $(\mu, \Sigma_{j=1}^{l} p_j \pi_j)$ for S, where $\pi_j \in !Y'$ is given by

$$\pi_j(z) = \begin{cases} \pi(z) & \text{if } z \notin \{y_1, \ldots, y_k\} \\ m_{ij} & \text{if } z = y_i \end{cases}$$

for each $j \in \{1, \ldots, l\}$. Indeed for $x \in X$ we have

$$\begin{aligned}
\sum_{z \in Y'} \tau(x, z) &= \sum_{z \in Y' - \{y_1, \ldots, y_k\}} \tau(x, z) + \sum_{i=1}^{k} \tau(x, y_i) \\
&= \sum_{z \in Y' - \{y_1, \ldots, y_k\}} \sigma(x, z) + \sum_{i=1}^{k} \delta_{x, x_i} \sigma(x_i, y) \\
&= \sum_{z \in Y' - \{z_1, \ldots, z_k\}} \sigma(x, z) + \sigma(x, y) \\
&= \sum_{y \in Y} \sigma(x, y) \\
&= \mu(x)
\end{aligned}$$

and for $z \in Y' - \{y_1, \ldots, y_k\}$ we have

$$\sum_{x \in X} \tau(x, z) = \sum_{x \in X} \sigma(x, z) = p\pi(z) = \sum_{j=1}^{l} p_j \pi_j(z)$$

since $\pi_j(z) = \pi(z)$ (for $j = 1, \ldots, l$) and $\Sigma_{j=1}^{l} p_j = p$ and last for $z = y_i$ (with $i = 1, \ldots, l$), we have

$$\sum_{x \in X} \tau(x, z) = \sigma(x_i, y) = \sum_{j=1}^{l} p_j m_{ij} = \sum_{j=1}^{l} p_j \pi_j(z)$$

By Lemma 8.3.4, since $(\mu, \Sigma_{j=1}^{l} p_j \pi_j) \in !S$, we can find $\mu_1, \ldots, \mu_l \in !X$ such that $\Sigma_{j=1}^{l} \mu_j = \mu$ and $(\mu_j, p_j \pi_j) \in !S$ for each $j = 1, \ldots, l$. Since $\deg(S) < \deg(R)$, we can apply the inductive hypothesis for each $j = 1, \ldots, l$. So we can find a family $(p_{js})_{j=1, s=1}^{l, q(S)}$ of elements of $\mathbf{A}$ such that such $p_j = \Sigma_{s=1}^{q(S)} p_{js}$ and we can find a family $(\mu_{js})_{j=1, h=1}^{l, q(S)}$ of elements of $!X$ such that $\mu_j = \Sigma_{s=1}^{q(S)} p_{js} \mu_{js}$ and moreover $(\mu_{js}, \pi_j) \in !S$ for each $j = 1, \ldots, l$ and $s = 1, \ldots, q(S)$. We conclude the proof by showing that $(\mu_{js}, \pi) \in !R$. Let $\tau_{js} \in \mathbf{A}\langle X \times Y' \rangle$ be a witness of (μ_{js}, π_j) for S. Let $\sigma_{js} \in \mathbf{A}\langle X \times Y \rangle$ be given by

$$\sigma_{js}(x, y') = \begin{cases} \tau_{js}(x, y') & \text{if } y' \neq y \\ \Sigma_{i=1}^{k} \tau_{js}(x, y_i) & \text{if } y' = y \end{cases}$$

For $y' \in Y - \{y\}$ we have $\Sigma_{x \in X} \sigma_{js}(x, y') = \Sigma_{x \in X} \tau_{js}(x, y') = \pi_j(y') = \pi(y')$. Next we have

$$\begin{aligned}
\sum_{x \in X} \sigma_{js}(x, y) &= \sum_{x \in X} \sum_{i=1}^{k} \tau_{js}(x, y_i) \\
&= \sum_{i=1}^{k} \sum_{x \in X} \tau_{js}(x, y_i) \\
&= \sum_{i=1}^{k} \pi_j(y_i) \\
&= \sum_{i=1}^{k} m_{ij} \\
&= \pi(y)
\end{aligned}$$

On the other hand we have

$$
\begin{aligned}
\sum_{y'\in Y} \sigma_{js}(x,y') &= \sum_{y'\in Y-\{y\}} \sigma_{js}(x,y') + \sigma_{js}(x,y) \\
&= \sum_{y'\in Y-\{y\}} \tau_{js}(x,y') + \sum_{i=1}^{k} \tau_{js}(x,y_i) \\
&= \sum_{y'\in Y} \tau_{js}(x,z) \\
&= \mu_{js}(x) \\
&= \pi(y)
\end{aligned}
$$

It remains to prove that $\mathsf{su}(\sigma_{js}) \subseteq R$, but this results immediately from the definition of σ_{js} and from the fact that $\mathsf{su}(\tau_{js}) \subseteq S$. Observe that we can take $q(R) = l \cdot q(S)$, so that in general $q(R) = 2^{\deg(R)}$. $\hfill\square$

Lemma 8.3.6. $\mathrm{dig} : \,! \Rightarrow\,!!$ *is a natural transformation.*

Proof. Let X, Y be sets and let $R \subseteq X \times Y$. Let $(\mu, \Pi) \in\,!X\times!!Y$.

Assume first that $(\mu, \Pi) \in\,!!R \cdot \mathrm{dig}_X$. Let $M \in\,!!X$ be such that $(M, \Pi) \in\,!!R$ and $(\mu, M) \in \mathrm{dig}_X$, that is $\Sigma(M) = \mu$. Let $\Theta \in \mathbf{A}\langle!X\times!Y\rangle$ be a witness of (M, Π) for $!R$. This means that

$$
\forall \mu' \in\,!X.\ M(\mu') = \sum_{\pi'\in !Y} \Theta(\mu', \pi')
$$

$$
\forall \pi' \in\,!Y.\ \Pi(\pi') = \sum_{\mu'\in !X} \Theta(\mu', \pi')
$$

Since $\mathsf{su}(\Theta) \subseteq\,!R$ by Lemma 8.3.2 we have

$$
\Big(\sum_{\mu'\in !X, \pi'\in !Y} \Theta(\mu', \pi')\mu',\ \sum_{\mu'\in !X, \pi'\in !Y} \Theta(\mu', \pi')\pi' \Big) \in\,!R
$$

that is $(\Sigma(M), \Sigma(\Pi)) \in\,!R$. Therefore $(\mu, \Pi) \in \mathrm{dig}_Y \cdot\,!R$ since $(\Sigma(\Pi), \Pi) \in \mathrm{dig}_Y$.

Conversely, assume that $(\mu, \Pi) \in \mathrm{dig}_Y \cdot\,!R$, that is $(\mu, \Sigma(\Pi)) \in\,!R$ and in turn $(\mu, \Sigma_{\pi\in !Y}\Pi(\pi)\pi) \in\,!R$. Let $R_0 \subseteq R$ be finite and such that $(\mu, \Sigma_{\pi\in !Y}\Pi(\pi)\pi) \in\,!R_0$. Such an R_0 exists because μ and Π have finite support. By Lemma 8.3.4, one can find a family $(\mu^\pi)_{\pi\in\mathsf{su}(\Pi)}$ of elements of $!X$ such that $\mu = \Sigma_{\pi\in\mathsf{su}(\Pi)}\mu^\pi$ and $\forall\pi \in \mathsf{su}(\Pi).\ (\mu^\pi, \Pi(\pi)\pi) \in\,!R_0$. Applying Lemma 8.3.5, for each $\pi \in \mathsf{su}(\Pi)$, we can find a family $(\mu_i^\pi)_{i=1}^{q(R_0)}$ of elements of $!X$ and a family $(p_i^\pi)_{i=1}^{q(R_0)}$ of elements of $\mathbf{A}$ such that

$$
\sum_{i=1}^{q(R_0)} p_i^\pi = \Pi(\pi) \text{ and } \sum_{i=1}^{q(R_0)} p_i^\pi \mu_i^\pi = \mu^\pi \text{ and } \forall i = 1, \ldots, q(R_0).\ (\mu_i^\pi, \pi) \in\,!R
$$

We define $M \in \mathbf{A}^{!X}$ by setting

$$
M = \sum_{i=1}^{q(R_0)} \sum_{\pi\in\mathsf{su}(\Pi)} p_i^\pi [\mu_i^\pi]
$$

This sum is finite because $\mathsf{su}(\Pi)$ is a finite set. We have

$$
\begin{aligned}
\Sigma(M) &= \sum_{i=1}^{q(R_0)} \sum_{\pi \in \mathsf{su}(\Pi)} p_i^\pi \mu_i^\pi \\
&= \sum_{\pi \in \mathsf{su}(\Pi)} \mu^\pi \\
&= \mu
\end{aligned}
$$

so that $(\mu, M) \in \mathrm{dig}_X$. Moreover we have $\forall \pi \in \mathsf{su}(\Pi). \forall i = 1, \ldots, q(R_0). \ (\mu_i^\pi, \pi) \in !R$ and hence by Lemma 8.3.2 we have

$$
\Big(M, \sum_{i=1}^{q(R_0)} \sum_{\pi \in \mathsf{su}(\Pi)} p_i^\pi [\pi] \Big) \in !!R
$$

and hence $(M, \Pi) \in !!R$ because

$$
\sum_{i=1}^{q(R_0)} \sum_{\pi \in \mathsf{su}(\Pi)} p_i^\pi [\pi] = \sum_{\pi \in \mathsf{su}(\Pi)} \Pi(\pi)[\pi] = \Pi
$$

This shows that $(\mu, \Pi) \in !!R \cdot \mathrm{dig}_X$ as announced. $\qquad \square$

Before proving that the above data define a comonad, we need the following auxiliary lemmas.

Lemma 8.3.7. *Let* $\mathcal{M} \in !!!X$. *Then* $\Sigma(\Sigma(\mathcal{M})) = \Sigma_{N \in !!X} \mathcal{M}(N)\Sigma(N)$.

Proof. We have

$$
\begin{aligned}
\Sigma(\Sigma(\mathcal{M})) &= \sum_{\nu \in !X} \Sigma(\mathcal{M})(\nu)\nu \\
&= \sum_{\nu \in !X} \Big(\sum_{N \in !!X} \mathcal{M}(N)N(\nu) \Big) \nu \\
&= \sum_{N \in !!X} \mathcal{M}(N) \Big(\sum_{\nu \in !X} N(\nu)\nu \Big)
\end{aligned}
$$

and we are done. $\qquad \square$

Lemma 8.3.8. $(!, \mathrm{der}, \mathrm{dig})$ *is a comonad over* **Rel**.

Proof. We prove the comonad equations, starting from $\mathrm{der}_{!X} \cdot \mathrm{dig}_X = \mathrm{id}_{!X}$. Let $(\mu, \mu') \in !X \times !X$. Assume first that $(\mu, \mu') \in \mathrm{der}_{!X} \cdot \mathrm{dig}_X$. Then we can find $M \in !!X$ such that $(\mu, M) \in \mathrm{dig}_X$ and $(M, \mu') \in \mathrm{der}_{!X}$. This means that $M = [\mu']$ and hence $\Sigma(M) = \mu'$, so that $\mu = \mu'$. Conversely, for $\mu \in !X$ we have $(\mu, [\mu]) \in \mathrm{dig}_X$, therefore $(\mu, \mu) \in \mathrm{der}_{!X} \cdot \mathrm{dig}_X$.

Next we prove that $!\mathrm{der}_X \cdot \mathrm{dig}_X = \mathrm{id}_{!X}$. Let $(\mu, \mu') \in \mathrm{der}_{!X} \cdot \mathrm{dig}_X$ and let $M \in !!X$ be such that $(\mu, M) \in \mathrm{dig}_X$, that is, $\Sigma(M) = \mu$, and $(M, \mu') \in !\mathrm{der}_X$. Let $\sigma \in \mathbf{A}\langle !X \times X \rangle$ be a witness of (M, μ') for der_X. This means that $\mu'(x) = \Sigma_{\nu \in !X}\sigma(\nu, x) = \sigma([x], x)$ since $\mathsf{su}(\sigma) \subseteq \mathrm{der}_X$, and that $M(\nu) = \sigma([x], x)$ if $\nu = [x]$ and $M(\nu) = 0$ if $\sharp\nu \neq 1$. It follows that $\Sigma(M) = \Sigma_{\nu \in !X}M(\nu)\nu = \Sigma_{x \in X}\sigma([x], x)[x] = \mu'$

and hence $\mu = \mu'$. Conversely one has $([\mu], \mu) \in !\mathrm{der}_X \cdot \mathrm{dig}_X$ because $M \in !!X$ defined by
$$M(\nu) = \begin{cases} \mu(x) & \text{if } \nu = [x] \\ 0 & \text{if } \sharp\nu \neq 0 \end{cases}$$
is such that $(\mu, M) \in \mathrm{dig}_X$ and $(M, \mu) \in !\mathrm{id}_X$.

We consider now the last comonad equation, namely $\mathrm{dig}_{!X} \cdot \mathrm{dig}_X = !\mathrm{dig}_X \cdot \mathrm{dig}_X$. Let $(\mu, \mathcal{M}) \in !X \times !!!X$. Assume first that $(\mu, \mathcal{M}) \in \mathrm{dig}_{!X} \cdot \mathrm{dig}_X$, that is $\Sigma(\Sigma(\mathcal{M})) = \mu$. We define $M \in M^{!X}$ as follows:

$$M(\nu) = \sum_{\substack{N \in !!X \\ \Sigma(N) = \nu}} \mathcal{M}(N)$$

Then $M \in !!X$. Indeed for each $\nu \in \mathsf{su}(M)$ we can find $N \in \mathsf{su}(\mathcal{M})$ such that $\nu \in \mathsf{su}(N)$, hence $\mathsf{su}(M) \subseteq \cup_{N \in \mathsf{su}(\mathcal{M})}\mathsf{su}(N)$ and this latter set is finite. We have

$$\begin{aligned} \Sigma(M) &= \sum_{\nu \in !X} M(\nu)\nu \\ &= \sum_{\nu \in !X} \left(\sum_{\Sigma(N) = \nu} \mathcal{M}(N) \right)\nu \\ &= \sum_{N \in !!X} \mathcal{M}(N)\Sigma(N) \\ &= \Sigma(\Sigma(\mathcal{M})) \\ &= \mu \end{aligned}$$

and hence $(\mu, \Sigma(M)) \in \mathrm{dig}_X$. Let $\Theta \in \mathbf{A}\langle !X \times !!X \rangle$ be defined by

$$\Theta(\nu, N) = \begin{cases} \mathcal{M}(N) & \text{if } \Sigma(N) = \nu \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $\mathsf{su}(\Theta) \subseteq \mathrm{dig}_X$. Moreover we have $\Sigma_{\nu \in !X}\Theta(\nu, N) = \mathcal{M}(N)$ for all $N \in !!X$ and $\Sigma_{N \in !!X}\Theta(\nu, N) = \Sigma_{\Sigma(N) = \nu}\mathcal{M}(N) = M(\nu)$ for all $\nu \in !X$, by definition of M. Thi shows that Θ is a witness of $(M, \mathcal{M})$ for dig_X. So we have shown that $(M, \mathcal{M}) \in !\mathrm{dig}_X$ and therefore $(\mu, \mathcal{M}) \in !\mathrm{dig}_X \cdot \mathrm{dig}_X$.

