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The equilibrium-value convergence for the multiple-partners game ☆

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ABSTRACT

We study the *assignment game* (Shapley and Shubik, 1972) and its generalization of the *multiple-partners game* (Sotomayor, 1992), the simplest many-to-many extension. Our main result is that the Shapley value of a replicated multiple-partners game converges to a competitive equilibrium payoff when the number of replicas tends to infinity. The result also holds for a large subclass of semivalues since we prove that they converge to the same value as the replica becomes large. Furthermore, in supermodular and monotonic assignment games, the asymptotic Shapley value coincides with the mean stable imputation. The proof of our theorem relies on Hall's theorem.

1. Introduction

Shapley and Shubik (1972) introduced the *assignment game*, a two-sided matching market, where a set of possibly heterogeneous players from one side meets with another set of possibly heterogeneous players from the other side. If two players form a partnership, they enjoy a surplus, which can be divided between them in any manner they decide. Sotomayor (1992) proposed the simplest many-to-many generalization of the assignment game, the *multiple-partners game*. Each player in a multiple-partners game has a quota and can have as many partnerships with different players from the other side as her quota allows. The total surplus of a player is the sum of the surpluses she obtains in all her partnerships. In this paper, *we study the Shapley value and competitive equilibria in large multiple-partners games.* Clearly, our results apply in particular to the assignment game.

The multiple-partners game can represent markets with sellers and buyers, firms and workers, or venture capital firms and startups. We will refer to the players as *buyers* and *sellers*. In this market, sellers own objects that they can sell to buyers to create a surplus. All the objects a seller owns are identical, and each buyer is interested in acquiring at most one object from each seller. If a buyer and a seller form a partnership, the seller transfers one of her items to the buyer. The surplus in the partnership, to be shared

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between the two agents, is the difference between the buyer's valuation of the object and the seller's valuation, which is normalized to zero.

An outcome of the multiple-partners game is a matching, which specifies a set of partners for each player respecting the quotas and a payoff vector that stipulates the sharing of the surplus in each partnership. Stability and competitive equilibrium are the main solution concepts in matching models.

An outcome is *stable* if a buyer and seller pair can increase their payoffs by forming a partnership and, if necessary for remaining within their quotas, by leaving some of their current partners. Sotomayor (1992) showed that the set of stable outcomes is always not empty. In a *competitive equilibrium outcome*, the objects a seller offers each have an associated non-negative price, and, given the price vector, each buyer chooses a bundle that maximizes her total surplus. Sotomayor (2007) proved that the set of competitive equilibrium outcomes is not empty and is a subset of the set of stable outcomes.[1] She characterized a competitive equilibrium outcome as a stable outcome where each seller obtains the same payoff in each transaction.

We can represent the multiple-partners game as a coalitional game with transferable utility (a *TU game*) and apply single-valued solution concepts for TU games. The *Shapley value* (Shapley, 1953) is the most popular value in TU games. It has been studied in assignment games by Hoffmann and Sudhölter (2007) and van den Brink and Pinter (2015).[2] While the competitive equilibrium is a notion based on individual optimization and stability on what pairs can extract, the Shapley value is a notion of fair division based on what individuals contribute.

A major deficiency of the Shapley value of an assignment game (hence, also of a multiple-partners game) is that it may not be a stable payoff.[3] Nevertheless, it has two main advantages relative to the concepts of stability and competitive equilibrium. First, each player's Shapley payoff takes into account their whole influence in the market.[4] Second, the Shapley value satisfies appealing normative properties such as null player, additivity, strong monotonicity, and balanced contributions.[5]

The main purpose of our paper is to show that replication resolves this deficiency: the Shapley value becomes stable. We prove that the Shapley value of a replicated multiple-partners game converges to a competitive equilibrium payoff (hence, to a stable payoff) when the number of replicas tends to infinity. Thus, our result generalizes Shapley and Shubik's (1969) and Liggett et al.'s (2009) theorems on the asymptotic behavior of the Shapley value of expanding 1-to-1 *glove markets* and expanding 1-to-k *glove markets*.

For the proof of our main theorem, we introduce and study the *multiple-partners game with types*, which is a game where several agents are identical, and we provide properties of *large multiple-partners games with types*, where the number of players of any type is larger than any players' quota. We show, for instance, that every competitive equilibrium outcome of a large game with types satisfies *equal treatment of equals* (that is, two identical players obtain the same vector of payoffs) and *equal treatment of partnerships* (that is, the payoff obtained by a player is the same in all her partnerships).[6]

We provide further results of the limit of the Shapley value of the replicated game for supermodular and monotonic assignment games. For this class of games, Schwarz and Yenmez (2011) found that the "mean stable imputation" (the average between the buyer-optimal and the seller-optimal stable payoff vectors) and the "median stable imputation" (where each player obtains the median of their payoff in the set of stable payoffs) coincide. We show that the Shapley value of the replicated (supermodular and monotonic) assignment game converges to this vector, representing a compromise between the two extreme competitive allocations.

Finally, we show that our convergence result extends to a large subclass of *semivalues* (Dubey et al., 1981), including the Banzhaf value (Banzhaf, 1964), for instance. Semivalues are single-valued solutions obtained by relaxing the axioms that characterize the Shapley value, mainly the efficiency of the value. All the semivalues of this subclass of a replicated game converge to the same competitive equilibrium payoff as the replica becomes large.

The seminal paper of Shapley and Shubik (1972) started a very large literature that studies and applies the assignment game. Crawford and Knoer (1981) and Kelso and Crawford (1982) introduced one-to-many matching models with side payments. Since then, extensions of the assignment game to environments where each agent of a one or two sided market can form several partnerships have been used in applied environments. For instance, the multiple-partners game has been applied to "mobility-as-a-service" networks for planning and evaluation of public agency platforms (Rasulkhani and Chow, 2019; Pantelidis et al., 2020; Yao and Zhang, 2023). Also, Fox et al. (2018) developed methods to identify the distribution of unobserved characteristics for this game. This allows the use of the model, for instance, in venture capital markets, where venture capitalists are on one side of the market, and entrepreneurial biotech and medical firms are on the other (Fox et al., 2012).

[1] The sets of stable and competitive equilibrium outcomes coincide in the assignment game.

[2] Núñez and Rafels (2019) reviewed the contributions that study the Shapley value in the assignment game. The nucleolus (Schmeidler, 1969) is another popular solution concept for TU games considered for the assignment game, e.g., Llerena et al. (2015).

[3] Moreover, the computation of the Shapley value requires the worth of all coalitions, which implies solving a combinatorial optimization problem for each. By contrast, computing the sets of stable allocations and competitive equilibria only requires the matrix of all the buyer-seller surpluses (see Núñez and Rafels, 2019).

[4] For example, Núñez and Rafels (2019) discuss an assignment game with two buyers and one seller, where the object's value for the first buyer is higher than for the second buyer. In a stable allocation, the first buyer acquires the object at a price between his and the second buyer's valuations, and the second buyer gets a zero payoff. However, the second buyer influences the game since, in his absence, the price paid by the first buyer could go down to zero. The Shapley value recognizes this influence and assigns a positive payoff to the second buyer.

[5] See, for instance, Winter (2002).

[6] The equal treatment of partnerships for the sellers holds in any competitive equilibrium outcome, by definition. Our results for large multiple-partners games with types generalize previous results by Sotomayor (2019), who showed that the replicated market satisfies equal treatment of equals and equal treatment of partnerships.

The line of research on the value-equilibrium convergence was initiated by Shapley (1964), who showed the convergence of the Shapley value of replicated exchange economies with transferable utility to a competitive allocation.[7] The assignment game, or its generalization of the multiple-partners game, is not a market game; hence, the two notions of competitive equilibrium differ. In particular, objects are not divisible in our setting, and transferring utility is only allowed between the partners. Our proof uses the celebrated theorem in combinatorics due to Hall (1935). This method allows us to prove the result not only for the assignment game but for the multiple-partners game and to extend it to a large family of semivalues.

Champsaur (1975) proved that the NTU Shapley value payoffs (Shapley, 1967) are asymptotically included in the set of competitive payoffs for exchange economies with production (see also Shapley and Shubik, 1969; Mas-Colell, 1977).[8] Relatedly, Wooders and Zame (1987) formulated a fairly general class of games with transferable utility and proved that the Shapley value payoff of a sufficiently large game is in the individually rational ε-core.

In the multiple-partners game, Sotomayor (2019) introduced multi-stage cooperative games, defined the concepts of sequential stability and perfect competitive equilibrium, and studied the effect of the replications of the market on the cooperative and the competitive structures of the extended markets. She proved, in particular, that the sets of stable and competitive equilibrium payoffs shrink to the set of stable payoffs that satisfy equal treatment of equals and equal treatment of partnerships when the replica is large.[9] In contrast, our focus is the convergence of the Shapley value and other semivalues to a competitive payoff.

The remainder of the paper is organized as follows. Section 2 describes the environment and the main solution concepts. Section 3 introduces the multiple-partners game with types. Section 4 states the convergence of the Shapley value to a competitive equilibrium payoff. Section 5 provides properties of the competitive equilibria of large multiple-partners game with types; they will be used to prove the convergence in Section 6. Section 7 characterizes the limit of the Shapley value of the replica game for supermodular and monotonic assignment games. Section 8 extends the convergence result to semivalues. Section 9 concludes the paper. All the proofs except for the main theorem are in the Appendix.

2. The multiple-partners game

2.1. The model

We study the *multiple-partners game*, introduced by Sotomayor (1992), a generalization of the assignment game (Shapley and Shubik, 1972). In this model, there are two finite and disjoint sets of players: a set of *buyers* $B = \{b_1, \dots, b_{n_b}\}$ and a set of *sellers* $S = \{s_1, \dots, s_{n_s}\}$. We use b and s to represent, generically, any element of B and S, respectively.

Each player has a *quota* representing the maximum number of partnerships they can enter. The quota $q(b) > 0$ of buyer $b \in B$ is an integer representing the maximum number of sellers he can partner with. Similarly, each seller $s \in S$ can partner with $q(s) > 0$ distinct buyers.[10] We denote by $\mathbf{q} := (q(i))_{i\in B\cup S}$ the vector of all the quotas.

Without loss of generality, we assume that every player has a reservation utility of 0; that is, a seller or a buyer obtains a worth of 0 from any unfilled spot in their quota. Players can obtain non-negative payoffs when they form partnerships. For each pair $(b, s) \in B \times S$, there is a non-negative number $a_{bs} \geq 0$, representing the *worth* (or *surplus*) generated from a partnership between the buyer b and the seller s. We denote by $\mathbf{a} \equiv (a_{bs})_{(b,s)\in B\times S}$ the matrix of all these numbers. The surplus a_{bs} can be shared between b and s any way they decide. Finally, we assume that players' preferences are separable across pairs in that the payoff from a partnership does not depend on the other partnerships formed.

The market described above is $M := \langle B, S, \mathbf{a}, \mathbf{q}\rangle$. It will often be denoted by M when this simplification does not lead to confusion.

A feasible matching for M assigns at most $q(b)$ distinct sellers to each buyer b and at most $q(s)$ distinct buyers to each seller s. We represent it through a matrix.

Definition 1. *A* ***feasible matching*** *for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$ *is a* $n_b \times n_s$ *matrix* $\mathbf{x} = (x_{bs})_{(b,s)\in B\times S}$ *of non-negative integer entries such that* $x_{bs} \in \{0, 1\}$ *for all* $b \in B$ *and all* $s \in S$*. Furthermore,* $\sum_{b\in B} x_{bs} \leq q(s)$ *for all* $s \in S$ *and* $\sum_{s\in S} x_{bs} \leq q(b)$ *for all* $b \in B$*.*

We denote by $\mathcal{A}(B, S, \mathbf{q})$ the set of feasible matchings between B and S.

For each feasible matching $\mathbf{x}$ for M, we denote by $C_b(\mathbf{x})$ the set of partners assigned to buyer b at $\mathbf{x}$: $C_b(\mathbf{x}) := \{s \in S \mid x_{bs} = 1\}$. Similarly, $C_s(\mathbf{x})$ is the set of partners assigned to seller s at $\mathbf{x}$. Therefore, $C_b(\mathbf{x})$ has a most $q(b)$ elements, for each $b \in B$, and $C_s(\mathbf{x})$ has at most $q(s)$ elements, for each $s \in S$. The set of pairs $(b, s) \in B \times S$ that are assigned to each other at $\mathbf{x}$ is denoted by $C(\mathbf{x})$. That is, $(b, s) \in C(\mathbf{x})$ if $x_{bs} = 1$. We say that buyer b and seller s are (respectively, are not) matched at $\mathbf{x}$ if $(b, s) \in C(\mathbf{x})$ (respectively, $(b, s) \notin C(\mathbf{x})$).

An outcome of the market M involves a matching and a vector of payoffs:

[7] See Hart (2002) for a survey on this topic.

[8] In parallel with the asymptotic approach, the equivalence between the Shapley value payoffs and the set of competitive payoffs is proved for economic environments with a continuum of players. In particular, Aumann and Shapley (1974) established the equivalence for market games with a continuum of traders. Aumann (1975) obtained a similar result for pure exchange economies with a continuum of traders.

[9] See Massó and Neme (2014) for a class of theorems regarding the finite convergence of the set of competitive allocations and other stability concepts for a different type of many-to-many assignment game.

[10] A multiple-partners game is an *assignment game* if $q(b) = 1$ for all $b \in B$ and $q(s) = 1$ for all $s \in S$.

Definition 2. *A feasible outcome for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$, denoted by $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$, is a feasible matching $\boldsymbol{x}$ and a pair of payoff vectors $(\mathbf{u}, \mathbf{v})$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^{C(\boldsymbol{x})}$ satisfy $u_{bs} \geq 0$, $v_{bs} \geq 0$, and $u_{bs} + v_{bs} = a_{bs}$ for all $(b,s) \in C(\boldsymbol{x})$.*

Definition 3. *A feasible payoff vector $(\mathbf{u}, \mathbf{v})$ for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$ is the projection of some feasible outcome $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ for M on $\mathbb{R}_+^{C(\boldsymbol{x})} \times \mathbb{R}_+^{C(\boldsymbol{x})}$.*

In a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$, the individual payoffs of each $b \in B$ and $s \in S$ are given by the arrays of numbers $u_{bs} \geq 0$ and $v_{bs} \geq 0$, respectively, only defined if and only if $x_{bs} = 1$.

Given a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$, we denote by $u_{b,\min}(\mathbf{x})$ and $v_{s,\min}(\mathbf{x})$ the *minimum payoff* of buyer b and seller s, respectively, among his/her payoffs. That is, $u_{b,\min}(\mathbf{x}) := \min_{s\in C_b(\mathbf{x})} u_{bs}$ if $|C_b(\mathbf{x})| = q(b)$ and $u_{b,\min}(\mathbf{x}) = 0$ otherwise.[11] Similarly, $v_{s,\min}(\mathbf{x}) := \min_{b\in C_s(\mathbf{x})} v_{bs}$ if $|C_s(\mathbf{x})| = q(s)$ and $v_{s,\min}(\mathbf{x}) = 0$ otherwise. The *total payoff* of buyer b and seller s is given by $\sum_{s\in C_b(\mathbf{x})} u_{bs}$ and $\sum_{b\in C_s(\mathbf{x})} v_{bs}$, respectively.

2.2. Stability and competitive equilibrium

Stability is a natural solution concept for the multiple-partners game. Sotomayor (1992) proved that the notion of setwise-stability[12] is equivalent to the following notion of pairwise stability, which we will refer to simply as *stability*. For a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$ to be stable, there cannot exist some pair of players (b, s) who are not matched at $\mathbf{x}$ but who could both get a higher payoff by forming a partnership while at the same time dissolving one of their current partnerships, if it is necessary to stay within their quotas (which would be the case if $|C_b(\mathbf{x})| = q(b)$ or $|C_s(\mathbf{x})| = q(s)$).

Definition 4. *A feasible outcome $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$ is stable if $u_{b,\min}(\mathbf{x}) + v_{s,\min}(\mathbf{x}) \geq a_{bs}$ for all $b \in B$ and all $s \notin C_b(\boldsymbol{x})$.*

Another natural solution concept for the multiple-partners game is the *competitive equilibrium* (Sotomayor, 2007). Under the competitive approach, we can interpret the number of partners $q(s)$ seller s can have as the number of identical objects she can sell. In this approach, every seller sells all her items at the same non-negative price p_s.[13] Hence, in this case, we can identify a seller with any of her objects and use s to refer to an object owned by seller s. We denote by $\mathbf{p} = (p_s)_{s\in S} \in \mathbb{R}_+^S$ a *price vector*.

In a competitive equilibrium, given a price vector $\mathbf{p}$, each buyer b maximizes his total payoff over the sets of objects that are feasible to him. We say that a *set* $Q \subseteq S$ is *feasible to buyer* b if it has at most $q(b)$ elements. Therefore, we define buyer b's *demand set* $D_b(\mathbf{p})$ as:

$$D_b(\mathbf{p}) := \underset{Q \text{ feasible to } b}{\operatorname{argmax}} \sum_{s\in Q} (a_{bs} - p_s).$$

In a competitive equilibrium, every agent is assigned a set of partners in their demand set, and every unsold object has a zero price:

Definition 5. *A competitive equilibrium (CE) of $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$ is a pair $(\mathbf{p}, \boldsymbol{x})$, where $\mathbf{p} \in \mathbb{R}_+^S$ and $\boldsymbol{x}$ is a feasible matching for M, such that:*

1. *$C_b(\boldsymbol{x}) \in D_b(\mathbf{p})$ for all $b \in B$ and*
2. *$p_s = 0$ if $|C_s(\boldsymbol{x})| < q(s)$.*

Each competitive equilibrium $(\mathbf{p}, \mathbf{x})$ of M is a projection of some feasible outcome $(\mathbf{u}, \mathbf{p}; \mathbf{x})$, where we only keep one copy of each seller's price p_s and $u_{bs} = a_{bs} - p_s$ if $s \in C_b(\mathbf{x})$. We refer to such an outcome as a *CE outcome* of M. Similarly, a *CE payoff vector* $(\mathbf{u}, \mathbf{p})$ is a projection of some CE outcome $(\mathbf{u}, \mathbf{p}; \mathbf{x})$. We say that the matching $\mathbf{x}$ is *compatible* with the payoff vector $(\mathbf{u}, \mathbf{p})$. The set of CE payoff vectors for M is denoted by $\mathcal{CE}(M)$.

