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Centralized and competitive extraction of distributed renewable resources on networks with nonlinear aggregate growth

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ABSTRACT

We study the optimal and strategic extraction of a renewable resource that is spatially distributed over a network, migrates across nodes (with no loss of biomass during transit), and evolves according to a nonlinear (concave) aggregate growth function. A subset of nodes hosts extractors, while the remaining ones serve as reserves. We analyze a centralized planner's problem and a non-cooperative game with stationary Markov strategies. For three canonical growth families (logistic, power, and log-type saturating laws) paired with corresponding utilities, we derive closed-form value functions and feedback rules for the planner, and we construct a symmetric Markov equilibrium for the game on strongly connected networks.

1. Introduction

Many renewable resources are distributed across space and move between locations, so that extraction at one site affects stocks and payoffs elsewhere. Classical bioeconomic models describe aggregate dynamics for a single homogeneous stock (e.g. Gordon, 1954; Schaefer, 1954; Clark, 2010). That benchmark is useful, but it abstracts from spatial heterogeneity and resource mobility. For fisheries, groundwater, transboundary wildlife, connected reservoirs, forests with dispersal, and other mobile stocks, both the aggregate mass of the resource and its spatial allocation matter for productivity, incentives, and welfare. Once movement links locations, harvesting in one place affects opportunities elsewhere, so spatial externalities become intrinsic to the problem.

The modern economic literature on mobile renewable resources starts from spatially explicit, non-strategic models. A model of spatial differentiation of resources can already be found in Kaitala (1986), in which the stock depends on a continuous spatial variable and its evolution is governed by a partial differential equation including diffusion terms. A more recent contribution is Sanchirico and Wilen (1999), who study exploitation in a patchy environment and show that equilibrium biomass and effort patterns depend jointly on local bioeconomic conditions and on the mechanism governing biological dispersal. Sanchirico and Wilen (2005) then analyze optimal spatial management and show that policy scope should be matched to ecosystem scale: when biological connectivity extends

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beyond the regulator's jurisdiction, spatially fragmented management can be inefficient. Related non-strategic contributions include Bhat and Huffaker (2007), Costello and Polasky (2008), Xepapadeas (2010), and Brock et al. (2014). The broad perspective in Smith et al. (2009) is especially important for our purposes: it clarifies how spatial-dynamic resource problems can be modeled either in discrete-space form or in continuous-space diffusion form, and it emphasizes that dispersal and connectivity shape both transitional dynamics and welfare.

Strategic interaction enters naturally once access is decentralized. In the aggregate, non-spatial literature, the classic benchmark is Levhari and Mirman (1980), while the differential-games literature on common-property resources is surveyed by Clemhout and Wan (1994); see also Clemhout and Wan (1985). These papers establish the importance of feedback behavior and of the contrast between cooperative and non-cooperative management, but they work essentially with aggregate-state models and do not incorporate spatially distributed stocks. In the spatial resource literature, Kaffine and Costello (2011) provide a pioneering game-theoretic analysis of a connected renewable resource under territorial use rights, and Costello et al. (2015) extend this perspective through the notion of partial enclosure. Further developments include Herrera et al. (2016), Costello and Kaffine (2018), and Viana (2019). It is also worth mentioning İlkı lıç (2011), who studies static equilibrium extraction on a network.

An emerging approach to modeling spatial heterogeneity relies on *explicit* network representations, in which the resource is distributed across a finite number of interconnected regions (or nodes) linked through stock flows represented as network connections. The network interpretation is flexible enough to cover patchy habitats, connected basins, corridors of movement, or other economically relevant systems in which flows redistribute biomass across locations. In the context of resource economics, this perspective has recently been developed by Fabbri et al. (2024). Related network-based approaches have also been applied to models of spatial economic growth (Calvia et al., 2024; Fabbri et al., 2025). The present study belongs to this strand of the literature and further develops the framework introduced by Fabbri et al. (2024), extending the Levhari and Mirman model to a spatially differentiated resource. [1]

Our model departs from the existing literature along at least two directions. First, we derive an *explicit stationary Markov-perfect equilibrium* in a dynamic game, in contrast to the spatial bioeconomic literature, which typically focuses on optimal management, static spatial allocation, or reduced-form equilibrium characterizations. In this respect, Kaffine and Costello (2011) is a key benchmark, because it is one of the few papers to derive a feedback equilibrium in a spatial setting with connected resource dynamics. Nonetheless, even there the structure remains quite restrictive for the type of question we ask here: while local biological growth is nonlinear, instantaneous returns are linear in harvest, which greatly simplifies the feedback rule and leaves little room for the kind of smooth interior transitional dynamics generated by nonlinear utility-growth pairings.

Second, unlike Fabbri et al. (2024) (and Fabbri et al., 2020 for the two-location case), we assume that stock dynamics exhibit a *nonlinear* dependence on the stock, with reproduction determined by the aggregate stock and redistributed across locations proportionally to each node's stock. The aggregate-growth specification is economically meaningful in settings where local productivity depends on total biomass through congestion, crowding, recruitment, or system-wide ecological conditions, while migration redistributes the stock across locations. For example, it provides a reduced-form representation of migratory species in which reproduction is concentrated in specific spawning habitats and subsequent migration leads to spatial mixing and redistribution across local stocks, as is the case for salmon (see, e.g., Quinn and Dittman, 1990, for a qualitative description of these processes).

The nonlinearity of the dynamics constitutes the main novelty of the paper. Indeed, much of the existing literature relies on linear or linearized stock dynamics, largely because they ensure analytical tractability in spatial control and game-theoretic problems, often allowing for closed-form or near closed-form characterizations of equilibrium policies when aggregation is feasible. However, many renewable resources exhibit intrinsically nonlinear dynamics due to density dependence and environmental constraints. Concave surplus–production functions capture diminishing marginal biological productivity at higher stock levels, as in the Schaefer framework and its successors (Schaefer, 1957; Hilborn and Walters, 1992; Clark, 2010). Carrying-capacity constraints generate saturation effects that anchor the long-run stock around a finite level (Hilborn and Walters, 1992; Clark, 2010). Moreover, threshold effects may arise at low densities, where recruitment or survival declines disproportionately (Allee effects), potentially leading to tipping points, multiple steady states, or extinction traps (Courchamp et al., 2008).

Although the assumption of aggregate reproduction may appear restrictive, and a local reproduction at each site seems more natural in many applications, the choice reflects a broader technical difficulty in the literature. Indeed, a setting in which reproduction is centralized allows for a full analytical solution. To date, analytical characterizations under fully localized reproduction remain elusive. A local specification was already considered in Kaitala (1986), as a discretization of a continuous-space model, but no explicit solution is available. More generally, much of the dynamic-game literature on common-property resources proceeds "by example": tractable Markov-perfect equilibria are typically obtained only under restrictive functional forms, while general existence and regularity results for Markovian equilibria in nonlinear differential games remain limited. In continuous time, Markov strategies are characterized by systems of Hamilton–Jacobi equations, whose well-posedness is delicate even in relatively simple settings (Bressan and Priuli, 2006). The use of convenient closed-form strategy classes (e.g., linear Markov rules) typically requires strong restrictions on the primitives (Gaudet and Lohoues, 2008), and available results are often extended only through perturbation methods (e.g., Bressan and Nguyen, 2018). This motivates the disciplined set of assumptions adopted here: the nonlinearity is of aggregate type and, in each case, is paired with a suitable utility specification.

[1] Our model is formulated in continuous time, whereas Levhari and Mirman (1980) analyze a discrete-time setting. Continuous-time counterparts of the Levhari–Mirman model are provided in Plourde and Yeung (1989) and, more generally, in Gaudet and Lohoues (2008).

To summarize, this paper studies a spatially distributed renewable resource on a strongly connected network with linear, mass-conserving migration and nonlinear aggregate growth. A subset of nodes hosts active extractors, while the remaining nodes operate as reserves. We analyze both a centralized benchmark, in which a planner maximizes discounted welfare, and a non-cooperative game with stationary Markov strategies.

Our contribution is threefold. First, we derive closed-form value functions and feedback extraction rules for the planner under three canonical concave growth specifications (logistic, power, and log-type saturating) paired with the corresponding utility functions. Second, in the same environment, we construct a symmetric stationary Markov equilibrium for the decentralized game on a general strongly connected network. To the best of our knowledge, this is the first paper to obtain closed-form Markov strategies for spatial extraction on a general network with both strictly concave utility functions and nonlinear aggregate growth. Third, we clarify how the network structure and the nonlinear growth term play distinct roles: migration governs the transport of shadow values and, hence, the long-run spatial allocation, whereas nonlinear growth determines aggregate dynamics and bounds the global motion of the system.

The remainder of the paper is organized as follows. Section 2 introduces the model setup, including the network structure, resource dynamics, and agents' optimization problems. Section 3 derives the socially optimal solution, presenting the planner's HJB equation and the closed-form optimal policy. Section 4 analyzes the noncooperative game, characterizes the stationary Markov perfect equilibrium, and provides an explicit formula for the equilibrium extraction strategy at each node. Section 5 concludes with a summary of findings and directions for future research. Appendix A collects technical lemmas and proofs.

2. The model

We consider a common property resource distributed across an area divided into subareas or regions. The local quantities of the resource change over time and *move* across regions in specified proportions. This model can represent various types of resources, such as livestock migrating across regional or national boundaries, or groundwater or oil supplies. For illustration, one can focus on fish that move between different regional waters. The overall area is modeled generically as a network, with n nodes, each representing a different region, connected to one another by suitable inflow/outflow functions. A number $f \le n$ of agents are each assigned to a single region for exclusive exploitation and decide the quantity of resource they extract in time.

State and control variables. Specifically, we denote by $N = \{1, 2, \dots, n\}$ the set of nodes, $F \subset N$ the subset where an agent is active, with the remaining $N \setminus F$ nodes used as reserves. Here e_i denotes the i-th vector of the canonical basis on $\mathbb{R}^n$, $\mathbf{e} = \sum_{i=1}^n e_i$ the unit vector, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$, and $\mathbb{R}_+$ the set of nonnegative real values. For any $x \in \mathbb{R}^n$, we write $x \ge 0$ to mean $x_i \ge 0$ for all $i \in N$, and $x > 0$ to mean $x_i > 0$ for all $i \in N$.

For all $i \in N$, $X_i(t)$ stands for the mass at node i at time t, and we set $X(t) = (X_1(t), \dots, X_n(t))^\top$. The evolution in time of mass $X_i(t)$ on region i depends on several factors: the natural growth $g_i(X(t))$ of the resource at node i; the net inflow $\alpha_i(X(t))$ of the resource stock exchanged with the other nodes; the rates of extraction $c_i(t)$ at time t from region i, namely the decision variables of the problem, chosen by the agents.

The evolution of the system is then described by the set of ODEs

$$\begin{cases} X_i'(t) = g_i(X(t)) + \alpha_i(X(t)) - c_i(t), \quad t > 0 \\ X_i(0) = x_i, \end{cases} \tag{1}$$

with g_i and α_i representing the biological growth and the net inflow at node i. Moreover, the local stocks and extractions are subject to the nonnegativity constraints $c_i(t) \ge 0$, $X_i(t) \ge 0$, for all $t \ge 0$.

We denote by $X(t; x, c(\cdot))$, or shortly by $X(t)$, the trajectory of the system starting at x and driven by control $c(t) = (c_1(t), c_2(t), \cdots, c_n(t))^\top$.

We now specify further g_i and α_i. For the net inflow α_i, we mainly consider the case of a diffusion linear in the regional stocks, more specifically we make the following assumption.

Hypothesis 1. The stock exchanges between two nodes of the network are represented by the network's adjacency matrix B, with entries $b_{ij} \ge 0$ for all i, j in N, and $b_{ii} = 0$. The outflow from node i to node j at time t is represented by $b_{ij} X_i(t)$.
The network is assumed *strongly connected*.