Assume conversely that $(\mu, \mathcal{M}) \in !\mathrm{dig}_X \cdot \mathrm{dig}_X$. So let $M \in !!X$ be such that $(\mu, M) \in \mathrm{dig}_X$ and $(M, \mathcal{M}) \in !\mathrm{dig}_X$. Let $\Theta \in \mathbf{A}\langle !X \times !!X \rangle$ be a witness of $(M, \mathcal{M})$ for dig_X. Since $\mathsf{su}(\Theta) \subseteq \mathrm{dig}_X$, there is a map $H : !!X \to \mathbf{A}$ such that

$$\Theta(\nu, N) = \begin{cases} H(N) & \text{if } \Sigma(N) = \nu \\ 0 & \text{otherwise} \end{cases}$$

For any $N \in !!X$ we must have $\mathcal{M}(N) = \Sigma_{\nu \in !X}\Theta(\nu, N) = H(N)$ so that $H = \mathcal{M}$. Therefore we have $M(\nu) = \Sigma_{\Sigma(N) = \nu}\mathcal{M}(N)$ for all $\nu \in !X$. By Lemma 8.3.7 we have

$$\begin{aligned} \Sigma(\Sigma(\mathcal{M})) &= \sum_{N \in !!X} \mathcal{M}(N)\Sigma(N) \\ &= \sum_{\nu \in !X} \left(\sum_{\Sigma(N) = \nu} \mathcal{M}(N)\nu \right) \\ &= \sum_{\nu \in !X} M(\nu)\nu, \text{ since } M(\nu) = \sum_{\Sigma(N) = \nu} \mathcal{M}(N), \\ &= \Sigma(M) \\ &= \mu \end{aligned}$$

Therefore $(\mu, \mathcal{M}) \in \mathrm{dig}_{!X} \cdot \mathrm{dig}_X$. $\qquad\qquad\Box$

Fundamental isomorphism. One of the most important property of the exponential is that it maps cartesian products to tensor products. Combined with the monoidal closure of **Rel**, this property leads to the cartesian closure of the Kleisli category **Rel**$_!$.

Proposition 8.3.9. *Given two sets* X_1, Y_2 *there is a natural bijection* n_{X_1, X_2} : $!X_1 \otimes !X_2 \to !(X_1 \& X_2)$ *and a bijection* $\mathrm{n}_0 : \mathbf{1} \to !\top$.

Proof. The second bijection is $\mathrm{n} = \{(*, [\,])\}$. The first one is

$$\mathrm{n}_{X_1, X_2} = \{((\mu_1, \mu_2), \mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2)) : \mu_i \in !X_i (i = 1, 2)\}$$

where $\mathrm{in}_i = \Sigma_{x \in X_i} \mu_i(x)[(i, x)]$ $(i = 1, 2)$. Let us check that this isomorphism is natural, so let $R_i \subseteq X_i \times Y_i$ $(i = 1, 2)$. We must check that $\mathrm{n}_{Y_1, Y_2} \cdot (!R_1 \otimes !R_2) = !(R_1 \& R_2) \cdot \mathrm{n}_{X_1, X_2}$. So let $\mu_i \in !X_i$ and $\nu_i \in Y_i$ $(i = 1, 2)$.

Assume first that $((\mu_1, \mu_2), \mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2)) \in \mathrm{n}_{Y_1, Y_2} \cdot (!R_1 \otimes !R_2)$. This means that one can find $\nu'_i \in !Y_i$ $(i = 1, 2)$ such that $(\mu_i, \nu'_i) \in !R_i$ $(i = 1, 2)$ and $((\nu'_1, \nu'_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in \mathrm{n}_{Y_1, Y_2}$. This means that $\nu_i = \nu'_i$ $(i = 1, 2)$. Since $(\mu_1, \nu_1) \in !R_1$ we have $(\mathrm{in}_1(\mu_1), \mathrm{in}_1(\nu_1)) \in !(R_1 \& R_2)$ and similarly $(\mathrm{in}_2(\mu_2), \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2)$ and hence $(\mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2)$ by Lemma 8.3.2. But $((\mu_1, \mu_2), \mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2)) \in \mathrm{n}_{X_1, X_2}$ and we have therefore $((\mu_1, \mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2) \cdot \mathrm{n}_{X_1, X_2}$.

Assume conversely that $((\mu_1, \mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2) \cdot \mathrm{n}_{X_1, X_2}$, so that there exist $\mu'_i \in !X_i$ $(i = 1, 2)$ with $((\mu_1, \mu_2), \mathrm{in}_1(\mu'_1) + \mathrm{in}_2(\mu'_2)) \in \mathrm{n}_{X_1, X_2}$ and $(\mathrm{in}_1(\mu'_1) + \mathrm{in}_2(\mu'_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2)$. Therefore $\mu'_i = \mu_i$ $(i = 1, 2)$ and hence $(\mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in !(R_1 \& R_2)$. Let φ be a witness of $(\mathrm{in}_1(\mu_1) + \mathrm{in}_2(\mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2))$ for $R_1 \& R_2$. Since $\mathsf{su}(\varphi) \subseteq R_1 \& R_2$, we have $(\mu_i, \nu_i) \in !R_i$ $(i = 1, 2)$: take $\varphi_i \in \mathbf{A}\langle X_i \times Y_i \rangle$ defined by $\varphi_i(x_i, y_i) = \varphi((i, x_i), (i, y_i))$. Then φ_i is a witness of (μ_i, ν_i) for R_i. It follows that $((\mu_1, \mu_2), (\nu_1, \nu_2)) \in !R_1 \otimes !R_2$ and therefore $((\mu_1, \mu_2), \mathrm{in}_1(\nu_1) + \mathrm{in}_2(\nu_2)) \in \mathrm{n}_{Y_1, Y_2} \cdot (!R_1 \otimes !R_2)$. $\qquad\Box$

Structural morphisms. They are used for interpreting the *structural rules* of linear logic, associated with the exponentials. The *weakening* morphism $\mathrm{wkn}_X : !X \to \mathbf{1}$ is given by $\mathrm{wkn}_X = \{([\,], *)\}$. The *contraction* morphism $\mathrm{con}_X : !X \to !X \otimes !X$ is obtained by applying the exponential functor $!$ to the diagonal map from X to $X \& X$, so that $\mathrm{con}_X = \{(\lambda + \rho, (\lambda, \rho)) : \lambda, \rho \in !X\}$.

There are other equations to be checked for proving that we have indeed defined *new-Seely category* (see [21]) but the corresponding verifications are straightforward.

8.3.2 The Kleisli cartesian closed category

The objects of the Kleisli category **Rel**$_!$ of the comonad "$!$" (recall that we dropped the subscript from "$!_\mathbf{A}$") are all the sets, and $\mathbf{Rel}_!(X, Y) = \mathbf{Rel}(!X, Y)$. In this

category the identity morphism of an object X is $\mathrm{der}_X \in \mathbf{Rel}_!(X,X)$ and the composition of $R \in \mathbf{Rel}_!(X,Y)$ with $S \in \mathbf{Rel}_!(Y,Z)$ is defined as $S \circ R = S \cdot !R \cdot \mathrm{dig}_X$.

We give a direct characterization of this composition law.

Proposition 8.3.10. *For $(\mu, y) \in !X \times Y$ we have $(\mu, y) \in S \circ R$ iff there exist $z_1, \ldots, z_n \in Y$ (not necessarily distinct), $p_1, \ldots, p_n \in \mathbf{A}$ and $\mu_1, \ldots, \mu_n \in !X$ such that $\forall i = 1, \ldots, n. \ (\mu_i, z_i) \in R$, and $(\sum_{i=1}^n p_i[z_i], y) \in S$ and $\mu = \sum_{i=1}^n p_i \mu_i$. In other words*

$$S \circ R = \{(\Sigma_{i=1}^n p_i \mu_i, y) : \exists z_1, \ldots, z_n \in Y. \ (\Sigma_{i=1}^n p_i[z_i], y) \in S, \ \forall i = 1, \ldots, n. \ (\mu_i, z_i) \in R\}$$

Proof. Assume first that $(\mu, z) \in S \circ R$. Let $M \in !!X$ be such that $(\mu, M) \in \mathrm{dig}_X$ and let $\nu \in !Y$ be such that $(\nu, z) \in S$ and $(M, \nu) \in !R$. We have $\Sigma(M) = \mu$. Let $\sigma \in \mathbf{A}\langle !X \times Y \rangle$ be a witness of (M, ν) for R and let $(\mu_1, y_1), \ldots, (\mu_n, y_n)$ be a repetition-free enumeration of the set $\mathsf{su}(\sigma) \subseteq R$. Taking $p_i = \sigma(\mu_i, y_i)$ we have $\Sigma_{i=1}^n p_i[y_i] = \nu$ and $\Sigma_{i=1}^n p_i[\mu_i] = M$, and therefore $\mu = \Sigma_{i=1}^n p_i \mu_i$.

Assume conversely that (μ, z) satisfies the conditions in the second part of the statement. Then we take $\nu = \Sigma_{i=1}^n p_i[y_i]$ and $M = \Sigma_{i=1}^n p_i[\mu_i]$. We have $(\nu, z) \in S$ and $(\mu, M) \in \mathrm{dig}_X$ and we have just to check that $(M, \nu) \in !R$. To this end we define $\sigma = \Sigma_{i=1}^n p_i[(\mu_i, y_i)]$; this is a witness of (M, ν) for R, as easily checked. $\qquad\square$

We recall that the cartesian product of X and Y in this category is $X \& Y$, with projections $\mathrm{der}_{X \& Y} \cdot \pi_1$ and $\mathrm{der}_{X \& Y} \cdot \pi_2$. The intuitionistic function space of X and Y is $!X \multimap Y$. Evaluation $\mathrm{ev} \in \mathbf{Rel}_!(X \& (!X \multimap Y), Y) \cong \mathbf{Rel}(!X \otimes !(!X \multimap Y), Y)$ is $\mathrm{ev} = \{((\mu, [(\mu, y)]), y) : \mu \in !X, \ y \in Y\}$.

Curryfication of $R \in \mathbf{Rel}_!(Z \& X, Y) \cong \mathbf{Rel}(!Z \otimes !X, Y)$ is defined as $\mathrm{cur}(R) = \{(\pi, (\mu, y)) : ((\pi, \mu), y) \in R\} \in \mathbf{Rel}_!(Z, !X \multimap Y)$.

Differential structure and the Taylor expansion. We sketch very briefly the differential structure of this category, which can be used for interpreting the differential lambda-calculus introduced in [44], or the various resource lambda-calculi based on this kind of differential structures [47, 112].

We introduce first the codereliction morphism $\mathrm{cod}_X \in \mathbf{Rel}(X, !X)$ by $\mathrm{cod}_X = \{(x, [x]) : x \in X\}$. Naturality is proved exactly as the naturality of dereliction in Subsection 8.3.1. *Coweakening* $\mathrm{cow}_X \in \mathbf{Rel}(\mathbf{1}, !X)$ and *cocontraction* $\mathrm{coc}_X \in \mathbf{Rel}(!X \otimes !X, !X)$ are obtained by applying the ! functor to the empty morphism $\emptyset \in \mathbf{Rel}(\top, X)$ and the "codiagonal" morphism $\pi_1 \cup \pi_2 \in \mathbf{Rel}(X \& X, X)$, respectively and by using the fundamental isomorphism. The equations involving dereliction and coweakening and cocontraction (see [45, 22, 52]) are satisfied by naturality of dereliction. Similarly for the equations involving codereliction, weakening and contraction. One should check that the chain rule holds: this is a bit long to express, but the proof is simple verification.

Using codereliction and cocontraction one defines easily a morphism $\delta_X \in \mathbf{Rel}(!X \otimes X, !X)$ with $\delta_X = \{((\mu, x), \mu + [x])\}$. Given $R \in \mathbf{Rel}(X, Y) = \mathbf{Rel}(!X, Y)$ one can define $R' \in \mathbf{Rel}_!(X, X \multimap Y)$ (by linear curryfication of $R \cdot \delta_X$) which can be

considered as the first derivative of R: this operation satisfies all the usual rules of differential calculus (linearity, Leibniz rule, chain rule, etc.). Iterating this operation and using also dereliction and contraction one can define the Taylor expansion of R as the infinite union $\cup_{n\geq 0} R_n$ where, for each $n \in \mathbb{N}$ the morphism $R_n \in \mathbf{Rel}_!(X,Y)$ is simply $R_n = \{(\mu, x) \in R : \sharp\mu = n\}$.

Note that if $\mathbf{A}$ has an infinite element, it is not true in general that R coincides with its Taylor expansion. As an example take $\omega \in \overline{\mathbb{N}}$, i.e., the infinite element in the completed set of natural numbers, and take $R = \{(\omega[*], *)\} \in \mathbf{Rel}_!(1,1)$. Then $R_n = \emptyset$ for all $n \in \mathbb{N}$.

So we have defined a model of differential linear logic which does not satisfy the Taylor formula.

8.4 Models of lambda calculus in the generalized relational semantics

Recall from §2 the definition of interpretation $|M|_{\bar{x}}^{\mathbf{U}} \in \mathbb{C}(U^{\bar{x}}, U)$ of an ordinary λ-term M with free variables $\mathrm{FV}(M) \subseteq \bar{x}$ in a reflexive object $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ of a ccc $\mathbb{C}$. In Section 8.1 we gave the explicit description of the interpretation function in the case of reflexive objects in $\mathbf{MRel}$. Here we accomplish this task for the category $\mathbf{Rel}_!$ (for any chosen comonad $!(\_) = \mathbf{A}\langle \_ \rangle$).

As a matter of notation by $\bar{\mu} = (\mu_1, \ldots, \mu_n)$ we denote a sequence of $\mathbf{A}$ multisets and for $p \in \mathbf{A}$ we write $p\bar{\mu}$ as a shorthand for the sequence $(p\mu_1, \ldots, p\mu_n)$. Moreover if $\bar{\mu}, \bar{\nu}$ are sequences of the same length, say n, then $p\bar{\mu} + q\bar{\nu} = (p\mu_1 + q\nu_1, \ldots, p\mu_n + q\nu_n)$.

$$|x_i|_{\bar{x}}^{\mathbf{U}} \quad = \{(([\,]\ldots,[\alpha],\ldots,[\,]),\alpha) : \alpha \in U\}$$

$$|\lambda y.M|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{j=1}^{k} p_j\bar{\mu}_j, \alpha) : \exists(\nu_1,\beta_1),\ldots,(\nu_k,\beta_k) \in U \Rightarrow U. \ ((\bar{\mu}_j,\nu_j),\beta_j) \in |M|_{\bar{x},y}^{\mathbf{U}}$$
$$(j=1,\ldots,k), \ (\Sigma_{j=1}^{k} p_j[(\nu_j,\beta_j)],\alpha) \in \mathrm{Lam}\}$$

$$|MN|_{\bar{x}}^{\mathbf{U}} = \{(\Sigma_{i=1}^{h} p_i\bar{\nu}_i + \Sigma_{j=1}^{k} q_j\bar{\mu}_j, \alpha) : \exists\beta_1,\ldots,\beta_k \in U.\exists\gamma_1,\ldots,\gamma_h \in U.$$
$$(\bar{\nu}_i,\gamma_i) \in |M|_{\bar{x}}^{\mathbf{U}} \ (i=1,\ldots,h), \ (\bar{\mu}_j,\beta_j) \in |N|_{\bar{x}}^{\mathbf{U}}$$
$$(j=1,\ldots,k), \ (\Sigma_{i=1}^{h} p_i[\gamma_i],(\Sigma_{j=1}^{k} q_j[\beta_j],\alpha)) \in \mathrm{Ap}\}$$

We also give the characterization of interpretation in terms of typing, as already done in Section 8.1.

$$\frac{}{x_1 : [\,],\ldots,x_i : [\alpha],\ldots,x_n : [\,] \rhd^{\mathbf{U}} x_i : \alpha} \ [\mathsf{var}]$$

$$\frac{\Gamma_j, y : \nu_j \rhd^{\mathbf{U}} M : \beta_j \ (j=1,\ldots,k) \quad (\Sigma_{j=1}^{k} p_j[(\nu_j,\beta_j)],\alpha) \in \mathrm{Lam}}{\Sigma_{j=1}^{k} p_j\Gamma_j \rhd^{\mathbf{U}} \lambda y.M : \alpha} \ [\mathsf{abs}]$$

$$\frac{\Gamma_i \rhd^{\mathbf{U}} M : \gamma_i \ (i = 1, \ldots, h) \quad \Delta_j \rhd^{\mathbf{U}} N : \beta_j \ (j = 1, \ldots, k) \quad (\Sigma_{i=1}^h p_i[\gamma_i], (\Sigma_{j=1}^k q_j[\beta_j], \alpha)) \in \mathrm{Ap}}{\Sigma_{i=1}^k p_i \Gamma_i + \Sigma_{i=1}^h q_i \Delta_i \rhd^{\mathbf{U}} MN : \alpha} \text{[app]}$$

We recall that the sum of context is defined when they have the same domain, so in the rule "app" it is implicitly required that the Δ_i's and the Γ_j's all have the same domain.