As Sotomayor (2007) proved, the set of CE outcomes is a subset of the set of stable outcomes. It corresponds to the set of stable outcomes where all the prices of the objects a seller owns are equal.[14]

2.3. Representation as a TU game and the Shapley value

A multiple-partners game can be represented as a coalitional game with transferable utilities (*TU game*). Therefore, solution concepts proposed for TU games can also be used in the multiple-partners game.

[11] $|A|$ denotes the number of elements in the set A.

[12] A feasible outcome is setwise-stable if there is no coalition of players who, by forming new partnerships only among themselves—possibly dissolving some partnerships to remain within their quotas and possibly keeping other partnerships—can all obtain a higher payoff.

[13] The prices of two objects owned by the same seller in a competitive equilibrium must be the same. If two objects owned by the same seller had different prices, no buyer would demand the more expensive one.

[14] Sotomayor (2007) also proved that the sets of stable outcomes and CE outcomes are endowed with a complete lattice structure.

A TU game is a pair (N, v) where N is the player set and the function $v : 2^N \to \mathbb{R}$ satisfies that $v(\emptyset) = 0$. A subset T of N is called a coalition, and $v(T)$ represents the worth of T. For any player $i \in N$ and coalition $T \subseteq N \setminus \{i\}$, player i's marginal contribution to T is $\mathrm{D}^i v(T) := v(T \cup \{i\}) - v(T)$.

The most important single-valued solution for TU games is the *Shapley value* (Shapley, 1953). It was originally defined as the unique single-valued solution satisfying efficiency, additivity, equal treatment, and null player.[15] The Shapley value of the TU game (N, v) can be written as follows:

$$Sh_i(N, v) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N| - |T| - 1)!}{|N|!} \mathrm{D}^i v(T).$$

Consider the multiple-partners game $M = \langle B, S, \mathbf{a}, \mathbf{q} \rangle$. We represent M as a TU game $(B \cup S, v^M)$ by letting

$$v^M(T) := \max_{\mathbf{x} \in \mathcal{A}(T \cap B, T \cap S, \mathbf{q}_T)} \sum_{b \in T \cap B} \sum_{s \in C_b(\mathbf{x})} a_{bs} \tag{1}$$

for all $T \in 2^{B \cup S} \setminus \{\emptyset\}$, setting $v^M(\emptyset) = 0$, and $\mathbf{q}_T$ is the restriction of $\mathbf{q}$ to T. That is, $v^M(T)$ is the maximum surplus the coalition T can obtain by forming feasible partnerships between the set of buyers in T (i.e., $T \cap B$) and the set of sellers in T (i.e., $T \cap S$).

The prescription of the Shapley value to the buyer $b \in B$ in M (denoted as $Sh_b(M)$) is defined as the Shapley value of the player b in the induced TU game $(B \cup S, v^M)$:

$$Sh_b(M) := Sh_b(B \cup S, v^M). \tag{2}$$

The prescription $Sh_s(M)$ to a seller s is defined similarly, for all $s \in S$.

We close this section by introducing and discussing the equal treatment properties in the multiple-partners game. We adapt the definition of the properties for TU games. In a TU game (N, v), two distinct players $i, j \in N$ are equal players if $v(T \cup \{i\}) = v(T \cup \{j\})$ for all $T \subseteq N \setminus \{i, j\}$. That is, two players are equal if they have the same vector of marginal contributions.

Two distinct buyers $b, b' \in B$ are said to be *equal buyers* in M if $q(b) = q(b')$ and $a_{bs} = a_{b's}$ for all $s \in S$. Similarly, two distinct sellers $s, s' \in S$ are said to be *equal sellers* in M if $q(s) = q(s')$ and $a_{bs} = a_{bs'}$ for all $b \in B$. Notice that the notions of equal players for TU games and multiple-partners games are compatible: If two distinct buyers (or sellers) are equal in M, they are also equal in $(B \cup S, v^M)$.

We introduce some notation to define equal treatment of equals. Consider a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$. By $\mathbf{u}_b$ we denote the buyer b's vector of payoffs. That is, $\mathbf{u}_b$ has $q(b)$ components, including the payoffs $(u_{bs})_{s \in C_b(\mathbf{x})}$ and as many zeros as unfilled slots in the buyer's quota, if any. For convenience, we list the individual payoffs in non-increasing order. Similarly, we denote by $\mathbf{v}_s$ the ($q(s)$-dimensional) seller s's vector of payoffs, where the individual payoffs are listed in non-increasing order.

Definition 6. *A feasible outcome* $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q} \rangle$ *satisfies equal treatment of equals if* $\boldsymbol{u}_b = \boldsymbol{u}_{b'}$ *for all equal buyers* b *and* b' *and* $\boldsymbol{v}_s = \boldsymbol{v}_{s'}$ *for all equal sellers* s *and* s'.

We also define equal treatment of partnerships. For the sellers, for instance, we say that a feasible outcome satisfies the property if every seller obtains the same payoff from all their sold and unsold objects.

Definition 7. *Consider a feasible outcome* $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q} \rangle$*:*

(a) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *satisfies equal treatment of partnerships among buyers if all the components of* $\mathbf{u}_b$ *are equal for all* $b \in B$.
(b) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *satisfies equal treatment of partnerships among sellers if all the components of* $\mathbf{v}_s$ *are equal for all* $s \in S$.
(c) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *satisfies equal treatment of partnerships if it satisfies equal treatment of partnerships among buyers and sellers.*

We note that CE outcomes satisfy equal treatment of partnerships among sellers.

3. The multiple-partners game with types

The main objective of this paper is to study the replicated multiple-partners game. In such a replicated game, for each player in the original game, we add new players who are equal to him/her. We say these equal players are of the same "type." In this section, we introduce a notation that conveniently represents multiple-partners games where several players are of the same type. In the replica of a game, the numbers of players of each type are the same. However, in the proof of our main result (Theorem 1), we require properties of games where the numbers of players of each type are different.

In a *multiple-partners game with types*, there are two finite and disjoint sets of types: a set of *buyer types* $\underline{B} = \{\underline{b}_1, \ldots, \underline{b}_{t_b}\}$ and a set of *seller types* $\underline{S} = \{\underline{s}_1, \ldots, \underline{s}_{t_s}\}$. Hence, the numbers of types in $\underline{B}$ and $\underline{S}$ are t_b and t_s, respectively. We use $\underline{b}$ and $\underline{s}$ to represent a

[15] Let ψ be a single-valued solution. Efficiency of ψ requires $\sum_{i \in N} \psi_i(N, v) = v(N)$ for any (N, v). The solution ψ is additive if $\psi(N, v + v') = \psi(N, v) + \psi(N, v')$ for any two games (N, v) and (N, v'). It satisfies the null player axiom if $\psi_i(N, v) = 0$ for any null player i in (N, v) (that is, for a player $i \in N$ such that $\mathrm{D}^i v(T) = 0$ for any $T \subseteq N \setminus \{i\}$), for any (N, v). We introduce the equal treatment property at the end of this section.

generic member of $\underline{B}$ and $\underline{S}$, respectively. There can be several buyers or sellers of the same type. If two distinct buyers are of type $\underline{b}$, they are equal: their quotas are equal, and their surplus with any seller is identical. Similarly, if two distinct sellers are of type $\underline{s}$, then the number of their objects are the same, and their surplus with any buyer is also the same. We indicate the number of buyers and sellers of a certain type through the function $y : \underline{B} \cup \underline{S} \to \mathbb{Z}_+$. Hence, $y(\underline{b})$ and $y(\underline{s})$ represent the number of type-$\underline{b}$ buyers and the number of type-$\underline{s}$ sellers, respectively.

The *multiple-partners game with types* is denoted as $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$, where $\mathbf{a} = (a_{\underline{bs}})_{(\underline{b},\underline{s})\in \underline{B}\times\underline{S}}$ is the matrix that represents the surplus from a partnership between any buyer of type $\underline{b}$ and any seller of type $\underline{s}$, $q(\underline{b}) > 0$ is the maximum number of objects each buyer of type $\underline{b}$ can acquire, and $q(\underline{s}) > 0$ is the number of identical objects owned by each seller of type $\underline{s}$.

In the multiple-partners game with types $\underline{M}$, each type-$\underline{b}$ buyer is denoted by $\underline{b}(h)$, where $h = 1, \dots, y(\underline{b})$, and each type-$\underline{s}$ seller is denoted by $\underline{s}(g)$, where $g = 1, \dots, y(\underline{s})$. We also denote $B_{\underline{b}} = \{\underline{b}(h) \mid h = 1, \dots, y(\underline{b})\}$ as the set of type-$\underline{b}$ buyers and $S_{\underline{s}} = \{\underline{s}(g) \mid g = 1, \dots, y(\underline{s})\}$ as the set of type-$\underline{s}$ sellers.

The simplest example of a multiple-partners game with types is a glove market:

Example 1. *A glove market satisfies $\underline{B} = \{\underline{b}_1\}$, $\underline{S} = \{\underline{s}_1\}$, $q(\underline{b}_1) = q(\underline{s}_1) = 1$, and $a_{\underline{b}_1\underline{s}_1} = 1$. The number of buyers of the unique buyer type is $y(\underline{b}_1)$, and the number of sellers of the unique seller type is $y(\underline{s}_1)$. The interpretation is that each buyer owns a left glove, while each seller owns a right glove. A single glove is worthless. Pairing a left glove with a right glove generates one unit of surplus.*

We notice that the definition of multiple-partners games with types does not introduce a new class of games but just a notation. A multiple-partners game with types can be easily written as a multiple-partners game where the set of players is larger: Given $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$, we can define the game $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$, with $B = \bigcup_{\underline{b}\in\underline{B}} B_{\underline{b}}$, $S = \bigcup_{\underline{s}\in\underline{S}} S_{\underline{s}}$, and $a_{bs} = a_{\underline{bs}}$, $q(b) = q(\underline{b})$, and $q(s) = q(\underline{s})$ if $b \in B_{\underline{b}}$ and $s \in S_{\underline{s}}$. On the other hand, a multiple-partners game with no equal players is a multiple-partners game with types where the number of players of each type is one.

4. The replicas of the multiple-partners game and the equilibrium-value convergence

In this section, we formally define the replicas of the multiple-partners game. Then, we will state the convergence of the Shapley value to a competitive equilibrium payoff as the number of replicas tends to infinity.

The replica of a multiple-partners game $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$ is a multiple-partners game where each buyer $b \in B$ and seller $s \in S$ is "replicated" the same number of times. Hence, the replica of M is a multiple-partners game with types where each type of buyer $\underline{b}_i$ has the quota and the vector of surpluses of buyer $b_i \in B$, and each type of seller $\underline{s}_j$ has the quota and the vector of surpluses of seller $s_j \in S$.

Definition 8. *Consider a multiple-partners game $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$, where $B = \{b_1, \dots, b_{n_b}\}$ and $S = \{s_1, \dots, s_{n_s}\}$. The k-fold replica M^k of M is a multiple-partners game with types $\langle \underline{B}, \underline{S}, \mathbf{y}^k, \mathbf{a}, \mathbf{q}\rangle$, where $\underline{B} = \{\underline{b}_1, \dots, \underline{b}_{n_b}\}$, $\underline{S} = \{\underline{s}_1, \dots, \underline{s}_{n_s}\}$, the characteristics of each buyer of type $\underline{b}$ (respectively, each seller of type $\underline{s}$) in M^k are the same as those of the buyer b (respectively, seller s) in M, and $y^k(\underline{b}) = y^k(\underline{s}) = k$ for all $\underline{b} \in \underline{B}$ and all $\underline{s} \in \underline{S}$.*

Consider the replica $M^k = \langle \underline{B}, \underline{S}, \mathbf{y}^k, \mathbf{a}, \mathbf{q}\rangle$ of the game $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$. Since the Shapley value Sh satisfies equal treatment of equals, the value of all the k replicas of a player in $B \cup S$ is the same. Hence, we simplify the notation and write $Sh_{\underline{b}}(M^k)$ to indicate the Shapley value of any buyer of type $\underline{b} \in \underline{B}$ (that is, any of the replicas of the buyer $b \in B$) in the game M^k and similarly for $Sh_{\underline{s}}(M^k)$.

Discussing the property of equal treatment of equals in the CE outcomes of M^k requires additional notation. Denote by $q^{\max}$ the greatest quota among the players in M, that is, $q^{\max} := \max\{q_b^{\max}, q_s^{\max}\}$, where $q_b^{\max} := \max\{q(\underline{b}) \mid \underline{b} \in \underline{B}\}$ and $q_s^{\max} := \max\{q(\underline{s}) \mid \underline{s} \in \underline{S}\}$. We say that a replica with $k > q^{\max}$ is *large*. Sotomayor (2019) showed that the CE outcomes of a large replica satisfy the property of equal treatment of partnerships (that is, not only all the objects of each seller receive the same payoff, but also each buyer obtains the same payoff in all his slots). As we will show in the next section, the CE outcomes of a large replica also satisfy equal treatment of equals; that is, two buyers or two sellers of the same type have the same CE payoff vectors.

Therefore, in a CE outcome of a large replica, we only need to identify the payoff $u_{\underline{b}} \in \mathbb{R}$ of each buyer of type $\underline{b} \in \underline{B}$ in each of his slots and the price $p_{\underline{s}} \in \mathbb{R}$ of each seller of type $\underline{s} \in \underline{S}$ for each of her objects. Given $u_{\underline{b}}$, the total payoff of any buyer of type $\underline{b}$ is $q(\underline{b})u_{\underline{b}}$, and similarly for the sellers. Hence, we denote a CE outcome of a large replica M^k by $(\underline{\mathbf{u}}, \underline{\mathbf{p}}; \mathbf{x})$, where $\underline{\mathbf{u}} = (u_{\underline{b}})_{\underline{b}\in\underline{B}} \in \mathbb{R}^{\underline{B}}$ and $\underline{\mathbf{p}} = (p_{\underline{s}})_{\underline{s}\in\underline{S}} \in \mathbb{R}^{\underline{S}}$. Similarly, we regard the set $\mathcal{CE}(M^k)$ of CE payoff vectors for M^k as a set of vectors in $\mathbb{R}^{\underline{B}} \times \mathbb{R}^{\underline{S}}$, when M^k is large.

Moreover, although the set of CE outcomes of the multiple-partners game is not replication invariant,[16] Sotomayor (2019) proved that it eventually becomes constant through replication: $\mathcal{CE}(M^k) = \mathcal{CE}(M^K)$ for all $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$ and $k \geq K = q^{\max} + 1$.

[16] To see that replication invariance does not hold, consider $M = \langle \{b\}, \{s\}, q(b) = q(s) = 2, a_{bs} = 1\rangle$. In the associated TU game, $v^M(\{b, s\}) = 1$ and $v^{M^2}(\{\underline{b}(1), \underline{b}(2), \underline{s}(1), \underline{s}(2)\}) = 4$. Therefore, the sum of the players' payoffs in a CE of M^2 is four times the sum of the payoffs in a CE of M, whereas it should be two times if the set of CE outcomes were replication invariant.

We can now state Theorem 1: the Shapley value $Sh(M^k)$ of the replica M^k of any multiple-partners game M converges to a CE payoff as the replica becomes large.

Theorem 1. *Consider a multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q}\rangle$ and denote $K \equiv r^{\max} + 1$. There exists $(\underline{\mathbf{u}}^K, \underline{\mathbf{p}}^K) \in \mathcal{CE}(M^K)$ such that:*

$$\lim_{k\to+\infty} Sh_{\underline{b}}(M^k) = q(\underline{b})\underline{u}^K_{\underline{b}} \text{ and } \lim_{k\to+\infty} Sh_{\underline{s}}(M^k) = q(\underline{s})\underline{p}^K_{\underline{s}}$$

for all $\underline{b} \in \underline{B}$ and $\underline{s} \in \underline{S}$.

The proof of the theorem is presented in Section 6 because it requires the analysis of large multiple-partners games with types. We briefly comment here on some of the key elements of the proof. We use that the prescription of the Shapley value to a player i in a TU game (N, v) can be expressed as i's expected marginal contribution as follows (Dubey et al., 1981):

$$Sh_i(N, v) = \mathbb{E}[\mathrm{D}^i\, v(\tilde{T})], \tag{3}$$

where the random coalition $\tilde{T}$ follows the probability distribution

$$P(\tilde{T} = T) = \int_{[0,1]} z^{|T|}(1-z)^{|N\setminus(T\cup\{i\})|}\lambda(dz) \tag{4}$$

for all $T \subseteq N \setminus \{i\}$, where λ is the Lebesgue measure on $[0,1]$. To interpret the previous expressions, suppose that a player $i \in N$ joins a random coalition $\tilde{T}$, whose composition is determined in two stages: first, choose $z \in [0,1]$ according to the uniform distribution on $[0,1]$; second, select each player $k \in N \setminus \{i\}$ independently as a member of $\tilde{T}$ with probability z. Then player i's prescription according to Sh is equal to the expectation of the increment to the worth of $\tilde{T}$ due to the arrival of i.