Under this assumption one has

$$\alpha_i(X(t)) = \sum_{j=1, j\neq i}^n b_{ji} X_j(t) - \sum_{j=1, j\neq i}^n b_{ij} X_i(t) = \langle Be_i, X(t) \rangle - \sum_{j=1, j\neq i}^n b_{ij}\, X_i(t),$$

If we set $\alpha(x) = (\alpha_1(x), \alpha_2(x), \cdots, \alpha_n(x))^\top$, and D the diagonal matrix of the outflows

$$D = \begin{pmatrix} -\sum_{j=1, j\neq 1}^n b_{1j} & 0 & 0 & \cdots & 0 \\ 0 & -\sum_{j=1, j\neq 2}^n b_{2j} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\sum_{j=1, j\neq n}^n b_{nj} \end{pmatrix}$$

then

$$\alpha(x) = (D + B^\top)x, \ x \in \mathbb{R}^n.$$

Note that, under Hypothesis 1, one has

$$\sum_{i=1}^{n} \alpha_i(X(t)) = \langle \alpha(X(t)), \mathbf{e} \rangle = 0, \quad \forall t \geq 0 \tag{2}$$

meaning that the migration process itself is purely redistributive. In other words, the spatial transport mechanism does not create or destroy biomass during transit between nodes. It is important to emphasize that while the net sum of migration flows is zero at all times, the total aggregate mass of the resource in the entire system, $m(t) = \sum_i X_i(t)$, will continuously change over time due to the local biological growth $g_i(X(t))$ and the extraction efforts $c_i(t)$.

Example 1 (Symmetric diffusion of Fick's type)**.** A useful special case of Hypothesis 1 is a symmetric diffusive coupling in the spirit of Fick's law. In this setting, the flow between two nodes is linear and proportional to the difference in their stocks. Equivalently, let $W = (w_{ij})_{i,j \in N}$ be a nonnegative weight matrix with $w_{ij} = w_{ji}$ for $i \neq j$ and $w_{ii} = 0$. Interpreting w_{ij} as the intensity of exchange along edge (i, j), the net inflow at node i reads

$$\alpha_i(x) \; = \; \sum_{j \neq i} w_{ji}(x_j - x_i).$$

Remark 1. In Remark 4, we will discuss a generalization of Hypothesis 1, for possibly nonlinear net flows α_i.

As for the *growth function* g_i, we assume g_i is a real-valued concave function belonging to one of the following classes:

Hypothesis 2. For all $i \in N$

$$g_i(x) = x_i \; \varphi\left(\sum_{i=1}^{n} x_i \right) = x_i \; \varphi(\langle x, \mathbf{e} \rangle)$$

where the *saturation function* φ can be chosen among the following:

(S1) $\varphi(m) = \Gamma\left(1 - \frac{1}{K} m^{\sigma - 1}\right)$, with $\sigma > 1$, $\Gamma > 0$;
(S2) $\varphi(m) = m^{\sigma - 1} - \delta$, with $0 < \sigma < 1$, and $\delta > 0$;
(S3) $\varphi(m) = \Gamma\left(1 - \frac{1}{K} \ln m\right)$, with $\Gamma, K \in \mathbb{R}$.

where $m \in (0, +\infty)$, and Γ, K, δ and σ are parameters denoting, respectively, the growth rate, the carrying capacity, the natural decay rate, and the inverse of the intertemporal elasticity of substitution.

Overall, if we set $c(t) = (c_1(t), c_2(t), \cdots, c_n(t))^{\top}$, the vector of extractions chosen by agents, system (1) can be rewritten in vector form as

$$\begin{cases} X'(t) = \varphi(\langle X(t), \mathbf{e} \rangle) X(t) + (D + B^{\top}) X(t) - c(t), & t \geq 0 \\ X(0) = x. \end{cases} \tag{3}$$

subject to the constraints

$$c_i(t) \geq 0, \quad X_i(t) \geq 0, \text{ for all } t \geq 0. \tag{4}$$

We finally describe the agents and their utility.

Hypothesis 3. Assume $\rho > 0$ is the discount rate. We suppose that at each location $i \in F$ there is an agent with the following instantaneous utility:

(U1) $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$ when the saturation function φ satisfies $(S1)$ or $(S2)$;
(U2) $u(c) = \ln(c)$ when the saturation function φ satisfies $(S3)$.

We will analyze and compare two frameworks:

(F1) In Section 3 we will study the case of a single "Benthamite" planner maximizing the sum of the utilities from the extraction of the resource at the different nodes

$$J(c; x) = \int_0^{+\infty} e^{-\rho t} \sum_{i=1}^{f} u(c_i(t)) dt \tag{5}$$

under the constraints (3) and (4);

(F2) In Section 4 we will look at the case of f competing agents in a dynamic game, with agent i choosing c_i so as to maximize

$$J_i\left(c_i; x, c_{-i}\right) = \int_0^{+\infty} e^{-\rho t} u(c_i(t)) dt \tag{6}$$

under (3) and (4).

Note that in (5) and (6), x is the initial stock and the notation $J(c; x)$, (respectively, $J_i\left(c_i; x, c_{-i}\right)$) emphasizes that J (respectively, J_i) depends implicitly on x (respectively, x, c_{-i}) through the constraint given by the state Eq. (3).

Remark 2. Our specification $g_i(x) = x_i\,\varphi(\langle x, \mathbf{e}\rangle)$ assumes that congestion or biological limits operate at the level of the *aggregate* stock rather than separately at each node. While many spatial bioeconomic models use patch-level crowding, there are relevant settings in which a nonlocal specification is more natural, that is, settings in which local growth depends not only on the stock at a given site, but also on biomass conditions at the system-wide level.

One example is provided by shallow estuaries and bays with benthic filter feeders, where grazing may affect phytoplankton dynamics at the scale of the whole system rather than only at individual locations (Officer et al., 1982). More generally, continuous-space population models often incorporate nonlocal effects, for instance through spatial averaging or dispersal-based interactions (see, e.g., Murray, 2003; Okubo and Levin, 2001).

Another relevant example involves migratory species such as salmon. In these populations, reproduction is concentrated in specific spawning habitats, while subsequent migration redistributes biomass across locations (see, e.g., Quinn and Dittman, 1990, for a qualitative description of these processes). Moreover, because adults tend to return to a limited set of natal spawning sites, reproductive success may ultimately be constrained by the finite capacity of those sites. In this sense, dependence on the aggregate stock can be interpreted as a reduced-form representation of basin-wide or region-wide crowding.

2.1. Primitives of the system

It will be convenient to represent the trajectory of the system $X(t)$ as the product

$$X(t) = m(t)Y(t) \tag{7}$$

where $m(t)$ is the *total mass* of the resource, namely, the sum at time t of the stocks at different nodes

$$m(t) \equiv \sum_{i\in N} X_i(t) = \langle \mathbf{e}, X(t)\rangle \tag{8}$$

while $Y(t)$ is the vector of the *shares of the total stock* in different regions, having the property

$$\sum_{i=1}^{n} Y(t) = \langle \mathbf{e}, Y_i(t)\rangle \equiv 1, \quad \forall t \geq 0,$$

that is, $Y(t)$ varies in time remaining on the simplex of the equation

$$y_1 + y_2 + \cdots + y_n = 1, \quad y_i \geq 0.$$

Hence $X(t)$ satisfies the equation in (3) if and only if $m(t)$ and $Y(t)$ satisfy

$$\begin{cases} m'(t) = \varphi(m(t))m(t) - \langle c(t), \mathbf{e}\rangle, & t \geq 0 \\ Y'(t) = \left(D + B^\top + \frac{\langle c(t), \mathbf{e}\rangle}{m(t)}\right)Y(t) - \frac{c(t)}{m(t)} & t \geq 0 \\ m(0) = \langle x, \mathbf{e}\rangle,\ Y(0) = \frac{1}{\langle x, \mathbf{e}\rangle}x \end{cases} \tag{9}$$

Evolution of the system in the absence of extraction

We now examine the relationship between the system's evolution and the network parameters.

Lemma 1. *Assume that Hypothesis 1 is satisfied. Then the matrix $D + B$ has 0 as an eigenvalue; moreover, all other eigenvalues have strictly negative real parts. More explicitly, if $\{0, \lambda_2, \cdots, \lambda_n\}$ are the eigenvalues of $D + B$, then*

$$0 > \mathrm{Re}\lambda_2 \geq \mathrm{Re}\lambda_3 \geq \cdots \geq \mathrm{Re}\lambda_n \tag{10}$$

The eigenspace associated with the zero eigenvalue has dimension 1, and is generated by the (dominant) eigenvector $\mathbf{e}$. Similarly, the transpose $D + B^\top$ has a (dominant) eigenvector ζ associated to the zero eigenvalue, having all positive coordinates. All eigenvectors associated with nonzero eigenvalues have at least a nonpositive coordinate.

The reader will find the proof of this lemma in the appendix.

The role of eigenvalues and eigenvectors is made immediately clear when analyzing the evolution of the system in the absence of extraction. By differentiating the product in (7), using (2), and setting $c(t) \equiv 0$ in (3), one sees that $m(t)$ and $Y(t)$ defined in (7) (8) satisfy the equations

$$\begin{cases} m'(t) = m(t)\varphi(m(t)), & t \geq 0 \\ Y'(t) = (D + B^\top)Y(t), & t \geq 0 \\ m(0) = \langle \mathbf{e}, x\rangle, \quad Y(0) = \langle \mathbf{e}, x\rangle^{-1}x \end{cases} \tag{11}$$

and the following lemma holds.

Lemma 2. *Assume that Hypothesis 1 is satisfied. Assume the initial stock $x \geq 0$, with x nonzero, and assume a null extraction $c(t) \equiv 0$. Then*

$$Y(t) = \langle \mathbf{e}, x\rangle^{-1} e^{(D+B^\top)t}x, \quad \textit{with} \quad Y_i(t) \geq 0, \quad \forall t \geq 0, \forall i \in N, \tag{12}$$

$$\lim_{t\to\infty} Y(t) = \langle \mathbf{e}, \zeta\rangle^{-1}\zeta, \tag{13}$$

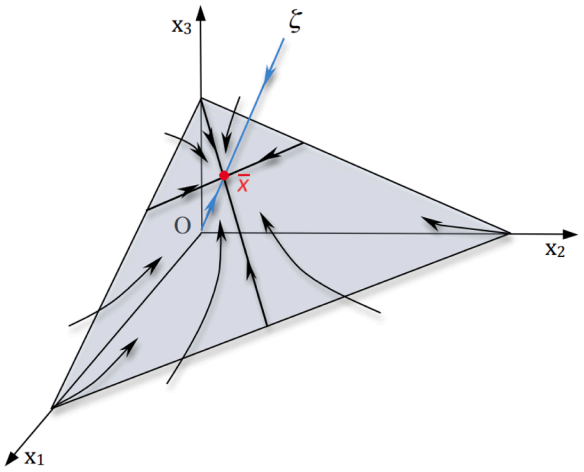

Fig. 1. Phase portrait for a 3-node network ($n = 3$) with real eigenvalues. The trajectories converge to a steady state $\bar{x}$ on the simplex representing the long-run total mass $\overline{m}$. The final spatial distribution is aligned with the direction of the dominant eigenvector ζ.

where ζ is the dominant eigenvector defined in Lemma 1. Moreover, $m(t) > 0$ for all $t \geq 0$ and:

(i) in assumptions (S1), $m(t) = m_0(t) := \left(e^{-\Gamma(\sigma-1)t}\left[(\sum_i x_i)^{1-\sigma} - \frac{1}{K}\right] + \frac{1}{K}\right)^{\frac{1}{1-\sigma}}$

(ii) in assumptions (S2), $m(t) = m_0(t) := \left(e^{-\delta(1-\sigma)t}\left[(\sum_i x_i)^{1-\sigma} - \frac{1}{\delta}\right] + \frac{1}{\delta}\right)^{\frac{1}{1-\sigma}}$

(iii) in assumptions (S3), $m(t) = m_0(t) := \exp\left(e^{-\frac{\Gamma}{K}t}(\log(\langle x, \mathbf{e}\rangle) - K) + K\right)$.

So that, in the long run, as $t \to \infty$, we have

$$\bar{m} := \lim_{t\to\infty} m(t) = \begin{cases} K^{\frac{1}{\sigma-1}}, & \text{under assumptions } (S1), \\ \delta^{\frac{1}{\sigma-1}}, & \text{under assumptions } (S2), \\ e^K, & \text{under assumptions } (S3). \end{cases}$$

The reader will find the proof of this lemma in the appendix, while the proof of the Corollary below is straightforward.

Corollary 1. *In the assumptions of Lemma 2, and for $\bar{m} = \lim_{t\to\infty} m(t)$, the trajectory of system (3), with null extraction $c(t) \equiv 0$, satisfies*

$$\bar{x} = \lim_{t\to\infty} X(t) = \bar{m}\langle \mathbf{e}, \zeta\rangle^{-1}\zeta$$

that is, it converges to a point of the simplex of the equation

$$x_1 + x_2 + \cdots + x_n = \bar{m}, \quad x_i \geq 0.$$

Fig. 1 illustrates this dynamic for a three-region network.