8.4.1   Graph models in $\mathbf{Rel}_!$

Graph models [10, 15] have been isolated by Engeler, Plotkin and Scott [48, 95, 107] in the continuous semantics. We develop here a similar construction in the generalized relational semantics.

In this subsection we let $!(\_) = \mathbf{A}\langle\_\rangle$ be a comonad implemented with a multiplicity semi-ring $\mathbf{A}$ and we denote by $\mathbf{Rel}_!$ the corresponding ccc.

Let S be a non-empty set whose elements are not pairs and let $j : (!S \multimap S) \hookrightarrow S$ be a partial injection.

We inductively define a sequence $\{U_n\}_{n \geq 0}$ of sets and a sequence of partial injections $\{i_n : (!U_n \multimap U_n) \hookrightarrow U_n\}_{n \geq 0}$ as follows:

- $U_0 := S$, $i_0 := j$;

- $U_{n+1} := U_n \cup ((!U_n \times U_n) - \mathsf{do}(i_n))$

- $i_{n+1}(\mu, \alpha) = \begin{cases} i_n(\mu, \alpha) & \text{if } (\mu, \alpha) \in \mathsf{do}(i_n) \\ (\mu, \alpha) & \text{otherwise} \end{cases}$

Finally one defines

$$U = \cup_{n \geq 0} U_n \qquad i = \cup_{n \geq 0} i_n$$

With these data one can define a reflexive object $\mathbf{U} = (U, \mathrm{Ap}, \mathrm{Lam})$ in $\mathbf{Rel}_!$ by setting

$$\mathrm{Ap} = \{([i(\mu, \alpha)], (\mu, \alpha)) : (\mu, \alpha) \in !U \times U\} \quad \mathrm{Lam} = \{([(\mu, \alpha)], i(\mu, \alpha)) : (\mu, \alpha) \in !U \times U\}$$

Clearly we have $\mathrm{Ap} \circ \mathrm{Lam} = \mathrm{id}_{!U \multimap U}$, whatever be the choice of the multiplicity semi-ring used to implement the functor "!".

Terminology. We call *relational graph model* generated by the pair (S, i) any model $\mathbf{U}$ constructed in the way just described, starting from a set S with a partial injection $j : (!S \multimap S) \hookrightarrow S$.

We now give the typing system that characterizes its interpretation function of a relational graph model $\mathbf{U}$. Given $\mu \in !U$ and $\alpha \in U$ we set $\mu \to \alpha = i(\mu, \alpha)$ (in order to achieve a more appealing presentation). Since the model $\mathbf{U}$ is fixed and clear from the context, we omit the superscript and write just $\Gamma \rhd M : \alpha$ in place of $\Gamma \rhd^{\mathbf{U}} M : \alpha$.

$$\frac{}{x_1 : [\,], \ldots, x_i : [\alpha], \ldots, x_n : [\,] \rhd x_i : \alpha} \text{[var]}$$

$$\frac{\Gamma, y : \mu \rhd M : \alpha}{\Gamma \rhd \lambda y.M : \mu \to \alpha} \text{ [abs]}$$

$$\frac{\Gamma \rhd M : (\Sigma_{i=1}^k p_i[\beta_i]) \to \alpha \qquad \Delta_i \rhd N : \beta_i \ (i = 1, \ldots, k)}{\Gamma + \Sigma_{i=1}^k p_i \Delta_i \rhd MN : \alpha} \text{ [app]}$$

Definition 8.4.1. *We define* **D** *as the relational graph model generated by the set* $A = \{\alpha\}$ *with partial injection* $\omega[\alpha] \to \alpha = \alpha$ *in the Cartesian closed category determined by choosing the multiplicity semi-ring* $\overline{\mathbb{N}}$ *for defining the exponential.*

Note that the model **D** of Definition 8.4.1 is the "quantitative" analogue of Park's graph model in the continuous semantics.

We recall that Ω is the looping λ-term $(\lambda x.xx)(\lambda x.xx)$. The next two propositions show that the model **D** is not sensible, since it does not equate the two unsolvable terms Ω and $\lambda y.\Omega$ (see Theorem 8.4.2).

Lemma 8.4.1. $|\Omega|^{\mathbf{D}} = \{\alpha\}$.

Proof. We have the following deduction tree (where we inserted equations between types or $\overline{\mathbb{N}}$-multisets of types that we use).

$$\frac{\dfrac{x : [\alpha] \rhd x : \alpha = \omega[\alpha] \to \alpha \quad x : [\alpha] \rhd x : \alpha}{\dfrac{x : [\alpha] + \omega[\alpha] = \omega[\alpha] \rhd xx : \alpha}{\rhd \lambda x.xx : \omega[\alpha] \to \alpha}} \qquad \dfrac{\text{(same derivation)}}{\rhd \lambda x.xx : \omega[\alpha] \to \alpha = \alpha}}{\rhd (\lambda x.xx)(\lambda x.xx) : \alpha}$$

Therefore indeed $\alpha \in |\Omega|^{\mathbf{D}}$.

Conversely, let $\gamma \in D$ and assume that $\rhd \Omega : \gamma$. There must exist $\mu \in {!}D$ such that $\rhd \lambda x.xx : \mu \to \gamma$ and for all $\beta \in \mathsf{su}(\mu) \ \rhd \lambda x.xx : \beta$. From the first of these two judgements we get $x : \mu \rhd xx : \gamma$ and hence there must exist $\nu \in {!}D$ such that $\mu = \nu + [\nu \to \gamma]$. From the second judgement we get $\rhd \lambda x.xx : \nu \to \gamma$ and for all $\beta \in \mathsf{su}(\nu) \ \rhd \lambda x.xx : \beta$. Iterating this process we build a sequence $(\mu_i)_{i \geq 1}$ of elements of ${!}D$ such that $\rhd \lambda x.xx : \mu_i \to \gamma$, for all $\beta \in \mathsf{su}(\mu_i) \ \rhd \lambda x.xx : \beta$ and $\mu_i = \mu_{i+1} + [\mu_{i+1} \to \gamma]$ for all $i \geq 1$. Let $\beta_i = \mu_i \to \gamma$; then for all $i \geq 1$ we have $\beta_i \in \mathsf{su}(\mu_1)$ and since $\mathsf{su}(\mu_1)$ is finite we can find an i, n such that $\beta_{i+n} = \beta_i$. We have $\beta_i = \mu_i \to \gamma = (\mu_{i+1} + [\beta_{i+1}]) \to \gamma = \cdots = (\mu_{i+n} + [\beta_{i+1}] + \cdots + [\beta_{i+n}]) \to \gamma$ and since $\beta_{i+n} = \mu_{i+n} \to \gamma$ we get $\mu_{i+n} = \mu_{i+n} + [\beta_{i+1}] + \cdots + [\beta_{i+n}]$ and hence $\beta_{i+n} \in \mathsf{su}(\mu_{i+n})$. But $\beta_{i+n} = \mu_{i+n} \to \gamma$ and hence we must have $\beta_{i+n} = \gamma$. Indeed, if $\beta_{i+n} \notin A$ then we have $\beta_{i+n} = (\mu_{i+n}, \gamma)$ and, if k is the least integer such that $\beta_{i+n} \in D_k$ (recall that D_k is the k-th step in the construction of D), we have $k > 0$ and $\beta \in D_{k-1}$ for all $\beta \in \mathsf{su}(\mu_{i+n})$. This is impossible since $\beta_{i+n} \in \mathsf{su}(\mu_{i+n})$. Since $\mu_{i+n} \to \gamma = \gamma$, we have $\gamma = \alpha$ and we are done. $\qquad \square$

Theorem 8.4.2. $|\lambda y.\Omega|^{\mathbf{D}} \neq |\Omega|^{\mathbf{D}}$.

Proof. By Lemma 8.4.1 we have $|\Omega|^{\mathbf{D}} = \{\alpha\}$. Now by definition of interpretation we also have $[\,] \to \alpha \in |\lambda y.\Omega|^{\mathbf{D}}$. We conclude since $\alpha \neq [\,] \to \alpha$. $\qquad\square$

9

The purely algebraic theory of resource lambda calculus

In this chapter we develop a purely algebraic study of Ehrhard and Regnier's resource λ-calculus. We follow the lines of the universal-algebraic tradition in the study of λ-calculi, exploring a number of varieties which can be considered as classes of algebraic models of resource λ-calculus. We axiomatize the variety of *resource combinatory algebras* (RCAs) which are to the resource λ-calculus what combinatory algebras are to the pure λ-calculus, in the sense that they contain basic combinators which allow to define an abstraction on polynomials and to obtain a combinatory completeness result. Then establishing a parallel with the work of Curry we isolate the subvariety of *resource lambda-algebras* (RLAs) and prove that the free extension of an RLA validates the so-called ξ-*rule* for the abstraction; this is done by a construction, analogue to that of Krivine [75] for lambda-algebras, which shows that the free extension of an RLA is, up to isomorphism, an object very similar to the graded algebras which appear in module theory. Along the line of the work of Pigozzi and Salibra, we axiomatize the variety of *resource λ-abstraction algebras*. We also establish the relations between these varieties, laying down foundations for a model theory of resource λ-calculus. We then show that the ideal completion of a resource combinatory (resp. lambda-, λ-abstraction) algebra determines a "classical" combinatory (resp. lambda-, λ-abstraction) algebra, and that any model of the pure λ-calculus raising from a resource lambda-algebra induces a λ-theory which equates all terms having the same Böhm tree.

9.1   Preliminaries

We identify every natural number $n \in \mathbb{N}$ with the set $n = \{0, \dots, n-1\}$. $\mathfrak{S}_n$ denotes the set of all permutations (i.e., bijections) of set $n \in \mathbb{N}$.

Sequences: The overlined letters $\bar{a}, \bar{b}, \bar{c}, \dots$ range over the set A^* of all finite sequences over A. The length of a sequence $\bar{a}$ is denoted by $|\bar{a}|$. If $\bar{a}$ is a sequence then a_i ($i \in \mathbb{N}$) denotes the i-th element of $\bar{a}$. For a sequence $\bar{a}$ of length n and a map $\sigma : k \to n$ ($k, n \in \mathbb{N}$), the composition $\sigma\bar{a}$ is the sequence $(a_{\sigma(0)}, \dots, a_{\sigma(k-1)})$. Given two sequences $\bar{a}$ and $\bar{b}$, their concatenation is denoted by $\bar{a} \cdot \bar{b}$. Sequences of

length one and elements of A are identified so that $a \cdot \bar{b}$ is the concatenation of $a \in A$ and $\bar{b} \in A^*$. If $a \in A$, then a^k denotes the sequence $(a, \ldots, a)$ of length k. If $\bar{i}$ is a sequence of natural numbers of length k then $\Sigma \bar{i}$ denotes $i_0 + \cdots + i_{k-1}$.

Sequences of sequences will be denoted by the double over-line. Thus, $\bar{\bar{a}}$ will be a sequence of sequences, whose elements are the sequences $\bar{a}_0, \ldots, \bar{a}_{|\bar{\bar{a}}|-1}$. We denote by $\prod \bar{\bar{a}}$ the sequence $\bar{a}_0 \cdot \bar{a}_1 \cdot \ldots \cdot \bar{a}_{|\bar{\bar{a}}|-1}$ that is the juxtaposition of the sequences $\bar{a}_i$.

Partitions of a sequence: Let $\bar{a} \in A^n$ and $\bar{i} \in \mathbb{N}^*$ be sequences. A $\bar{i}$-*partition* of $\bar{a}$ is a sequence $\bar{\bar{b}}$ of sequences such that $|\bar{\bar{b}}| = |\bar{i}| = k + 1$, $|\bar{b}_0| = i_0, \ldots, |\bar{b}_{k-1}| = i_{k-1}$ and there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma \bar{a} = \prod \bar{\bar{b}}$. We write $\mathcal{Q}_{\bar{a}, \bar{i}}$ to denote the set of all $\bar{i}$-partitions of $\bar{a}$ and we agree that $\mathcal{Q}_{\bar{a}, \bar{i}} \neq \emptyset$ if, and only if, $\Sigma \bar{i} = |\bar{a}|$. Moreover by $\mathcal{Q}_{\bar{a}, k}$ we indicate the set $\bigcup_{\bar{i} \in \mathbb{N}^*, |\bar{i}| = k} \mathcal{Q}_{\bar{a}, \bar{i}}$. Let $\bar{x}, \bar{y}$ be sequences of the same length and let $\bar{\bar{a}} \in \mathcal{Q}_{\bar{x}, \bar{n}}$. We say that $\bar{\bar{b}} \in \mathcal{Q}_{\bar{y}, \bar{n}}$ is the *partition of $\bar{y}$ induced by* $\bar{\bar{a}}$ iff $\prod \bar{\bar{a}} = \sigma \bar{x}$ and $\prod \bar{\bar{b}} = \sigma \bar{y}$.

Kronecker's delta: In order to give concise axiomatic presentations, we will use the Kronecker function $\delta_{n,m} : A \to A$ with values in a pointed set A (with distinguished element 0) given by $\delta_{n,m}(a) = a$ if $n = m$ and $\delta_{n,m}(a) = 0$ otherwise. In particular for $\bar{a} \in A^*$ we will adopt the convention that the value of the expression $\delta_{|\bar{a}|,m}(a_0)$ is 0 if $|\bar{a}| = 0$.

Direct sums of join-semilattices: Let $(\mathbf{A}_i)_{i \in I}$ be a family of join-semilattices. We say that $\mathbf{B}$ is the *direct sum* of the family $(\mathbf{A}_i)_{i \in I}$, notation $\mathbf{B} = \oplus_{i \in I} \mathbf{A}_i$, if $\mathbf{B} \leq \prod_{i \in I} \mathbf{A}_i$ is the subalgebra of the sequences $(a_i \in A_i : i \in I)$ such that $\{i : a_i \neq 0\}$ is finite.