With the aid of equation (4), we can connect the distribution of this random coalition to a binomial distribution with unknown parameter z. It will follow from the law of large numbers and the central limit theorem that this random coalition converges to a random large "sufficiently uneven" game as the original game expands through replication. We introduce and study the properties of such games in Section 5. One of these properties uses Hall's theorem, which is also instrumental in other steps in the proof of Theorem 1. We provide a version of this theorem as Lemma 1.[17]

Lemma 1 *(Hall, 1935). Given B and S such that $|B| \leq |S|$, let $\varphi : B \rightsquigarrow S$ be a correspondence. φ satisfies the Hall condition: $|P| \leq |\bigcup_{b\in P}\varphi(b)|$ for all $P \in 2^B \setminus \{\varnothing\}$ if and only if there exists a function $\eta : B \to S$ satisfying (i) $\eta(b) \in \varphi(b)$ for all $b \in B$; (ii) $\eta(b) \neq \eta(b')$ for all $b, b' \in B$ such that $b \neq b'$.*

5. Properties of the competitive equilibria of the large multiple-partners game with types

In this section, we provide properties of the CE of the multiple-partners game with types when the number of buyers and sellers of each type is large.

We say that $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$ is large if the number of players of any type is higher than the greatest quota of all the players. Formally, recall that we denote $q^{\max}$ as the greatest quota among the types in $\underline{M}$. We denote by $y^{\min}$ the smallest number of players of a given type among the types in $\underline{M}$,[18] that is, $y^{\min} := \min\{y^{\min}_b, y^{\min}_s\}$, where $y^{\min}_b := \min\{y(\underline{b}) \mid \underline{b} \in \underline{B}\}$ and $y^{\min}_s := \min\{y(\underline{s}) \mid \underline{s} \in \underline{S}\}$. Then,

Definition 9. *A multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{q}\rangle$ is large if $y^{\min} > q^{\max}$.*

Two players of the same type can obtain different payoff vectors in a CE. However, Proposition 1 ensures that this can only happen if the game is not large: Two buyers or two sellers of the same type have the same payoff vectors if the multiple-partners game is large. Moreover, each buyer obtains the same payoff in all his slots. This result generalizes Lemmas 3.1 and 3.2 in Sotomayor (2019) for games where the number of players of each type is not necessarily the same.

Proposition 1. *Every CE outcome of a large multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{q}\rangle$ satisfies equal treatment of equals and equal treatment of partnerships.*

Proposition 1 allows a characterization of the CE payoff vectors of the large multiple-partners game with types, which we state in Remark 1:

[17] Demange et al. (1986) used Hall's theorem to characterize stable payoff vectors for an assignment game.

[18] Remember that, contrary to the replicas, the number of players of a type can differ across types.

Remark 1. *Combining Proposition 1 and Sotomayor's (2007) characterization of a CE payoff vector as a stable payoff vector satisfying equal treatment of partnerships among sellers, we obtain a symmetric characterization of CE payoff vectors for large games: every CE payoff vector of a large multiple-partners game with types is a stable payoff vector satisfying equal treatment of equals and equal treatment of partnerships.*

As a consequence of Proposition 1, we can simplify the notation for a CE outcome of any large multiple-partners game with types and denote a CE outcome of $\underline{M}$ simply by $(\underline{\mathbf{u}}, \underline{\mathbf{p}}; \mathbf{x})$, with $\underline{\mathbf{u}} = (u_{\underline{b}})_{\underline{b} \in \underline{B}} \in \mathbb{R}^{\underline{B}}$ and $\underline{\mathbf{p}} = (p_{\underline{s}})_{\underline{s} \in \underline{S}} \in \mathbb{R}^{\underline{S}}$.

To introduce our following result, we go back to the example of the glove market (Example 1). Remark 2 provides helpful information about the CE outcomes of some glove markets.

Remark 2. *Let us call a glove market asymmetric if $y(\underline{b}_1) \neq y(\underline{s}_1)$; that is, the number of buyers (of the unique type) is different from the number of sellers. It is easy to check that the CE payoff vector of an asymmetric glove market is unique. Moreover, the dependence of the CE payoff vector on $y(\underline{b}_1)$ and $y(\underline{s}_1)$ is ordinal rather than cardinal: $(u_{\underline{b}_1}, v_{\underline{s}_1}) = (1, 0)$ if $y(\underline{b}_1) < y(\underline{s}_1)$ and $(u_{\underline{b}_1}, v_{\underline{s}_1}) = (0, 1)$ if $y(\underline{b}_1) > y(\underline{s}_1)$.*

In the example of the glove market, we could say that the glove market with a number of buyers and sellers given by $\mathbf{y}$ is "equivalent" to the glove market characterized by $\mathbf{y}'$ if $y(\underline{b}_1) < y(\underline{s}_1)$ and $y'(\underline{b}_1) < y'(\underline{s}_1)$ because their CE payoff vector is the same. Our following result generalizes this idea to multiple-partners games with types. We first define an equivalence relation on the set of multiple-partners games.

We partition the set of multiple-partners games with a fixed set of buyer and seller types into equivalence classes using inequalities on the total number of partnerships that can be formed by any two sets of buyer types and seller types. Formally, fix $\underline{B}$, $\underline{S}$, and $\mathbf{q}$. Two distinct games $\underline{M}$ and $\underline{M}'$ may differ on the number of each type of buyer and seller, that is, $\mathbf{y}$ and $\mathbf{y}'$ may be different (differences in the matrixes of worth $\mathbf{a}$ and $\mathbf{a}'$ are not relevant for our next definition). We define the equivalence relation on multiple-partners games as follows:

Definition 10. *Let $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{q} \rangle$ and $\underline{M}' = \langle \underline{B}, \underline{S}, \boldsymbol{y}', \boldsymbol{a}', \boldsymbol{q} \rangle$ be two multiple-partners games with types. We say that $\underline{M}$ and $\underline{M}'$ are equivalent, and we write $\underline{M} \sim \underline{M}'$, if:*

$$\sum_{\underline{b} \in H} y(\underline{b}) q(\underline{b}) \leq \sum_{\underline{s} \in G} y(\underline{s}) q(\underline{s}) \iff \sum_{\underline{b} \in H} y'(\underline{b}) q(\underline{b}) \leq \sum_{\underline{s} \in G} y'(\underline{s}) q(\underline{s})$$

for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$.

The equivalence relation $\sim$ induces a partition on the set of multiple-partners games with types, given $(\underline{B}, \underline{S}, \mathbf{q})$. We denote this partition by $T(\underline{B}, \underline{S}, \mathbf{q})$. Each cell $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{q})$ is referred to as a *class of games*.

Proposition 2 uses Hall's theorem to show that large multiple-partners games with types of the same class have the same CE payoff vectors if their matrices of worths are the same.

Proposition 2. *Consider two large multiple-partners games with types $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q} \rangle$ and $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}, \mathbf{q} \rangle$ of the same class. Then $\mathcal{CE}(\underline{M}) = \mathcal{CE}(\underline{M}')$.*

We now introduce the concept of an "uneven game," which plays a key role in proving our convergence result. This concept facilitates the decomposition of the asymptotic Shapley value because, as we will show, every asymptotic Shapley value is representable as a convex combination of marginal contributions to uneven games.

Definition 11. *A multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{q} \rangle$ is uneven if*

$$\sum_{\underline{b} \in H} y(\underline{b}) q(\underline{b}) \neq \sum_{\underline{s} \in G} y(\underline{s}) q(\underline{s}) \tag{5}$$

for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$.

In an uneven game, for any sets of buyer and seller types, the total number of partnerships the buyers (of those buyer types) can make differs from the total number of partnerships that the sellers (of those seller types) can make.

Remark 3. *For fixed sets of buyer and seller types $\underline{B}$ and $\underline{S}$, the unevenness of a multiple-partners game is determined by at most $(2^{n_b} - 1)(2^{n_s} - 1)$ inequalities.*

Not every multiple-partners game has a unique CE payoff vector. However, we have seen that asymmetric glove markets, which are uneven games, only have one CE outcome. Proposition 3 shows that if an uneven multiple-partners game with types is large, it has a unique CE as an asymmetric glove market.

Proposition 3. *The CE payoff vector $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ of a large uneven multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{q} \rangle$ is unique.*

Using Proposition 3, we know that the correspondence $\mathcal{CE}$ restricted to large uneven multiple-partners games with types is a function. Hence, we can write the CE payoff vector of such a game $\underline{M}$ as $(\underline{\mathbf{u}}(\underline{M}), \underline{\mathbf{p}}(\underline{M}))$. Moreover, Proposition 2 states that this function does not make full use of the information in $\mathbf{y}$. It suffices to know, for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$, whether $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ or $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ (remember that the equality is not possible if the game is uneven). Then, we can view a class $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{q})$, if restricted to uneven games, as a specification of $(2^{n_b} - 1)(2^{n_s} - 1)$ inequalities. We state this fact in Remark 4.

Remark 4. *Fix the sets of buyer and seller types $\underline{B}$ and $\underline{S}$ and the vector $\mathbf{q}$. A class $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{q})$, restricted to uneven games, specifies, for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$, either $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ or $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$.*

We close this section by studying the effect of the entrance of a new player of an existing type in a game. Take a large uneven game $\underline{M}$. We know that the $\mathcal{CE}(\underline{M})$ is a singleton. Consider the entrance of one player with an existing type (who can be matched with at most $q^{\max}$ players from the other side of the market); call $\underline{M}'$ the new game. According to Proposition 2, the unique CE outcome of $\underline{M}$ coincides with the unique CE outcome of $\underline{M}'$ if the games are equivalent. This is certainly the case (that is, the equivalence in Definition 10 holds) if $\underline{M}$ is "sufficiently uneven":

Definition 12. *A multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \boldsymbol{a}, \boldsymbol{q}\rangle$ is sufficiently uneven if*

$$\min_{\substack{H\in 2^{\underline{B}}\setminus\{\emptyset\},\\ G\in 2^{\underline{S}}\setminus\{\emptyset\}}} \left| \sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s}) \right| > q^{\max}.$$

The game is sufficiently uneven if, after including at most $q^{\max}$ slots/objects of an existing type, the new game belongs to the same class as the original game. Therefore, if $\underline{M}$ is sufficiently uneven, the entrance of a player of an existing type does not change the original CE payoff vector. We state this comparative statics phenomenon in Corollary 1 when the additional player is a buyer; a similar corollary holds for a seller.

Corollary 1. *Let $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$ be a large sufficiently uneven game and take $\underline{b} \in \underline{B}$. Consider $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}, \mathbf{q}\rangle$ that satisfies $y'(\underline{b}) = y(\underline{b}) + 1$, $y'(\underline{b}') = y(\underline{b}')$ for all $\underline{b}' \in \underline{B} \setminus \{\underline{b}\}$, and $y'(\underline{s}) = y(\underline{s})$ for all $\underline{s} \in \underline{S}$. Then $\mathcal{CE}(\underline{M}') = \mathcal{CE}(\underline{M})$ and it is a singleton.*

Corollary 2 expresses a useful implication of Corollary 1 in the framework of TU games: The marginal contribution of an entrant (a buyer, in this case) to a large sufficiently uneven multiple-partners game is the CE payoff of a player of the same type as the entrant in the uneven game.

Corollary 2. *Let $\underline{b}$, $\underline{M}$, and $\underline{M}'$ be the same as in Corollary 1. Denote by $\bar{B}$ and $\bar{S}$ the sets of buyers and sellers in $\underline{M}$, respectively. Let $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ be the CE payoff vector for $\underline{M}$. Then the marginal contribution of the buyer $\underline{b}(y(\underline{b}) + 1)$ to the game $\underline{M}'$ is $\mathrm{D}^{\underline{b}(y(\underline{b})+1)}\, v^{\underline{M}'}(\bar{B} \cup \bar{S}) = q(\underline{b})\underline{u}_{\underline{b}}$.*

Corollary 2 holds because the players in $\underline{M}$ keep their CE payoffs in $\underline{M}'$ and the new buyer $\underline{b}$ obtains his CE payoff in each of his $q(\underline{b})$ slots. Hence, the additional surplus in the game is $q(\underline{b})\underline{u}_{\underline{b}}$.

6. Proof of the equilibrium-value convergence

This section uses the properties developed in Section 5 to prove Theorem 1.

Proof of Theorem 1. Given the game $M = \langle B, S, \mathbf{a}, \mathbf{q}\rangle$, consider the k-fold replica $M^k = \langle \underline{B}, \underline{S}, \mathbf{y}^k, \mathbf{a}, \mathbf{q}\rangle$, with $k \geq K$. Let $B^k_{\underline{b}} = \{\underline{b}(h) \mid h = 1, \dots, k\}$ be the set of buyers of type $\underline{b} \in \underline{B}$ and $S^k_{\underline{s}} = \{\underline{s}(g) \mid g = 1, \dots, k\}$ the set of sellers of type $\underline{s} \in \underline{S}$. Moreover, denote by B^k and S^k the sets of buyers and sellers in M^k, respectively, that is, $B^k = \bigcup_{\underline{b}\in\underline{B}} B^k_{\underline{b}}$ and $S^k = \bigcup_{\underline{s}\in\underline{S}} S^k_{\underline{s}}$.

Without loss of generality, choose an arbitrary buyer type $\underline{b}^* \in \underline{B}$ and take the type-$\underline{b}^*$ buyer $\underline{b}^*(1)$. We want to compute $\lim_{k\to+\infty} Sh_{\underline{b}^*}(M^k)$, which, because of the equal treatment property of the Shapley value, is equal to $\lim_{k\to+\infty} Sh_{\underline{b}^*(1)}(M^k)$. First,

$$\begin{aligned} Sh_{\underline{b}^*(1)}(M^k) =& Sh_{\underline{b}^*(1)}(B^k \cup S^k, v^{M^k}) \\ =& \sum_{T\subseteq (B^k\setminus\{\underline{b}^*(1)\})\cup S^k} \left[\int_{[0,1]} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1} \lambda(dz)\right] \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T) \\ =& \int_{[0,1]} \sum_{T\subseteq (B^k\setminus\{\underline{b}^*(1)\})\cup S^k} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1}\, \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T)\lambda(dz), \end{aligned} \tag{6}$$

where the first equality follows from equation (2), with λ being the Lebesgue measure, the second from equations (3) and (4), and the third from the linearity of the integration.

We now construct, for any parameter $z \in (0,1)$ and for each type of buyer and each type of seller, a coalition-valued random variable using binomial distributions on the sets of players of that type, excluding the player $\underline{b}^*(1)$. The probability of each player's presence in the random coalition is z. That is, we define the random variable $\tilde{B}^k_{\underline{b}^*}$ by $P(\tilde{B}^k_{\underline{b}^*} = T) = z^{|T|}(1-z)^{k-|T|-1}$ for all $T \subseteq B^k_{\underline{b}} \setminus \{\underline{b}^*(1)\}$; the random variable $\tilde{B}^k_{\underline{b}}$ by $P(\tilde{B}^k_{\underline{b}} = T) = z^{|T|}(1-z)^{k-|T|}$ for all $T \subseteq B^k_{\underline{b}}$ and buyer type $\underline{b} \in \underline{B} \setminus \{\underline{b}^*\}$; and the random variable $\tilde{S}^k_{\underline{s}}$ by $P(\tilde{S}^k_{\underline{s}} = T) = z^{|T|}(1-z)^{k-|T|}$ for all $T \subseteq S^k_{\underline{s}}$ and seller type $\underline{s} \in \underline{S}$.[19] Moreover, we use the previous random variables on the sets of players of the same type to construct a new random variable, which we denote $\tilde{N}^{k,\underline{b}^*(1)}$, on the subsets of the set of all the players except $\underline{b}^*(1)$, that is, on the subsets of $(B^k \setminus \{\underline{b}^*(1)\}) \cup S^k$. We take the players' presences in any of the previous random coalitions as mutually independent; hence, the random variable is

$$\tilde{N}^{k,\underline{b}^*(1)} = \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right). \tag{7}$$

Its probability distribution is

$$\begin{aligned} P(\tilde{N}^{k,\underline{b}^*(1)} = T) =& \left(\prod_{\underline{b}\in\underline{B}} P(\tilde{B}^k_{\underline{b}} = T \cap B^k_{\underline{b}}) \right) \left(\prod_{\underline{s}\in\underline{S}} P(\tilde{S}^k_{\underline{s}} = T \cap S^k_{\underline{s}}) \right) \\ =& z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1}, \end{aligned} \tag{8}$$

for all $T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k$. Given this probability distribution, we proceed to the analysis of $\lim_{k\to+\infty} Sh_{\underline{b}^*}(M^k)$, using (6):

$$\begin{aligned} &\lim_{k\to+\infty} \int_{[0,1]} \sum_{T\subseteq(B^k\setminus\{\underline{b}^*(1)\})\cup S^k} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1}\, \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T)\lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \sum_{T\subseteq(B^k\setminus\{\underline{b}^*(1)\})\cup S^k} P(\tilde{N}^{k,\underline{b}^*(1)} = T)\, \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T)\lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(\tilde{N}^{k,\underline{b}^*(1)})]\lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}\left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \lambda(dz) \\ =& \int_{[0,1]} \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}\left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \lambda(dz) \\ =& \int_{(0,1)} \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}\left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \lambda(dz) \end{aligned}$$

where the first equality follows from equation (8), the second from the definition of the expectation operator over the random variable $\tilde{N}^{k,\underline{b}^*(1)}$, the third from equation (7), the fourth from the uniform convergence of the functions $z \mapsto \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}))]$ as $k \to +\infty$,[20] and the last from $\lambda(\{0,1\}) = 0$. Thus, we have

$$\lim_{k\to+\infty} Sh_{\underline{b}^*}(M^k) = \int_{(0,1)} \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}\left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \lambda(dz). \tag{9}$$

Next, we use the coalition-valued random variable $(\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}))$ to define the game-valued random variable $\tilde{M}^k$, which is a multiple-partners game with a random population:

$$\tilde{M}^k := \langle \underline{B}, \underline{S}, \tilde{\mathbf{y}}^k, \mathbf{a}, \mathbf{q} \rangle,$$

where the random vector $\tilde{\mathbf{y}}^k = (\tilde{y}^k(\underline{b}_1), \ldots, \tilde{y}^k(\underline{b}_{n_b}); \tilde{y}^k(\underline{s}_1), \ldots, \tilde{y}^k(\underline{s}_{n_s}))$ is defined by $\tilde{y}^k(\underline{b}) = |\tilde{B}^k_{\underline{b}}|$ for all $\underline{b} \in \underline{B}$ and $\tilde{y}^k(\underline{s}) = |\tilde{S}^k_{\underline{s}}|$ for all $\underline{s} \in \underline{S}$. The components in $\tilde{\mathbf{y}}^k$ are mutually independent, and their probability distributions are

$$P(\tilde{y}^k(\underline{b}^*) = y) = \binom{k-1}{y} z^y (1-z)^{k-y-1}, \tag{10}$$

[19] To lighten the notation, we do not indicate that the random variables depend on z.