Note that in the case of linear increments of the stock treated in Fabbri et al. (2024), the trajectory was either diverging or converging to zero along the direction of the dominant eigenvector. In the nonlinear cases treated here, the trajectory converges to a vector $\bar{x}$ with all finite components, is scaled by the total mass $m(t)$, and it is distributed in the long run proportionally to the coordinates of the dominant eigenvector ζ.

That also establishes the relationship between the trajectory of the system and the network: in the absence of extraction, the stock distributes proportionally to the node's eigenvector centrality of the matrix $D + B$. Namely, it is the linear part of the system, embodied by $Y(t)$, that dictates the proportions between the components of the limiting stock.

3. The social planner problem

We study a centralized benchmark in which a benevolent decision maker chooses the entire extraction vector $c(t)$ to maximize the discounted sum of agents' instantaneous utilities:

$$\max_{c(\cdot)\geq 0} \int_0^\infty e^{-\rho t} \sum_{i\in F} u(c_i(t))\, dt \quad \text{subject to} (3) - -(4).$$

In other words, we consider the problem of a utilitarian (Benthamite) social planner.[2]

We address the problem by means of Bellman's Dynamic Programming. To this extent, if $L^1_{loc}(0,+\infty;\mathbb{R}^n_+)$ denotes the set of non-negative vector-valued and locally integrable functions on $[0,+\infty)$ (we choose this space to guarantee that the state equation admits a well-defined absolutely continuous solution while allowing for flexible, possibly discontinuous, extraction profiles), and $X(t) = X(t;c,x)$ denotes the trajectory starting at x and driven by a control c, we set

$$\mathbb{A} := \{c \in L^1_{loc}(0,+\infty;\mathbb{R}^n_+) : c(t) \geq 0, X(t;c,x) \geq 0, \forall t \geq 0\} \tag{14}$$

as the set of admissible strategies for the social planner. We will call any control $c \in \mathbb{A}$ and the associated trajectory an *admissible couple*. Then the *value function* for the social planner is defined as

$$V(x) = \sup_{c\in\mathbb{A}} J(c;x).$$

The Hamilton-Jacobi-Bellman (briefly, HJB) equation associated to the problem by Dynamic Programming is

$$\rho v(x) = \max_{c\geq 0}\left\{\sum_{i=1}^{f}\left(u(c_i) - c_i\frac{\partial v}{\partial x_i}(x)\right)\right\} + \langle \nabla v(x), \alpha(x) + \varphi(\langle x,\mathbf{e}\rangle)x\rangle. \tag{15}$$

Note that $\operatorname{argmax}_{a\geq 0}\{u(a) - ap\} = (p)^{-\frac{1}{\sigma}}$, $p > 0$, with $\sigma = 1$ in the case of a logarithmic utility. Hence, if a classical solution v of the HJB equation exists with $\frac{\partial v}{\partial x_i} > 0$, for all $i \in F$, the maximizing control is given by

$$c_i^*(x) = \left(\frac{\partial v}{\partial x_i}(x)\right)^{-\frac{1}{\sigma}}, \forall i \in F, \quad c_i^* \equiv 0, \forall i \in N \setminus F, \tag{16}$$

with $\sigma = 1$ in the case of a logarithmic utility. The reader will find in Section A.2 in the Appendix how (15) can be explicitly specified in the different assumptions on the data.

Now, the saturation functions introduced in Hypothesis 2 cannot be combined arbitrarily with the utility functions in Hypothesis 3. An indication of how these assumptions should be paired already emerges from Table 1 below, where such couplings are displayed in the left column. Although restrictive, this disciplined pairing is what makes it possible to derive explicit formulas for the optimal strategies and the value function. For this reason, the result should be understood as a benchmark rather than a claim of full generality for problems with more general nonlinear dynamics. Such explicit solutions may also prove useful in future investigations of the stability of trajectories under perturbations of either the dynamics or the utility. This is expected to be even more relevant in competitive settings, where general existence theorems for Nash equilibria are typically unavailable.

Therefore, consider the following set of parameters and quantities.

Table 1
The first two columns summarize the choice of the saturation and utility functions defining the chosen framework, while the last three specify the corresponding values of θ^*, A, and B to be used in the formulas contained in Theorem 1.

framework	$\varphi(m)$	$u(c)$	θ^*	A	B
(S1)(U1)	$\Gamma\left(1-\frac{1}{K}m^{\sigma-1}\right)$	$-\frac{c^{1-\sigma}}{\sigma-1}$, $\sigma>1$	$\frac{\rho+\Gamma(\sigma-1)}{\sigma f}$	$(\theta^*)^{-\sigma}$	$-\frac{\Gamma(\theta^*)^{-\sigma}}{K\rho}$
(S2)(U1)	$m^{\sigma-1}-\delta$	$\frac{c^{1-\sigma}}{1-\sigma}$, $\sigma<1$	$\frac{\rho+\delta(1-\sigma)}{\sigma f}$	$(\theta^*)^{-\sigma}$	$\frac{(\theta^*)^{-\sigma}}{\rho}$
(S3)(U2)	$\Gamma\left(1-\frac{1}{K}\log m\right)$	$\log c$	$\frac{1}{f}(\rho+\frac{\Gamma}{K})$	$(\theta^*)^{-1}$	$\frac{\Gamma-f\theta^*+f\theta^*\log\theta^*}{\theta^*\rho}$

Theorem 1 (Existence and Uniqueness of the Optimal Strategy)**.** *In each of the assumption frameworks described in Table 1, and assuming $\theta^* > 0$, the following facts hold:*

(i) when admissible, the only closed-loop optimal strategy ψ^* is given by

$$\psi_i^* = 0, \quad \forall i \notin F \quad \text{and} \quad \psi_i^*(x) = \theta^*\langle x,\mathbf{e}\rangle, \ \forall i \in F; \tag{17}$$

(ii) in such a case, the value function of the problem is

$$V(x) = A\,u(\langle \mathbf{e}, x\rangle) + B. \tag{18}$$

(iii) moreover, the value function $V(x)$ is the unique classical solution of the HJB Eq. (15), in the sense of Theorems 3 and 4 in Appendix A.

The proof of Theorem 1 constitutes the contents of Appendix A.2. Note that Theorems 3 and 4 and their proofs provide sufficient conditions for the value function to be the unique solution to the HJB Eq. (15) under different assumptions. It should also be noted that such results rely on the proof of existence and uniqueness of the solution of the *closed-loop equation*, as worked out in the next Section 3.1.

[2] From a game-theoretic perspective, this setup coincides with a fully cooperative scenario where all active agents ($i \in F$) form a grand coalition and strictly commit to a joint extraction strategy $c(t)$. Under the assumption of transferable utility with uniform weights, the agents can frictionlessly redistribute the total payoff via side payments. Consequently, the grand coalition seeks to maximize the sum of the individual payoffs, which exactly coincides with the planner's aggregate objective defined in (5): $\sum_{i\in F} J_i(c_i;x,c_{-i}) = \int_0^{+\infty} e^{-\rho t}\sum_{i=1}^{f} u(c_i(t))dt = J(c;x)$.

Remark 3.

The spatial uniformity of both the optimal extraction policy and the value function follows from the interplay of preferences, biology, and network geometry. On the demand side, the social planner assigns uniform weights to all agents with identical, strictly concave utility functions, creating a motive to equalize the marginal utility of extraction across space. On the supply side, the per-unit biological productivity is symmetric across nodes (as it depends on the aggregate mass rather than local densities), and the network redistributes the biomass without loss.

From a network theory perspective, this result reflects a fundamental duality between two distinct measures of centrality. The long-run physical distribution of the resource is governed by the (right) eigenvector ζ of the migration matrix, which captures a heterogeneous quantity (or "reception") centrality. Conversely, the spatial distribution of shadow prices is governed by the corresponding left eigenvector $\mathbf{e}$, capturing a value (or "emission") centrality. Because migration is strictly mass-conserving, this value centrality is perfectly uniform. Consequently, the shadow price of the resource is equalized across all active regions, making the value function dependent solely on the aggregate stock rather than its specific spatial distribution. Therefore, the planner extracts uniformly, allowing the network's natural migration dynamics to redistribute the resource and sustain this symmetric harvesting.

This mechanism mirrors some findings in the continuous-space literature; notably, Boucekkine et al. (2013) show (see their Theorem 3.1) that in a spatial AK model (with homogeneous productivity and trade-diffusion), the planner's objective to perfectly smooth consumption across locations yields a spatially uniform optimal control. Our discrete-network setting recovers a strictly analogous property: just as in our model, their value function weights all local stocks equally. In their infinite-dimensional setting, the spatial distribution of shadow prices is governed by the adjoint of the growth-and-diffusion operator, and the spatial flatness of prices perfectly corresponds to the fact that the constant function is an eigenfunction of this adjoint operator, acting as the continuous counterpart to our uniform left eigenvector.

Remark 4. Let $\tilde{\alpha} : \mathbb{R}^n_+ \to \mathbb{R}^n$ be locally Lipschitz and mass-preserving, i.e. $\langle \mathbf{e}, \tilde{\alpha}(x)\rangle = 0$ for all x (and inward-pointing on $\partial\mathbb{R}^n_+$). Replacing the linear migration $\alpha(x) = (D + B^\top)x$ by $\tilde{\alpha}(x)$ leaves $V(x) = A\,u(\langle \mathbf{e}, x\rangle) + B$ a good candidate solution of the HJB equation in Theorem 1. Indeed, in this case $\nabla V(x) = A\,u'(\langle \mathbf{e}, x\rangle)\mathbf{e}$ so $\langle \nabla V(x), \tilde{\alpha}(x)\rangle = 0$ and the migration term drops out, hence the maximizer of the Hamiltonian coincides with c^* in Theorem 1, and V satisfies the same scalar identity in the aggregate $m = \langle \mathbf{e}, x\rangle$ as in the linear case. Conditions to characterize the admissibility of the candidate optimal trajectory and, later, the candidate Markovian Nash equilibrium are harder, and the dynamics of the "direction" Y of the trajectory can no longer be described by properties of suitable matrices.

3.1. The closed-loop equation

As we will see in the next sections, both the optimal control for the cooperating agents and the equilibrium strategy for the competing agents in the game described in the forthcoming Section 4 share the same structure

$$c_i^*(t) = \psi_i(X(t)) = \theta\langle X(t), \mathbf{e}\rangle, \quad i \in F; \quad c_i^*(t) = 0, \quad i \notin F. \tag{19}$$

If E is the $n \times n$ matrix having the i-th row equal to $\mathbf{e}$ when $i \in F$, and equal to the zero vector when $i \notin F$, namely

$$E = (e_{ij}), \quad \text{with } e_{ij} = 1, \forall i \in F, \text{ and } e_{ij} = 0, \forall i \notin F, \quad \forall j \in N. \tag{20}$$

then such optimal control can be written in vector form as $c^*(t) = \theta E X(t)$. In particular, for the case of cooperating agents, one chooses $\theta = \theta^*$ to have the optimal feedback control. What we say next will hold for all θ unless otherwise specified.

When we insert a closed-loop control of type (19) into the state Eq. (3), we derive the following closed-loop equation, whose existence and uniqueness we need to discuss

$$\begin{cases} X'(t) = \varphi(\langle \mathbf{e}, X(t)\rangle)X(t) + (D + B^\top - \theta E)X(t) \quad t \geq 0 \\ X(0) = x. \end{cases} \tag{21}$$

To do so, we note that (21) is equivalent to the coupled systems for the evolution of the total mass and the stock shares, namely

$$\begin{cases} m'(t) = \big(\varphi(m(t)) - f\theta\big)m(t), & t \geq 0 \\ Y'(t) = (D + B^\top - \theta E + f\theta I)Y(t), & t \geq 0 \\ (m(0), Y(0)) = (\langle \mathbf{e}, x\rangle, \frac{x}{\langle \mathbf{e}, x\rangle}). \end{cases} \tag{22}$$

Clearly, the (linear) equation for $Y(t)$ has a unique solution, and the same holds for the equation for $m(t)$. This implies the following fact.