9.2   Bag-applicative algebras

Let $\mathbf{R}$ be a semiring with unit. We introduce an *algebraic signature* Γ constituted by a binary operator "$+$", a nullary operator "0", a family of unary operators r ($r \in \mathbf{R}$), and a family of operators $\cdot_k$ ($k \in \mathbb{N}$) of arity $k + 1$, called collectively *applications*.

The prefix notation for application is indeed cumbersome for common use so that each operation $\cdot_n(a, b_0, \ldots, b_{n-1})$ will be replaced by the lighter $a[b_0, \ldots, b_{n-1}]$, so that, for example, $\cdot_0(a) = a[\;]$ and $\cdot_2(a, b, c) = a[b, c]$. Another reason for this choice is that, when we write $a[b_0, \ldots, b_{n-1}]$, we think to the element a applied to the "bag" $[b_0, \ldots, b_{n-1}]$. We will also adopt the usual convention that application associates to the left. For a sequence $\bar{b}$ of length n, we adopt the further notational simplification to write $a\bar{b}$ instead of $a[b_0, \ldots, b_{n-1}]$. By a^k we indicate the sequence $(a, \ldots, a)$ of length k, thus $ba^k = b[a, \ldots, a]$ (a repeated k times), with the convention that $ba^0 = b[\;]$. Note that the above conventions lead to write just ab for $a[b]$: clearly in case b is itself an application cd, we are obliged to write $a[cd]$ in order to avoid any ambiguity.

Definition 9.2.1. *A Γ-algebra is called a* bag-applicative algebra *if it satisfies the*

following axioms, which are universally quantified.

Commutative Monoid Axioms :
$(x + y) + z = x + (y + z);$ $\quad$ $x + y = y + x;$ $\quad$ $0 + x = x$
Module Axioms $(r, s \in R)$:
$r(x + y) = rx + ry;$ $\quad$ $(r + s)x = rx + sx;$ $\quad$ $(rs)x = r(sx);$ $\quad$ $1x = x;$ $\quad$ $0x = 0$
Multiset Axiom : $x[y_0, \ldots, y_{k-1}] = x[y_{\sigma(0)}, \ldots, y_{\sigma(k-1)}]$ $\quad$ $(\sigma \in \mathfrak{S}_k)$
Multilinearity Axioms :
$x[0, y_0, \ldots, y_{k-1}] = 0;$ $\quad$ $0[y_0, \ldots, y_{k-1}] = 0$
$(ax + by)[y_0, \ldots, y_{k-1}] = a(x[y_0, \ldots, y_{k-1}]) + b(y[y_0, \ldots, y_{k-1}])$
$x[\ldots, ay + bz, \ldots] = a(x[\ldots, y, \ldots]) + b(x[\ldots, z, \ldots])$

If a signature Δ extends Γ, we say that a Δ-algebra **A** is a bag-applicative Δ-algebra if it is so the Γ-reduct of **A**.

9.3 The resource lambda calculus from the algebraic point of view

The variable-binding properties of λ-abstraction prevent names in $r\lambda$-calculus from operating as real algebraic variables. The same problem occurs in classic λ-calculus and was solved by Pigozzi and Salibra [93] by introducing the variety of λ-abstraction algebras. We adopt here their solution and transform the names (i.e., elements of V) into constants.

Definition 9.3.1. *The signature Γ_λ is an extension of the signature Γ of bag-applicative algebra by a family of nullary operators $x \in V$, one for each element of V, and a family of unary operators λx $(x \in V)$, called collectively λ-abstractions.*

The $r\lambda$-terms are just the Γ_λ-terms without occurrences of algebraic variables. The absolutely free Γ_λ-algebra is the algebra $\mathbf{\Lambda^r} = (\Lambda^r, +, 0, \cdot_k, \lambda x, x)_{x \in V, k \in \mathbb{N}}$, where Λ^r is the set of $r\lambda$-terms and the operations are just the syntactical operations of construction of the $r\lambda$-terms.

Definition 9.3.2. *A $r\lambda$-theory is any congruence on $\mathbf{\Lambda^r}$ (with respect to all the involved operations) including all the identities of Figure 1.*

The least $r\lambda$-theory, denoted by $\lambda\beta r$, is consistent by Theorem 7.2.2 (§7). If T is a $r\lambda$-theory, we denote by $\mathbf{\Lambda^r_T} \equiv \mathbf{\Lambda^r}/T$ the quotient of the absolutely free Γ_λ-algebra $\mathbf{\Lambda^r}$ modulo the $r\lambda$-theory T. $\mathbf{\Lambda^r_T}$ is called the *term algebra* of T.

We now abstract the notion of term algebra by introducing the variety of resource λ-abstraction algebras as a pure algebraic theory of $r\lambda$-calculus. The term algebras of $r\lambda$-theories are the first example of resource λ-abstraction algebra. Another example will be given in Subsection 9.5.1.

Definition 9.3.3. *A* resource λ-abstraction algebra *(*RLAA, *for short) is a bag-applicative Γ_λ-algebra satisfying the following identities (for all $a \in A, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in A^*$, and $x \neq y \in V$):*

(rβ_1) $(\lambda x.x)\bar{a} = \delta_{|\bar{a}|,1}(a_0)$

(rβ_2) $(\lambda x.y)\bar{a} = \delta_{|\bar{a}|,0}(y)$

(rβ_3) $(\lambda x.\lambda x.a)\bar{b} = \delta_{|\bar{b}|,0}(\lambda x.a)$

(rβ_4) $(\lambda y.b_i)[\,] = b_i,$ *for all* $i < |\bar{b}| \Rightarrow (\lambda x.\lambda y.a)\bar{b} = \lambda y.(\lambda x.a)\bar{b}$

(rβ_5) $(\lambda x.a\bar{b})\bar{c} = \Sigma_{\bar{\bar{d}} \in \mathcal{Q}_{\bar{c},k+1}}(\lambda x.a)\bar{d}_0[(\lambda x.b_0)\bar{d}_1, \ldots, (\lambda x.b_{k-1})\bar{d}_k]$ $(|\bar{b}| = k)$

(rα) $(\lambda x.a)x^k = a,\ (\lambda y.a)[\,] = a \Rightarrow \lambda x.a = \lambda y.(\lambda x.a)y^k$

(rγ) $(\lambda x.a)x^n + a = a$

(rλ) $\lambda x.0 = 0;\quad \lambda x.(a+b) = \lambda x.a + \lambda x.b$

   Some of the axioms above are not pure identities, though they can be turned into such as it is done in the case of λ-abstraction algebras [93].

   An element a is *finite-dimensional* if there exists a finite subset $X \subseteq V$ such that $(\lambda x.a)[\,] = a$ for all $x \in V - X$ and, for all $x \in X$, there is exactly one $n \neq 0$ such that $(\lambda x.a)x^n = a$, and in such a case $(\lambda x.a)x^k = 0$ for all $k \neq n$; this last statement follows from $(r\beta_5),(r\beta_3)$ and $(r\beta_1)$. Finite-dimensional elements are a generalization of the $r\lambda$-terms. In particular $a \in A$ is *zero-dimensional* if $(\lambda x.a)[\,] = a$ for all $x \in V$. We say that a *name $x \in V$ does not occur free in $a \in A$* if $(\lambda x.a)[\,] = a$. The set of zero-dimensional elements, which generalizes closed $r\lambda$-terms, will be denoted by Zd $\mathbf{A}$. In general a RLAA may have elements where all the names occur free; these elements are a generalization of infinite λ-terms. A RLAA $\mathbf{A}$ is called *locally finite* if it is generated by its finite-dimensional elements (through the join/sum operator). Every RLAA $\mathbf{A}$ contains a canonical locally finite RLAA, which is the subalgebra of $\mathbf{A}$ generated by all its finite-dimensional elements. This algebra will be denoted by Lf$\mathbf{A}$.

Proposition 9.3.1.   *(i) For any $r\lambda$-theory T the term algebra $\mathbf{\Lambda_T^r}$ is a locally finite* RLAA.

   (ii) The minimal subalgebra of a RLAA $\mathbf{A}$ *is isomorphic to $\mathbf{\Lambda_T^r}$ for some T.*

9.4   Resource combinatory algebras

In this section we introduce a class of algebras which are to the $r\lambda$-calculus what combinatory algebras are to the pure λ-calculus. The signature Γ_c of is an extension

of the signature Γ of bag-applicative algebras by a nullary operator K and a family of nullary operators $S_{\bar{n}}$ ($\bar{n} \in \mathbb{N}^*$). Recall the definition of the set $\mathcal{Q}_{\bar{z},\bar{n}}$ from the preliminaries.

Definition 9.4.1. *A* resource combinatory algebra *(*RCA*, for short) is a bag-applicative* Γ_c*-algebra satisfying the following identities:*

(K) $K\bar{x}\bar{y} = \delta_{|\bar{y}|,0}(\delta_{|\bar{x}|,1}(x_0))$

($S_{\bar{n}}$) $S_{\bar{n}}\bar{x}\bar{y}\bar{z} = \delta_{|\bar{x}|,1}(\delta_{k,|\bar{y}|}(\delta_{|\bar{y}|,|\bar{n}|-1}(\delta_{|\bar{z}|,\Sigma\bar{n}}(\sum_{\bar{\bar{c}} \in \mathcal{Q}_{\bar{z},\bar{n}}} x_0\bar{c}_0[y_0\bar{c}_1,\dots,y_{k-1}\bar{c}_k]))))$

The variety of resource combinatory algebras is denoted by RCA. We secretly think of K and $S_{\bar{n}}$ as the following $r\lambda$-terms:

$$K_\lambda \equiv \lambda xy.x; \quad S_{\bar{n},\lambda} \equiv \lambda xyz.xz^{n_0}[yz^{n_1},\dots,yz^{n_k}] \quad (|\bar{n}|-1=k) \tag{9.1}$$

We define *(resource) monomials* with names in V and constant in A by the following grammar: $t ::= x \mid c_a \mid K \mid S_{\bar{n}} \mid t_0[t_1,\dots t_n]$ ($\bar{n} \in \mathbb{N}, a \in A$). A *(resource) polynomial* is a finite sum of monomials: $t_1 + \dots + t_n$. We denote by $P(A)$ the set of all polynomials with names in V and constant in A. For a monomial t we define the *degree* $deg_x(t)$ of $x \in V$ in t as the number of occurrences of the name x in t.

We define an abstraction operation on polynomials, with which the abstraction of $r\lambda$-calculus can be simulated. First of all we need to define the combinator $I \equiv S_{(1)}K[\,]$. It is immediate to see that $I\bar{x} = \delta_{|\bar{x}|,1}(x_0)$.

Definition 9.4.2. *Let* $t, t_1, \dots, t_n$ *be monomials. We define a new monomial* $\lambda^*x.t$ *as follows:*

(i) $\lambda^*x.t \equiv Kt$ *if* $deg_x(t) = 0$

(ii) $\lambda^*x.x \equiv I$

(iii) $\lambda^*x.t_0[t_1,\dots,t_k] \equiv S_{\bar{n}}[\lambda^*x.t_0][\lambda^*x.t_1,\dots,\lambda^*x.t_k]$ ($\bar{n} = (deg_x(t_0),\dots,deg_x(t_k))$) *if* $\exists i \; deg_x(t_i) \neq 0$.

We extend the definition of abstraction to polynomials: $\lambda^*x.\Sigma_{i=1}^n t_i = \Sigma_{i=1}^n \lambda^*x.t_i$.

Let t be a monomial with $deg_x(t) = n$, $\bar{p}$ be a sequence of n polynomials and $\sigma \in \mathfrak{S}_n$ be a permutation. Then the expression $t\{\bar{x} := \sigma\bar{p}\}$ denotes the *simultaneous* substitution of the i-th occurrence x^i of x in t by the polynomial $p_{\sigma(i)}$ ($i = 1,\dots,n$). As usual we write $\mathbf{A} \vDash t = u$ to express the fact the equation $t = u$ holds under any valuation in the algebra $\mathbf{A}$.

Lemma 9.4.1. *Let* $\mathbf{A}$ *be a* RCA*. For any monomial* t*, any sequence* $\bar{p}$ *of polynomials, and any name* x *we have:* $\mathbf{A} \vDash (\lambda^*x.t)\bar{p} = \delta_{deg_x(t),|\bar{p}|}(\Sigma_{\sigma \in \mathfrak{S}_{deg_x(t)}} t\{\bar{x} := \sigma\bar{p}\})$.

Proof. The proof is an easy induction. The only interesting case is that in which $t \equiv t_0[t_1, \dots, t_k]$ and there exists i such that $deg_x(t_i) \neq 0$.

$$
\begin{aligned}
& (\lambda^* x.t_0[t_1, \dots, t_k])\bar{p} \\
= \ & S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \dots, \lambda^* x.t_k]\bar{p}, & \text{by Def. 9.4.2(iii)} \\
= \ & \Sigma_{\bar{\bar{p}} \in \mathcal{Q}_{\bar{p}, \bar{n}}}(\lambda^* x.t_0)\bar{p}_0 \big[(\lambda^* x.t_1)\bar{p}_1, \dots, (\lambda^* x.t_k)\bar{p}_k \big] & \text{by Def. 9.4.1} \\
= \ & \Sigma_{\bar{\bar{p}} \in \mathcal{Q}_{\bar{p}, \bar{n}}}(\Sigma_{\sigma_0 \in \mathfrak{S}_{n_0}} t_0\{\bar{x} := \sigma_0 \bar{p}_0\})[\Sigma_{\sigma_1 \in \mathfrak{S}_{n_1}} t_1\{\bar{x} := \sigma_1 \bar{p}_1\} \dots] & \text{by ind. hyp.} \\
= \ & \Sigma_{\bar{\bar{p}} \in \mathcal{Q}_{\bar{p}, \bar{n}}} \Sigma_{(\sigma_0, \dots, \sigma_k) \in \prod_{j=0}^{k} \mathfrak{S}_{n_j}} t_0\{\bar{x} := \sigma_0 \bar{p}_0\}[t_1\{\bar{x} := \sigma_1 \bar{p}_1\} \dots] \\
= \ & \Sigma_{\bar{\bar{p}} \in \mathcal{Q}_{\bar{p}, \bar{n}}} t_0\{\bar{x} := \bar{p}_0\}[t_1\{\bar{x} := \bar{p}_1\} \dots] \quad \text{because } \sigma_0 \bar{p}_0 \cdot \sigma_1 \bar{p}_1 \cdot \dots \in \mathcal{Q}_{\bar{p}, \bar{n}} \\
= \ & \Sigma_{\sigma \in \mathfrak{S}_m}(t_0[t_1, \dots, t_k])\{\bar{x} := \sigma \bar{p}\} \ (m = \Sigma \bar{n})
\end{aligned}
$$

Of course if $|\bar{p}| \neq \Sigma \bar{n}$, then the above calculation yields 0 as a result. $\square$

Let $\mathbf{A}$ be a RLAA. The *combinatory reduct* of $\mathbf{A}$ is defined as the algebra $\mathrm{Cr}\,\mathbf{A} = (A, \cdot_k, K_\lambda^{\mathbf{A}}, S_{\bar{n}, \lambda}^{\mathbf{A}})$, where the $r\lambda$-terms K_λ and $S_{\bar{n}, \lambda}$ are defined in (9.1) above. The subalgebra of $\mathrm{Cr}\,\mathbf{A}$ constituted by the zero-dimensional elements of $\mathbf{A}$ will be denoted by $\mathbf{Zd}\,\mathbf{A}$.