[20] Notice that the k-th function $z \mapsto \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}))]$ is continuous and defined on a compact set $[0,1]$ for every $k \in \mathbb{Z}_+$.

for all $y = 0, \dots, k-1$, and

$$P(\tilde{y}^k(\underline{b}) = y) = P(\tilde{y}^k(\underline{s}) = y) = \binom{k}{y} z^y (1-z)^{k-y}, \tag{11}$$

for all $\underline{b} \in \underline{B} \setminus \{\underline{b}^*\}$, $\underline{s} \in \underline{S}$, and $y = 0, \dots, k$.

Consider a realization $(\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}})$ of the random variable $(\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}})$ and let $\bar{\mathbf{y}}^k$ be the corresponding realization of the variable $\tilde{\mathbf{y}}^k$. We construct the game $\bar{M}^k$, which is the game with the set of players $(\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}})$ with the addition of the player $\underline{b}^*(1)$. That is, $\bar{M}^k := \langle \underline{B}, \underline{S}, (\bar{\mathbf{y}}^k)', \mathbf{a}, \mathbf{q} \rangle$ is the game corresponding to $(\bar{\mathbf{y}}^k)'$, where $(\bar{\mathbf{y}}^k)'(\underline{b}^*) = \bar{\mathbf{y}}^k(\underline{b}^*) + 1$, $(\bar{\mathbf{y}}^k)'(\underline{b}) = \bar{\mathbf{y}}^k(\underline{b})$ for all $\underline{b} \in \underline{B} \setminus \{\underline{b}^*\}$, and $(\bar{\mathbf{y}}^k)'(\underline{s}) = \bar{\mathbf{y}}^k(\underline{s})$ for all $\underline{s} \in \underline{S}$.[21] The TU game associated with $\bar{M}^k$ is a subgame of the game associated with M^k. Hence, $\mathrm{D}^{\underline{b}^*(1)} v^{\bar{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}}) \right) = \mathrm{D}^{\underline{b}^*(1)} v^{M^k} \left((\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}}) \right)$. We rewrite equation (9) as:

$$\lim_{k\to+\infty} Sh_{\underline{b}^*}(M^k) = \int_{(0,1)} \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \lambda(dz), \tag{12}$$

where we write $v^{\tilde{M}^k}$ to indicate that $\tilde{M}^k$ is a random game whose players are derived from the random vector that determines the composition of buyers and sellers.

To separate the games $\bar{M}^k$ that are large and sufficiently uneven from those that are not, we define the indicator function $I_{\bar{M}^k}$:

$$I_{\bar{M}^k} = \begin{cases} 1 & \text{if } \bar{M}^k \text{ is large and sufficiently uneven,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, denote by $I_{\tilde{M}^k}$ the random indicator function, depending on the realization of the random variable $\tilde{M}^k$. Then, inspecting the integrand in (12), we have

$$\begin{aligned}
&\lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right] \\
&= \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) I_{\tilde{M}^k} \right. \\
&\qquad \left. + \mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) (1 - I_{\tilde{M}^k}) \right] \\
&= \lim_{k\to+\infty} \mathbb{E}[q(\underline{b}^*) \underline{u}_{\underline{b}^*}(\tilde{M}^k) I_{\tilde{M}^k}] \\
&\qquad + \lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) (1 - I_{\tilde{M}^k}) \right],
\end{aligned} \tag{13}$$

where the first equality follows from additivity of the expectation operator and the second from Corollary 2, which allows replacing the buyer $\underline{b}^*(1)$'s marginal contribution to a sufficiently large uneven assignment game with $q(\underline{b}^*) \underline{u}_{\underline{b}^*}(\tilde{M}^k)$.

We separately analyze the two addends of (13).

Concerning the second addend, we claim that

$$\lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) (1 - I_{\tilde{M}^k}) \right] = 0. \tag{14}$$

To prove (14), we first note that a player's marginal contribution to any coalition is bounded: $0 \le \mathrm{D}^{\underline{b}^*(1)} v^{\bar{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}}) \right) \le q(\underline{b}^*) \max_{\underline{s}\in\underline{S}} a_{\underline{b}^*\underline{s}}$, for any realization of the random variable. Therefore, (14) holds if

$$\lim_{k\to+\infty} \mathbb{E}[1 - I_{\tilde{M}^k}] = 0. \tag{15}$$

To put it differently, the probability that $\tilde{M}^k$ is not sufficiently uneven converges to 0 as $k \to +\infty$. To show this property, it suffices to verify that

$$\lim_{k\to+\infty} P\left(\Big| \sum_{\underline{b}\in H} \tilde{y}^k(\underline{b}) q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s}) q(\underline{s}) \Big| > q^{\mathrm{Max}} \right) = 1, \tag{16}$$

[21] We denote the game $\bar{M}^k$ instead of $(\bar{M}^k)'$, as in Corollary 1, for notational simplicity.

for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and $G \in 2^{\underline{S}} \setminus \{\emptyset\}$, where we write $q^{\text{Max}} = q^{\max} + q(\underline{b}^*)$.[22] We distinguish two cases for the pairs (H, G) to prove (16): Case (a), when $\sum_{\underline{b}\in H} q(\underline{b}) \neq \sum_{\underline{s}\in G} q(\underline{s})$ and Case (b), when $\sum_{\underline{b}\in H} q(\underline{b}) = \sum_{\underline{s}\in G} q(\underline{s})$.

For Case (a), assume, without loss of generality, that $\sum_{\underline{b}\in H} q(\underline{b}) > \sum_{\underline{s}\in G} q(\underline{s})$.[23] By the Chebyshev's inequality,[24] we have

$$P\left(\left|\frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b})}{k\sum_{\underline{b}\in H} q(\underline{b})} - z\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2 \sum_{\underline{b}\in H} q(\underline{b})}, \tag{17}$$

for all $\epsilon \in \mathbb{R}_{++}$ and all $k \in \mathbb{Z}_+$. Similarly, $P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{k\sum_{\underline{s}\in G} q(\underline{s})} - z\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2\sum_{\underline{s}\in G} q(\underline{s})}$ for all $\epsilon \in \mathbb{R}_{++}$ and all $k \in \mathbb{Z}_+$. Denote by $c := \frac{\sum_{\underline{s}\in G} q(\underline{s})}{\sum_{\underline{b}\in H} q(\underline{b})} \in (0,1)$. Notice that $\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{k\sum_{\underline{s}\in G} q(\underline{s})} - z\right| \geq \epsilon$ if and only if $\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{k\sum_{\underline{b}\in H} q(\underline{b})} - cz\right| \geq c\epsilon$. Then,

$$P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{k\sum_{\underline{b}\in H} q(\underline{b})} - cz\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2\sum_{\underline{s}\in G} q(\underline{s})}, \tag{18}$$

for all $\epsilon \in \mathbb{R}_{++}$ and all $k \in \mathbb{Z}_+$.

We use (17) and (18) to prove (16) in Case (a). Choose $\epsilon < z - cz$ and pick an arbitrary $\delta \in \mathbb{R}_{++}$. Using (17) and (18), there is $D \in \mathbb{Z}_+$[25] such that $P\left(\left|\frac{\sum_{\underline{b}\in H} \tilde{y}^d(\underline{b})q(\underline{b})}{d\sum_{\underline{b}\in H} q(\underline{b})} - z\right| \geq \frac{\epsilon}{2}\right) \leq \frac{\delta}{2}$, $P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^d(\underline{s})q(\underline{s})}{d\sum_{\underline{b}\in H} q(\underline{b})} - cz\right| \geq \frac{\epsilon}{2}\right) \leq \frac{\delta}{2}$, and $\frac{q^{\text{Max}}}{d\sum_{\underline{b}\in H} q(\underline{b})} < z - cz - \epsilon$, for all $d \geq D$. This implies that $P(\sum_{\underline{b}\in H} \tilde{y}^d(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^d(\underline{s})q(\underline{s}) > q^{\text{Max}}) = P\left(\frac{\sum_{\underline{b}\in H} \tilde{y}^d(\underline{b})q(\underline{b})}{d\sum_{\underline{b}\in H} q(\underline{b})} - \frac{\sum_{\underline{s}\in G} \tilde{y}^d(\underline{s})q(\underline{s})}{d\sum_{\underline{b}\in H} q(\underline{b})} > \frac{q^{\text{Max}}}{d\sum_{\underline{b}\in H} q(\underline{b})}\right) \geq P\left(\frac{\sum_{\underline{b}\in H} \tilde{y}^d(\underline{b})q(\underline{b})}{d\sum_{\underline{b}\in H} q(\underline{b})} - \frac{\sum_{\underline{s}\in G} \tilde{y}^d(\underline{s})q(\underline{s})}{d\sum_{\underline{b}\in H} q(\underline{b})} > z - cz - \epsilon\right) \geq 1 - \delta$. Since this inequality holds for all sufficiently small $\epsilon > 0$, then,

$$\lim_{k\to+\infty} P(|\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})| > q^{\text{Max}}) = 1$$

when $\sum_{\underline{b}\in H} q(\underline{b}) \neq \sum_{\underline{s}\in G} q(\underline{s})$.

For Case (b), where $\sum_{\underline{b}\in H} q(\underline{b}) = \sum_{\underline{s}\in G} q(\underline{s})$, let d−lim be the limit operator with respect to convergence in distribution. By the de Moivre-Laplace central limit theorem,

$$\text{d−lim}_{k\to+\infty} \frac{\tilde{y}^k(\underline{b}) - kz}{\sqrt{kz(1-z)}} = \tilde{\xi}_{\underline{b}} \text{ and } \text{d−lim}_{k\to+\infty} \frac{\tilde{y}^k(\underline{s}) - kz}{\sqrt{kz(1-z)}} = \tilde{\xi}_{\underline{s}},$$

where $\tilde{\xi}_{\underline{b}}$ and $\tilde{\xi}_{\underline{s}}$ follow the standard normal distribution for all $\underline{b} \in \underline{B}$ and all $\underline{s} \in \underline{S}$. Since the components in $\tilde{\xi} = (\tilde{\xi}_{\underline{b}_1}, \ldots, \tilde{\xi}_{\underline{b}_{n_b}}; \tilde{\xi}_{\underline{s}_1}, \ldots, \tilde{\xi}_{\underline{s}_{n_s}})$ are mutually independent,

$$\begin{aligned}
&\text{d−lim}_{k\to+\infty} \frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{\sqrt{kz(1-z)}} \\
&= \text{d−lim}_{k\to+\infty} \frac{\sum_{\underline{b}\in H}(\tilde{y}^k(\underline{b}) - kz)q(\underline{b}) - \sum_{\underline{s}\in G}(\tilde{y}^k(\underline{s}) - kz)q(\underline{s})}{\sqrt{kz(1-z)}} \\
&= \sum_{\underline{b}\in H} q(\underline{b})\, \text{d−lim}_{k\to+\infty} \frac{(\tilde{y}^k(\underline{b}) - kz)}{\sqrt{kz(1-z)}} - \sum_{\underline{s}\in G} q(\underline{s})\, \text{d−lim}_{k\to+\infty} \frac{(\tilde{y}^k(\underline{s}) - kz)}{\sqrt{kz(1-z)}} \\
&= \sum_{\underline{b}\in H} q(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} q(\underline{s})\tilde{\xi}_{\underline{s}}.
\end{aligned} \tag{19}$$

The random variable $\sum_{\underline{b}\in H} q(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} q(\underline{s})\tilde{\xi}_{\underline{s}}$ follows the normal distribution with mean equal to 0 and variance equal to $\sum_{\underline{b}\in H} q(\underline{b})^2 + \sum_{\underline{s}\in G} q(\underline{s})^2$. Therefore,

$$\lim_{k\to+\infty} P\left(|\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})| \leq q^{\text{Max}}\right)$$

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[22] We use q^{Max} instead of $q^{\max}$ in equation (16) because the buyer $\underline{b}^*(1)$ is also in the game.

[23] We do not distinguish further between $\underline{b}^* \in H$ and $\underline{b}^* \notin H$ because k and $k-1$ are of the same order as $k \to +\infty$ when applying the Chebyshev's inequality and the central limit theorem.

[24] See, e.g., Shiryaev (2016).

[25] Such a D exists because, given ϵ, we can select k large enough so that the right-hand side of equations (17) and (18) are arbitrarily small.

$$= \lim_{k\to+\infty} P\left(-\frac{q^{\text{Max}}}{\sqrt{kz(1-z)}} \le \frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})}{\sqrt{kz(1-z)}} \le \frac{q^{\text{Max}}}{\sqrt{kz(1-z)}} \right)$$
$$= P\left(\sum_{\underline{b}\in H} q(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} s_{\underline{s}}\tilde{\xi}_{\underline{s}} = 0 \right) = 0.$$

Thus, the probability that $\tilde{M}^k$ is not sufficiently uneven also converges to 0 in Case (b). Therefore, (14) holds, and the second addend of (13) is equal to zero.

We now analyze the first addend of (13):

$$\lim_{k\to+\infty} \mathbb{E}[q(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)I_{\tilde{M}^k}] = \lim_{k\to+\infty} \mathbb{E}[q(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)]$$
$$= \lim_{k\to+\infty} \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})q(\underline{b}^*)\underline{u}_{\underline{b}^*}(\mathcal{M})$$
$$= q(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}),$$

where the first equality holds because the probability that $\tilde{M}^k$ is sufficiently uneven converges to 1 and $q(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)$ is bounded, the second (where we denote $P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})$ the probability that $\tilde{M}^k$ is in the class $\mathcal{M} \in T(\underline{B},\underline{S},\mathbf{q})$) follows from the finiteness of $T(\underline{B},\underline{S},\mathbf{q})$ (Remark 3), by which we can take the summation ranging over each class of games, and the last equality follows from the additivity of the limit operator.

Therefore, going back to equation (13), we write

$$\lim_{k\to+\infty} \mathbb{E}\left[\mathrm{D}^{\underline{b}^*(1)} v^{\tilde{M}^k} \left((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \right) \right]$$
$$= q(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}). \tag{20}$$

We now discuss about $\left(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\right)_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})}$, which is a probability distribution over the elements of the partition $T(\underline{B},\underline{S},\mathbf{q})$. Remember that an element of $T(\underline{B},\underline{S},\mathbf{q})$ is an equivalence class characterized by inequalities over the pairs (H,G), where $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ (see Remark 4).

Some equivalence classes have a zero probability in $\left(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\right)_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})}$. To see this, consider a pair (H,G) such that $\sum_{\underline{b}\in H} q(\underline{b}) > \sum_{\underline{s}\in G} q(\underline{s})$. According to the weak law of large numbers, the probability of $\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) > \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})$ converges to 1 as k tends to $+\infty$. Therefore, in the distribution $\left(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\right)_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})}$, the probability of being in an equivalence class where $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) \le \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ if $\sum_{\underline{b}\in H} q(\underline{b}) > \sum_{\underline{s}\in G} q(\underline{s})$ is zero. A similar argument applies for those pairs (H,G) for which $\sum_{\underline{b}\in H} q(\underline{b}) < \sum_{\underline{s}\in G} q(\underline{s})$.

Consider now a pair (H,G) for which $\sum_{\underline{b}\in H} q(\underline{b}) = \sum_{\underline{s}\in G} q(\underline{s})$. Following the equation (19), $\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})q(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})q(\underline{s})$ converges in distribution to $\sum_{\underline{b}\in H} q(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} q(\underline{s})\tilde{\xi}_{\underline{s}}$, which follows the normal distribution with mean equal to 0 and variance equal to $\sum_{\underline{b}\in H} q(\underline{b})^2 + \sum_{\underline{s}\in G} q(\underline{s})^2$ as $k \to +\infty$.

Therefore, the equivalence class that $\tilde{M}^k$ belongs to converges in distribution to $\tilde{\mathcal{M}}^{\tilde{\xi}}$, which is determined by the random vector $\tilde{\xi}$ and it is independent of both z and the choice of player $\underline{b}^*(1)$. Moreover, the $T(\underline{B},\underline{S},\mathbf{q})$-valued random variable $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is defined as follows. If the realization of $\tilde{\xi}$ is $\bar{\xi}$, then the realization of $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is the class $\mathcal{M}$ given by:

1. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $\sum_{\underline{b}\in H} q(\underline{b}) < \sum_{\underline{s}\in G} q(\underline{s})$;
2. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $\sum_{\underline{b}\in H} q(\underline{b}) > \sum_{\underline{s}\in G} q(\underline{s})$;
3. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $\sum_{\underline{b}\in H} q(\underline{b}) = \sum_{\underline{s}\in G} q(\underline{s})$ if and only if $\sum_{\underline{b}\in H} \bar{\xi}_{\underline{b}} q(\underline{b}) < \sum_{\underline{s}\in G} \bar{\xi}_{\underline{s}} q(\underline{s})$.