Corollary 2. *In the assumptions of Theorem 1, and when $\theta = \theta^*$, the closed-loop Eq. (21) has a unique solution $X^*(t)$.*

3.2. Total mass dynamics

Proposition 1 (Evolution of the Total Mass). *Assume Hypothesis 1, $x \geq 0$ with $x \neq 0$. A solution $m(t)$ of (22) is:*

(i) in assumptions $(S1)(U1)$, for $\Gamma > f\theta$,

$$m(t) = \left[e^{-(\Gamma - f\theta)(\sigma-1)t}\Big(\langle \mathbf{e}, x\rangle^{1-\sigma} - \frac{1}{K}\frac{\Gamma}{\Gamma - f\theta}\Big) + \frac{1}{K}\frac{\Gamma}{\Gamma - f\theta}\right]^{\frac{1}{1-\sigma}}$$

(ii) in assumptions $(S2)(U1)$*, for* $f(1-\sigma)<1$*,*

$$m(t)=\left[e^{-(\delta+f\theta)(1-\sigma)t}\big(\langle \mathbf{e},x\rangle^{1-\sigma}-\frac{1}{\delta+f\theta}\big)+\frac{1}{\delta+f\theta}\right]^{\frac{1}{1-\sigma}}$$

(iii) in assumptions $(S3)(U2)$*,*

$$m(t)=\exp\left[e^{-\frac{\Gamma}{K}t}\left(\log\langle \mathbf{e},x\rangle-K+\frac{Kf\theta}{\Gamma}\right)+K-\frac{Kf\theta}{\Gamma}\right]$$

Moreover, $m(t)>0$ *at all times* $t\geq 0$ *and converges to a finite positive number* $\hat{m}$*.*

The proof of this fact is a straightforward adaptation of the proof of Lemma 2.

Taking limits in the explicit formulas of Proposition 1 and plugging the planner's coefficient θ^* from Table 1 yields the steady-state values for m:

$$\begin{aligned}&\text{(S1)(U1):}\quad m^*=\left[\tfrac{K}{\sigma}\left(1-\tfrac{\rho}{\Gamma}\right)\right]^{\frac{1}{\sigma-1}},\\&\text{(S2)(U1):}\quad m^*=\left[\tfrac{\rho+\delta}{\sigma}\right]^{\frac{1}{\sigma-1}},\\&\text{(S3)(U2):}\quad m^*=\exp\Big(K-1-\tfrac{K\rho}{\Gamma}\Big).\end{aligned}$$

In particular, in the planner benchmark, both m^* and the optimal aggregate extraction $f\theta^*$ (being θ^* proportional to $1/f$, see Table 1) are independent of the network and of the number of active sites f.

Observe that in all three frameworks, the socially optimal steady-state aggregate mass exactly satisfies the generalized Hotelling rule for renewable resources:

$$g'(m^*)=\rho, \tag{23}$$

where $g(m):=m\varphi(m)$ is the aggregate biological growth function. Because the social planner perfectly internalizes all spatial externalities and the migration process is mass-conserving, the spatial network behaves in the aggregate exactly like a single-node model, optimally equalizing the marginal biological productivity of the entire stock to the discount rate.

Remark 5. The previous formulas imply that:

$$\frac{\partial m^*}{\partial\rho}<0,\qquad \frac{\partial m^*}{\partial\Gamma}>0\,,\qquad \frac{\partial m^*}{\partial K}>0\ \text{(assuming }\Gamma>\rho)$$

with the last two holding when the constants Γ and K appear in the formulas. Indeed, as one would expect, a higher discount rate ρ lowers the present value of future extraction, shifting extraction upward and the steady state downward. Conversely, a larger intrinsic growth Γ or carrying capacity K raises the regeneration curve, allowing a higher sustainable stock (where the condition $\Gamma>\rho$ conventionally ensures that the intrinsic biological growth outpaces the financial discount rate).

3.3. Stock shares dynamic

Clearly, the components of the limit value of the trajectory, when existing, may bear different signs although their sum m is positive. We then proceed to study the stock shares vector $Y(t)$. To this extent, the following lemma is fundamental.

Lemma 3. *Let* E *be the matrix defined in* (20)*, and* $\theta\in\mathbb{R}$*. Then:*

(i) the matrix $(D+B-\theta E^\top+f\theta I)$ *has eigenvector* $\mathbf{e}$ *associated with the null eigenvalue; hence, there exists a real eigenvector* ζ_θ *of* $(D+B-\theta E^\top+f\theta I)^\top$ *associated with the null eigenvalue.*

(ii) Consider a basis $\{\zeta,v_2,\dots,v_n\}$ *of generalized eigenvectors of* $D+B^\top$*, associated with the eigenvalues* $\{0,\lambda_2,\dots,\lambda_n\}$ *described in Lemma 1. Then* $\{\zeta_\theta,v_2,\dots,v_n\}$ *is a basis of generalized eigenvectors for* $(D+B-\theta E^\top+f\theta I)$ *associated with eigenvalues* $\{0,\lambda_2+f\theta,\dots,\lambda_n+f\theta\}$*.*

The proof of this fact can be found in Appendix A.3.

Note that, in comparison with the system with null extraction, the equation for $Y(t)$ is modified by the linear term $(-\theta E+f\theta I)Y(t)$, so that

$$Y'(t)=\big(D+B^\top-\theta E+f\theta I\big)Y(t).$$

In general, the matrix $D+B^\top-\theta E+f\theta I$ is no longer Metzler, and the associated positivity properties may fail. Therefore, the closed-loop trajectory need not remain in the positive orthant, nor converge to the direction of a positive dominant eigenvector.

Lemma 3 shows more precisely how extraction affects the share dynamics. Relative to the no-extraction case in Lemma 2, the dominant eigenvector changes from ζ to ζ_θ, while the remaining generalized eigenvectors $v_2,\dots,v_n$ are unchanged. Only their eigenvalues are shifted, from λ_i to $\lambda_i+f\theta$, for $i\geq 2$.

To understand this fact, observe that the non-dominant modes correspond to pure redistributions of biomass across nodes at a given aggregate stock, namely to directions v such that $e^\top v=0$.

Since the feedback rule is

$$c_i = \theta\langle e, X\rangle, \qquad i \in F,$$

extraction depends only on aggregate biomass and is the same across active nodes. Hence it is insensitive to changes in spatial composition that leave total biomass unchanged. Formally, writing $E = \xi e^\top$, one has

$$Ev = \xi\, e^\top v = 0 \qquad \text{whenever } e^\top v = 0.$$

For this reason, the transitional redistributive directions remain those induced by migration alone. By contrast, the dominant eigenvector determines the stationary spatial composition, and this changes under extraction because harvesting creates a persistent sink on the active nodes.

Note also that $\mathrm{Re}\,\lambda_i + f\theta^*$ may become positive, so that 0 ceases to be the eigenvalue of maximal real part; in addition, the matrix $D + B - \theta E^\top + f\theta^* I$ may cease to be Metzler, so that without further assumptions we cannot establish whether ζ_θ remains a positive (dominant) eigenvector. Both issues disappear when players are sufficiently "patient", meaning that θ is small enough. To this extent, we set

$$\theta_1 \equiv \frac{|\lambda_2|}{f}$$

and note that $\theta < \theta_1$ if and only if $\lambda_2 + f\theta$ is smaller than the dominant eigenvalue 0. Moreover, since ζ_θ depends continuously on θ, and $\zeta > 0$, there exists $\theta_2 > 0$ such that ζ_θ is positive for all $\theta < \theta_2$. We then set

$$\theta_2 = \sup\{r : \zeta_\theta > 0,\ \forall \theta \in [0, r]\}.$$

This discussion therefore leads us to the first result concerning the admissibility of the candidate-optimal control identified in feedback form in Theorem 1.

Proposition 2. *Assume the hypotheses of Theorem 1 hold and suppose that* $0 < \theta < \min\{\theta_1, \theta_2\}$. *Let* X, Y *be the state variables described in (22). Then:*

(i) $\zeta_\theta > 0$ *and*

$$\lim_{t\to\infty} Y(t) = \frac{\zeta_\theta}{\langle e, \zeta_\theta\rangle}, \qquad \lim_{t\to\infty} X(t) = \hat{m}\,\frac{\zeta_\theta}{\langle e, \zeta_\theta\rangle},$$

where $\hat{m} = \lim_{t\to\infty} m(t)$ *is given in Proposition 1.*

(ii) There exists a constant L *such that, for every* x *in the cone*

$$C := \left\{ x \in \mathbb{R}^n_+ \ : \ \left| \frac{x}{\langle x, \mathbf{e}\rangle} - \frac{\zeta_\theta}{\langle \zeta_\theta, \mathbf{e}\rangle} \right| < L \right\},$$

the candidate-optimal trajectory is admissible and therefore optimal.

The proof of this fact, containing a more explicit estimate of the constant L, can be found in Appendix A.3.

Remark 6. The estimate of the constant L can be more specific. For instance, if we assume that $M := D + B^\top - \theta E + f\theta I$ is diagonalizable, then we can write $M = S\,\mathrm{diag}\,(0, \mu_2, \dots, \mu_n)S^{-1}$ for eigenvalues $\{\mu_i\}_{1\le i\le n}$, where $\mu_1 = 0$ is simple and $\mathrm{Re}\,(\mu_k) < 0$, for every $k \ge 2$, and matrix of the corresponding eigenvectors S. Then, in the proof of Proposition 2, $\eta := \min_{k\ge 2}\left(-\mathrm{Re}\,\mu_k\right) > 0$, $C := \|S\|_\infty \left\|S^{-1}\right\|_\infty$, and

$$L = \frac{\min_i \tilde{\zeta}_i}{2\|S\|_\infty \left\|S^{-1}\right\|_\infty}, \quad \textit{where } \tilde{\zeta} = \zeta_\theta / \langle e, \zeta_\theta\rangle.$$

The result of Proposition 2 ensures that, for a certain subset of initial data, the candidate-optimal trajectory is admissible. One may, on the other hand, ask whether there exist conditions under which the candidate-optimal trajectory is admissible starting from any initial data in $\mathbb{R}^+_n$. This question will be related to the conditions used in Section 4 to ensure that the game is well defined and the Nash equilibrium subgame-perfect.

The idea is that the points where one actually needs to check that the system remains positive are those on the boundary of $\mathbb{R}^+_n$. If, at such points, the direction of the trajectory predicted by the candidate-optimal control is inward, then there is no possibility of leaving the positive orthant. We only need to check the nodes with extraction, since for the others, when the stock is zero, there is no outflow of the resource. Hence, for every $i \in F$, it is required that Y_i', as predicted by the second equation in (22), be positive starting from any Y such that $Y_i = 0$ (and the other components non-negative, with at least one positive). Since the second equation in (22) is linear, this condition is equivalent to requiring that

$$\left\langle (D + B^\top - \theta E + f\theta I)e_j, e_i \right\rangle > 0$$

for every $j \ne i$. In other words, every element of the matrix $(D + B^\top - \theta E + f\theta I)$ must be positive for all $i \in F$ and $j \ne i$, that is,

$$(0 <)\theta^* < \min_{i\in F, j\ne i} b_{ij}$$

(so this implies, in particular, that $\min_{i\in F, j\ne i} b_{ij} > 0$).

We can summarize this discussion in the following proposition.

Proposition 3. *Suppose that the conditions of Theorem 1 are satisfied and that*

$$\theta^* < \min_{i\in F, j\neq i} b_{ij}.$$

Then the candidate-optimal trajectory is admissible for any initial data in $\mathbb{R}^n_+$ and is optimal.

4. Strategic interaction

In the investigation of the Nash equilibria of the game, we restrict our attention to stationary Markovian equilibria. Specifically, the extraction rates c_i of the agents are modeled as reaction maps to the observed stock level $X(t)$ at time t:

$$c(t) = \psi(X(t)), \quad \text{with } c_i(t) = \psi_i(X(t)),\ \forall i \in N$$

(where $\psi_i \equiv 0$ for all $i \in N \setminus F$). Consequently, the system evolves according to the *closed-loop equation* (CLE):

$$\begin{cases} \dot{X}(t) = \varphi(\langle X(t), \mathbf{e}\rangle)X(t) + (D + B^\top)X(t) - \psi(X(t)), \quad t > 0 \\ X(0) = x, \end{cases} \tag{24}$$

provided that such an equation admits a unique solution. More precisely,

$$\psi = (\psi_1, \psi_2, \dots, \psi_n), \quad \psi_i : \mathbb{R}^n_+ \to [0, +\infty).$$

We denote the set of admissible strategy profiles by

$$\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

where $\mathbb{A}_i$ represents the collection of all (re)actions ψ_i of Player i (or a null reaction at nodes with reserves). We denote the solution of Eq. (24) by $X(t; x, \psi)$. We also follow the convention of denoting by ψ_{-i} all components of ψ other than the i-th, so that $\psi = (\psi_i, \psi_{-i})$.