Proposition 9.4.2. *Let $\mathbf{A}$ be a locally finite RLAA. Then, $\mathrm{Cr}\,\mathbf{A}$ is a RCA.*

The proof of the above proposition is trivial because of the hypothesis of locally finiteness. If we drop this hypothesis, then we cannot always apply α-conversion because elements may exist where all variables occur free.

The $r\lambda$-term t_λ associated with a polynomial t can be easily defined by induction: $K, S_{\bar{n}}$ are respectively translated into K_λ and $S_{\bar{n}, \lambda}$ (see (9.1) above); $(c_a)_\lambda = c_a$; $(t[s_1, \dots, s_n])_\lambda = t_\lambda[s_{1, \lambda}, \dots, s_{n, \lambda}]$; $(\Sigma t_i)_\lambda = \Sigma t_{i, \lambda}$.

The following lemma can be shown by induction over the complexity of the polynomial p. If $\mathbf{A}$ is a RLAA, then $p^{\mathrm{Cr}\,\mathbf{A}}$ denotes the interpretation of p into $\mathrm{Cr}\,\mathbf{A}$.

Lemma 9.4.3. *Let $\mathbf{A}$ be a RLAA and p be a polynomial. Then, $p^{\mathrm{Cr}\,\mathbf{A}} = p_\lambda^{\mathbf{A}}$ and $(\lambda^* x.p)^{\mathrm{Cr}\,\mathbf{A}} = \lambda x^{\mathbf{A}}.p_\lambda^{\mathbf{A}}$.*

9.5 Resource lambda-algebras

In this section we axiomatize the variety of resource λ-algebras ($r\lambda$-algebras for short), and prove that the free extension of an $r\lambda$-algebra in the variety of $r\lambda$-algebras can be turned into a RLAA, so that it validates all the rules of $r\lambda$-calculus. For the subsequent developments, it turns out very important to isolate a particular family of combinators: for $n \in \mathbb{N}$, the *n-homogenizer* is the combinator $H_n \equiv S_{(0,n)}[KI]$. Using the equation schemata of RCAs we obtain that $H_n \bar{x} \bar{y} = \delta_{|\bar{x}|, 1}(\delta_{|\bar{y}|, n}(x_0 \bar{y}))$. The elements of the form $H_n a$ are the semantical counterpart of monomials of the form $\lambda^* x.t$, with $deg_x(t) = n$. Via H_n it is in fact possible to give a semantical notion of degree: $a \in A$ is called *homogeneous of degree n* iff $H_n a = a$.

We now define $r\lambda$-algebras. We advice the reader that some identities defining $r\lambda$-algebras are difficult to read, nonetheless they still resemble those for λ-algebras.

Definition 9.5.1. *A* RCA **A** *is a $r\lambda$-algebra if it satisfies the λ^*-closure of the following identities:*

(R0) $H_n[H_m x] = \delta_{n,m}(H_m x)$

(R1) $K = H_1 K; \qquad Kx = H_0[Kx]$

(R2) $S_{\bar{n}} = H_1 S_{\bar{n}}; \qquad S_{\bar{n}} x = H_{|\bar{n}|-1}[S_{\bar{n}} x]; \qquad S_{\bar{n}} x \bar{y} = H_{\Sigma \bar{n}}[S_{\bar{n}} x \bar{y}]$

(R3) $S_{\bar{m}}[S_{\bar{n}}[KK]\bar{x}]\bar{y} = \begin{cases} H_{n_1} x_0 & \text{if } |\bar{x}| = 1, |\bar{y}| = 0, \bar{n} = (0, n_1), \bar{m} = (n_1) \\ 0 & \text{otherwise} \end{cases}$

(R4) $S_{\bar{n}}[S_{\bar{m}}[S_{\bar{p}}[KS_{\bar{l}}]\bar{x}]\bar{y}]\bar{z} =$
$$\begin{cases} \Sigma_{\bar{\bar{s}} \in \mathcal{Q}_{\bar{z},\bar{l}}} S_{(\Sigma(m_0 \cdot \bar{o}_0), \ldots, \Sigma(m_k \cdot \bar{o}_k))}[S_{m_0 \cdot \bar{o}_0} x_0 \bar{s}_0][S_{m_1 \cdot \bar{o}_1} y_0 \bar{s}_1, \ldots, S_{m_k \cdot \bar{o}_k} y_{k-1} \bar{s}_k] \\ \text{if } |\bar{x}| = 1, \bar{p} = (0, p_1), |\bar{y}| + 1 = |\bar{m}| = k, \bar{n} = (\Sigma \bar{m}) \cdot \bar{n}', |\bar{z}| = |\bar{n}'| = \Sigma \bar{l}, \\ \bar{m} = p_1 \cdot \bar{l}, \text{ and, for each } \bar{\bar{s}} \in \mathcal{Q}_{\bar{z},\bar{l}}, \bar{o} \in \mathcal{Q}_{\bar{n}',\bar{l}} \text{ is the partition of } \bar{n}' \text{ induced by } \bar{\bar{s}}; \\ 0 \qquad \text{otherwise.} \end{cases}$$

(R5) $K[x\bar{y}] = S_{0^{k+1}}[Kx][Ky_0, \ldots, Ky_{k-1}] \ (|\bar{y}| = k)$

(R6) $H_k x = S_{0 \cdot 1^k}[Kx]I^k$

The variety of $r\lambda$-algebras will be denoted by RLA. The next lemma shows the aforementioned connection between homogenizers and the induced λ-abstraction on polynomials. As a side comment, we remark the similarity between the axioms given in Definition 9.5.1 and those of Definition 2.3.5 §2. Roughly speaking the combinator 1 of lambda algebras could be thought of as the series $\sum_{n \geq 0} H_n$: this is the leading intuition of the forthcoming Section 9.6, and in particular of Theorem 9.6.1.

Lemma 9.5.1. *Let **A** be a* RLA *and t be a monomial. Then $\mathbf{A} \vDash H_n[\lambda^* x.t] = \delta_{n, \deg_x(t)}(\lambda^* x.t)$.*

Proof. If $\deg_x(t) = 0$, then $\lambda^* x.t = Kt = H_0[Kt]$ (by (R1)) $= H_0[\lambda^* x.t]$. If $t \equiv x$, then $\lambda^* x.x = I = S_1 K[\ ] = H_1[S_1 K[\ ]]$ by (R2) $= H_1[\lambda^* x.x]$. Let $t \equiv t_0[t_1, \ldots, t_k]$ with $\deg_x(t_i) \neq 0$ for some i, and $\bar{n} = (\deg_x(t_0), \ldots, \deg_x(t_k))$. Then we have:

$$\begin{aligned} \lambda^* x.t_0[t_1, \ldots, t_k] &= S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \ldots, \lambda^* x.t_k] \\ &= H_{\Sigma \bar{n}}[S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \ldots, \lambda^* x.t_k]] \quad \text{by (R2)} \\ &= H_{\deg_x(t)}[\lambda^* x.t_0[t_1, \ldots, t_k]]. \end{aligned}$$

$\square$

The following theorem is the main result of the section; its proof, divided into five lemmas, occupies the rest of this section and involves the explicit construction of the free extension of a RLA by one name as a graded algebra (Lemmas 9.5.6-9.5.9) and two key observations (Lemmas 9.5.10, 9.5.11).

Theorem 9.5.2. *The free extension $\mathbf{A}[V]$ of a $r\lambda$-algebra $\mathbf{A}$ by the set V of names in the variety* RCA *satisfies the following ξ-rule, for all polynomials $p, q \in P(A)$:*

$$(\xi) \qquad \mathbf{A}[V] \vDash p = q \;\Rightarrow\; \mathbf{A}[V] \vDash \lambda^* x.p = \lambda^* x.q.$$

We apply the above theorem to define λ-abstraction operators on $A[V]$. For any $e \in A[V]$, we define $\lambda x.e = \lambda^* x.p$, for some polynomial $p \in e$. Rule ξ validates the above definition of λx. Define the algebra $\mathbf{A}[V]_\lambda = (A[V], +, 0, \cdot_k, \lambda x, x)_{x \in V}$, where $(A[V], +, 0, \cdot_k,)$ is the Γ-reduct of the free extension $\mathbf{A}[V]$, λx is defined as above and the name $x \in V$ is viewed as a nullary operator; $\mathbf{A}[V]_\lambda$ is called the RLAA *freely generated by the $r\lambda$-algebra* $\mathbf{A}$.

Corollary 9.5.3. $\mathbf{A}[V]_\lambda$ *is a locally finite* RLAA *such that* $K_\lambda^{\mathbf{A}[V]_\lambda} = K^{\mathbf{A}}$ *and* $S_{\bar{n},\lambda}^{\mathbf{A}[V]_\lambda} = S_{\bar{n}}^{\mathbf{A}}$.

Proof. It is straightforward to check the axioms of RLAA. We now prove the last part of the corollary.

$$
\begin{aligned}
\lambda^* xy.x \;\; &= S_{(0,1)}[KK]I, \text{ by definition} \\
&= H_1 K, \text{ by (R6)} \\
&= K, \text{ by (R1)}.
\end{aligned}
$$

Let $\bar{p} = (p_0, \ldots, p_n)$ and $t \equiv \lambda^* xyz.xz^{p_0}[yz^{p_1}, \ldots, yz^{p_n}]$. Then we have:

$$
\begin{aligned}
t \;\; &= \lambda^* xy.S_{\bar{p}}[S_{0 \cdot 1^{p_0}}[Kx]I^{p_0}][S_{0 \cdot 1^{p_1}}[Ky]I^{p_1}, \ldots, S_{0 \cdot 1^{p_n}}[Ky]I^{p_n}] \quad \text{by definition} \\
&= \lambda^* xy.S_{0 \cdot 1^{\Sigma\bar{p}}}[S_{0^{|\bar{p}|}}[S_{(0,0)}[KS_{\bar{p}}][Kx]](Ky)^{|\bar{p}|-1}]I^{\Sigma\bar{p}} \quad \text{by (R4)} \\
&= \lambda^* xy.S_{0 \cdot 1^{\Sigma\bar{p}}}[S_{0^{|\bar{p}|}}[K[S_{\bar{p}}x]](Ky)^{|\bar{p}|-1}]I^{\Sigma\bar{p}} \text{ by (R5)} \\
&= \lambda^* xy.S_{0 \cdot 1^{\Sigma\bar{p}}}[K[S_{\bar{p}}xy^{|\bar{p}|-1}]]I^{\Sigma\bar{p}} \text{ by (R5)} \\
&= \lambda^* xy.H_{\Sigma\bar{p}}[S_{\bar{p}}xy^{|\bar{p}|-1}] \text{ by (R6)} \\
&= \lambda^* xy.S_{\bar{p}}xy^{|\bar{p}|-1} \text{ by (R2)} \\
&= \lambda^* x.S_{(0,1,\ldots,1)}[K[S_{\bar{p}}x]]I^{|\bar{p}|-1} \text{ by definition} \\
&= \lambda^* x.H_{|\bar{p}|-1}[S_{\bar{p}}x] \text{ by (R6)} \\
&= \lambda^* x.S_{\bar{p}}x \text{ by (R2)} \\
&= S_{(0,1)}[KS_{\bar{p}}]I \text{ by definition} \\
&= H_1 S_{\bar{p}} \text{ by (R6)} \\
&= S_{\bar{p}} \text{ by (R2).}
\end{aligned}
$$

$\square$

Theorem 9.5.4. *Let $\mathbf{A}$ be a locally finite* RLAA *of dimension V. Then its combinatory reduct $Cr\mathbf{A}$ is a* RLA.

Proof. We start recalling that $K^{Cr\mathbf{A}} = \lambda xy.x$ and $S_{\bar{p}}^{Cr\mathbf{A}} = \lambda xyz.xz^{p_0}[yz^{p_1}, \ldots, yz^{p_n}]$. Now using axioms $(r\beta_1)$-$(r\beta_5)$ we can prove that $S_{(1)}^{Cr\mathbf{A}}K^{Cr\mathbf{A}}[\,] = \lambda z.z$; define $I^{Cr\mathbf{A}} = \lambda z.z$. Again by axioms $(r\beta_1)$-$(r\beta_5)$ we can prove that $S_{(0,n)}^{Cr\mathbf{A}}[K^{Cr\mathbf{A}}I^{Cr\mathbf{A}}] = \lambda yz.yz^n$; define $H_n^{Cr\mathbf{A}} = \lambda yz.yz^n$.

Before proving the RLA axioms, we remark tha it is possible to achieve "α-renaming" for each one of the combinators $H_n^{Cr\mathbf{A}}$, $K^{Cr\mathbf{A}}$, and $S_{\vec{p}}^{Cr\mathbf{A}}$. As an example, we do that for $K^{Cr\mathbf{A}}$. Let $x \neq y \neq z \neq w \in V$.

$$
\begin{aligned}
\lambda xy.x &= \lambda z.(\lambda xy.x)z, \text{ by } (r\alpha), \text{ since } (\lambda zy.x)[\,] = \lambda y.x \text{ and } (\lambda xy.x)x = \lambda y.x \\
&= \lambda z.\lambda y.z \\
&= \lambda z.\lambda w.(\lambda y.z)[\,], \text{ by } (r\beta_4) \text{ and } (r\beta_2), \\
&= \lambda zw.z
\end{aligned}
$$

The following axioms are then proved following this pattern: whenever we have arbitrary elements $a, b, c, ovb, \ldots$ first "α-rename" the combinators in such a way that all the (bound) names occurring in them are out of the dimension set of $a, b, c, ovb, \ldots$ and then use axioms $(r\beta_1)$-$(r\beta_5)$ plus commutativity of bags, idempotence of sums and linearity axioms.