Then, going back to equation (12):

$$\lim_{k\to+\infty} Sh_{\underline{b}^*}(M^k) = \int_{(0,1)} q(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M})\lambda(dz)$$
$$= q(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} P(\tilde{\mathcal{M}}^{\tilde{\xi}} = \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}) = q(\underline{b}^*)\mathbb{E}[\underline{u}_{\underline{b}^*}(\tilde{\mathcal{M}}^{\tilde{\xi}})], \tag{21}$$

where the first equality uses (13) and (20) in equation (12) and the second holds because the equivalence class that $\tilde{M}^k$ belongs to converges in distribution to $\tilde{\mathcal{M}}^{\tilde{\xi}}$.

The proof for the convergence of the payoff of an arbitrary seller is the same. Therefore, we have proven the following:

$$\lim_{k\to+\infty} Sh_{\underline{b}}(M^k) = q(\underline{b})\mathbb{E}[\underline{u}_{\underline{b}}(\tilde{\mathcal{M}}^{\tilde{\xi}})] \quad \text{and} \quad \lim_{k\to+\infty} Sh_{\underline{s}}(M^k) = q(\underline{s})\mathbb{E}[\underline{p}_{\underline{s}}(\tilde{\mathcal{M}}^{\tilde{\xi}})], \tag{22}$$

for all $\underline{b} \in \underline{B}$, $\underline{s} \in \underline{S}$.

Finally, we show that $\lim_{k\to+\infty} Sh(M^k) \in \mathcal{CE}(M^K)$, where $K = q^{\max} + 1$. We know that an outcome is a CE outcome if and only if it is stable and all the prices of the objects every seller owns are equal (Sotomayor, 2007). Since the Shapley value satisfies the equal treatment property, we prove that the $\lim_{k\to+\infty} Sh(M^k)$ is a CE outcome showing it is feasible and stable.

We use the linear programming approach introduced by Sotomayor (1992). Let $k \geq K$. The total payoff of any CE outcome is equal to $v^{M^k}(B^k \cup S^k)$, which can be computed through the following primal problem:

$$\begin{aligned} v^{M^k}(B^k \cup S^k) = \max_{\mathbf{x}\in\mathbb{R}_+^{B^k\times S^k}} & \sum_{(\underline{b}(h),\underline{s}(g))\in B^k\times S^k} a_{\underline{b}\underline{s}} x_{\underline{b}(h)\underline{s}(g)} \\ \text{s.t.} & \sum_{\underline{s}(g)\in S^k} x_{\underline{b}(h)\underline{s}(g)} \leq q(\underline{b}) \text{ for all } \underline{b} \in \underline{B} \text{ and } h = 1,\dots,k, \\ & \sum_{\underline{b}(h)\in B^k} x_{\underline{b}(h)\underline{s}(g)} \leq q(\underline{s}), \text{ for all } \underline{s} \in \underline{S} \text{ and } g = 1,\dots,k, \\ & x_{\underline{b}(h)\underline{s}(g)} \leq 1 \text{ for all } \underline{b} \in \underline{B}, \underline{s} \in \underline{S}, \text{ and } h,g = 1,\dots,k. \end{aligned}$$

Its dual problem is:

$$\begin{aligned} \min_{\mathbf{y}\in\mathbb{R}_+^{B^k},\mathbf{z}\in\mathbb{R}_+^{S^k},\mathbf{w}\in\mathbb{R}_+^{B^k\times S^k}} & \sum_{\underline{b}(h)\in B^k} q(\underline{b})y_{\underline{b}(h)} + \sum_{\underline{s}(g)\in S^k} q(\underline{s})z_{\underline{s}(g)} + \sum_{\underline{b}(h)\in B^k}\sum_{\underline{s}(g)\in S^k} w_{\underline{b}(h)\underline{s}(g)} \\ \text{s.t. } & y_{\underline{b}(h)} + z_{\underline{s}(g)} + w_{\underline{b}(h)\underline{s}(g)} \geq a_{\underline{b}\underline{s}} \text{ for all } \underline{b}(h) \in B^k \text{ and } \underline{s}(g) \in S^k. \end{aligned} \tag{23}$$

In this dual problem, $y_{\underline{b}(h)}$ (resp., $z_{\underline{s}(g)}$) is the utility that each buyer (resp., seller) obtains in each transaction. Then, by equation (22), we show that $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$, defined by $y^*_{\underline{b}(h)} = (\lim_{k\to+\infty} Sh_{\underline{b}}(M^k))/q(\underline{b})$ for every $\underline{b}(h) \in B^k$, $z^*_{\underline{s}(g)} = (\lim_{k\to+\infty} Sh_{\underline{s}}(M^k))/q(\underline{s})$ for every $\underline{s}(g) \in S^k$, and $\mathbf{w}^* = \mathbf{0}$, constitutes a solution to the dual problem.

To this end, let $y^k_{\underline{b}(h)} := (Sh_{\underline{b}}(M^k))/q(\underline{b})$, $z^k_{\underline{s}(g)} := (Sh_{\underline{s}}(M^k))/q(\underline{s})$. Then,

$$\begin{aligned} & \sum_{\underline{b}(h)\in B^k} q(\underline{b})y^k_{\underline{b}(h)} + \sum_{\underline{s}(g)\in S^k} q(\underline{s})z^k_{\underline{s}(g)} + \sum_{\underline{b}(h)\in B^k}\sum_{\underline{s}(g)\in S^k} w^*_{\underline{b}(h)\underline{s}(g)} \\ = & \sum_{\underline{b}\in\underline{B}} kq(\underline{b})\frac{Sh_{\underline{b}}(M^k)}{q(\underline{b})} + \sum_{\underline{s}\in\underline{S}} kq(\underline{s})\frac{Sh_{\underline{s}}(M^k)}{q(\underline{s})} \\ = & \sum_{\underline{b}\in\underline{B}}\sum_{h=1}^{k} Sh_{\underline{b}(h)}(B^k\cup S^k, v^{M^k}) + \sum_{\underline{s}\in\underline{S}}\sum_{g=1}^{k} Sh_{\underline{s}(g)}(B^k\cup S^k, v^{M^k}) = v^{M^k}(B^k\cup S^k), \end{aligned}$$

where the last equality holds because the Shapley value is efficient. It is evident that the equality also holds when $k \to +\infty$, that is, for $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$. Hence, we have shown that the payoff vector $\lim_{k\to+\infty} Sh(M^k)$ is feasible. We also need to show that $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$ satisfies the stability constraints (23). For each $\underline{b} \in \underline{B}$ and each $\underline{s} \in \underline{S}$,

$$\begin{aligned} & y^*_{\underline{b}(h)} + z^*_{\underline{s}(g)} + w^*_{\underline{b}(h)\underline{s}(g)} = \lim_{k\to+\infty}\frac{Sh_{\underline{b}}(M^k)}{q(\underline{b})} + \lim_{k\to+\infty}\frac{Sh_{\underline{s}}(M^k)}{q(\underline{s})} = \mathbb{E}[\underline{u}_{\underline{b}}(\tilde{\mathcal{M}}^{\tilde{\xi}})] + \mathbb{E}[\underline{p}_{\underline{s}}(\tilde{\mathcal{M}}^{\tilde{\xi}})] \\ = & \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}}(\mathcal{M}) + \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M})\underline{p}_{\underline{s}}(\mathcal{M}) \\ = & \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M})\left[\underline{u}_{\underline{b}}(\mathcal{M}) + \underline{p}_{\underline{s}}(\mathcal{M})\right] \\ \geq & \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M})a_{\underline{b}\underline{s}} = a_{\underline{b}\underline{s}}, \end{aligned}$$

where the first equality follows the definition of $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$, the second from equation (22), the third from equation (21), the fifth from $\underline{u}_{\underline{b}}(\mathcal{M}) + \underline{p}_{\underline{s}}(\mathcal{M}) \geq a_{\underline{b}\underline{s}}$ for all $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{q})$ because $(\underline{\mathbf{u}}(\mathcal{M}), \underline{\mathbf{p}}(\mathcal{M}))$ is the CE payoff vector of any game in $\mathcal{M}$, and the sixth holds because $(\lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}))_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{q})}$ constitutes a probability distribution. Therefore, we have proven that the constraints (23) are satisfied.

We note that individual rationality means that $Sh_{\underline{b}}(M^k)/q(\underline{b}) \geq 0$ for all $\underline{b} \in \underline{B}$, and $Sh_{\underline{s}}(M^k)/q(\underline{s}) \geq 0$ for all $\underline{s} \in \underline{S}$. It holds because the Shapley value can be represented as an expected marginal contribution, and a buyer's or seller's marginal contribution is always greater than zero in any multiple-partners game.

Therefore, we have shown that the limit of the Shapley value of any replicated multiple-partners game is a stable outcome of a sufficiently large replicated game. Moreover, consider an outcome that satisfies equal treatment of equals and equal treatment of partnerships so that we can write the constraints of the dual problem as in (23). Then, by inspecting the dual problem, the outcome satisfies (23) for $k \geq K$ if it satisfies the constraints for K. Since the set of CE satisfies the equal treatment properties if $k \geq K$ (Proposition 1), and it is the set of stable outcomes satisfying the equal treatment properties (see Remark 1), then the set of CE is the same for all $k \geq K$.

Finally, we claim that there is a matching that supports the payoff vector, i.e., the payoff vector is feasible. Indeed, let x^* be an integer solution to the primal problem when $k = K$.[26] Then x^* is a feasible matching for M^K. Moreover, $y^*_{\underline{b}(h)} + z^*_{\underline{s}(g)} > a_{\underline{bs}}$ implies $x^*_{\underline{b}(h)\underline{s}(g)} = 0$ for every buyer and seller by the complementary slackness theorem (see, e.g., Vanderbei, 2021). Thus $(\mathbf{y}^*, \mathbf{z}^*)$ is compatible with x^*; hence it is a feasible payoff vector for M^K. □

7. The asymptotic value in supermodular and monotonic assignment games

Theorem 1 ensures that the Shapley value of the replica of any multiple-partners game converges to a CE payoff. Since the set of CE payoffs of a multiple-partners game is often not a singleton, it is natural to enquire which equilibrium the Shapley value converges to. Although we cannot answer this question for the general multiple-partners game, we build on the analysis developed by Schwarz and Yenmez (2011) to identify the CE payoff that is the limit of the Shapley value of the replica market for supermodular and monotonic assignment games.

In the assignment game, there are two stable payoff vectors (i.e., CE payoff vectors) that are the best one for one side of the market and simultaneously the worst for the other (Shapley and Shubik, 1972). These two extreme points are called the buyer-optimal and the seller-optimal stable payoff vectors. Schwarz and Yenmez (2011) studied the mean stable imputation, that is, the vector of the averages between the buyer-optimal and the seller-optimal stable payoffs. They also defined (using an appropriate Lebesgue measure) the "median stable imputation," where each player simultaneously obtains the median of their utility in the set of stable payoffs. They proved that the mean and the median payoff vectors coincide when the assignment game is supermodular and monotonic. We show that, within this class of assignment games, the mean (hence, also the median) stable payoff vector coincides with the limit of the Shapley value of its k-fold replica.

We briefly introduce the necessary elements and state and prove the result.

Consider $M = \langle B, S, \mathbf{a} \rangle$ an assignment game, with $B = \{b_1, \ldots, b_{n_b}\}$ and $S = \{s_1, \ldots, s_{n_s}\}$. Hence, we assume $q(b) = q(s) = 1$ for all $b \in B$ and $s \in S$. Moreover, let $n_b = n_s = n$.[27]

Definition 13. *The assignment game $M = \langle B, S, \mathbf{a} \rangle$ is supermodular if $a_{b_i s_j} + a_{b_{i'} s_{j'}} \geq a_{b_{i'} s_j} + a_{b_i s_{j'}}$ for all $b_i, b_{i'} \in B$ and $s_j, s_{j'} \in S$ such that $i \geq i'$ and $j \geq j'$.*

Definition 14. *The assignment game $M = \langle B, S, \mathbf{a} \rangle$ is monotonic if $a_{b_i s_j} \geq a_{b_{i'} s_{j'}}$ for all $b_i, b_{i'} \in B$ such that $i \geq i'$ and all $s_j, s_{j'} \in S$ such that $j \geq j'$.*

The supermodularity and monotonicity of an assignment game imply that there is an assortative matching where the buyers with the higher indexes are matched with the sellers with the higher indexes.[28]

We denote by $(\mathbf{u}^{bo}, \mathbf{p}^{bo})$ and $(\mathbf{u}^{so}, \mathbf{p}^{so})$ the buyer-optimal and the seller-optimal stable payoff vectors, respectively. Computing these two extreme payoff vectors for games with stable assortative matching is standard by constructing a chain of indifference relations. We state it without proof.[29]

Lemma 2. *If the assignment game $M = \langle B, S, \mathbf{a} \rangle$ is supermodular and monotonic, then*

$$u^{bo}_{b_i} = \begin{cases} a_{b_1 s_1} & i = 1 \\ a_{b_i s_i} - \sum_{m=2}^{i} (a_{b_{(m-1)} s_m} - a_{b_{(m-1)} s_{(m-1)}}) & i = 2, \ldots, n \end{cases} \text{ and}$$

$$p^{bo}_{s_j} = \begin{cases} 0 & j = 1 \\ \sum_{m=2}^{j} (a_{b_{(m-1)} s_m} - a_{b_{(m-1)} s_{(m-1)}}) & j = 2, \ldots, n \end{cases};$$

$$u^{so}_{b_i} = \begin{cases} 0 & i = 1 \\ \sum_{m=2}^{i} (a_{b_m s_{(m-1)}} - a_{b_{(m-1)} s_{(m-1)}}) & i = 2, \ldots, n \end{cases} \text{ and}$$

$$p^{so}_{s_j} = \begin{cases} a_{b_1 s_1} & j = 1 \\ a_{b_j s_j} - \sum_{m=2}^{j} (a_{b_m s_{(m-1)}} - a_{b_{(m-1)} s_{(m-1)}}) & j = 2, \ldots, n \end{cases}.$$

[26] It is well-known that such a solution can be found using the simplex method.

[27] We can take $n_b = n_s = n$ since we can add $|n_b - n_s|$ null players to the shorter side of the game without consequences for the set of stable payoffs.

[28] See, for instance, Chade et al. (2017) for an introduction of this result in the assignment game.

[29] See Eriksson et al. (2000).

Denote by $(\mathbf{u}^{mean}, \mathbf{p}^{mean})$ the mean of $(\mathbf{u}^{bo}, \mathbf{p}^{bo})$ and $(\mathbf{u}^{so}, \mathbf{p}^{so})$. Corollary 3 is immediate after Lemma 2:

Corollary 3. *If the assignment game $M = \langle B, S, \boldsymbol{a}\rangle$ is supermodular and monotonic, then*

$$u_i^{mean} = \begin{cases} \frac{a_{b_1 s_1}}{2} & i = 1 \\ \frac{a_{b_i s_i} + \sum_{m=2}^{i}(a_{b_m s_{(m-1)}} - a_{b_{(m-1)} s_m})}{2} & i = 2, \dots, n \end{cases} \quad \text{and}$$

$$p_j^{mean} = \begin{cases} \frac{a_{b_1 s_1}}{2} & j = 1 \\ \frac{a_{b_j s_j} + \sum_{m=2}^{j}(a_{b_{(m-1)} s_m} - a_{b_m s_{(m-1)}})}{2} & j = 2, \dots, n \end{cases}.$$

We now state the main result of this section. Let M^k be the k-fold replica of the assignment game M. Denote $u_b^{asy} := \lim_{k\to+\infty} Sh_{\underline{b}}(M^k)$ and $p_s^{asy} := \lim_{k\to+\infty} Sh_{\underline{s}}(M^k)$.

Proposition 4. *If the assignment game M is supermodular and monotonic, then $(\mathbf{u}^{asy}, \mathbf{p}^{asy}) = (\mathbf{u}^{mean}, \mathbf{p}^{mean})$.*

Proposition 4 highlights that, for supermodular and monotonic assignment games, the Shapley value of the replica game converges to the CE payoff vector that can be thought of as the "fairest," as it corresponds to the mean of the players' best and worst CE payoffs and the median of all equilibrium payoffs. Proposition 4, combined with Schwarz and Yenmez's, suggests that, restricting to monotonic and supermodular assignment games, this payoff vector may be the most reasonable solution.

The mean and the median payoff vectors may differ when the assignment game is not monotonic or supermodular. Similarly, the asymptotic value does not necessarily coincide with either of them, as we show through the following numerical example.

Example 2. *Consider the assignment game $M = \langle B, S, \boldsymbol{a}\rangle$, where $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$, $a_{b_1 s_1} = a_{b_1 s_2} = 2$, $a_{b_2 s_1} = 1$, and $a_{b_2 s_2} = 3$. It is easy to check that the example satisfies supermodularity but not monotonicity.*

The limit of the Shapley value for this game is the CE payoff vector $(\mathbf{u}^{asy}, \mathbf{p}^{asy}) = (\frac{9}{8}, \frac{11}{8}, \frac{7}{8}, \frac{13}{8}) = (1.125, 1.375, 0.875, 1.625)$.

The optimal (hence, the stable) matching in M matches buyer b_1 with seller s_1 and buyer b_2 with seller s_2. Beyond their non-negativity, the constraints that the CE payoffs satisfy are the following. First, $u_{b_1} + p_{s_1} = 2$ and $u_{b_2} + p_{s_2} = 3$, i.e., $p_{s_1} = 2 - u_{b_1}$ and $p_{s_2} = 3 - u_{b_2}$. Second, $u_{b_1} + p_{s_2} \geq 2$ and $u_{b_2} + p_{s_1} \geq 1$. Therefore, $u_{b_1} + 3 - u_{b_2} \geq 2$ (i.e., $u_{b_1} - u_{b_2} \geq -1$) and $u_{b_2} + 2 - u_{b_1} \geq 1$ (i.e., $u_{b_2} - u_{b_1} \geq -1$). Hence, $-1 \leq u_{b_2} - u_{b_1} \leq 1$.