Definition 1 (Markovian Perfect Equilibrium)**.** We define a strategy profile $\psi \in \mathbb{A}$ as a *Markovian Perfect Equilibrium (MPE)* if, for all $x_0 \in \mathbb{R}^n_+$ and $i \in F$, the control $c_i(t) = \psi_i(X^{\psi,x_0}(t))$ is optimal for Player i's problem, where $c_j(t) = \psi_j(X(t))$ for every $j \neq i$, the nonnegative state and extraction rate constraints (4), and the discounted total payoff $J_i(c_i)$ given by (6), which is to be maximized over the set of admissible controls $\mathbb{A}_i$.

We focus on networks with the following connectivity structure: all nodes in F have positive inward edges from all other nodes: $b_{ij} > 0$, $\forall i \in F$, $j \neq i$.

We consider the following set of strategies $\mathbb{A}_i$ for player i, $i \in F$, given by

$$\mathbb{A}_i := \left\{ \psi_i : \mathbb{R}^n_+ \to [0, +\infty) \ : \ \begin{array}{c} (i)\ \psi_i \text{ is Lipschitz continuous} \\ (ii)\ \psi_i(x) \le \left\langle (D + B^\top)x, e_i \right\rangle \\ \text{for all } x \in \mathbb{R}^n_+ \text{ such that } x_i = 0. \end{array} \right\} \tag{25}$$

When $i \notin F$, we assume $\mathbb{A}_i$ contains only the null strategy. Note that the Lipschitz continuity of ψ_i implies that the CLE (24) has a unique solution, $X(t)$, whereas the condition (ii) ensures that, when the stock at node i is null, the extraction ψ_i can be, at most, as much as the inflow at i from the other nodes, so that the stock remains nonnegative at all times. For this reason a Markovian Nash equilibrium ψ within $\mathbb{A}$ is inherently subgame-perfect because if a player deviates (whether intentionally or by mistake) from ψ, they cannot exit the set $\mathbb{R}^n_+$, and the strategy profile ψ remains valid as a Nash equilibrium from any state reached within $\mathbb{R}^n_+$.

4.1. Existence of symmetric Markovian equilibria

In Theorem 2, we provide an existence result for a Markovian equilibrium (i.e. a Nash equilibrium in closed-loop form) for the game where all players extract the same resource quantity and have the same utility. Explicit formulas will also be provided. The general statement of the theorem will refer to any of the frameworks described in Table 2 below.

Table 2
The first two columns summarize the choice of the saturation and utility functions defining the chosen framework, while the last three specify the corresponding values of $\hat{\theta}$, A, and B to be used in the formulas contained in Theorem 2.

framework	$\varphi(m)$	$u(c)$	$\hat{\theta}$	A	B
(S1)(U1)	$\Gamma\left(1 - \frac{1}{K} m^{\sigma-1}\right)$	$-\frac{c^{1-\sigma}}{\sigma-1}$, $\sigma > 1$	$\frac{\rho+\Gamma(\sigma-1)}{1+f(\sigma-1)}$	$\hat{\theta}^{-\sigma}$	$-\frac{\Gamma\hat{\theta}^{-\sigma}}{K\rho}$
(S2)(U1)	$m^{\sigma-1} - \delta$	$\frac{c^{1-\sigma}}{1-\sigma}$, $\sigma < 1$	$\frac{\rho+\delta(1-\sigma)}{1-f(1-\sigma)}$	$\hat{\theta}^{-\sigma}$	$\frac{\hat{\theta}^{-\sigma}}{\rho}$
(S3)(U2)	$\Gamma\left(1 - \frac{1}{K}\log m\right)$	$\log c$	$\rho + \frac{\Gamma}{K}$	$\hat{\theta}^{-1}$	$\frac{\Gamma - f\hat{\theta} + \hat{\theta}\log\hat{\theta}}{\hat{\theta}\rho}$

Theorem 2 (Existence of Markovian Equilibria). *In each of the assumption frameworks described in Table 2, and assuming*[3]

$$0 < \hat{\theta} < \min_{i \in F,\, j \neq i} b_{ij},$$

(being b_{ij} the components of B) the following facts hold:

(i) the set of strategies ψ^* given by

$$\psi_i^* = 0, \quad \forall i \notin F \quad \text{and} \quad \psi_i^*(t) = \hat{\theta}\langle \mathbf{e}, X(t)\rangle, \ \forall i \in F; \tag{26}$$

is admissible (i.e. it belongs to $\mathbb{A}$)
(ii) the set of strategies ψ^* described above is a Markovian Perfect Equilibrium
(iii) the utility of each player at the equilibrium is the same

$$V^i(x) = V(x) := A\, u(\langle \mathbf{e}, x\rangle) + B \tag{27}$$

The proof of the theorem can be found in Appendix A.3.

We can compare extraction in the planner's case and in the decentralized game, and obtain a classic "tragedy of the commons" result. Indeed, if we denote by θ^* the per–site feedback under the planner (Table 1) and by $\hat{\theta}$ the per-site coefficient in the symmetric Markov equilibrium (Table 2), the difference in aggregate extraction is

$$\Delta_f := f\hat{\theta} - f\theta^*.$$

Under the interiority requirements of each framework, one obtains the closed forms:

$$\text{(S1)(U1)} \qquad \Delta_f = \frac{(\rho + \Gamma(\sigma-1))(f-1)}{\sigma\,[1 + f(\sigma-1)]},$$

$$\text{(S2)(U1)} \qquad \Delta_f = \frac{(\rho + \delta(1-\sigma))(f-1)}{\sigma\,[1 - f(1-\sigma)]},$$

$$\text{(S3)(U2)} \qquad \Delta_f = (f-1)\Big(\rho + \frac{\Gamma}{K}\Big),$$

which is positive in all three cases.

Similarly, let m^* denote the long-run stock under the planner's solution and let $\hat{m}$ denote the long-run stock in the decentralized game. We have in the steady state $\varphi(m) = f\hat{\theta}$ so, using the corresponding expressions for φ and $\hat{\theta}$, we obtain:

$$\text{(S1)(U1)} \quad \hat{m} = \left[\frac{K}{1+f(\sigma-1)}\Big(1 - \frac{f\rho}{\Gamma}\Big)\right]^{\frac{1}{\sigma-1}};$$

$$\text{(S2)(U1)} \quad \hat{m} = \left[\frac{\delta + f\rho}{1 - f(1-\sigma)}\right]^{\frac{1}{\sigma-1}};$$

$$\text{(S3)(U2)} \quad \hat{m} = \exp\Big(K - f - \frac{fK\rho}{\Gamma}\Big).$$

It is then easy to verify that, in all three cases, $\hat{m} < m^*$ for $f > 1$ (and they coincide for $f = 1$).

The direct comparison derived above between the centralized planner's solution and the non-cooperative Markov equilibrium provides a formal characterization of the spatial tragedy of the commons. Indeed, because the resource migrates across the network, an agent faces the risk that unharvested biomass will diffuse to rival nodes so that the competitive extraction policy is strictly more aggressive than the socially optimal one.

A direct consequence of the strategic over-extraction is the strict loss of social utility. The difference $\Delta W(x) := V^{SP}(x) - \sum_{i \in F} V^i(x) > 0$ can be expressed in closed form using the previous results. Under logarithmic utility (S3-U2), $\Delta W(x)$ reduces to a state-independent constant:

$$\Delta W(x) = \frac{f}{\rho}(f - 1 - \log f). \tag{28}$$

(which is strictly positive for $f > 1$).

In the power utility cases (S1-U1 and S2-U1), the loss is state-dependent:

$$\Delta W(x) = \Delta A\big[u(\langle e, x\rangle) + C\big],$$

with $C = -\frac{\Gamma}{K\rho}$ in (S1) and $C = \frac{1}{\rho}$ in (S2), where the multiplier $\Delta A := A^{SP} - fA^{NE} > 0$ is given by:

$$\Delta A = \frac{(\sigma f)^\sigma - f(1 + f(\sigma-1))^\sigma}{[\rho + \Gamma(\sigma-1)]^\sigma} \quad \text{for (S1),}$$

and

$$\Delta A = \frac{(\sigma f)^\sigma - f(1 - f(1-\sigma))^\sigma}{[\rho + \delta(1-\sigma)]^\sigma} \quad \text{for (S2).}$$

[3] Note that in order for $\hat{\theta}$ to be strictly positive, in framework (S2)(U1) it must be assumed that $f < \frac{1}{1-\sigma}$. Conversely, in frameworks (S1)(U1) and (S3)(U2) $\hat{\theta}$ is positive without further assumptions.

5. Conclusions

This paper develops a tractable framework to study the joint determination of spatial allocation and aggregate dynamics in a network model of a renewable resource. By combining aggregate nonlinear growth with migration across interconnected locations, the model delivers an explicit characterization of equilibrium outcomes in a spatial differential game. A central insight is that migration and biological growth play distinct roles: while nonlinear growth governs the evolution and steady state of aggregate biomass, migration determines how this biomass is distributed across space.

Beyond its theoretical contribution, the model is relevant for a broad class of spatially structured resource systems. Natural applications include fisheries with migration across fishing grounds, wildlife populations moving across ecological corridors, and interconnected environmental systems such as groundwater basins or grazing areas. The network structure provides a flexible representation of spatial linkages, while the distinction between exploited locations and reserves allows the analysis of spatial management policies within a unified framework.

An important implication of the model is that the long-run spatial distribution of the resource depends on the location of reserves within the network. This feature makes the framework particularly suitable for the design of policies aimed at regulating not only the aggregate level of the resource, but also its spatial allocation. In this respect, the model can be used to study how the placement of protected areas, conservation zones, or restricted-access nodes affects equilibrium outcomes through the interaction between migration flows and network structure. It therefore provides a tractable framework for evaluating spatial policies that seek to concentrate or disperse biomass across locations, depending on ecological or economic objectives.

Several extensions offer promising directions for future research. A first avenue is to relax the assumption of aggregate-dependent growth and allow for localized biological dynamics, bringing the model closer to many ecological applications. A second is to consider more general migration mechanisms, including nonlinear or density-dependent dispersal. A third extension is to introduce heterogeneity across nodes and agents, for example through location-specific productivity, harvesting costs, or institutional asymmetries.

Further developments could enrich the strategic environment by incorporating stochastic shocks, time-varying networks, or partial cooperation among agents. Another natural extension concerns the endogenous design of reserves and spatial connectivity, where a planner jointly determines harvesting rules and the structure of the network. More broadly, the analysis suggests that network-based approaches provide a fruitful route toward a richer theory of spatial differential games with mobile resources, particularly when one can identify specifications that preserve analytical tractability while capturing key ecological and economic mechanisms.

Declaration of competing interest

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Appendix A. Technical proofs

A.1. Proofs for Section 2

Proof of Lemma 1. The matrix $D + B$ has (at least) one zero eigenvalue. In fact, the sum of the elements along the rows is 0, which implies that the columns are linearly dependent: their sum is indeed the zero vector. Consequently, the vector $\mathbf{e}$ is a (positive) eigenvector associated to the eigenvalue 0. Note also that the Gershgorin Circle Theorem (Theorem 6.1.1 in Horn and Johnson, 2013) implies that all other eigenvalues λ_i satisfy $\mathrm{Re}\lambda_i \leq 0$. Now since the network is strongly connected and the matrix $D + B$ is Metzler, the Perron-Frobenius theorem implies $D + B$ has a dominant real eigenvalue λ associated with a dominant eigenvector w, such that $\lambda > \mathrm{Re}\lambda_i$ for all other eigenvalues λ_i, and with w the only eigenvector whose coordinates can be chosen all strictly positive. Then necessarily (10) holds with $\lambda = 0$, and w is proportional to $\mathbf{e}$. Since the transpose has the same eigenvalues, there exists a dominant eigenvector ζ for $D + B^T$ for the dominant eigenvalue 0. □

Proof of Lemma 2. The proof of the explicit formula for $Y(t)$ in (12) is a direct application of the variation of constants formula. The positivity of the Y_i's is a consequence of the matrix $D + B^\top$ being Metzler (see for example Farina and Rinaldi (2000)).