Now we calculate the expressions in the RLA axioms.

$$
\begin{aligned}
(\lambda yz.yz^n)[(\lambda yz.yz^m)a] &= (\lambda yz.yz^n)[\lambda z.az^m], \text{ by } (r\beta_1)\text{-}(r\beta_5), \\
&= \lambda z.(\lambda z.az^m)z^n, \text{ by } (r\beta_1)\text{-}(r\beta_5), \\
&= \lambda z.az^n, \text{ if } m = n \text{ and } 0 \text{ otherwise}, \\
&\qquad \text{ by } (r\beta_1)\text{-}(r\beta_5) \text{ and the idempotence of sum.}
\end{aligned}
$$

This proves (R0). Here to be able to apply $(r\beta_4)$ we use the hypothesis of local finiteness of the RLA $\mathbf{A}$.

$$
\begin{aligned}
(\lambda yz.yz)[\lambda xy.x] &= \lambda z.(\lambda xy.x)z \\
&= \lambda zy.z \\
&= K^{Cr\mathbf{A}}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda yz.y[\,])[(\lambda xy.x)a] &= (\lambda z.(\lambda y.a)[\,]) \\
&= \lambda z.a \\
&= (\lambda yz.y)a
\end{aligned}
$$

This proves (R1). Here to be able to apply $(r\beta_4)$ we use the hypothesis of local finiteness of the RLA $\mathbf{A}$.

$$
\begin{aligned}
(\lambda xyz.xz^{p_0}[yz^{p_1}, \ldots, yz^{p_n}])a &= \lambda yz.az^{p_0}[yz^{p_1}, \ldots, yz^{p_n}] \\
&= \lambda y.(\lambda yz.az^{p_0}[yz^{p_1}, \ldots, yz^{p_n}])y^{n-1} \\
&= (\lambda xy.xy^{n-1})[\lambda yz.az^{p_0}[yz^{p_1}, \ldots, yz^{p_n}] \\
&= (\lambda xy.xy^{n-1})[\lambda xyz.xz^{p_0}[yz^{p_1}, \ldots, yz^{p_n}])a]
\end{aligned}
$$

$$
\begin{aligned}
S_{\bar{p}}a\bar{b} &= \sum_{\sigma \in \mathfrak{S}_n} \lambda y.ay^{p_0}[b_{\sigma(0)}y^{p_1}, \ldots, b_{\sigma(n-1)}y^{p_n}] \\
&= \sum_{\sigma \in \mathfrak{S}_n} \lambda y.(\lambda z.az^{p_0}[b_{\sigma(0)}z^{p_1}, \ldots, b_{\sigma(n-1)}z^{p_n}])y^{\Sigma\bar{p}} \\
&= (\lambda xy.xy^{\Sigma\bar{p}})[\lambda z.\sum_{\sigma \in \mathfrak{S}_n} az^{p_0}[b_{\sigma(0)}z^{p_1}, \ldots, b_{\sigma(n-1)}z^{p_n}]] \\
&= (\lambda xy.xy^{\Sigma\bar{p}})[S_{\bar{p}}a\bar{b}]
\end{aligned}
$$

This proves (R2). Here to be able to apply $(r\beta_4)$ we use the hypothesis of local finiteness of the RLA $\mathbf{A}$.

$$
\begin{aligned}
(\lambda xy.x)[a\bar{b}] &= \lambda y.a\bar{b} \\
&= \sum_{\sigma \in \mathfrak{S}_k} \lambda z.a[b_{\sigma(0)}, \ldots, b_{\sigma(k-1)}], \text{ by commutativity,} \\
&= \sum_{\sigma \in \mathfrak{S}_k} \lambda z.(\lambda y.a)[\,][(\lambda y.b_{\sigma(0)})[\,], \ldots, (\lambda y.b_{\sigma(k-1)})[\,]] \\
&= (\lambda xyz.x[\,](y[\,])^k)[\lambda y.a][\lambda y.b_0, \ldots, \lambda y.b_{k-1}]
\end{aligned}
$$

This proves (R5). Here to be able to apply $(r\beta_4)$ we use the hypothesis of local finiteness of the RLA $\mathbf{A}$.

$$
\begin{aligned}
(\lambda xy.xy^k)a &= \lambda y.ay^k \\
&= \lambda z.az^k \\
&= \lambda z.a(Iz)^k \\
&= (\lambda yz.a(yz)^k)I^k \\
&= (\lambda yz.(\lambda y.a)[\,](yz)^k)I^k \\
&= (\lambda xyz.x[\,](yz)^k)[Ka]I^k
\end{aligned}
$$

This proves (R6). Here to be able to apply $(r\beta_4)$ we use the hypothesis of local finiteness of the RLA $\mathbf{A}$.

$$
\begin{aligned}
S_{(n)}[S_{0,n}[KK]a][\,] &= S_{(n)}[\lambda z.K[az^n]][\,] \\
&= (\lambda xyz.xz^n[\,])[\lambda z.K[az^n]][\,] \\
&= (\lambda yz.(\lambda z.K[az^n])z^n[\,])[\,] \\
&= (\lambda yz.K[az^n][\,])[\,] \\
&= (\lambda yz.az^n)[\,] \\
&= \lambda z.az^n \\
&= H_n a
\end{aligned}
$$

This proves the "positive" branch of (R3).

$$
\begin{aligned}
&S_{(p+\Sigma_i m_i, n_1, \ldots, n_{\Sigma\bar{l}})}[S_{(p,m_1,\ldots,m_k)}[S_{(0,p)}[KS_{\bar{l}}]a]\bar{b}]\bar{c} \\
={}&S_{(p+\Sigma_i m_i, n_1, \ldots, n_{\Sigma\bar{l}})}[S_{(p,m_1,\ldots,m_k)}[\lambda z.S_{\bar{l}}[az^p]]\bar{b}]\bar{c} \\
={}&\sum_{\sigma \in \mathfrak{S}_k} S_{(p+\Sigma_i m_i, n_1, \ldots, n_{\Sigma\bar{l}})}[\lambda z.(\lambda z.S_{\bar{l}}[az^p])z^p[b_{\sigma(0)}z^{m_1}, \ldots, b_{\sigma(k-1)}z^{m_k}]]\bar{c} \\
={}&\sum_{\sigma \in \mathfrak{S}_k} S_{(p+\Sigma_i m_i, n_1, \ldots, n_{\Sigma\bar{l}})}[\lambda z.S_{\bar{l}}[az^p][b_{\sigma(0)}z^{m_1}, \ldots, b_{\sigma(k-1)}z^{m_k}]]\bar{c} \\
={}&\sum_{\sigma \in \mathfrak{S}_k} (\lambda xyz.xz^{p+\Sigma_i m_i}[yz^{n_1}, \ldots, yz^{n_{\Sigma\bar{l}}}])[\lambda z.S_{\bar{l}}[az^p][b_{\sigma(0)}z^{m_1}, \ldots, b_{\sigma(k-1)}z^{m_k}]]\bar{c} \\
={}&\sum_{\tau \in \mathfrak{S}_{\Sigma\bar{l}}} \sum_{\sigma \in \mathfrak{S}_k} \lambda z.(\lambda z.S_{\bar{l}}[az^p][b_{\sigma(0)}z^{m_1}, \ldots, b_{\sigma(k-1)}z^{m_k}])z^{p+\Sigma_i m_i}[c_{\tau(0)}z^{n_1}, \ldots, c_{\tau(\Sigma\bar{l}-1)}z^{n_{\Sigma\bar{l}}}] \\
={}&\sum_{\tau \in \mathfrak{S}_{\Sigma\bar{l}}} \sum_{\sigma \in \mathfrak{S}_k} \lambda z.S_{\bar{l}}[az^p][b_{\sigma(0)}z^{m_1}, \ldots, b_{\sigma(k-1)}z^{m_k}][c_{\tau(0)}z^{n_1}, \ldots, c_{\tau(\Sigma\bar{l}-1)}z^{n_{\Sigma\bar{l}}}] \\
={}&\sum_{\bar{\bar{s}} \in \mathcal{Q}_{\bar{c}, \bar{l}}} S_{(\Sigma(m_0 \cdot \bar{o}_0) \ldots \Sigma(m_k \cdot \bar{o}_k))}[S_{m_0 \cdot \bar{o}_0} a \bar{s}_0][S_{m_1 \cdot \bar{o}_1} b_0 \bar{s}_1, \ldots, S_{m_k \cdot \bar{o}_k} b_{k-1} \bar{s}_k]
\end{aligned}
$$

This proves the "positive" branch of (R4). $\square$

Corollary 9.5.5. *Let $\mathbf{A}$ be a* RCA*. Then, $\mathbf{A}$ is a $r\lambda$-algebra iff $\mathbf{A}$ can be embedded into the combinatory reduct of some* RLAA $\mathbf{B}$.

Proof. $(\Rightarrow)$ Consider the RLAA $\mathbf{A}[V]_\lambda$ freely generated by $\mathbf{A}$. We show that $\mathbf{A} \cong \mathbf{Zd}\,\mathbf{A}[V]_\lambda$.

$(A \subseteq \mathrm{Zd}\,\mathbf{A}[V]_\lambda)$: For all $a \in A$ and all $x \in V$ we have: $(\lambda x.a)[\,] = (\lambda^* x.c_a)[\,] = Kc_a[\,] = c_a$, so that every $a \in A$ is zero-dimensional.

(Zd $\mathbf{A}[V]_\lambda \subseteq A$): Let $e \in \text{Zd}\,\mathbf{A}[V]_\lambda$, let $p = \sum_{i \in I} t_i \in e$ be a polynomial and let $x_1, \ldots, x_n$ be all names occurring in p. Let y be one of the above names. Since e is zero-dimensional, then $(\lambda^* y.p)[\ ] = \sum_{i \in I}(\lambda^* y.t_i)[\ ] = p$. By Lemma 9.4.1 we have that either $(\lambda^* y.t_i)[\ ] = t_i$ or $(\lambda^* y.t_i)[\ ] = 0$. Let $J = \{i \in I : (\lambda^* y.t_i)[\ ] = t_i\} = \{i \in I : deg_y(t_i) = 0\}$. Then we have: $p = \sum_{j \in J}(\lambda^* y.t_j)[\ ] = \sum_{j \in J} t_j$. Since $deg_y(t_i) = 0$, then p is equivalent to a polynomial without free occurrences of name y. By iterating the reasoning with all other names, at the end of the process we get that $p = \sum_{r \in K} t_r$ is equivalent to a polynomial without free names, whose interpretation is of course in A.

($\Leftarrow$) By Lemma 9.4.3 is sufficient to verify that the RLAA $\mathbf{A}[V]$ freely generated by $\mathbf{A}$ satisfies all identities $t_\lambda = u_\lambda$, where $t = u$ is one of the axioms (R0)-(R6). This is a tedious but straightforward verification. $\qquad\square$

This corollary is very useful to prove when a RCA is a RLA (see Subsection 9.5.1). The construction of the free extension will turn out to be the construction of a graded algebra as a direct sum of specific join semilattices. We now provide the proof of Theorem 9.5.2. The proof is inspired by a construction by Krivine [75].

Lemma 9.5.6. *Let $\mathbf{A}$ be a $r\lambda$-algebra and set $B_n = \{a \in A : H_n a = a\}$. Then $(B_n, +, 0)$ is a join sub-semilattice of $(A, +, 0)$ such that $B_n \cap B_m = \{0\}$ if $n \neq m$.*

We now define an algebra $\mathbf{B} = (B, +, 0, \bullet_k, KK, KS_{\bar{n}})_{k \in \mathbb{N}, \bar{n} \in \mathbb{N}^*}$, called the $\mathbb{N}$-*graded algebra generated by* $\mathbf{A}$ in the similarity type of RCA by setting:

1. $(B, +, 0) = \oplus_{n \in \mathbb{N}}(B_n, +, 0)$ is the direct sum of the join semilattices $(B_n, +, 0)$.

2. each application "$\bullet_k$" is the extension by linearity of the following operation: $a_0 \bullet_k [a_1, \ldots, a_k] = S_{\bar{p}} a_0 [a_1, \ldots, a_k]$, with $a_i \in B_{p_i}$.

Lemma 9.5.7. *The $\mathbb{N}$-graded algebra $\mathbf{B}$ is a RCA which satisfies the following conditions:*

(i) KK and $KS_{\bar{n}}$ are elements of B_0;

(ii) $B_{d_0} \bullet_k [B_{d_1}, \ldots, B_{d_{k-1}}] \subseteq B_{\Sigma\bar{d}}$, for all $\bar{d} \in \mathbb{N}^{k+1}$.

Proof. The algebra B is closed under applications by axiom (R2). By axiom (R1) KK and $KS_{\bar{n}}$ are elements of B_0. We now show that $\mathbf{B}$ is RCA. Let $H_{n_{i+1}} a_i = a_i$.

$$
\begin{aligned}
(KK) \bullet \bar{a} \bullet \bar{b} &= (S_{\bar{n}}[KK]\bar{a}) \bullet \bar{b}, & n_0 = 0,\ H_{n_{i+1}} a_i = a_i \\
&= S_{\bar{m}}[S_{\bar{n}}[KK]\bar{a}]\bar{b}, & m_0 = \Sigma\bar{n},\ H_{m_{j+1}} b_j = b_j \\
&= H_{n_1} a_0, \quad \text{by (R3), if } |\bar{a}| = 1 \text{ and } |\bar{b}| = 0. \\
&= a_0, \quad \text{by assumption.}
\end{aligned}
$$

The axiom $(S_{\bar{l}})$ of RCA follows directly from an application of (R4). In fact

$$
\begin{aligned}
(KS_{\bar{l}}) \bullet \bar{a} \bullet \bar{b} \bullet \bar{c} \;=\; & S_{\bar{n}}[S_{\bar{m}}[S_{\bar{p}}[KS_{\bar{l}}]\bar{a}]\bar{b}]\bar{c} \\
=\; & \begin{cases}
\Sigma_{\bar{\bar{s}}\in\mathcal{Q}_{\bar{c},\bar{l}}} S_{(\Sigma(m_0,\cdot,\bar{o}_0)\ldots\Sigma(m_k\cdot\bar{o}_k))}[S_{m_0\cdot\bar{o}_0}a_0\bar{s}_0][S_{m_1\cdot\bar{o}_1}b_0\bar{s}_1,\ldots,S_{m_k\cdot\bar{o}_k}b_{k-1}\bar{s}_k] \\
\quad \text{if } |\bar{a}|=1,\ k=|\bar{b}|=|\bar{l}|-1,\ |\bar{c}|=\Sigma\bar{l} \\
\quad \text{where for each } \bar{\bar{s}}\in\mathcal{Q}_{\bar{c},\bar{l}},\ \bar{\bar{o}}\in\mathcal{Q}_{\bar{n}',\bar{l}} \text{ is the partition of } \bar{n}' \text{ induced by } \bar{\bar{s}} \\
0 \quad\quad \text{otherwise}
\end{cases} \\
=\; & \begin{cases}
\Sigma_{\bar{\bar{s}}\in\mathcal{Q}_{\bar{c},\bar{l}}} a_0 \bullet \bar{s}_0 \bullet [b_0 \bullet \bar{s}_1,\ldots,b_{k-1}\bullet \bar{s}_k] \\
\quad\quad \text{if } |\bar{a}|=1,\ k=|\bar{b}|=|\bar{l}|-1,\ |\bar{c}|=\Sigma\bar{l} \\
0 \quad\quad \text{otherwise}
\end{cases}
\end{aligned}
$$

$\square$

Lemma 9.5.8. *The map ι, defined by $\iota(a) = Ka$, is an embedding of $\mathbf{A}$ into $\mathbf{B}$.*

Proof. By (R5) $\iota(a_0[a_1,\ldots,a_n]) = K[a_0[a_1,\ldots,a_n]] = S_{0^{n+1}}[Ka_0][Ka_1,\ldots,Ka_n] = \iota(a_0) \bullet [\iota(a_1),\ldots,\iota(a_n)]$. The other properties are trivial. $\square$ $\square$

By Lemma 9.5.8 $\mathbf{B}$ is a RLA. We are now going to show the connection between the $\mathbb{N}$-graded algebra $\mathbf{B}$ and the free extension $\mathbf{A}[x]$ by one name x.

Lemma 9.5.9. *The $\mathbb{N}$-graded algebra $\mathbf{B}$ is the free extension of $\mathbf{A}$ by one name in the variety RLA. Consequently, $\mathbf{B} \cong \mathbf{A}[x]$.*

Proof. We prove that $\mathbf{B} \cong \mathbf{A}[x]$. Let $\mathbf{C}$ be a RCA, $c \in C$ and $f : \mathbf{A} \to \mathbf{C}$ be a homomorphism. Define a family of functions $f_k : B_k \to C$ $(k \in \mathbb{N})$ as follows: $f_k(a) = f(a)\cdot^{\mathbf{C}}c^k$ for all $a \in B_k$. Let $f^* : B \to C$ be the unique extension by linearity of the family of functions f_k, that is, $f^*(0) = 0$ and $f^*(\Sigma_{i=1}^m a_i) = \Sigma_{i=1}^m f_{d_i}(a_i)$ for $a_i \in B_{d_i}$.