The shadow area in Fig. 1 represents the set of the buyers' CE payoffs in the space (u_{b_1}, u_{b_2}). The buyer-optimal and seller-optimal stable payoff vectors are $(2, 3, 0, 0)$ and $\mathbf{u}^{so} = (0, 0, 2, 3)$, respectively. Hence, the mean stable imputation is $(\mathbf{u}^{mean}, \mathbf{p}^{mean}) = (1, 1.5, 1, 1.5)$ (marked as the gray circle). The median stable imputation is $(9/8, 3 - (\sqrt{14}/2), 7/8, \sqrt{14}/2)$ (marked as the asterisk).

Example 2 highlights that the asymptotic value, the mean stable imputation, and the median stable imputation can be three different payoff vectors in general.

8. Extension of the convergence result to the semivalues

Dubey et al. (1981) relaxed the axiom system of the Shapley value and defined the class of semivalues. A *semivalue* is a single-valued solution satisfying positivity, additivity, equal treatment, null player, and null player out.[30] Interestingly, each semivalue can be uniquely identified by a probability distribution λ over $[0, 1]$ using the same equations (3) and (4) as the Shapley value (we recall that the Shapley value corresponds to the case where λ is the Lebesgue measure). That is, the prescription of the semivalue ψ^λ to a player i in a TU game (N, v) is i's expected marginal contribution, which depends on the distribution λ over $[0, 1]$:

$$\psi_i^\lambda(N, v) = \mathbb{E}[\mathrm{D}^i\, v(\tilde{T})], \tag{24}$$

where the random coalition $\tilde{T}$ follows the probability distribution $P(\tilde{T} = T)$, as defined in (4), for all $T \subseteq N \setminus \{i\}$. Clearly, the prescription of ψ^λ hinges on the choice of λ.

Any semivalue ψ^λ satisfies equal treatment of equals. Hence, we write $\psi_{\underline{b}}^\lambda(M^k)$ and $\psi_{\underline{s}}^\lambda(\underline{M^k})$ to indicate the semivalue of any of the players of the type $\underline{b} \in \underline{B}$ and $\underline{s} \in \underline{S}$.

Theorem 2 generalizes Theorem 1 to all the semivalues $\psi^\lambda(M^k)$ with $\lambda(\{0, 1\}) = 0$. It states that all these semivalues of the replica M^k converge to the same value as the replica becomes large. Therefore, the players' payoffs in these semivalues converge to a CE payoff.

[30] The single-valued solution ψ satisfies the null player out property if $\psi_j(N \setminus \{i\}, v) = \psi_j(N, v)$ for any $j \in N \setminus \{i\}$ if i is a null player in (N, v), for any game (N, v). The Shapley value satisfies the null player out property.

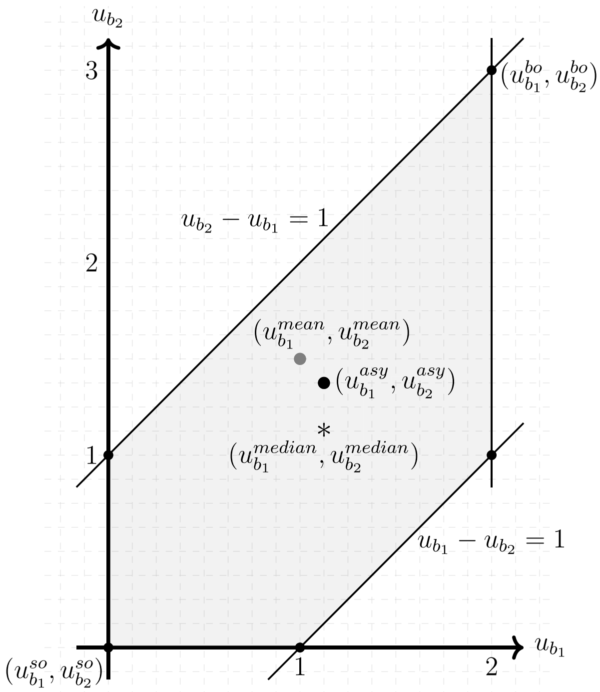

Fig. 1. The asymptotic value, the mean stable imputation, and the median stable imputation of Example 2.

Theorem 2. *Consider the multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{q} \rangle$ and denote $K \equiv r^{\max} + 1$. There exists $(\underline{\mathbf{u}}^K, \underline{\mathbf{p}}^K) \in \mathcal{CE}(M^K)$ such that:*

$$\lim_{k\to+\infty} \psi_{\underline{b}}^{\lambda}(M^k) = q(\underline{b})\underline{u}_{\underline{b}}^K \text{ and } \lim_{k\to+\infty} \psi_{\underline{s}}^{\lambda}(M^k) = q(\underline{s})\underline{p}_{\underline{s}}^K$$

for all $\underline{b} \in \underline{B}$, $\underline{s} \in \underline{S}$, and $\lambda \in \Delta([0,1])$ such that $\lambda(\{0,1\}) = 0$.

We do not write the proof of Theorem 2 because the proof of Theorem 1 did not rely on λ being the Lebesgue measure; all we need is that $\lambda(\{0,1\}) = 0$. Therefore, the limit is the same for all the semivalues that satisfy this condition.

Remark 5. *The proviso that $\lambda(\{0,1\}) = 0$ does not rule out the possibility that a semivalue with $\lambda(\{0,1\}) > 0$ converges to the same limit for some games. For instance, consider an asymmetric glove market (Example 1). The semivalue with $\lambda(\{1\}) = 1$ coincides with the CE outcome for every k-fold replica.*

We end this section with a short discussion on the type of convergence of the semivalues, including the Shapley value. In the framework of replicated multiple-partners games, Sotomayor (2019) proved that the set of stable outcomes shrinks finitely to the set of CE payoff vectors, which in turn shrinks finitely to the set of stable outcomes satisfying equal treatment of equals and equal treatment of partnerships. By contrast, the semivalues converge differently to a CE payoff vector, as stated in Theorem 1. Although the limit of a semivalue is a CE payoff vector, the semivalue of an arbitrarily large finite replica may not be. For example, consider an asymmetric glove market. On the one hand, any semivalue prescribes a strictly positive payoff to each player from the long side since, for every finite replica, there is always a non-negligible chance that this player joins a coalition with a majority of players from the short side. On the other hand, the unique CE outcome prescribes a zero payoff to players from the long side.

The example of an asymmetric glove market does not preclude the possibility of finding, for each game, a distinct semivalue that finitely converges to a CE outcome. To illustrate this point, we go back to Example 2. The limit of the semivalues for this game is the CE payoff vector $(1.125, 1.375, 0.875, 1.625)$. For the k-fold replica of the game, we compute some semivalues determined by a distribution over $[0,1]$ given by a Beta distribution, characterized by two parameters $\alpha, \beta > 0$, and those given by a Dirac probability measure concentrated on a point $q \in [0,1]$. The semivalue corresponding to $\alpha = \beta = 1$ is the Shapley value, and the one with $q = 0.5$ is the Banzhaf value.

Parameters	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
Beta distribution $\alpha = 1$, $\beta = 1$	(1.083, 1.417, 0.917, 1.583)	(1.098, 1.402, 0.902 ,1.598)	(1.104, 1.396, 0.896, 1.604)	(1.108, 1.392, 0.892, 1.608)	(1.110, 1.390, 0.890, 1.610)	(1.112, 1.388, 0.888, 1.612)
Beta distribution $\alpha = 0.5$, $\beta = 0.5$	(1.063, 1.438, 0.938, 1.563)	(1.076, 1.424, 0.924, 1.576)	(1.083, 1.417, 0.917, 1.583)	(1.088, 1.412, 0.912, 1.588)	(1.091, 1.409, 0.901, 1.591)	(1.094, 1.406, 0.906, 1.594)
Dirac measure $q = 0.5$	(1.125, 1.375, 0.875, 1.625)	(1.125, 1.375, 0.875, 1.625)	(1.125, 1.375, 0.875, 1.625)	(1.125, 1.375, 0.875, 1.625)	(1.125, 1.375, 0.875, 1.625)	(1.125, 1.375, 0.875, 1.625)
Dirac measure $q = 0.4$	(0.992, 1.152, 0.752, 1.392)	(1.050, 1.250, 0.802, 1.498)	(1.067, 1.280, 0.819, 1.529)	(1.076, 1.295, 0.827, 1.544)	(1.082, 1.305, 0.833, 1.554)	(1.312 ,1.086, 0.837, 1.561)
Dirac measure $q = 0$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$

The Banzhaf value ($q = 0.5$) attains the limit from the beginning and stays constant henceforth. The other semivalues approach the CE payoff vector but do not reach it. Finally, the semivalue with $q = 0$, which is outside the subclass that we study, does not converge to any CE payoff vector.

9. Conclusion

The classic solution concepts for the multiple-partners game, as for matching games in general, are stability and competitive equilibrium. Single-valued solutions concepts, such as the Shapley value, are not well-studied. In this paper, we have contributed to a better understanding of the behavior of the Shapley value and many other semivalues in the multiple-partners game. We have shown that when the game is replicated, they all converge to the same competitive equilibrium payoff vector.

Sotomayor's (2019) analysis of the replicated multiple-partners game concluded that the sets of stable payoff vectors, CE payoff vectors, and stable payoff vectors that satisfy equal treatment of equals and equal treatment of partnerships converge finitely to the same set. By contrast, the convergence of the Shapley value and the other semivalues is generally not finite.

CRediT authorship contribution statement

Chenghong Luo: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **David Pérez-Castrillo:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Chaoran Sun:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A

Proof of Proposition 1. Take any CE outcome of $\underline{M}$ and denote by $p_{\underline{s}(g)}$ the price set by the seller $\underline{s}(g)$ of type $\underline{s}$ for her objects in that outcome. First, we show the equal treatment of equals for the sellers, that is, for any type $\underline{s} \in \underline{S}$, the prices $p_{\underline{s}(g)}$ and $p_{\underline{s}(g')}$ set by two equal sellers of type $\underline{s}$ are the same if $y^{\min} > q^{\max}$. We prove this property by contradiction.

Let $\underline{s}$ be a seller's type such that $p_{\underline{s}(g)} \neq p_{\underline{s}(g')}$, for some g and g'. Denote by $p_{\underline{s},\min} := \min\{p_{\underline{s}(g)} \mid g = 1, \ldots, y(\underline{s})\}$ and split the set of type-$\underline{s}$ sellers $S_{\underline{s}}$ into two subsets, L and F, with $L := \{\underline{s}(g) \mid p_{\underline{s}(g)} = p_{\underline{s},\min}\}$ and $F := \{\underline{s}(g) \mid p_{\underline{s}(g)} > p_{\underline{s},\min}\}$. Both sets are non-empty.

Let $\mathbf{x}$ be the matching in this CE outcome and take $\underline{s}(g) \in F$. Given that $p_{\underline{s}(g)} > p_{\underline{s},\min} \geq 0$, $\underline{s}(g)$ sells all her objects in the CE outcome; hence, the set $C_{\underline{s}(g)}(\mathbf{x})$ of partners of $\underline{s}(g)$ contains exactly $q(\underline{s})$ buyers. Moreover, it must hold that $L \subseteq C_{\underline{b}(h)}(\mathbf{x})$ for every buyer $\underline{b}(h) \in C_{\underline{s}(g)}(\mathbf{x})$ because otherwise $\underline{b}(h)$ would have an incentive to swap the object from $\underline{s}(g)$ with some identical object owned by a seller in $L \setminus C_{\underline{b}(h)}(\mathbf{x})$. Since the quota of the sellers in L is also $q(\underline{s})$, each of them sells all her objects in the CE outcome to the buyers in $C_{\underline{s}(g)}(\mathbf{x})$.

We claim that it must also be the case that $F \subseteq C_{\underline{b}(h)}(\mathbf{x})$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(\mathbf{x})$. If not, there is at least one $\underline{s}(g') \in F$ who does not sell any object to a buyer in $C_{\underline{s}(g)}(\mathbf{x})$. Given that $p_{\underline{s}(g')} > 0$, the seller $\underline{s}(g')$ sells all her objects, which implies that there is a buyer $\underline{b}'(h') \in C_{\underline{s}(g')}(\mathbf{x})$ such that $\underline{b}'(h') \notin C_{\underline{s}(g)}(\mathbf{x})$. However, $\underline{b}'(h')$ has an incentive to swap the object from $\underline{s}(g')$ with an object initially owned by a seller in L that he does not buy since the sellers in L sell all their objects to the buyers in $C_{\underline{s}(g')}(\mathbf{x})$. This is a contradiction, hence, $F \subseteq C_{\underline{b}(h)}(\mathbf{x})$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(\mathbf{x})$.

Hence, we have proven that $S_{\underline{s}} = L \cup F \subseteq C_{\underline{b}(h)}(\mathbf{x})$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(\mathbf{x})$. Since $S_{\underline{s}}$ has at least $y^{\min}$ elements and $C_{\underline{b}(h)}(\mathbf{x})$ has at most $q^{\max}$ elements, $S_{\underline{s}} \subseteq C_{\underline{b}(h)}(\mathbf{x})$ can only happen if $y^{\min} \le q^{\max}$, which leads to a contradiction. Therefore, a CE outcome satisfies equal treatment of equals among sellers if $y^{\min} > q^{\max}$.

Second, we show the equal treatment of partnerships among buyers (the property holds among sellers by the definition of a CE). Hence, we show that $u_{\underline{b}(h)\underline{s}(g)} = u_{\underline{b}(h)\underline{s}'(g')}$ for all $\underline{b} \in \underline{B}$, $h = 1, \ldots, y(\underline{b})$, and $\underline{s}(g), \underline{s}'(g') \in C_{\underline{b}(h)}(\mathbf{x})$. Suppose otherwise, that is, $u_{\underline{b}(h)\underline{s}(g)} > u_{\underline{b}(h)\underline{s}'(g')}$ for some $\underline{s}(g), \underline{s}'(g') \in C_{\underline{b}(h)}(\mathbf{x})$. There exists $\underline{s}(g^*) \notin C_{\underline{b}(h)}(\mathbf{x})$ because $y^{\min} > q^{\max}$, there are at least $y^{\min}$ sellers of type $\underline{s}$, and $C_{\underline{b}(h)}(\mathbf{x})$ has at most $q(\underline{b}) \le q^{\max}$ elements. Then, buyer $\underline{b}(h)$ has an incentive to swap the object from $\underline{s}'(g')$ with an object initially owned by $\underline{s}(g^*)$ because $u_{\underline{b}(h)\underline{s}(g^*)} = a_{\underline{b}\underline{s}} - p_{\underline{s}(g^*)} = a_{\underline{b}\underline{s}} - p_{\underline{s}(g)} = u_{\underline{b}(h)\underline{s}(g)} > u_{\underline{b}(h)\underline{s}'(g')}$, where we have used the property $p_{\underline{s}(g^*)} = p_{\underline{s}(g)}$ that we have proven above. Therefore, a CE outcome satisfies equal treatment of partnerships if $y^{\min} > q^{\max}$.

Finally, once we have proven the equal treatment of partnerships among buyers, we can use an argument similar to the proof of the equal treatment of equals for the sellers to prove that two equal buyers (not necessarily of the same type) attain the same payoff vector in a CE outcome. Therefore, a CE outcome satisfies equal treatment of equals if $y^{\min} > q^{\max}$. □

Proof of Proposition 2. In light of Remark 1, the characteristics of the CE outcomes of the sellers and the buyers are similar in a large multiple-partners game with types (in particular, the CE payoff vectors satisfy equal treatment of partnerships among the buyers). Therefore, we can assume, without loss of generality, that $\sum_{\underline{b}\in\underline{B}} y(\underline{b})q(\underline{b}) \le \sum_{\underline{s}\in\underline{S}} y(\underline{s})q(\underline{s})$ because the proof of the other case is symmetric. Moreover, it suffices to show that $(\underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbb{R}^{\underline{B}} \times \mathbb{R}^{\underline{S}}$ is a CE payoff vector of $\underline{M}'$ if it is a CE payoff vector of $\underline{M}$.

We follow Sotomayor (1992) and connect the multiple-partners game (with types) $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$ and a simple (one-to-one) assignment game $\hat{M}$. We split each buyer into several agents with unitary demand and each seller into several indivisible objects. We denote $\hat{B} := \{(\underline{b}(h), l) \mid \underline{b} \in \underline{B}, l = 1, \ldots, q(\underline{b}), \text{ and } h \in \{1, ..., y(\underline{b})\}\}$ and $\hat{S} := \{(\underline{s}(g), f) \mid \underline{s} \in \underline{S}, f = 1, \ldots, q(\underline{s}), \text{ and } g \in \{1, ..., y(\underline{s})\}\}$ the sets of agents and objects, respectively, of the assignment game connected with $\underline{M}$. Each agent $(\underline{b}(h), l)$ is identified by the buyer $\underline{b}(h)$ and her index l in $\underline{b}(h)$'s quota. Similarly, each object $(\underline{s}(g), f)$ is identified by its owner $\underline{s}(g)$ and its index f. Clearly, $|\hat{B}| \le |\hat{S}|$ follows from $\sum_{\underline{b}\in\underline{B}} y(\underline{b})q(\underline{b}) \le \sum_{\underline{s}\in\underline{S}} y(\underline{s})q(\underline{s})$.