By Lemma 1, $D + B^\top$ and $(D + B^\top)^\top$ share the spectrum $\{0, \lambda_2, \dots, \lambda_n\}$ with $\mathrm{Re}\,\lambda_i < 0$ for $i \geq 2$. Let $\zeta \gg 0$ be the right eigenvector of $D + B^\top$ for eigenvalue 0, and note that $\mathbf{e}$ is a left eigenvector of $D + B^\top$ for eigenvalue 0. The spectral projection onto $\ker(D + B^\top)$ is

$$P = \frac{\zeta\, \mathbf{e}^\top}{\langle \mathbf{e}, \zeta \rangle}.$$

Since 0 is simple and all other eigenvalues have strictly negative real part, the Jordan decomposition (see, e.g., Horn and Johnson, 2013) yields

$$e^{(D+B^\top)t} = P + R(t), \qquad \|R(t)\| \le Ce^{-\eta t}$$

for some $C, \eta > 0$. Therefore,

$$e^{(D+B^\top)t}x \longrightarrow Px = \frac{\langle \mathbf{e}, x\rangle}{\langle \mathbf{e}, \zeta\rangle}\zeta, \quad \text{and thus} \quad Y(t) = \langle \mathbf{e}, x\rangle^{-1} e^{(D+B^\top)t}x \longrightarrow \langle \mathbf{e}, \zeta\rangle^{-1}\zeta,$$

which is (13).

To derive the formulas for $m(t)$, we apply the following change of variables: for (i) and (ii), $\mu(t) = m(t)^{1-\sigma}$; for (iii), $\mu(t) = \log(m(t))$. Hence note that $\mu(t)$ satisfies a linear equation that can be solved by the variation of constants formula, thus the thesis follows. □

A.2. Proof of Theorem 1. Verification and uniqueness for the HJB equation

For the proof of Theorem 1, we make use of several preliminary results, some of which bear independent mathematical interest, and which we collect in this section.

We start by making the HJB Eq. (15) explicit in the different assumption frameworks:

(a) if $\sigma \neq 1$, and (U1) (S1) hold, then

$$\rho v(x) = \frac{\sigma}{1-\sigma}\sum_{i\in F}\left(\frac{\partial v}{\partial x_i}(x)\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(x), \alpha(x)\rangle + \Gamma\big(1 - K^{-1}\langle x, \mathbf{e}\rangle^{\sigma-1}\big)\langle \nabla v(x), x\rangle$$

(b) if $\sigma \neq 1$, and (U1) (S2) hold,

$$\rho v(x) = \frac{\sigma}{1-\sigma}\sum_{i=1}^{f}\left(\frac{\partial v}{\partial x_i}(x)\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(x), \alpha(x)\rangle + \big(\langle x, \mathbf{e}\rangle^{\sigma-1} - \delta\big)\langle \nabla v(x), x\rangle$$

(c) if $\sigma = 1$, and (U2)(S3) hold,

$$\rho v(x) = -\sum_{i=1}^{f}\ln\left(\frac{\partial v}{\partial x_i}\right) - f + \langle \nabla v(x), \alpha(x)\rangle + \Gamma\big(1 - K^{-1}\ln(\langle x, \mathbf{e}\rangle)\big)\langle \nabla v(x), x\rangle$$

The next step is guessing a solution to the HJB equation.

Proposition 4. *In each of the assumption frameworks in Table 1*

$$w(x) = A\,u(\langle \mathbf{e}, x\rangle) + B. \tag{A.1}$$

is a classical solution in $\mathbb{R}^n_+ - \{0\}$ of the HJB Eq. (15).

Proof. Note that $\nabla w(x) = Au'(\langle x, \mathbf{e}\rangle)\mathbf{e}$. We plug such a formula into the associated HJB equation, use (16), and note that $(D + B)\mathbf{e} = 0$ implies

$$\langle \nabla w(x), \alpha(x)\rangle = Au'(\langle x, \mathbf{e}\rangle)\langle (D + B)\mathbf{e}, x\rangle = 0.$$

Thus, with easy but tedious calculations one finds that w is a solution to the HJB equation if and only if A and B are those described in Table 1. □

For the reader's convenience we state here the well-known fundamental identity on which verification techniques are based. To this extent, we set

$$h(c, p) = \sum_{i=1}^{f}\big(u(c_i) - c_i p_i\big), \quad H(p) = \sup_{c>0} h(c, p).$$

Lemma 4 (The Fundamental Identity)**.** *Assume $v(x)$ is any classical solution to the HJB Eq. (15). Then*

$$v(x) = \int_0^t \big[H(\nabla v(X(s))) - h(c(s), \nabla v(X(s)))\big]ds + \int_0^t e^{-\rho s}\sum_{i\in F}u(c_i(s))ds + e^{-\rho t}v(X(t)),$$

for every t, for every control $c \in \mathbb{A}$ and trajectory $X(t)$.

Proof. One has

$$\begin{aligned}\frac{d}{ds}(e^{-\rho s}v(X(s))) &= -\rho e^{-\rho s}v(X(s)) + e^{-\rho s}\langle \nabla v(X(s)), \dot{X}(s)\rangle \\ &= e^{-\rho s}[-H(\nabla v(X(s))) + h(c(s), \nabla v(X(s)))] - e^{-\rho s}\sum_{i\in F}u(c_i(s))\end{aligned}$$

and, integrating on $[0, t]$, obtains the sought identity. □

In the theorems below, we implement verification techniques, proving alongside the uniqueness of the solution of the HJB Eq. (15). We first note that there exists a constant $C > 0$ such that, for every initial condition $x \in \mathbb{R}^n - \{0\}$ and for every admissible control $c \in \mathbb{A}$, the associated trajectory $X(t) = X(t; x, c)$ satisfies

$$|X(t)| \leq C(1 + |x|). \tag{A.2}$$

Indeed, the solution $m(t)$ of (9) and the corresponding no-extraction path $m_0(t)$, defined in Lemma 2 satisfy

$$m(t) \leq m_0(t) \leq \max\left\{\sum_i x_i, \bar{m}\right\} \leq C(1 + |x|) \tag{A.3}$$

where the second inequality is implied by the monotonic convergence, from above or below, of $m_0(t)$ starting at $m_0(0) = \sum_i x_i$ to the long-run mass $\bar{m}$ (where in (S1), $\bar{m} = K^{\frac{1}{\sigma-1}}$; in (S2), $\bar{m} = \delta^{-\frac{1}{\sigma-1}}$); in (S3), $\bar{m} = e^K$, while the last inequality is true with $C = \sqrt{n}\max\{1, \bar{m}\}$. Hence, since $X_i(t) \geq 0$ for all i,

$$|X(t)| \leq \sum_{i=1}^{n} X_i(t) = m(t) \leq C(1 + |x|).$$

Theorem 3. *Consider the framework* $(U1)(S2)$.

(i) The value function $V(x)$ is the unique classical solution of the HJB Eq. (15) in the set $C = \{v \in C^1(\mathbb{R}^n_+ - \{0\}) \cap C^0(\mathbb{R}^n_+)\}$.
(ii) Consequently, $V(x) = w(x) \equiv Au(\langle x, \mathbf{e}\rangle) + B$, where $w(x)$ is given by (A.1), and the unique optimal feedback map is given by (17).

Proof. Let $v \in C$ be any solution of (15). Let x be any initial condition, $c \in \mathbb{A}$ any admissible control, and $X(t)$ the associated trajectory. Using (A.2) and the fact that v is locally bounded on $\mathbb{R}^n_+$, we have $|v(X(t))| \leq C_x$, for all $t \geq 0$. It follows

$$e^{-\rho t} v(X(t)) \xrightarrow{t\to\infty} 0. \tag{A.4}$$

Since the fundamental identity applies and $H(p) \geq h(p)$ for every $p \in \mathbb{R}^n$, we have

$$v(x) \geq \int_0^t e^{-\rho s} \sum_{i\in F} u(c_i(s))ds + e^{-\rho t} v(X(t)), \quad \forall t \geq 0$$

so that, letting $t \to \infty$ and using (A.4), we obtain

$$v(x) \geq \int_0^\infty e^{-\rho s} \sum_{i=1}^{f} u(c_i(s))ds = J(c; x) \tag{A.5}$$

implying $v(x) \geq V(x)$. On the other hand, $H(\nabla v(x)) - h(\nabla v(x), c) = 0$ if and only if $c_i(x) = c_i^*(x)$ where c_i^* is given by (16). If we set $X^*(t) = X(t; x, c^*)$ then

$$v(x) = \int_0^t e^{-\rho s} \sum_{i=1}^{f} u(c_i^*(s))ds + e^{-\rho t} v(X^*(t)), \tag{A.6}$$

and letting $t \to \infty$ we have

$$v(x) = \int_0^\infty e^{-\rho s} \sum_{i=1}^{f} u(c_i^*(s))ds = J\left(c^*; x\right). \tag{A.7}$$

This implies that c^* is optimal and $v(x) = V(x)$. In particular, we have uniqueness of the classical solution to the HJB equation in the set C. Since $w(x)$ defined by (A.1) is a solution in C, then $v(x) = V(x) = w(x)$, and c^* is an optimal feedback control.

Moreover, c^* is the unique optimal control. Indeed, let $(\tilde{c}, \tilde{X})$ be another optimal couple, where $\tilde{X}(t) = X(t; x, \tilde{c})$. Then $V(x) = J(\tilde{c}; x)$. Now choose $v = V, X = \tilde{X}, c = \tilde{c}$ in the fundamental identity, let $t \to \infty$, and use (A.4). We derive

$$\int_0^\infty \left[H(\nabla V(\tilde{X}(s))) - h(\tilde{c}(s), \nabla V(\tilde{X}(s)))\right]ds = 0,$$

and, since $H(p) \geq h(p, c)$, this implies

$$H(\nabla V(\tilde{X}(s))) - h(\tilde{c}(s), \nabla V(\tilde{X}(s))) = 0, \quad \textit{for a.e. } s \geq 0.$$

But then $(\tilde{c}, \tilde{X})$ must satisfy (16) with $v = V$, so that $\tilde{c}$ has to coincide with the unique maximizer c^*. □

Our target set of solutions v to the HJB Eq. (15) changes in frameworks (U1)(S1) and (U2)(S3). We will make the following assumptions on v:

(V1) For any admissible couple (X, c) such that $J(x, c) > -\infty$, it is

$$\limsup_{t\to\infty} e^{-\rho t} v(X(t)) \geq 0, \tag{A.8}$$

(V2) If (X^*, c^*) is the admissible couple satisfying (16), then

$$\lim_{t\to\infty} e^{-\rho t} v(X^*(t)) = 0, \tag{A.9}$$

Theorem 4. *Consider the frameworks* $(U1)(S1)$ *and* $(U2)(S3)$.

(i) The value function $V(x)$ *is the unique classical solution of the HJB Eq. (15) in the set*

$$\tilde{C} = \{v \in C^1(\mathbb{R}^n_+ - \{0\}) : (V1) \text{ and } (V2) \text{ hold}\}.$$

(ii) Consequently, $V(x) = w(x) \equiv Au(\langle x, \mathbf{e}\rangle) + B$, *where* $w(x)$ *is given by (A.1), and the unique optimal feedback map is given by (17).*

Proof. Let v be any classical solution of the HJB equation in $\tilde{C}$. By proceeding as in the proof of Theorem 3, but using (A.8), (A.9) in place of (A.4), we get that $v = V$ and that the control $c^* = \psi(X^*(\cdot))$ and the associated trajectory X^*, as defined respectively in (17) and (21), are optimal. Notice that if $J(x,c) = -\infty$, we always have the trivial inequality $v(x) \geq -\infty = J(x,c)$.