We now prove that f^* is a homomorphism. It is immediate to check that f^* is a monoid homomorphism, using multi-linearity of application. Since f^* extends f by linearity, it suffices to prove the following:

$$
\begin{aligned}
f_{\Sigma\bar{e}}(a \bullet \bar{b}) \;&= f(S_{\bar{e}}a\bar{b})c^{\Sigma\bar{e}} = S_{\bar{e}}^{\mathbf{C}}f(a)[f(b_0),\ldots,f(b_{n-1})]c^{\Sigma\bar{e}} \text{ , since } f \text{ is hom,} \\
&= f(a)c^{e_0}[f(b_0)c^{e_1},\ldots,f(b_{n-1})c^{e_n}], \text{ by } (S_{\bar{n}}) \text{ and idempotence of sum,} \\
&= f_{e_0}(a)[f_{e_1}(b_0),\ldots,f_{e_n}(b_{n-1})].
\end{aligned}
$$

We have: $f_0(K^{\mathbf{A}}K^{\mathbf{A}}) = f(K^{\mathbf{A}}K^{\mathbf{A}})[\,] = K^{\mathbf{C}}K^{\mathbf{C}}[\,] = K^{\mathbf{C}}$. A similar argument shows that $f^*(K^{\mathbf{A}}S_{\bar{n}}^{\mathbf{A}}) = S_{\bar{n}}^{\mathbf{C}}$. This shows that f^* is a homomorphism.

We have: $f^*(\iota(a)) = f_0(Ka) = f(Ka)[\,] = K^{\mathbf{C}}f(a)[\,] = f(a)$; and $f^*(I) = f_1(I) = f(I)c = I^{\mathbf{C}}c = c$. This proves $f^* \circ \iota = f$ and $f^*(I) = c$.

Finally suppose $h : \mathbf{B} \to \mathbf{C}$ is another homomorphism satisfying $h \circ \iota = f$ and $h(I) = c$. The uniqueness of f^* is shown as follows:

$$
\begin{aligned}
h(a) \;&= h(H_k a), \text{for some } k \in \mathbb{N}, \\
&= h(S_{0\cdot 1^k}[Ka]I^k), \text{by axiom (R6)} \\
&= h((Ka) \bullet I^k) = h(Ka)(h(I))^k = h(\iota(a))c^k = f_k(a)c^k = f^*(a). \quad \square
\end{aligned}
$$

$\square$

We denote by ι^* the unique isomorphism from $\mathbf{A}[x]$ onto $\mathbf{B}$ extending the embedding $\iota : \mathbf{A} \to \mathbf{B}$ defined in Lemma 9.5.8, and such that $\iota^*(x) = I$.

Lemma 9.5.10. *For all $a, b \in A$ we have $\mathbf{A}[x] \vDash ax^n = bx^k$ iff $\mathbf{A} \vDash H_n a = H_k b$.*

Proof. $\iota^*(ax^n) = \iota(a) \bullet I^n = S_{0 \cdot 1^n}[Ka]I^n = H_n a$, by (R6). We conclude since ι^* is an isomorphism. Of course, if $n \neq k$, then $H_n a = H_k b = 0$.   □   □

Lemma 9.5.11. *For all polynomials p, q with at most the name x we have that $\mathbf{A}[x] \vDash p = q$ implies $\mathbf{A}[x] \vDash \lambda^* x.p = \lambda^* x.q$.*

Proof. First we prove the result for monomials t, u. Let $n = deg_x(t)$ and $k = deg_x(u)$. By Lemma 9.4.1 $\mathbf{A}[x] \vDash (\lambda^* x.t)x^n = t = u = (\lambda^* x.u)x^k$. Now by Lemma 9.5.10 and by Lemma 9.5.1 it follows that $\mathbf{A} \vDash \lambda^* x.t = H_n[\lambda^* x.t] = H_k[\lambda^* x.s] = \lambda^* x.s$; therefore trivially $\mathbf{A}[x] \vDash \lambda^* x.t = \lambda^* x.s$. Now for polynomials p, q such that $\mathbf{A}[x] \vDash p = q$, we have that $\iota^{*-1}(\iota^*(\lambda^* x.p)) = \Sigma_{i \in I} \lambda^* x.t_i$ and $\iota^{*-1}(\iota^*(\lambda^* x.q)) = \Sigma_{i \in I} \lambda^* x.u_i$, where I is finite and for each $i \in I$, t_i and u_i are monomials and $\mathbf{A}[x] \vDash t_i = u_i$; this allows to conclude, using the previous result.   □   □

The extension of the above lemma to polynomials with an arbitrary number of names is standard, because $\mathbf{A}[x, y] \cong \mathbf{A}[x][y]$ and $\mathbf{A}[x]$ is a RLA.

9.5.1 An example

Multisets: $\mathcal{M}_f(D)$ is the set of all finite multisets with elements in D, where $m \in \mathcal{M}_f(D)$ is a function from D into $\mathbb{N}$ such that $m(a) = 0$ for all a belonging to a cofinite subset of D. The natural number $\sharp m = \Sigma_{a \in D} m(a)$ is the cardinality of m. The union $m \uplus p$ of two finite multisets is defined by $(m \uplus p)(a) = m(a) + p(a)$ for all $a \in D$.

Let D be a set together with an injection $\to : \mathcal{M}_f(D) \times D \to D$. We adopt the convention that the operator "$\to$" associates to the right, i.e., $p \to (q \to \gamma)$ is abbreviated by $p \to q \to \gamma$. We define an algebra $\mathbf{D} = (\mathcal{P}(D), \cup, \emptyset, \cdot_n, K, S_{\bar{k}})_{n \in \mathbb{N},\ \bar{k} \in \mathbb{N}^*}$ in the similarity type of RCA, where $K = \{[\alpha] \to [\ ] \to \alpha : \alpha \in D\}$, $S_{\bar{k}} = \{[p_0 \to [\beta_1, \ldots, \beta_n] \to \beta_0] \to [p_1 \to \beta_1, \ldots, p_n \to \beta_n] \to (\uplus_{i=0}^n p_i) \to \beta_0 : \beta_i \in D,\ p_i \in \mathcal{M}_f(D),\ \sharp p_i = k_i, |\bar{k}| = n + 1\}$, and application is the extension by linearity of the following map on singleton sets (we write γ for $\{\gamma\}$, etc.): $\gamma[\beta_1, \ldots, \beta_n] = \alpha$ if $\gamma = [\beta_1, \ldots, \beta_n] \to \alpha$; it is equal to $\emptyset$, otherwise. It is an easy calculation to show that $\mathbf{D}$ is a RCA. To prove that $\mathbf{D}$ is indeed a RLA, by Corollary 9.5.5 it is sufficient to embed $\mathbf{D}$ into the combinatory reduct of a suitable RLAA $\mathbf{E}$ that we define here:

(i) $\mathcal{M}_f(D)^{(V)} = \{\rho : V \to \mathcal{M}_f(D) : \rho(x) = [\ ]$ for cofinitely many $x \in V\}$ is the set of *environments*;

(ii) ε, defined by $x \mapsto [\ ]$, is the *empty environment*, while, for an environment ρ and a finite multiset m, we define a new environment $\rho\{x := m\}$ as follows: $\rho\{x := m\}(x) = m$ and $\rho\{x := m\}(y) = \rho(y)$ if $y \neq x$.

We now construct the algebra $\mathbf{E} = (\mathcal{P}(\mathcal{M}_f(D)^{(V)} \times D), \cup, \emptyset, \cdot_k, \lambda x^{\mathbf{E}}, x^{\mathbf{E}})_{x \in V, k \in \mathbb{N}}$ by defining application and abstraction as the extension by linearity of the following family of functions defined over the singletons (we write (ρ, α) for $\{(\rho, \alpha)\}$):

- $\lambda x^{\mathbf{E}}(\rho, \alpha) = (\rho\{x := [\,]\}, \rho(x) \to \alpha)$

- $x^{\mathbf{E}} = \{(\varepsilon\{x := [\alpha]\}, \alpha) : \alpha \in D\}$

- $(\rho_0, \alpha_0)[(\rho_1, \alpha_1) \ldots, (\rho_n, \alpha_n)] = \begin{cases} (\uplus_{i=0}^n \rho_i, \alpha) & \alpha_0 = [\alpha_1, \ldots, \alpha_n] \to \alpha \\ \emptyset & \text{otherwise} \end{cases}$

Notice that $(\lambda x.(\rho, \alpha))x^n = (\rho, \alpha)$ if, and only if, $\sharp\rho(x) = n$.

Theorem 9.5.12. *The algebra $\mathbf{E}$ is a* RLAA *and the map* $h : \mathcal{P}(D) \to \mathrm{Zd}\, E$, *defined by $h(X) = \{(\varepsilon, \alpha) : \alpha \in X\}$ is an embedding from $\mathbf{D}$ into $\mathrm{Cr}\,\mathbf{E}$, making $\mathbf{D}$ an* RLA.

Proof. The proof that $\mathbf{E}$ is a RLAA is a verification of axioms $(r\beta_1)$-$(r\beta_5)$, $(r\alpha)$, $(r\gamma)$, $(r\delta)$. As an example, we show the calculation for $(r\beta_4)$. By linearity it is sufficient to verify for singleton sets that we write without braced parenthesis. The assumption $(\lambda y.(\rho_i, \beta_i))[\,] = (\rho_i, \beta_i)$ means that $\rho(y) = [\,]$. Then we have

$$
\begin{aligned}
(\lambda xy(\sigma, \alpha))[(\rho_1, \beta_1), \ldots] &= (\sigma\{x, y := [\,]\}, \sigma(x) \to \sigma(y) \to \alpha)[(\rho_1, \beta_1), \ldots] \\
&= (\uplus_{i=1}^n \rho_i \uplus \sigma\{x, y := [\,]\}, \sigma(y) \to \alpha) \text{ assuming } \sigma(x) = [\beta_1, \ldots, \beta_n] \\
&= \lambda y.(\uplus_{i=1}^n \rho_i \uplus \sigma\{x := [\,]\}, \alpha) \text{ because } \rho_i(y) = [\,] \\
&= \lambda y.(\lambda x(\sigma, \alpha))[(\rho_1, \beta_1), \ldots]
\end{aligned}
$$

If $\sigma(x) \neq [\beta_1, \ldots, \beta_n]$ then both the expression give $\emptyset$ as result.

Now by direct calculations we observe that in the algebra $\mathbf{E}$:

$$
\begin{aligned}
(\lambda xy.x)^{\mathbf{E}} &= \{(\bot, \alpha) : \alpha \in K^{\mathbf{D}}\} \\
(\lambda xyz.xz^{n_0}[yz^{n_1}, \ldots, yz^{n_k}])^{\mathbf{E}} &= \{(\bot, \alpha) : \alpha \in S_{\bar{n}}^{\mathbf{D}}\}
\end{aligned}
$$

The second part of the theorem is trivial. $\qquad\square$

9.6 From resource to pure lambda calculus

After having introduced a number of structures which algebrize the resource λ-calculus, we show how, by some standard constructions, we can recover the algebraic models of pure λ-calculus. This is done, as often happens in mathematics, by the method of *ideal completion*. Let $\mathbf{A}$ be a bag-applicative Γ-algebra. An *ideal* is a downward closed subset X of A closed under join. For a subset $X \subseteq A$, $\downarrow X = \{b : \exists a_1, \ldots, a_n \in X.\ b \leq \sum_{i=1}^n a_i\}$ (where $a \leq b \Leftrightarrow a + b = b$) is the *ideal generated* by X. We denote by $Ide(\mathbf{A})$ the collection of all ideals of $\mathbf{A}$. Let $\mathbf{A}$ be a RCA. Define an algebra $Ide(\mathbf{A}) = (Ide(A), *, \underline{K}, \underline{S})$ by setting $\underline{K} = \downarrow \{K\}$; $\underline{S} = \downarrow \{S_{\bar{n}} : \bar{n} \in \mathbb{N}^*\}$; $X * Y = \downarrow \{a\bar{b} : a \in X,\ \bar{b} \in Y^*\}$. If $\mathbf{B}$ is a RLAA we define the structure $Ide(\mathbf{B}) = (Ide(B), *, \underline{\lambda}x, \underline{x})_{x \in V}$ by setting $\underline{x} = \downarrow \{x\}$; $\underline{\lambda}x.X = \downarrow \{\lambda x.a : a \in X\}$ and the application $*$ as above.

Theorem 9.6.1.   *(i) If $\mathbf{A}$ is a* RCA*, then $Ide(\mathbf{A})$ is a combinatory algebra.*

(ii) Let $\mathbf{A}$ be a RLAA *and* Lf$\mathbf{A}$ *be the subalgebra of $\mathbf{A}$ generated by its locally finite elements. Then $Ide(\,$Lf$\mathbf{A})$ is a λ-abstraction algebra.*

(iii) If $\mathbf{A}$ is a RLA*, then $Ide(\mathbf{A})$ is a λ-algebra.*

Proof. (i) We prove the axioms of a combinatory algebra.

$$\underline{K} * X * Y = \downarrow \{K\bar{b}\bar{c} : \bar{b} \in X^*,\ \bar{c} \in Y^*\} = \downarrow \{b : b \in X\} = X,\ \text{by axiom } (K) \text{ of RCA.}$$

$$
\begin{aligned}
\underline{S} * X * Y * Z &= \downarrow \{S_{\bar{n}}a\bar{b}\bar{c} : a \in X, \bar{b} \in Y^*, \bar{c} \in Z^*, \bar{n} \in \mathbb{N}^*\} \\
&= \downarrow \{\textstyle\sum_{\bar{d} \in \mathcal{Q}_{\bar{c},\bar{n}}} a\bar{d}_0[b_0\bar{d}_1, \ldots, b_{k-1}\bar{d}_k] : a \in X,\ \bar{b} \in Y^k,\ k = |\bar{n}| - 1,\ \bar{c} \in Z^{\Sigma\bar{n}}\} \\
&\qquad \text{by axiom } (S_{\bar{n}}), \\
&= \downarrow \{a\bar{d}_0[b_0\bar{d}_1, \ldots, b_{k-1}\bar{d}_k] : a \in X,\ \bar{b} \in Y^k,\ \bar{d}_0, \ldots, \bar{d}_k \in Z^*,\ k \in \mathbb{N}\} \\
&\qquad \text{since ideals are closed under joins and downward closed,} \\
&= X * Z * (Y * Z)
\end{aligned}
$$

(ii) We prove the axioms of a λ-abstraction algebra.