Moreover, given a feasible matching $\mathbf{x} \in \mathcal{A}(\underline{B}, \underline{S}, \mathbf{y}, \mathbf{q})$, it is possible to construct a one-to-one feasible matching $\hat{\mathbf{x}} \in \mathcal{A}(\hat{B}, \hat{S}, \mathbf{q})$.[31] Then, given $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q}\rangle$ and $\mathbf{x} \in \mathcal{A}(\underline{B}, \underline{S}, \mathbf{y}, \mathbf{q})$, we define the simple assignment game $\hat{M} = \langle \hat{B}, \hat{S}, \hat{\mathbf{a}}\rangle$, where:

$$\hat{a}_{(\underline{b}(h),l)(\underline{s}(g),f)} = \begin{cases} 0 & \text{if } x_{\underline{b}(h)\underline{s}(g)} = 1 \text{ and } \hat{x}_{(\underline{b}(h),l)(\underline{s}(g),f)} = 0 \\ a_{\underline{b}\underline{s}} & \text{otherwise.} \end{cases}$$

We first prove by contradiction that there exists a feasible matching $\mathbf{x}$ for $\underline{M}$ compatible with $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ such that every agent in $\hat{B}$ acquires an object in $\hat{S}$. Suppose that some agent $(\underline{b}(h), l) \in \hat{B}$ does not acquire any object at $\hat{\mathbf{x}}$. Since $|\hat{B}| \le |\hat{S}|$, there exists an unsold object $(\underline{s}(g), f) \in \hat{S}$.

Then we claim that we can construct a new matching $\bar{\mathbf{x}}$ by adding a partnership between $(\underline{b}(h), l)$ and $(\underline{s}(g), f)$ to $\hat{\mathbf{x}}$. First, assume that $a_{\underline{b}\underline{s}} > 0$. Then, it must be the case that $q(\underline{s}) > 1$ and that an object $(\underline{s}(g), f')$, with $f' \ne f$, is acquired by an agent $(\underline{b}(h), l')$, with $l' \ne l$, because otherwise $(\underline{b}(h), l)$ would acquire $(\underline{s}(g), f)$ at a price in the interval $(0, a_{\underline{b}\underline{s}})$. However, by equal treatment of partnerships, $\underline{u}_{\underline{b}} = 0$ (because $\underline{b}(h)$'s quota is not full), hence $\underline{p}_{\underline{s}} = a_{\underline{b}\underline{s}} - \underline{u}_{\underline{b}} > 0$, which contradicts the existence of an unsold type-$\underline{s}$ object $(\underline{s}(g), f)$. Second, if $a_{\underline{b}\underline{s}} = 0$, the existence of such a matching also holds because the set of matchings compatible with a competitive equilibrium is upper hemi-continuous with respect to the surplus matrix $\mathbf{a}$.[32]

Take $\mathbf{x}$ compatible with $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ such that every agent in $\hat{B}$ acquires an object in $\hat{S}$, and let $\hat{\mathbf{x}}$ be the one-to-one matching between the set of agents $\hat{B}$ and the set of objects $\hat{S}$ of $\underline{M}$ induced by $\mathbf{x}$. By Proposition 1 and the observation that every buyer's quota is full, we can define the following correspondence $\varphi : \hat{B} \rightsquigarrow \hat{S}$:

$$\varphi(\underline{b}(h), l) := \bigcup_{\substack{\underline{s}:\underline{u}_{\underline{b}}+\underline{p}_{\underline{s}}=a_{\underline{b}\underline{s}} \\ \text{for some } \underline{b}\in\underline{B}}} \{(\underline{s}(g), f)) \mid g = 1, \ldots, y(\underline{s}) \text{ and } f = 1, \ldots, q(\underline{s})\} \tag{25}$$

for all $(\underline{b}(h), l) \in \hat{B}$. That is, $\varphi(\underline{b}(h), l)$ is the set of all the objects $(\underline{s}(g), f)$ the agent $(\underline{b}(h), l)$ could acquire to obtain his equilibrium utility $\underline{u}_{\underline{b}}$, given the equilibrium price $\underline{p}_{\underline{s}}$ and the surplus $a_{\underline{b}\underline{s}}$ of the partnership.

We can consider the one-to-one matching $\hat{\mathbf{x}}$ as a function from $\hat{B}$ to $\hat{S}$ that assigns different objects to different agents $(\underline{b}(h), l), (\underline{b}'(h'), l') \in \hat{B}$. Hence, by Hall's theorem (Lemma 1), we have $|P| \le |\varphi(P)|$ for all $P \in 2^{\hat{B}} \setminus \{\varnothing\}$. This implies, in particular,

[31] See Sotomayor (1992) for details.

[32] To see it, notice that the set of matchings compatible with a competitive equilibrium is a finite subset of the set of solutions to a linear programming problem. Then, it follows from Berge's theorem of maximum (see, e.g., Kreps, 2013) that the set of solutions, and, hence, the set of matchings compatible with a competitive equilibrium, is upper hemi-continuous with respect to the surplus matrix $\mathbf{a}$.

$$\sum_{\underline{b}\in\underline{P}} q(\underline{b})y(\underline{b}) \leq \sum_{\substack{\underline{s}:\underline{u}_{\underline{b}}+\underline{p}_{\underline{s}}=a_{\underline{bs}} \\ \text{for some } \underline{b}\in\underline{P}}} q(\underline{s})y(\underline{s}) \tag{26}$$

for all $\underline{P} \in 2^{\underline{B}} \setminus \{\emptyset\}$. We note that the right-hand side of the equation (26) corresponds to the sum over some subset $G \subseteq \underline{S}$.

By the definition of the equivalence relation $\sim$ (Definition 10), Condition (26) also holds for the game $\underline{M}'$. That is,

$$\sum_{\underline{b}\in\underline{P}} q(\underline{b})y'(\underline{b}) \leq \sum_{\substack{\underline{s}:\underline{u}_{\underline{b}}+\underline{p}_{\underline{s}}=a_{\underline{bs}} \\ \text{for some } \underline{b}\in\underline{P}}} q(\underline{s})y'(\underline{s}) \tag{27}$$

for all $\underline{P} \in 2^{\underline{B}} \setminus \{\emptyset\}$.

Following equation (25), define a correspondence φ' between the set of agents $\hat{B}'$ and the set of objects $\hat{S}'$ of $\underline{M}'$ by

$$\varphi'(\underline{b}(h), l) := \bigcup_{\substack{\underline{s}:\underline{u}_{\underline{b}}+\underline{p}_{\underline{s}}=a_{\underline{bs}} \\ \text{for some } \underline{b}\in\underline{B}}} \{(\underline{s}(g), f)) \mid g = 1, \ldots, y'(\underline{s}) \text{ and } f = 1, \ldots, q(\underline{s})\}$$

for all $(\underline{b}(h), l) \in \hat{B}'$. The correspondence φ' is well-defined because the matrix $\mathbf{a}$ in the game $\underline{M}'$ is the same as in $\underline{M}$. Moreover, φ' satisfies $|P| \leq |\varphi'(P)|$ for all $P \in 2^{\hat{B}'} \setminus \{\emptyset\}$ because of Condition (27).

Using Hall's theorem again, there is a one-to-one matching $\hat{\mathbf{x}}'$ between $\hat{B}'$ and $\hat{S}'$. Moreover, each buyer $(\underline{b}(h), l) \in \hat{B}'$ maximizes her utility, given the price vector, since she obtains the same utility as in the market $\underline{M}$ and the prices are the same. However, $\hat{\mathbf{x}}'$ may not correspond to a feasible matching of $\underline{M}'$ because we cannot rule out the possibility that multiple agents of the same buyer are assigned to multiple objects from the same seller at $\hat{\mathbf{x}}'$. We modify $\hat{\mathbf{x}}'$ to construct a new matching that solves this problem.

Suppose that there exists a buyer $\underline{b}(h)$ and a seller $\underline{s}(g)$ such that $\hat{x}'_{(\underline{b}(h),l)(\underline{s}(g),f)} = 1$ and $\hat{x}'_{(\underline{b}(h),l')(\underline{s}(g),f')} = 1$, with $l \neq l'$ and $f \neq f'$ (otherwise, we are done). Then, there exists a buyer $\underline{b}(h') \neq \underline{b}(h)$ such that $\hat{\mathbf{x}}'$ does not assign any agent of $\underline{b}(h')$ to an object from the seller $\underline{s}(g)$. Such a buyer exists because the number of objects initially owned by $\underline{s}(g)$ is $q(\underline{s})$ and $y'^{\min} > q^{\max}$. Then, let $(\underline{b}(h'), l''')$ be such an agent of the buyer $\underline{b}(h')$. $(\underline{b}(h'), l''')$ is assigned to an object $(\underline{s}'(j), q)$ at $\hat{\mathbf{x}}'$. Notice that $\underline{s}' \neq \underline{s}$. Then, we can swap the assigned object to $(b(h), l')$ with the assigned object to $(\underline{b}(h'), l''')$. Moreover, since the buyers $\underline{b}(h)$ and $\underline{b}(h')$ are equal, their surplus vectors are equal, and their payoffs are identical under the CE payoff vector $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$. Hence, the CE payoff obtained by all the players is compatible with the new matching.

We continue this swapping procedure until no pair of agents of the same buyer are assigned to objects from the same seller. Denote by $\hat{\mathbf{x}}''$ this resulting function. It corresponds to a feasible matching of $\underline{M}'$. Moreover, given the price vector, each buyer maximizes her utility, and the prices are zero if a seller does not sell all her objects. Therefore, $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ is a CE payoff vector of $\underline{M}'$. □

Proof of Proposition 3. We prove the result by contradiction. Let $(\underline{\mathbf{u}}, \underline{\mathbf{p}}; \mathbf{x})$ and $(\underline{\mathbf{u}}', \underline{\mathbf{p}}'; \mathbf{x}')$, with $(\underline{\mathbf{u}}, \underline{\mathbf{p}}) \neq (\underline{\mathbf{u}}', \underline{\mathbf{p}}')$, be two CE outcomes of some large uneven game $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{q} \rangle$. Let $\hat{\mathbf{x}}, \hat{\mathbf{x}}' \in \mathcal{A}(\hat{B}, \hat{S}, \mathbf{q})$ be the one-to-one matchings constructed from $\mathbf{x}$ and $\mathbf{x}'$, respectively. Let $\hat{M} = \langle \hat{B}, \hat{S}, \hat{\mathbf{a}} \rangle$ be the simple game of $\underline{M}$, as defined in the proof of Proposition 2.

Suppose that $\underline{u}_{\underline{b}} > \underline{u}'_{\underline{b}}$ for some $\underline{b} \in \underline{B}$. Given that $\underline{u}_{\underline{b}} > 0$, any agent $(\underline{b}(h), l)$ of the buyer of type $\underline{b}$ is matched with an object $(\underline{s}(g), f)$ from some seller of type $\underline{s}$ at $\hat{\mathbf{x}}$. Thus $\underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}}$, for any such seller type $\underline{s}$. Moreover, it must be the case $\underline{u}'_{\underline{b}} \geq a_{\underline{bs}} - \underline{p}'_{\underline{s}}$ because $(\underline{\mathbf{u}}', \underline{\mathbf{p}}'; \mathbf{x}')$ is a CE outcome. Hence, $\underline{p}'_{\underline{s}} \geq a_{\underline{bs}} - \underline{u}'_{\underline{b}} > a_{\underline{bs}} - \underline{u}_{\underline{b}} = \underline{p}_{\underline{s}}$. In fact, we have constructed a one-to-one function from $O := \{(\underline{b}(h), l) \in \hat{B} \mid \underline{u}_{\underline{b}} > \underline{u}'_{\underline{b}}\}$ to $R := \{(\underline{s}(g), f) \in \hat{S} \mid \underline{p}_{\underline{s}} < \underline{p}'_{\underline{s}}\}$ (because $\hat{\mathbf{x}}$ is one-to-one). This implies that $|O| \leq |R|$.

A symmetric argument allows to construct a one-to-one function from R to O at $\hat{\mathbf{x}}'$, which implies that $|O| \geq |R|$. Therefore, $|O| = |R|$.

Since all the agents of all the buyers of type $\underline{b}$ attain the same utility $u_{\underline{b}}$ and all the objects of all the sellers of type $\underline{s}$ are sold at the same price $p_{\underline{s}}$, it is the case that $O = \bigcup_{\underline{b}\in H} B_{\underline{b}}$ and $R = \bigcup_{\underline{s}\in G} S_{\underline{s}}$ for some $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and some $G \in 2^{\underline{B}} \setminus \{\emptyset\}$. However, this implies that $\sum_{\underline{b}\in H} y(\underline{b})q(\underline{b}) = |O| = |R| = \sum_{\underline{s}\in G} y(\underline{s})q(\underline{s})$, in contradiction with the definition of an uneven game (condition (5) of Definition 11). □

Proof of Proposition 4. Consider $M = \langle B, S, \mathbf{a} \rangle$ with $n_b = n_s = n$. In the proof of Theorem 1, we show:

$$u_b^{asy} = \mathbb{E}[\underline{u}_{\underline{b}}(\tilde{\mathcal{M}}^{\tilde{\xi}})] \text{ and } p_s^{asy} = \mathbb{E}[\underline{p}_{\underline{s}}(\tilde{\mathcal{M}}^{\tilde{\xi}})],$$

where the random variable $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is defined as follows (we adapt the result in the proof to the case where $q(\underline{b}_i) = q(\underline{s}_j) = 1$ for all $\underline{b}_i \in \underline{B}$ and $\underline{s}_j \in \underline{S}$). If the realization of the vector of standard normal distributions $\tilde{\xi} = (\tilde{\xi}_{\underline{b}_1}, \ldots, \tilde{\xi}_{\underline{b}_n}; \tilde{\xi}_{\underline{s}_1}, \ldots, \tilde{\xi}_{\underline{s}_n})$ is ξ, then the realization of $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is the class $\mathcal{M}$ given by:

1. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $|H| < |G|$;
2. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $|H| > |G|$;

3. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\emptyset\}$ and all $G \in 2^{\underline{S}} \setminus \{\emptyset\}$ such that $|H| = |G|$ if and only if $\sum_{\underline{b}\in H} \bar{\xi}_{\underline{b}} < \sum_{\underline{s}\in G} \bar{\xi}_{\underline{s}}$.

Moreover, we can focus on large uneven assignment games (equation (15)). Hence, take an arbitrary large uneven assignment game with types $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}\rangle$ that is a realization of $\tilde{\mathcal{M}}^{\bar{\xi}}$. Define $\underline{M}'' = \langle \underline{B}, \underline{S}, \mathbf{y}'', \mathbf{a}\rangle$ such that $y''(\underline{b}_i) = y'(\underline{s}_i)$ and $y''(\underline{s}_j) = y'(\underline{b}_j)$ for all $i, j = 1 \dots n$. Denote $\mathcal{M}'$ and $\mathcal{M}''$ the classes $\underline{M}'$ and $\underline{M}''$ belong to.

The classes $\mathcal{M}'$ and $\mathcal{M}''$ are chosen with equal probability according to the limit distribution $\tilde{M}^{\bar{\xi}}$. Hence, to show that $(\mathbf{u}^{asy}, \mathbf{p}^{asy}) = (\mathbf{u}^{mean}, \mathbf{p}^{mean})$, it suffices to prove that $\underline{\mathbf{u}}(\mathcal{M}') + \underline{\mathbf{u}}(\mathcal{M}'') = 2\mathbf{u}^{mean}$ and $\underline{\mathbf{p}}(\mathcal{M}') + \underline{\mathbf{p}}(\mathcal{M}'') = 2\mathbf{p}^{mean}$ for any uneven assignment game with types $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}\rangle$ that is a realization of $\tilde{\mathcal{M}}^{\bar{\xi}}$. In addition, it is convenient to jointly prove that $(\underline{\mathbf{u}}(\mathcal{M}'), \underline{\mathbf{p}}(\mathcal{M}'))$ and $(\underline{\mathbf{u}}(\mathcal{M}''), \underline{\mathbf{p}}(\mathcal{M}''))$ are monotonic; that is, $\underline{u}_{b_i}(\mathcal{M}') \ge \underline{u}_{b_{i'}}(\mathcal{M}')$, $\underline{p}_{s_j}(\mathcal{M}') \ge \underline{p}_{s_{j'}}(\mathcal{M}')$, $\underline{u}_{b_i}(\mathcal{M}'') \ge \underline{u}_{b_{i'}}(\mathcal{M}'')$, and $\underline{p}_{s_j}(\mathcal{M}'') \ge \underline{p}_{s_{j'}}(\mathcal{M}'')$ if $i \ge i'$ and $j \ge j'$. We prove them by an induction on n. Clearly, the equalities and the monotonicity hold for $n = 1$.

We hypothesize that the properties hold for $n = t \ge 1$. Now consider the case where $n = t + 1$, that is, take any $\underline{M}'$ and $\underline{M}''$ with $t + 1$ types of buyers and sellers.

Take the subgames $\underline{M}'^r$ and $\underline{M}''^r$, which correspond to $\underline{M}'$ and $\underline{M}''$ restricted to the types of buyers $\underline{B} \setminus \{\underline{b}_1\}$ and the types of sellers $\underline{S} \setminus \{\underline{s}_1\}$. By the induction hypothesis and Corollary 3, $\underline{u}_{\underline{b}_2}(\underline{M}'^r) + \underline{u}_{\underline{b}_2}(\underline{M}''^r) = 2u^{mean}_{\underline{b}_2} = a_{\underline{b}_2\underline{s}_2}$ and $\underline{p}_{\underline{s}_2}(\underline{M}'^r) + \underline{p}_{\underline{s}_2}(\underline{M}''^r) = 2p^{mean}_{\underline{s}_2} = a_{\underline{b}_2\underline{s}_2}$. Similarly, $\underline{u}_{\underline{b}_i}(\underline{M}'^r) + \underline{u}_{\underline{b}_i}(\underline{M}''^r) = 2u^{mean}_{\underline{b}_i} = a_{\underline{b}_i\underline{s}_i} + \sum_{m=3}^{i}(a_{\underline{b}_m\underline{s}_{(m-1)}} - a_{\underline{b}_{(m-1)}\underline{s}_m})$ and $\underline{p}_{\underline{s}_j}(\underline{M}'^r) + \underline{p}_{\underline{s}_j}(\underline{M}''^r) = a_{\underline{b}_j\underline{s}_j} + \sum_{m=3}^{j}(a_{\underline{b}_{(m-1)}\underline{s}_m} - a_{\underline{b}_m\underline{s}_{(m-1)}})$, for $i, j = 3, \dots, t + 1$. Moreover, $\underline{u}_{b_i}(\mathcal{M}'^r) \ge \underline{u}_{b_{i'}}(\mathcal{M}'^r)$, $\underline{p}_{s_j}(\mathcal{M}'^r) \ge \underline{p}_{s_{j'}}(\mathcal{M}'^r)$, $\underline{u}_{b_i}(\mathcal{M}''^r) \ge \underline{u}_{b_{i'}}(\mathcal{M}''^r)$, and $\underline{p}_{s_j}(\mathcal{M}''^r) \ge \underline{p}_{s_{j'}}(\mathcal{M}''^r)$ for $i, i', j, j' = 2, \dots t + 1$ such that $i \ge i'$ and $j \ge j'$.