Then, to prove that the value function can be expressed as the solution $w(x)$ of the HJB Eq. (A.1), it is enough to show that $w \in \tilde{C}$. Clearly (V2) holds as, for m^* given by Lemma 1 with $\theta = \theta^*$, one has

$$w(X^*(t)) = Au(m^*(t)) + B \xrightarrow{t\to\infty} Au(\hat{m}) + B.$$

Next we prove that $w(x)$ satisfies (V1) in the case of framework (U1)(S1) (the case (U2)(S3) can be treated similarly). We consider an initial state x and an admissible control $c(t)$, with associated state $X(t)$ and mass $m(t) = \langle X(t), \mathbf{e}\rangle$, such that $J(x,c) > -\infty$, and prove

$$\limsup_{t\to\infty} e^{-\rho t}\Big(\frac{A}{1-\sigma}m(t) + B\Big) \geq 0 \iff \liminf_{t\to\infty} e^{-\rho t} m(t) \leq 0. \tag{A.10}$$

with $\sigma > 1$ and A, B as in Table 1. We start by rewriting $J(x,c)$ as

$$\begin{aligned} J(x,c) &= -\frac{1}{\sigma-1}\sum_{T=0}^{\infty}\int_T^{T+1} e^{-\rho t}\sum_{i\in F}(c_i(t))^{1-\sigma}dt \leq -\frac{f^\sigma}{\sigma-1}\sum_{T=0}^{\infty}\int_T^{T+1} e^{-\rho t}\Big(\sum_{i\in F} c_i(t)\Big)^{1-\sigma}dt \\ &\leq -\frac{f^\sigma}{\sigma-1}\sum_{T=0}^{\infty}\delta_T\Big(\int_T^{T+1} e^{-\rho t}\delta_T^{-1}\sum_{i\in F} c_i(t)dt\Big)^{1-\sigma} = -\frac{f^\sigma}{\sigma-1}\sum_{T=0}^{\infty}\delta_T\Big(\int_T^{T+1} e^{-\rho t}\delta_T^{-1}\sum_{i\in F} c_i(t)dt\Big)^{1-\sigma} \end{aligned}$$

where the first inequality follows by Jensen's inequality and in the last one we have again used Jensen's inequality with unitary measure on $[T, T+1]$ of density $f(t) = e^{-\rho t}{\delta_T}^{-1}$ with $\delta_T = \int_T^{T+1} e^{-\rho t}dt = \frac{e^\rho - 1}{\rho e^\rho}e^{-\rho T}$.

Now, integrating the equation for $m(t)$ in (9) over $[T, T+1]$, and using the fact that $m(t) \geq 0$ and $\varphi(m(t)) \leq \Gamma$, we derive

$$\begin{aligned} \int_T^{T+1}\langle c(t), \mathbf{e}\rangle dt &\leq \int_T^{T+1}\varphi(m(t))m(t)dt - m(T+1) + m(T) \\ &\leq m(T) + \Gamma\int_T^{T+1} m(t)dt \leq m(T) + (e^\Gamma - 1)m(T) = e^\Gamma m(T), \end{aligned}$$

where in the last inequality we used $m'(t) \leq \Gamma m(t)$ and Gronwall's Lemma on $[T, T+1]$. Combining the two chains of inequalities above, and recalling that $c_i = 0$ for all $i \notin F$, and that $x \mapsto x^{1-\sigma}$ with $\sigma > 1$ is decreasing, we obtain

$$\begin{aligned} J(x,c) &\leq -\frac{f^\sigma}{\sigma-1}\sum_{T=0}^{\infty}\delta_T^\sigma\Big(e^{-\rho T}\int_T^{T+1}\langle c(t), \mathbf{e}\rangle dt\Big)^{1-\sigma} \leq -\frac{f^\sigma}{\sigma-1}\sum_{T=0}^{\infty}\delta_T^\sigma\big(e^{-\rho T}e^\Gamma m(T)\big)^{1-\sigma} \\ &= -\frac{e^{\Gamma(1-\sigma)}}{\sigma-1}\Big(\frac{f(e^\rho-1)}{\rho e^\rho}\Big)^\sigma\sum_{T=0}^{\infty} e^{-\rho T}(m(T))^{1-\sigma}. \end{aligned}$$

Note that the series above needs to be finite (and positive) as $J(x,c) > -\infty$. Thus, necessarily

$$\lim_{T\to\infty} e^{-\rho T}(m(T))^{1-\sigma} = 0,$$

and the proof of (V1) is complete.

Lastly, we prove that c^* is the unique optimal control. Indeed, let $(\tilde{c}, \tilde{X})$ be another optimal couple, where $\tilde{X}(t) = X(t; x, \tilde{c})$. Then $V(x) = J(\tilde{c}; x)$. Now choose $v = V, X = \tilde{X}, c = \tilde{c}$ in the fundamental identity, take $\limsup_{t\to\infty}$, and use (A.8), then

$$\int_0^\infty \big[H(\nabla V(\tilde{X}(s))) - h(\tilde{c}(s), \nabla V(\tilde{X}(s)))\big]ds \leq 0.$$

Thus, we must have $\tilde{c} = c^*$. □

A.3. Other proofs

Proof of Lemma 3. The matrix E can also be written as the product of the two vector matrices $\xi = \sum_{i\in F} e_i$ (where $\{e_i\}$ is the canonical base in $\mathbb{R}^n$) and $\mathbf{e}^\top$, namely $E = \xi\mathbf{e}^\top$. Moreover, the range of $E^\top$ is $\mathrm{Im}E = \{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$, $\mathbf{e}$ is an eigenvector of $E^\top$ with eigenvalue $\langle \mathbf{e}, \xi\rangle = f$. Thus $\mathbf{e}$ is an eigenvector of $D + B - \theta E^\top + f\theta I$ with eigenvalue 0. This implies the existence of an eigenvector ζ_θ of the transpose matrix associated to the same eigenvalue 0, and (i) is proved.

To prove (*ii*) we first show that any generalized eigenvector v of $D + B - \theta E^\top + f\theta I$ with an eigenvalue $\lambda_i \neq 0$ is orthogonal to $\mathbf{e}$. Thus, consider an eigenvalue $\lambda_i \neq 0$ ($i \geq 2$) and v_i as an element of the generalized eigenspace $V(\lambda_i)$. Hence, there exists a strictly positive integer m such that $(D + B - \theta E^\top + f\theta I - \lambda_i I)^m v_i = 0$. Therefore,

$$0 = \mathbf{e}^\top \left[(D + B - \theta E^\top + f\theta I - \lambda_i I)^m v_i\right] = (0 - \lambda_i)^m \mathbf{e}^\top v_i,$$

and given that $\lambda_i \neq 0$, then $\mathbf{e}^\top v_i = 0$, implying e is orthogonal to v_i.
Thus, since $Ev_i = \xi\langle \mathbf{e}, v_i\rangle = 0$, v_i is also a generalized eigenvector for $D + B - \theta E^\top + f\theta I$ associated with the eigenvalue λ_i, implying the second assertion. □

Proof of Proposition 2.

(i) As discussed immediately before the proposition, we know that for $0 < \theta < \min\{\theta_1, \theta_2\}$ the matrix $M := D + B^\top - \theta E + f\theta I$ has a simple eigenvalue 0 with left eigenvector e and right eigenvector $\zeta_\theta \gg 0$, and all other eigenvalues have strictly negative real part. Hence the direction of the trajectory converges to the (unique) dominant eigenvector: $Y(t) \to \zeta_\theta/\langle e, \zeta_\theta\rangle$, and ζ_θ is strictly positive.
(ii) We look at $Y(t)$. Note first that if $Y(0) = \zeta_\theta/\langle e, \zeta_\theta\rangle$, then $Y(t) \equiv \zeta_\theta/\langle e, \zeta_\theta\rangle$ for all $t \geq 0$.

Let $S := \{y \in \mathbb{R}^n : \langle e, y\rangle = 1\}$ and set $\tilde{\zeta} := \zeta_\theta/\langle e, \zeta_\theta\rangle$. Since M has a simple eigenvalue 0 with left eigenvector e and right eigenvector $\zeta_\theta \gg 0$, while all other eigenvalues have strictly negative real part, defining $P := \tilde{\zeta} e^\top$ for the projector onto the direction of ζ_θ (with $e^\top \tilde{\zeta} = 1$), there exists $C, \eta > 0$ depending on the asymptotic properties of the matrix M such that

$$\|e^{Mt} - P\|_\infty \leq Ce^{-\eta t} \qquad \text{for all } t \geq 0.$$

Since $M\tilde{\zeta} = 0$, we have $e^{Mt}\tilde{\zeta} = \tilde{\zeta}$; together with $P(y - \tilde{\zeta}) = \tilde{\zeta} e^\top (y - \tilde{\zeta}) = \tilde{\zeta}\left(e^\top y - e^\top \tilde{\zeta}\right) = 0$ for $y \in S$, it implies $e^{Mt} y = \tilde{\zeta} + \left(e^{Mt} - P\right)\left(y - \tilde{\zeta}\right)$, hence

$$\left\|\tilde{\zeta} - e^{Mt} y\right\|_\infty \leq C\|y - \tilde{\zeta}\|_\infty,$$

So if $C\|y - \tilde{\zeta}\|_\infty \leq \hat{\zeta}/2$, where $\hat{\zeta} := \min_i \tilde{\zeta}_i > 0$, we get $\left(e^{Mt} y\right)_i \geq \hat{\zeta}/2 > 0$ for all i, t. Therefore taking $K_L := \{y \in S : \|y - \tilde{\zeta}\| \leq L\}$, for $L := \hat{\zeta}/(2C)$, one obtains $e^{Mt} y \in \mathbb{R}^n_{++}$ for all $t \geq 0$ and all $y \in K_L$, which is exactly the cone condition stated for x (via $Y(0) = x/\langle x, e\rangle$). The claim follows. □

Proof of Theorem 2. Regarding (*i*), the Lipschitz continuity is obvious since ψ^* is linear. For the boundary inequality we proceed as follows. If i is not in F then $\psi_i^* = 0$ and there the inequality is verified. So fix $i \in F$ and $x \in \mathbb{R}^n_+$ with $x_i = 0$. Denote by $\underline{b}$ the $min_{i \neq j} b_{ij}$,

$$\left\langle (D + B^\top)x, e_i \right\rangle = \sum_{j \neq i} b_{ji} x_j - \Big(\sum_{j \neq i} b_{ij} \Big) x_i = \sum_{j \neq i} b_{ji} x_j \;\geq\; \underline{b} \sum_{j \neq i} x_j.$$

Since $x_i = 0$, $\sum_{j \neq i} x_j = \langle \mathbf{e}, x\rangle$. Therefore,

$$\left\langle (D + B^\top)x, e_i \right\rangle \;\geq\; \underline{b}\,\langle \mathbf{e}, x\rangle \;>\; \hat{\theta}\,\langle \mathbf{e}, x\rangle \;=\; \psi_i(x),$$

where the strict inequality is implied by $0 < \hat{\theta} < \underline{b}$ since $\langle \mathbf{e}, x\rangle > 0$.

Now we prove (*ii*) and (*iii*).

Fix $x \in \mathbb{R}^n_+$. To show the claim, it is enough to suppose all players $j \in F$ other than i use the affine stationary feedback $\psi_j(x) = \hat{\theta}\,\langle \mathbf{e}, x\rangle = \hat{\theta}\, m$, where $\hat{\theta}$ is as in Table 2, to show that player i's best response is $\psi_i(x) = \hat{\theta} m$, and that the associated value function has the form $V^i(x) = A\,u(m) + B$, with the corresponding constants A, B in Table 2.

Let us consider the problem of player i against ψ_{-i}, and denote by v^i the value function of player i. The HJB reads

$$\rho\, v^i(x) = \max_{c_i \geq 0} \left\{ u(c_i) - c_i\, \partial_{x_i} v^i(x) \right\} + \left\langle \nabla v^i(x), (D + B^\top)x + \varphi(\langle \mathbf{e}, x\rangle)\, x \right\rangle - \sum_{j \in F \setminus \{i\}} \partial_{x_j} v^i(x)\, \hat{\theta}\, \langle \mathbf{e}, x\rangle. \tag{A.11}$$

We verify that, for a suitable choice of the constants A and B, the function

$$v^i(x) = A\,u(\langle \mathbf{e}, x\rangle) + B = A\,u(m) + B$$

is a solution of such an equation. Indeed, we have

$$\nabla v^i(x) = A\,u'(m)\,\mathbf{e}, \quad p_i := \partial_{x_i} v^i = A\,u'(m).$$

So the above HJB Eq. (A.11) is rewritten as

$$\rho\, v^i(x) = \max_{c_i \geq 0} \{u(c_i) - c_i\, p_i\} + \left\langle \nabla v^i, (D + B^\top)x + \varphi(m)\, x \right\rangle - \sum_{j \in F \setminus \{i\}} \partial_{x_j} v^i\, \hat{\theta}\, m. \tag{A.12}$$

Since $(D + B)\mathbf{e} = \mathbf{0}$, we have $\langle \nabla v^i, (D + B^\top)x\rangle = 0$, and

$$\left\langle \nabla v^i, \varphi(m)\, x \right\rangle = A\,u'(m)\,\varphi(m)\, m, \qquad \sum_{j \neq i} \partial_{x_j} v^i\, \hat{\theta}\, m = (f - 1)\, A\,u'(m)\,\hat{\theta}\, m.$$

All in all, in each case we need to prove that

$$\rho\left[A\,u(m) + B\right] \;=\; H\left(A\,u'(m)\right) \;+\; A\,u'(m)\,\varphi(m)\, m \;-\; (f - 1)\, A\,u'(m)\,\hat{\theta}\, m, \tag{A.13}$$

where $H(p) = \sup_{c \geq 0}\{u(c) - cp\}$. We remark that the maximizer is $c_i^* = (u')^{-1}(p_i)$.