$$
\begin{aligned}
(\beta_1)\ (\underline{\lambda}a.\underline{a})X &= \downarrow \{(\lambda a.a)\bar{x} : \bar{x} \in X^*\},\ \text{by linearity} \\
&= \downarrow \{(\lambda a.a)x : x \in X\}, \text{by } (r\beta_1) \\
&= X,\ \text{by } (r\beta_1)
\end{aligned}
$$

$$
\begin{aligned}
(\beta_2)\ (\underline{\lambda}a.\underline{b})X &= \downarrow \{(\lambda a.b)\bar{x} : \bar{x} \in X^*\} \\
&= \downarrow \{b\}, \text{by } (r\beta_2) \\
&= \underline{b}
\end{aligned}
$$

$$
\begin{aligned}
(\beta_3)\ (\underline{\lambda}a.X)\underline{a} &= \downarrow \{(\lambda a.x)a^n : x \in X, n \in \mathbb{N}\} \\
&= X, \text{by } (r6) \text{ and the hypothesis that } (\lambda a.x)a^n = x \text{ for some } n
\end{aligned}
$$

$$
\begin{aligned}
(\beta_4)\ (\underline{\lambda}a.\underline{\lambda}a.X)Y &= \downarrow \{(\lambda a.\lambda a.x)\bar{y} : x \in X,\ \bar{y} \in Y^*\} \\
&= \downarrow \{\lambda a.x : x \in X\},\ \text{by } (r\beta_3) \\
&= \underline{\lambda}a.X
\end{aligned}
$$

$$
\begin{aligned}
(\beta_5)\ (\underline{\lambda}a.XY)Z &= \downarrow \{(\lambda a.x\bar{y})\bar{z} : x \in X, \bar{y} \in Y^*, \bar{z} \in Z^*\} \\
&= \downarrow \{\textstyle\sum_{\sigma \in \mathfrak{S}_{|\bar{y}|}} \sum_{\sigma\bar{z} = \bar{z}_0 \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_{|\bar{y}|}} (\lambda a.x)\bar{z}_0[(\lambda a.y_0)\bar{z}_1, \ldots, (\lambda a.y_{|\bar{y}|-1})\bar{z}_{|\bar{y}|}] \\
&\qquad : x \in X, \bar{y} \in Y^*, \bar{z} \in Z^*\},\ \text{by } (r\beta_5) \\
&= \downarrow \{(\lambda a.x)\bar{z}_0[(\lambda a.y_0)\bar{z}_1, \ldots, (\lambda a.y_{|\bar{y}|-1})\bar{z}_{|\bar{y}|}] : x \in X,\ \bar{y} \in Y^*,\ \bar{z}_0, \ldots, \bar{z}_{|\bar{y}|} \in Z^*\} \\
&= (\underline{\lambda}a.X)Z((\underline{\lambda}a.Y)Z)
\end{aligned}
$$

Suppose now $(\underline{\lambda}b.Y)\underline{a} = Y$, that is, $Y = \downarrow \{(\lambda b.y)a^n : y \in Y, n \in \mathbb{N}\}$. Then it is easy to show that $(\lambda b.(\lambda b.y)a^n)[\ ] = (\lambda b.y)a^n$.

$$
\begin{aligned}
(\beta_6)\ (\underline{\lambda}a.\underline{\lambda}b.X)Y &= \downarrow \{(\lambda a.\lambda b.x)\bar{y} : x \in X,\ \bar{y} \in Y^*\} \\
&= \downarrow \{(\lambda a.\lambda b.x)[(\lambda b.y_1)a^{n_1}, \ldots, (\lambda b.y_k)a^{n_k}] : x \in X,\ y_i \in Y\} &&\text{by hyp.} \\
&= \downarrow \{\lambda b.(\lambda a.x)[(\lambda b.y_1)a^{n_1}, \ldots, (\lambda b.y_k)a^{n_k}] : x \in X,\ y_i \in Y\} &&\text{by } (\beta_4) \\
&= \downarrow \{\lambda b.(\lambda a.x)[y_1, \ldots, y_k] : x \in X,\ y_i \in Y\} &&\text{by hyp.} \\
&= \underline{\lambda}b.(\underline{\lambda}a.X)Y.
\end{aligned}
$$

Suppose now $(\underline{\lambda}b.X)\underline{c} = X$, that is, $X =\downarrow \{(\lambda b.x)c^n : x \in X, n \in \mathbb{N}\}$. It is easy to check that the hypotheses of rule $(r\alpha)$ are satisfied for the element $(\lambda b.x)c^n$, that is, $(\lambda b.(\lambda b.x)c^n)[\ ] = (\lambda b.x)c^n$ and there exists k such that $(\lambda a.(\lambda b.x)c^n)a^k = (\lambda b.x)c^n$. Then by $(r\alpha)$ we have: $\lambda a.(\lambda b.x)c^n = \lambda b.(\lambda a.(\lambda b.x)c^n)b^k$.

$$
\begin{aligned}
(\alpha) \ \underline{\lambda}b.(\underline{\lambda}a.X)\underline{b} \ &= \ \downarrow \{\lambda b.(\lambda a.x)b^k : x \in X, \ k \in \mathbb{N}\} \\
&= \ \downarrow \{\lambda b.(\lambda a.(\lambda b.x)c^n)b^k : x \in X, \ k \in \mathbb{N}\} \\
&= \ \downarrow \{\lambda a.(\lambda b.x)c^n : x \in X\} \\
&= \ \underline{\lambda}a.X
\end{aligned}
$$

(iii) Assume now that $\mathbf{A}$ is a RLA. Then $\mathbf{A}$ can be embedded into the resource combinatory reduct of $\mathbf{A}[V]$. Consider the RLAA $\mathbf{A}[V]_\lambda$ freely generated by $\mathbf{A}$. It is an easy matter to show that $Ide(\mathbf{A})$ can be embedded into the combinatory reduct of $Ide(\mathbf{A}[V]_\lambda)$, that we know to be an LAA from item (ii) of this theorem. $\qquad\square$

Note that the λ-abstraction algebra $Ide(\mathsf{LfA})$ of point (ii) is not necessarily locally finite: for example the element $\downarrow V$ when V is infinite breaks the property.

According to [46], we now define a translation of ordinary λ-terms into sets of $r\lambda$-terms, which also extends to a translation of Böhm trees. Let $\Lambda^\perp$ be the set normal terms in the λ-calculus extended with a constant $\perp$; as customary $\Lambda^\perp$ is endowed with a partial order whose bottom element is $\perp$ and where "less or equal" means "possibly more defined". Following [10], we identify the Böhm tree $BT(M)$ of a λ-term M with an ideal (downwards closed and directed subset) of $\Lambda^\perp$ quotiented by the equations $\perp N = \perp$ and $\lambda x.\perp = \perp$. This way we can also translate Böhm trees into subsets of Λ^r.

As a matter of terminology, a $r\lambda$-term t is: *simple* if none of its subterms (including t) contains either "+" or "0"; *normal* if none of its subterms (including t) is of the form $(\lambda x.t')\bar{s}$; in *canonical form* if it is a sum of simple terms. By an easy argument involving the multilinearity axioms of the $r\lambda$-calculus and Theorem 7.2.2, we can argue that for every term $t \in \Lambda^r$, there exists a unique normal term s in canonical form which is equal to t and we let $\mathrm{NF}(t)$ be the (finite) set of all simple terms whose sum is the normal canonical form of t.

Definition 9.6.1 ([46]). *Let $M \in \Lambda$ be a λ-term, possibly containing $\perp$. The set $\mathcal{T}(M) \subseteq \Lambda^r$ is defined inductively by the clauses: $\mathcal{T}(x) = \{x\}$, $\mathcal{T}(\perp) = \emptyset$, $\mathcal{T}(\lambda x.N) = \{\lambda x.t : t \in \mathcal{T}(N)\}$, and $\mathcal{T}(PQ) = \{t\bar{s} : t \in \mathcal{T}(P), \ \bar{s} \in \mathcal{T}(Q)^*\}$. Then $\mathcal{T}(N)$ happens to be the support of the Taylor expansion (see [46]) of the λ-term N. Now $\mathcal{T}(BT(M)) \subseteq \Lambda^r$ is defined as $\cup\{\mathcal{T}(B) : B \in BT(M)\}$.*

Recall now that λ-terms are ground terms in the similarity type of LAAs: hence for a LAA $\mathbf{B}$ it makes sense to write $M^{\mathbf{B}}$ to indicate the interpretation of M in $\mathbf{B}$. Similarly the notation $t^{\mathbf{A}}$ can be used to denote the interpretation of a resource λ-term t in a RLAA $\mathbf{A}$.

Lemma 9.6.2. *Let $\mathbf{A}$ be a locally finite RLAA and let $\mathbf{B} = Ide(\mathbf{A})$ be the LAA built over $\mathbf{A}$. Then for all $M \in \Lambda$, $M^{\mathbf{B}} = \downarrow \{u^{\mathbf{A}} : u \in \mathrm{NF}(t), \ t \in \mathcal{T}(M)\}$.*

Proof. The proof is by induction on $M \in \Lambda$ to show that $M^{\mathbf{B}} = \downarrow \{t^{\mathbf{A}} : t \in \mathcal{T}(M)\}$. Then the statement of the lemma follows from the simple observation that $\downarrow \{t^{\mathbf{A}} : t \in \mathcal{T}(M)\} = \downarrow \{u^{\mathbf{A}} : u \in \mathrm{NF}(t), \ t \in \mathcal{T}(M)\}$. The case in which M is a variable is trivial.

$$
\begin{aligned}
(\lambda x.N)^{\mathbf{B}} &= \downarrow \{\lambda x^{\mathbf{A}}.a : a \in N^{\mathbf{B}}\} \\
&= \downarrow \{\lambda x^{\mathbf{A}}.a : \exists \bar{t} \in \mathcal{T}(N)^*. \ a \leq \sum_i t_i^{\mathbf{A}}\}, \text{ using the ind. hyp.,} \\
&= \downarrow \{\lambda x^{\mathbf{A}}.t^{\mathbf{A}} : t \in \mathcal{T}(N)\} \\
&= \downarrow \{s^{\mathbf{A}} : s \in \mathcal{T}(\lambda x.N)\}
\end{aligned}
$$

$$
\begin{aligned}
(PQ)^{\mathbf{B}} &= \downarrow \{a\bar{b} : a \in P^{\mathbf{B}}, \ \bar{b} \in (Q^{\mathbf{B}})^*\} \\
&= \downarrow \{a\bar{b} : \exists \bar{t} \in \mathcal{T}(P)^*. \exists \bar{s}^1, \ldots, \bar{s}^n \in \mathcal{T}(Q)^*. \ a \leq \sum_i t_i^{\mathbf{A}}, \ b_i \leq \sum_h (s_h^j)^{\mathbf{A}}\} \\
&\quad \text{using the ind. hyp.,} \\
&= \downarrow \{(t'\bar{s})^{\mathbf{A}} : t' \in \mathcal{T}(P), \ \bar{s} \in (\mathcal{T}(Q))^*\} \text{ , using the linearity of operations in } \mathbf{A}, \\
&= \downarrow \{s^{\mathbf{A}} : s \in \mathcal{T}(PQ)\}.
\end{aligned}
$$

$\square$

Theorem 9.6.3 ([46]). *Let M be a λ-term and let u be a normal simple $r\lambda$-term. Then $u \in \mathcal{T}(BT(M))$ iff there exists $s \in \mathcal{T}(M)$ such that $u \in \mathrm{NF}(s)$.*

Lemma 9.6.4. *Let $\mathbf{A}$ be a locally finite* RLAA. *Then for all terms $M, N \in \Lambda$ we have $BT(M) = BT(N)$ implies $M^{Ide(\mathbf{A})} = N^{Ide(\mathbf{A})}$. In particular all λ-theories induced by the ideal completions of* RLAA*s are sensible.*

Proof. Suppose $BT(M) = BT(N)$. Then obviously $\mathcal{T}(BT(M)) = \mathcal{T}(BT(N))$. We conclude by applying Theorem 9.6.3 and Lemma 9.6.2 as follows: $M^{\mathbf{B}} = \downarrow \{u^{\mathbf{A}} : u \in \mathrm{NF}(t), \ t \in \mathcal{T}(M)\} = \downarrow \{u^{\mathbf{A}} : u \in \mathrm{NF}(t), \ t \in \mathcal{T}(N)\} = N^{\mathbf{B}}.$ $\quad \square$ $\qquad\qquad\qquad \square$

Conclusions

C.1 Part I

We faced the question of the existence of a reflexive **Cpo** model whose equational theory is $\lambda\beta$ (or $\lambda\beta\eta$). This problem, Problem 22 of the TLCA list, is still unresolved. Our work proves that this is not the case if we restrict our attention to reflexive objects in the category of algebraic lattices: this is the closest answer to the original problem, to date. To prove our results we combine the representation theory of Scott domains and the recursion theory developed so far over Scott domains. We axiomatize a class of webbed models which encompasses all existing ones in the literature and we develop for them constructions like extensions and completions which generalize those developed for graph models. Using our general theory we are in the position of answer another problem of the TLCA list: Problem 19.

At the time it was posted, Problem 19 seemed tremendously complicated. The answer that we are able to give it follows as an easy corollary of our work. We think that when new tools allow to easily solve problems that were considered difficult in the past, this is a sign of progress in science.

As a future work we mean to pursue our road towards the solution of Problem 22 of the TLCA list. We think that our techniques can be further pushed (to their limit) in order to prove that no reflexive object in **Sd** can have $\lambda\beta$ (or $\lambda\beta\eta$) as equational theory. This would put us in a good position for trying to "fill the gap" from **Sd** to **Cpo**. We hope this can be done by using techniques from universal (order) algebra, like those established by Adamek, Nelson and Reiterman [86, 1, 2, 3, 87].

At the same time, while trying to prove the above mentioned incompleteness results, we also aim at facing Selinger's *order-incompleteness* problem [109] which is related to our area of research and asks the following question: "does there exist a lambda theory (possibly with constants) which does not arise as the theory of an ordered model?"

C.2 Part II

We introduced the concept of multiplicity semi-ring, which can be used for generalizing the standard exponential construction of the relational model of Linear Logic. Such a semi-ring must contain $\mathbb{N}$ as a sub-semi-ring but can also have infinite elements ω satisfying $\omega + 1 = \omega$. In that case the corresponding model of Linear Logic is a model of the differential lambda calculus which does not satisfy the Taylor expansion formula, and it is possible to build non sensible models of the lambda calculus in the corresponding Kleisli Cartesian closed category. This shows

that models of the pure differential lambda calculus can have non sensible theories and provides a new way of building models of the pure lambda calculus where non termination is taken into account in a quantitative way by means of these infinite multiplicities.

The generalized relational semantics is then more expressive than the standard one, form the point of view of equational theories. So one may start wonder whether this new semantics is or not incomplete. We intend to prove, in the near future, the incompleteness of such semantics, when it is obtained by an "effective" multiplicity semiring. The idea is to adapt the techniques of the first part of this thesis, motivated by the observation that the relational models are a quantitative version of the webbed ones.

We set forth a purely algebraic study of Ehrhard and Regnier's resource λ-calculus, by introducing three equational classes of algebras: resource combinatory algebras, resource lambda-algebras and resource lambda-abstraction algebras. We established the relations between them, laying down foundations for a model theory of resource λ-calculus. We also showed that the ideal completion of a resource combinatory (resp. lambda-, lambda-abstraction) algebra induces a "classical" combinatory (resp. lambda-, lambda-abstraction) algebra, and that any model of the pure λ-calculus raising from a resource lambda-algebra determines a λ-theory which equates all terms having the same Böhm tree.

Let $\mathbf{U}$ be a reflexive object in the Kleisli ccc $\mathbf{Rel}_!$ obtained by a comonad ! implemented with some multiplicity semi-ring. Such object is at the same time a model of the resource lambda calculus to considering and a model of the ordinary lambda calculus and it can be endowed both by a structure of resource lambda-algebra or by a structure of lambda-algebra.

If the comonad happens to be the standard one (i.e. the multiplicity semi-ring is $\mathbb{N}$), then the passage from the first structure to the second one is captured at the algebraic level by means of the ideal completion. If instead the comonad is obtained via a multiplicity semi-ring like $\overline{\mathbb{N}}$, the ideal completion cannot reflect anymore the passage from the resource lambda-algebra structure to the lambda-algebra structure, since ideal completion only gives sensible models. It is very interesting for us to try to understand this kind of phenomena more deeply, also on the algebraic level.

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