Without loss of generality, assume that $\sum_{i=2}^{t+1} y'(\underline{b}_i) > \sum_{j=2}^{t+1} y'(\underline{s}_j)$. Since $\underline{M}'^r$ is supermodular and monotonic, it has an assortative stable matching.[33] Also, since there are more buyers than sellers in $\underline{M}'^r$, it is the case that $\underline{u}_{\underline{b}_2}(\underline{M}'^r) = 0$ and $\underline{p}_{\underline{s}_2}(\underline{M}'^r) = a_{\underline{b}_2\underline{s}_2}$. We distinguish two cases: (i) $\sum_{i=1}^{t+1} y'(\underline{b}_i) < \sum_{j=1}^{t+1} y'(\underline{s}_j)$ and; (ii) $\sum_{i=1}^{t+1} y'(\underline{b}_i) > \sum_{j=1}^{t+1} y'(\underline{s}_j)$. For each case, we will propose a payoff vector for $\underline{M}'$ and $\underline{M}''$ and show that they are stable, respectively. Since $\underline{M}'$ and $\underline{M}''$ are large and uneven, their sets of stable payoff vectors coincide with their sets of CE payoff vectors (Remark 1). Moreover, by Proposition 3, the set of CE payoff vectors in either game is a singleton.

For Case (i), define $\underline{u}_{\underline{b}_1}(\underline{M}') = a_{\underline{b}_1\underline{s}_1}$, $\underline{p}_{\underline{s}_1}(\underline{M}') = 0$, $\underline{u}_{\underline{b}_i}(\underline{M}') = \underline{u}_{\underline{b}_i}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1}$, and $\underline{p}_{\underline{s}_j}(\underline{M}') = \underline{p}_{\underline{s}_j}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1}$ for $i, j = 2, \dots, t + 1$. Similarly, define $\underline{u}_{\underline{b}_1}(\underline{M}'') = 0$, $\underline{p}_{\underline{s}_1}(\underline{M}'') = a_{\underline{b}_1\underline{s}_1}$, $\underline{u}_{\underline{b}_i}(\underline{M}'') = \underline{u}_{\underline{b}_i}(\underline{M}''^r) - a_{\underline{b}_1\underline{s}_2}$, and $\underline{p}_{\underline{s}_j}(\underline{M}'') = \underline{p}_{\underline{s}_j}(\underline{M}''^r) + a_{\underline{b}_1\underline{s}_2}$ for $i, j = 2, \dots, t + 1$. It is immediate to check that $\underline{\mathbf{u}}(\mathcal{M}') + \underline{\mathbf{u}}(\mathcal{M}'') = \underline{\mathbf{u}}(M') + \underline{\mathbf{u}}(M'') = 2\mathbf{u}^{mean}$ and $\underline{\mathbf{p}}(\mathcal{M}') + \underline{\mathbf{p}}(\mathcal{M}'') = \underline{\mathbf{p}}(M') + \underline{\mathbf{p}}(M'') = 2\mathbf{p}^{mean}$. Moreover, the monotonicity of $(\underline{\mathbf{u}}(\mathcal{M}'^r), \underline{\mathbf{p}}(\mathcal{M}'^r))$ and $(\underline{\mathbf{u}}(\mathcal{M}''^r), \underline{\mathbf{p}}(\mathcal{M}''^r))$ implies the monotonicity of $(\underline{\mathbf{u}}(\mathcal{M}'), \underline{\mathbf{p}}(\mathcal{M}'))$ and $(\underline{\mathbf{u}}(\mathcal{M}''), \underline{\mathbf{p}}(\mathcal{M}''))$ for all the players except possibly type-$\underline{b}_1$ buyers or type-$\underline{s}_1$ sellers.

We now verify that $(\underline{\mathbf{u}}(\underline{M}'), \underline{\mathbf{p}}(\underline{M}'))$ is a stable payoff vector. First, $(\underline{\mathbf{u}}(\underline{M}'), \underline{\mathbf{p}}(\underline{M}'))$ is individually rational. The proof that $\underline{u}_{\underline{b}} \ge 0$ for all $\underline{b} \in \underline{B}$ is immediate by induction because it is the sum of non-negative terms. Concerning the prices, $\underline{p}_{\underline{s}_1}(\underline{M}') = 0$ and $\underline{p}_{\underline{s}_j}(\underline{M}') = \underline{p}_{\underline{s}_j}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1} \ge \underline{p}_{\underline{s}_2}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1} = a_{\underline{b}_2\underline{s}_2} - a_{\underline{b}_2\underline{s}_1} \ge 0$ for $j = 2, \dots, t + 1$, where the second inequality follows from the monotonicity of $(\underline{p}_{\underline{s}_j}(\underline{M}'))_{j=2,\dots,t+1}$ and the last inequality follows from the monotonicity of $\underline{M}'$.

Second, we show that $(\underline{\mathbf{u}}(\underline{M}'), \underline{\mathbf{p}}(\underline{M}'))$ is not blocked by a buyer-seller pair $(\underline{b}_i, \underline{s}_j)$. We distinguish three possibilities: (a) If $i \ge 2$ and $j \ge 2$, then $\underline{u}_{\underline{b}_i}(\underline{M}') + \underline{p}_{\underline{s}_j}(\underline{M}') = \underline{u}_{\underline{b}_i}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1} + \underline{p}_{\underline{s}_j}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1} \ge a_{\underline{b}_i\underline{s}_j}$ because of the induction hypothesis. (b) If $i = 1$, using the induction hypothesis, supermodularity of $\underline{M}'$, and $\underline{u}_{\underline{b}_2}(\underline{M}'^r) = 0$, we have $\underline{u}_{\underline{b}_1}(\underline{M}') + \underline{p}_{\underline{s}_j}(\underline{M}') - a_{\underline{b}_1\underline{s}_j} = a_{\underline{b}_1\underline{s}_1} + (\underline{p}_{\underline{s}_j}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1}) - a_{\underline{b}_1\underline{s}_j} = \underline{u}_{\underline{b}_2}(\underline{M}'^r) + \underline{p}_{\underline{s}_j}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_1} + a_{\underline{b}_1\underline{s}_1} - a_{\underline{b}_1\underline{s}_j} \ge a_{\underline{b}_2\underline{s}_j} - a_{\underline{b}_2\underline{s}_1} + a_{\underline{b}_1\underline{s}_1} - a_{\underline{b}_1\underline{s}_j} = (a_{\underline{b}_2\underline{s}_j} - a_{\underline{b}_1\underline{s}_j}) - (a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_1\underline{s}_1}) \ge 0$, for $j = 2, \dots, t + 1$. (c) If $j = 1$, using the induction hypothesis, supermodularity of $\underline{M}'$, $\underline{p}_{\underline{s}_1}(\underline{M}') = 0$, and $\underline{p}_{\underline{s}_2}(\underline{M}'^r) = a_{\underline{b}_2\underline{s}_2}$, we have $\underline{u}_{\underline{b}_i}(\underline{M}') + \underline{p}_{\underline{s}_1}(\underline{M}') - a_{b_i s_1} = (\underline{u}_{\underline{b}_i}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1}) + \underline{p}_{\underline{s}_1}(\underline{M}') - a_{\underline{b}_i\underline{s}_1} = \underline{u}_{\underline{b}_i}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_i\underline{s}_1} = \underline{u}_{\underline{b}_i}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_i\underline{s}_1} + \underline{p}_{\underline{s}_2}(\underline{M}'^r) - a_{\underline{b}_2\underline{s}_2} = \underline{u}_{\underline{b}_i}(\underline{M}'^r) + \underline{p}_{\underline{s}_2}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_i\underline{s}_1} - a_{\underline{b}_2\underline{s}_2} \ge a_{\underline{b}_i\underline{s}_2} + a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_i\underline{s}_1} - a_{\underline{b}_2\underline{s}_2} = (a_{\underline{b}_i\underline{s}_2} - a_{\underline{b}_i\underline{s}_1}) - (a_{\underline{b}_2\underline{s}_2} - a_{\underline{b}_2\underline{s}_1}) \ge 0$, for $i = 2, \dots, t + 1$.

Third, we show that $(\underline{\mathbf{u}}(\underline{M}'), \underline{\mathbf{p}}(\underline{M}'))$ is monotonic. It suffices to show $\underline{u}_{\underline{b}_2}(\underline{M}') \ge \underline{u}_{\underline{b}_1}(\underline{M}')$ and $\underline{p}_{\underline{s}_2}(\underline{M}') \ge \underline{p}_{\underline{s}_1}(\underline{M}')$. They hold because $\underline{u}_{\underline{b}_2}(\underline{M}') - \underline{u}_{\underline{b}_1}(\underline{M}') = \underline{u}_{\underline{b}_2}(\underline{M}'^r) + a_{\underline{b}_2\underline{s}_1} - a_{\underline{b}_1\underline{s}_1} \ge \underline{u}_{\underline{b}_2}(\underline{M}'^r) \ge 0$ and $\underline{p}_{\underline{s}_2}(\underline{M}') - \underline{p}_{\underline{s}_1}(\underline{M}') = \underline{p}_{\underline{s}_2}(\underline{M}') - 0 \ge 0$.

We can verify that $(\underline{\mathbf{u}}(\underline{M}''), \underline{\mathbf{p}}(\underline{M}''))$ is also a stable payoff vector in the same manner.

[33] An assignment game with types $\underline{M}$ is supermodular if $a_{\underline{b}_i\underline{s}_j} + a_{\underline{b}_{i'}\underline{s}_{j'}} \ge a_{\underline{b}_{i'}\underline{s}_j} + a_{\underline{b}_i\underline{s}_{j'}}$ for all $\underline{b}_i, \underline{b}_{i'} \in \underline{B}$ and $\underline{s}_j, \underline{s}_{j'} \in \underline{S}$ such that $i \ge i'$ and $j \ge j'$. It is monotonic if $a_{\underline{b}_i\underline{s}_j} \ge a_{\underline{b}_{i'}\underline{s}_{j'}}$ for all $\underline{b}_i, \underline{b}_{i'} \in \underline{B}$ such that $i \ge i'$ and all $\underline{s}_j, \underline{s}_{j'} \in \underline{S}$ such that $j \ge j'$. The supermodularity and monotonicity of an assignment game with types imply the existence of an assortative matching.

For Case (ii), define $\underline{u}_{\underline{b}_1}(\underline{M}')=0$, $\underline{p}_{\underline{s}_1}(\underline{M}')=a_{\underline{b}_1\underline{s}_1}$, $\underline{u}_{\underline{b}_i}(\underline{M}')=\underline{u}_{\underline{b}_i}(\underline{M}'^r)+a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_1\underline{s}_1}$, $\underline{p}_{\underline{s}_j}(\underline{M}')=\underline{p}_{\underline{s}_j}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}$, for $i,j=2,\dots,t+1$. Define also $\underline{u}_{\underline{b}_1}(\underline{M}'')=a_{\underline{b}_1\underline{s}_1}$, $\underline{p}_{\underline{s}_1}(\underline{M}'')=0$, $\underline{u}_{\underline{b}_i}(\underline{M}'')=\underline{u}_{\underline{b}_i}(\underline{M}''^r)-a_{\underline{b}_1\underline{s}_2}+a_{\underline{b}_1\underline{s}_1}$, $\underline{p}_{\underline{s}_j}(\underline{M}'')=\underline{p}_{\underline{s}_j}(\underline{M}''^r)+a_{\underline{b}_1\underline{s}_2}-a_{\underline{b}_1\underline{s}_1}$, for $i,j=2,\dots,t+1$. It is immediate that $\underline{\mathbf{u}}(\mathcal{M}')+\underline{\mathbf{u}}(\mathcal{M}'')=2\mathbf{u}^{mean}$ and $\underline{\mathbf{p}}(\mathcal{M}')+\underline{\mathbf{p}}(\mathcal{M}'')=2\mathbf{p}^{mean}$. We verify that $(\underline{\mathbf{u}}(\underline{M}'),\underline{\mathbf{p}}(\underline{M}'))$ and $(\underline{\mathbf{u}}(\underline{M}''),\underline{\mathbf{p}}(\underline{M}''))$ are stable and monotonic payoff vectors; hence, each is the only CE payoff vector for the corresponding game.

Concerning the individual rationality of $(\underline{\mathbf{u}}(\underline{M}'),\underline{\mathbf{p}}(\underline{M}'))$, $\underline{u}_{\underline{b}}\geq 0$ for all $\underline{b}\in\underline{B}$ because it is the sum of non-negative terms and $a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_1\underline{s}_1}\geq 0$ by the monotonicity of $\underline{M}'$. As for the prices, $\underline{p}_{\underline{s}_j}(\underline{M}')=\underline{p}_{\underline{s}_j}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}\geq\underline{p}_{\underline{s}_2}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}=a_{\underline{b}_2\underline{s}_2}-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}\geq a_{\underline{b}_1\underline{s}_1}\geq 0$, for $j=2,\dots,t+1$.

We now check that $(\underline{\mathbf{u}}(\underline{M}'),\mathbf{p}(\underline{M}'))$ is not blocked by a new buyer-seller pair. Again we distinguish three possibilities, which are the same as those for Case (i). The argument for Possibility (a) is identical. (b) $\underline{u}_{\underline{b}_i}(\underline{M}')+\underline{p}_{\underline{s}_1}(\underline{M}')-a_{b_i s_1}=(\underline{u}_{\underline{b}_i}(\underline{M}'^r)+a_{b_2 s_1}-a_{\underline{b}_1\underline{s}_1})+\underline{p}_{\underline{s}_1}(\underline{M}')-a_{\underline{b}_i\underline{s}_1}=(\underline{u}_{\underline{b}_i}(\underline{M}'^r)+a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_1\underline{s}_1})+a_{\underline{b}_1\underline{s}_1}-a_{\underline{b}_i\underline{s}_1}=\underline{u}_{\underline{b}_i}(\underline{M}'^r)+a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_i\underline{s}_1}=\underline{u}_{\underline{b}_i}(\underline{M}'^r)+a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_i\underline{s}_1}+(\underline{p}_{\underline{s}_2}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_2})=\underline{u}_{\underline{b}_i}(\underline{M}'^r)+\underline{p}_{\underline{s}_2}(\underline{M}'^r)-a_{\underline{b}_i\underline{s}_1}+a_{\underline{b}_2\underline{s}_1}-a_{\underline{b}_2\underline{s}_2}\geq(a_{\underline{b}_i\underline{s}_2}-a_{\underline{b}_i\underline{s}_1})-(a_{\underline{b}_2\underline{s}_2}-a_{\underline{b}_2\underline{s}_1})\geq 0$, for $i=2,\dots,t+1$; and (c) $\underline{u}_{\underline{b}_1}(\underline{M}')+\underline{p}_{\underline{s}_j}(\underline{M}')=\underline{u}_{\underline{b}_1}(\underline{M}')+(\underline{p}_{\underline{s}_j}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1})=\underline{p}_{\underline{s}_j}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}=\underline{p}_{\underline{s}_j}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}+\underline{u}_{\underline{b}_2}(\underline{M}'^r)=(\underline{p}_{\underline{s}_j}(\underline{M}'^r)+\underline{u}_{\underline{b}_2}(\underline{M}'^r))-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1}\geq(a_{\underline{b}_2\underline{s}_j}-a_{\underline{b}_2\underline{s}_1})+a_{\underline{b}_1\underline{s}_1}\geq(a_{\underline{b}_1\underline{s}_j}-a_{\underline{b}_1\underline{s}_1})+a_{\underline{b}_1\underline{s}_1}=a_{\underline{b}_1\underline{s}_j}\geq 0$, for $j=2,\dots,t+1$.

It remains to verify that $\underline{u}_{\underline{b}_2}(\underline{M}')\geq\underline{u}_{\underline{b}_1}(\underline{M}')$ and $\underline{p}_{\underline{s}_2}(\underline{M}')\geq\underline{p}_{\underline{s}_1}(\underline{M}')$. Clearly, $\underline{u}_{\underline{b}_2}(\underline{M}')-\underline{u}_{\underline{b}_1}(\underline{M}')=\underline{u}_{\underline{b}_2}(\underline{M}')-0\geq 0$ and $\underline{p}_{\underline{s}_2}(\underline{M}')-\underline{p}_{\underline{s}_1}(\underline{M}')=(\underline{p}_{\underline{s}_2}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}+a_{\underline{b}_1\underline{s}_1})-a_{\underline{b}_1\underline{s}_1}=\underline{p}_{\underline{s}_2}(\underline{M}'^r)-a_{\underline{b}_2\underline{s}_1}=a_{\underline{b}_2\underline{s}_2}-a_{\underline{b}_2\underline{s}_1}\geq 0$.

Finally, we can verify that $(\underline{\mathbf{u}}(\underline{M}''),\underline{\mathbf{p}}(\underline{M}''))$ is also a stable payoff vector in the same manner. □

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