We now verify, case by case, that:

(a) $c_i^*(x) = \hat{\theta}\, m$ (best response exactly as claimed);
(b) the identity (A.13) holds with $v^i(x) = A\, u(m) + B$,

using the given $(\hat{\theta}, A, B)$ from Table 2.

Case (U1)-(S1): $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$, $\sigma \neq 1$ and $\varphi(m) = \Gamma\Big(1 - \frac{1}{K} m^{\sigma-1}\Big)$.
Here $u'(c) = c^{-\sigma}$, $(u')^{-1}(p) = p^{-1/\sigma}$. Then

$$c_i^* = (A\, m^{-\sigma})^{-1/\sigma} = A^{-1/\sigma} m.$$

With the prescribed $A = \hat{\theta}^{-\sigma}$ we get *(a)* $c_i^* = \hat{\theta}\, m$. The maximized Hamiltonian equals $H(p_i) = \frac{\sigma}{1-\sigma} p_i^{1-1/\sigma} = \frac{\sigma}{1-\sigma} A^{1-\frac{1}{\sigma}} m^{1-\sigma} = \frac{\sigma}{1-\sigma} \hat{\theta}^{1-\sigma} m^{1-\sigma}$. Moreover

$$A\, u'(m)\, \varphi(m)\, m = A\, m^{-\sigma} \Big[\Gamma m - \frac{\Gamma}{K} m^{\sigma}\Big] = A\, \Gamma\, m^{1-\sigma} - \frac{A\Gamma}{K},$$

and

$$(f-1)\, A\, u'(m)\, \hat{\theta} m = (f-1)\, A\, \hat{\theta}\, m^{1-\sigma}.$$

The HJB (A.12) reduces to two scalar identities:

$$\frac{\rho A}{1-\sigma} = \frac{\sigma}{1-\sigma} \hat{\theta}^{1-\sigma} + A\Gamma - (f-1) A \hat{\theta} \quad \text{(coeff. of } m^{1-\sigma}\text{)},$$

and

$$\rho B = -\frac{A\Gamma}{K} \quad \text{(constant term)}.$$

Substituting $A = \hat{\theta}^{-\sigma}$ and $\hat{\theta} = \dfrac{\rho + \Gamma(\sigma-1)}{1 + f(\sigma-1)}$ (Table 2) both equalities hold identically, hence (A.12) is satisfied and (b) follows.

Case (U1) - (S2): $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$, $\sigma \in (0,1)$; and $\varphi(m) = m^{\sigma-1} - \delta$. Again $(u')^{-1}(p) = p^{-1/\sigma}$, hence with $A = \hat{\theta}^{-\sigma}$ we get *(a)* $c_i^* = \hat{\theta}\, m$. Here

$$H(p_i) = \frac{\sigma}{1-\sigma} \hat{\theta}^{1-\sigma} m^{1-\sigma},$$

$$A\, u'(m)\, \varphi(m)\, m = A\, [1 - \delta\, m^{1-\sigma}],$$

and

$$(f-1)\, A\, u'(m)\, \hat{\theta} m = (f-1)\, A\, \hat{\theta}\, m^{1-\sigma}.$$

Thus (A.12) is equivalent to the system of the two following equations:

$$\frac{\rho A}{1-\sigma} = \frac{\sigma}{1-\sigma} \hat{\theta}^{1-\sigma} - \delta A - (f-1) A \hat{\theta} \quad \text{(from the coeff. of } m^{1-\sigma}\text{)},$$

$$\rho B = A \quad \text{(from the constant term)}.$$

Substituting $A = \hat{\theta}^{-\sigma}$ and $\hat{\theta} = \dfrac{\rho + \delta(1-\sigma)}{1 - f(1-\sigma)}$ (Table 2) the two identities are verified, hence (b) holds.

Case (U2) - (S3): $u(c) = \ln c$; $\varphi(m) = \Gamma\Big(1 - \frac{1}{K} \ln m\Big)$.
Here $u'(c) = 1/c$, $(u')^{-1}(p) = 1/p$, so $c_i^* = \dfrac{m}{A} = \hat{\theta}\, m$ with the prescribed $A = \hat{\theta}^{-1}$, giving *(a)*.
Moreover

$$H(p_i) = -\ln p_i - 1 = \ln m - \ln A - 1,$$

$$A\, u'(m)\, \varphi(m)\, m = A\, \Gamma - \frac{A\Gamma}{K} \ln m,$$

and

$$\sum_{j \neq i} \partial_{x_j} v^i\, \hat{\theta} m = (f-1)\, A\, \hat{\theta} = (f-1).$$

Matching the coefficients of $\ln m$ and the constants in (A.12) yields

$$\rho A = 1 - \frac{A\Gamma}{K} \quad \text{(coeff. of } \ln m\text{)}, \qquad \rho B = -\ln A - 1 + A\Gamma - (f-1) \quad \text{(constant term)}.$$

With $A = \hat{\theta}^{-1}$ and $\hat{\theta} = \rho + \Gamma/K$ (Table 2), both identities are satisfied, hence (b) holds.

Verification and conclusion. In each case above, the candidate $v^i(x) = A\, u(\langle \mathbf{e}, x \rangle) + B$ with the corresponding A, B solves (A.11) with maximizer $c_i^*(x) = (u')^{-1}(\partial_{x_i} v^i(x)) = \hat{\theta}\, \langle \mathbf{e}, x \rangle$. Arguing exactly as in Theorems 3 and 4 (applied with $(V1)$–$(V2)$ where needed) this v^i

coincides with the value function and c_i^* is the unique optimal Markovian feedback for player i against ψ_{-i}. Since i was arbitrary and all players share the same primitives, the strategy profile

$$\psi_i^*(x) = \begin{cases} \hat{\theta}\,\langle \mathbf{e}, x\rangle, & i \in F, \\ 0, & i \notin F, \end{cases}$$

is a Markovian equilibrium, and all players share the same value $V^i(x) = A\,u(\langle \mathbf{e}, x\rangle) + B$. This proves items (i)–(ii). □

References

Bhat, M.G., Huffaker, R.G., 2007. Management of a transboundary wildlife population: a self-enforcing cooperative agreement with renegotiation and variable transfer payments. J. Environ. Econ. Manag. 53 (1), 54–67.

Boucekkine, R., Camacho, C., Fabbri, G., 2013. Spatial dynamics and convergence: the spatial ak model. J. Econ. Theory 148 (6), 2719–2736.

Bressan, A., Nguyen, K.T., 2018. Stability of feedback solutions for infinite horizon noncooperative differential games. Dyn. Games Appl. 8 (1), 42–78.

Bressan, A., Priuli, F.S., 2006. Infinite horizon noncooperative differential games. J. Differ. Equ. 227, 230–257.

Brock, W.A., Xepapadeas, A., Yannacopoulos, A.N., 2014. Optimal control in space and time and the management of environmental resources. Annu. Rev. Resour. Econ. 6, 33–68.

Calvia, A., Gozzi, F., Leocata, M., Papayiannis, G.I., Xepapadeas, A., Yannacopoulos, A.N., 2024. An optimal control problem with state constraints in a spatio-temporal economic growth model on networks. J. Math. Econ. 113, 102991.

İlkı lıç, R., 2011. Networks of common property resources. Econ. Theory 47 (1), 105–134.

Clark, C.W., 2010. Mathematical Bioeconomics: The Mathematics of Conservation. Wiley-Blackwell, Hoboken, NJ. 3rd ed.

Clemhout, S., Wan, H.Y., 1985. Dynamic common property resources and environmental problems. J. Optim. Theory Appl. 46 (4), 471–481.

Clemhout, S., Wan, H.Y., 1994. Differential games – economic applications. In: Aumann, R., Hart, S., (Eds.), Handbook of Game Theory. Elsevier, Amsterdam. Vol. 2, pp. 801–825.

Costello, C., Kaffine, D., 2018. Natural resource federalism: preferences versus connectivity for patchy resources. Environ. Resour. Econ. 71 (1), 99–126.

Costello, C., Polasky, S., 2008. Optimal harvesting of stochastic spatial resources. J. Environ. Econ. Manag. 56, 1–18.

Costello, C., Quérou, N., Tomini, A., 2015. Partial enclosure of the commons. J. Public Econ. 121, 69–78.

Courchamp, F., Berec, L., Gascoigne, J., 2008. Allee Effects in Ecology and Conservation. Oxford University Press, Oxford.

Fabbri, G., Faggian, S., Freni, G., 2020. Policy effectiveness in spatial resource wars: a two-region model. J. Econ. Dyn. Control 111, 103818.

Fabbri, G., Faggian, S., Freni, G., 2024. On competition for spatially distributed resources in networks. Theor. Econ. 19 (2), 743–781.

Fabbri, G., Faggian, S., Freni, G., 2025. Growth models with externalities on networks. Decis. Econ. Finance 48 (1), 415–436.

Farina, L., Rinaldi, S., 2000. Positive Linear Systems: Theory and Applications. Wiley-Interscience, New York.

Gaudet, G., Lohoues, H., 2008. On limits to the use of linear markov strategies in common property natural resource games. Environ. Model. Assess. 13 (4), 567–574.

Gordon, H.S., 1954. The economic theory of a common-property resource: the fishery. J. Polit. Econ. 62 (2), 124–142.

Herrera, G.E., Moeller, H.V., Neubert, M.G., 2016. High-seas fish wars generate marine reserves. PNAS 113 (14), 3767–3772.

Hilborn, R., Walters, C.J., 1992. Quantitative Fisheries Stock Assessment: Choice, Dynamics and Uncertainty. Chapman & Hall, London.

Horn, R.A., Johnson, C.R., 2013. Matrix Analysis. Cambridge University Press, Cambridge. 2nd ed.

Kaffine, D.T., Costello, C., 2011. Unitization of spatially connected renewable resources. B. E. J. Econ. Anal. Policy 11 (1), 1–29.

Kaitala, V., 1986. Game theory models of fisheries management – a survey. In: Beckmann, M., Krelle, W., (Eds.), Dynamic Games and Applications in Economics. Springer, Berlin. Vol. 265 of Lecture Notes in Economics and Mathematical Systems, pp. 252–266.

Levhari, D., Mirman, L.J., 1980. The great fish war: an example using a dynamic cournot-nash solution. Bell J. Econ. 11 (1), 322–334.

Murray, J.D., 2003. Mathematical Biology II: Spatial Models and Biomedical Applications. Springer, New York. 3rd ed.

Officer, C.B., Smayda, T.J., Mann, R., 1982. Benthic filter feeding: a natural eutrophication control. Mar. Ecol. Prog. Ser. 9, 203–210.

Okubo, A., Levin, S.A., 2001. Diffusion and Ecological Problems: Modern Perspectives. Springer, New York. 2nd ed.

Plourde, C., Yeung, D., 1989. Harvesting of a transboundary replenishable fish stock: a noncooperative game solution. Mar. Resour. Econ. 6 (1), 57–70.

Quinn, T.P., Dittman, A.H., 1990. Pacific salmon migrations and homing: mechanisms and adaptive significance. Trends Ecol. Evol. 5 (6), 174–177.

Sanchirico, J.N., Wilen, J.E., 1999. Bioeconomics of spatial exploitation in a patchy environment. J. Environ. Econ. Manag. 37 (2), 129–150.

Sanchirico, J.N., Wilen, J.E., 2005. Optimal spatial management of renewable resources: matching policy scope to ecosystem scale. J. Environ. Econ. Manag. 50, 23–46.

Schaefer, M.B., 1954. Some aspects of the dynamics of populations important to the management of commercial marine fisheries. Inter-Am. Trop. Tuna Comm. Bull. 1 (2), 27–56.

Schaefer, M.B., 1957. Some considerations of population dynamics and economics in relation to the management of commercial marine fisheries. J. Fish. Res. Board Can. 14 (5), 669–681.

Smith, M.D., Sanchirico, J.N., Wilen, J.E., 2009. The economics of spatial-dynamic processes: applications to renewable resources. J. Environ. Econ. Manag. 57, 104–121.

Viana, D.F., 2019. Management of mobile species with spatial property rights: challenges and opportunities. Fish. Res. 218, 29–34.

Xepapadeas, A., 2010. The spatial dimension in environmental and resource economics. Environ. Dev. Econ. 15 (6), 747–758.