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Distorted copulas

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ABSTRACT

In this paper, we deal with the problem of preserving the copula property by means of a transformation called distortion where automorphisms of the real unit interval are involved. A new methodological approach is followed by resorting to a special branch of group theory. The crucial role of absolute continuity of one-dimensional sections of a distorted copula is illustrated. Some necessary and sufficient conditions concerning the automorphisms which preserve the copula property for a large class of bivariate copulas are proved. Several examples are presented.

1. Introduction

Copulas have become a widespread tool successfully applied in diverse fields where multivariate dependence is of interest (see [9,17,23,26] for a thorough exposition). For this reason, it is desirable to have a large collection of methods able to produce new families of copulas. Probably, the most popular family of copulas is that of *Archimedean copulas* (see e.g. [9]). A d-dimensional *strict* Archimedean copula C may be represented in terms of a *multiplicative* generator h in the form

$$C(u_1, \dots, u_d) = h(\Pi(h^{-1}(u_1), \dots, h^{-1}(u_d))),$$

where h is an automorphism of the real unit interval, h^{-1} is its inverse and Π is the *product* copula. A possible way to generalize this representation is to replace the product with an arbitrary copula. In other terms, given a copula C and an automorphism h, we can produce a new function $C_h : \mathbb{I}^d \to \mathbb{I}$ defined as

$$C_h(u_1, \dots, u_d) = h(C(h^{-1}(u_1), \dots, h^{-1}(u_d))).$$

This type of construction has its origins in the study of *distorted* probability distribution functions (see [5] and the references therein). For this reason, in the literature the function C_h is often referred to as *distorted* copula (by means of h). The study of distorted copulas, almost exclusively devoted to the bivariate case, can boast a long list of contributions: see, among others, [2–4,7,8,11,12,14,18,21].

In this paper, we deal with the problem of determining the set $\Theta(C)$ of automorphisms of the real unit interval which ensure that the distortion of a fixed copula C by means of any $h \in \Theta(C)$ is still a copula. As far as we know, such problem has never received any systematic treatment in the literature. We highlight the methodological novelty of using some tools strictly connected to a special branch of group theory, illustrated in Section 2. In Section 3, we introduce the notion of distortion with regard to semi–copulas. In Section 4, we tackle our problem by employing classical results recalled in Section 2. In Section 5, the problem of the absolute continuity of one-dimensional sections of a distorted copula is discussed. Finally, in Section 6, we analyze some properties which ensure that the distortion preserves the property of being a copula for a relevant class of bivariate copulas. Particularly, when we consider the family of Marshall-Olkin copulas (see, e.g., [23]) and a large subclass of *semilinear* copulas (see [10,19]), we provide a

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characterization theorem of the automorphisms by which the distortion preserves the copula property. The proofs of the main results may be found in the appendices.

2. Group action

In this section, we recall the notion of *group action* and some related aspects (see, for instance, [25]). The reader is assumed to be acquainted with the basic notions of *group* and *semigroup*. We warn the reader that, for any pair (A, B) of arbitrary sets, the notation $A \subset B$ means that A is a proper subset of B.

Definition 2.1. Let G be a group and X a non-empty set. A left action is a function $\alpha : G \times X \to X$, denoted by $\alpha(g, x) = g \cdot x$, such that:

(i) $e \cdot x = x$ for every $x \in X$, where e denotes the identity element of the group G;
(ii) $(gh) \cdot x = g \cdot (h \cdot x)$, for every $x \in X$ and for any $g, h \in G$, where gh denotes the product in G of the elements g and h.

One also says that G *acts* on X.

In Definition 2.1, the elements of G "act" on the left. A right action of G on X may be defined analogously. In both cases, X is called a G-set. In the following, we shall reserve the symbol X exclusively for a G-set. To avoid a cumbersome notation, we will omit the "dot" to denote the image under the action map α of a pair (g, x) and write just gx instead. With this "abuse of notation", condition (ii) of Definition 2.1 becomes $(gh)x = g(hx)$, which just resembles the "associativity" for the multiplication operation in the definition of a group. Be warned however that this simplified notation may in some contexts be ambiguous and confusing.

Definition 2.2. Fixing any $x \in X$, the stabilizer of x, denoted by $St(x)$, is the subset of G given by

$$St(x) = \{g \in G : gx = x\}.$$

It is not difficult to see that the stabilizer of any $x \in X$ is a subgroup of G.

Definition 2.3. Fixing any $x \in X$, the orbit of x, denoted by $\mathcal{O}(x)$, is the subset of X given by

$$\mathcal{O}(x) = \{gx : g \in G\}.$$

It is well-known that the orbits of X form a partition whose associated equivalence relation on X is given by $x \sim y$ if, and only if, $y = gx$ for some $g \in G$.

The next elementary property concerns the stabilizer of two elements belonging to the same orbit (see, for instance, [25, Exercise 3.37]).

Proposition 2.4. *If $x \in X$ and $g \in G$, then $St(gx) = \{ghg^{-1} : h \in St(x)\}$.*

Corollary 2.5. *If the stabilizer of an element $x \in X$ is trivial, i.e. it reduces to the identity element, then every $y \in \mathcal{O}(x)$ has a trivial stabilizer.*

As far as we know, the concepts and results below presented are new. In the sequel, let Y be a non-empty subset of X.

Definition 2.6. Fixing any $y \in Y$, the Y-orbit of y, denoted by $\mathcal{O}_Y(y)$, is defined as $\mathcal{O}_Y(y) = \mathcal{O}(y) \cap Y$.

Remark 2.7. Notice that the family of Y-orbits is a partition of Y as a straightforward consequence of the fact that the orbits of X form a partition.

We are particularly interested to a special class of subsets of G strictly connected to Y.

Definition 2.8. Given any $y \in Y$, the *generalized stabilizer* of y (with respect to Y), denoted by $G_Y(y)$, is the subset of G given by

$$G_Y(y) = \{g \in G : gy \in Y\}.$$

It is immediate to see that

$$St(y) \subseteq G_Y(y) \quad \text{for all } y \in Y. \tag{2.1}$$

Given any $y \in Y$, let $\Psi_y : G_Y(y) \to Y$ be defined as $\Psi_y(g) = gy$. Denote by $Ran(\Psi_y)$ the range of Ψ_y.

Lemma 2.9. *Let $y \in Y$. Then, $\mathcal{O}_Y(y) = Ran(\Psi_y)$.*

Proof. If $z \in \mathcal{O}_Y(y)$, then, by Definition 2.6, we have that both $z \in Y$ and $z = gy$ for some $g \in G$, which amounts to $g \in G_Y(y)$, and, hence, $\mathcal{O}_Y(y) \subseteq Ran(\Psi_y)$. The converse is immediate. □

Let A be a non-empty subset of G. Given any $g \in G$, denote by Ag the subset of G given by $Ag = \{ag : a \in A\}$.

Proposition 2.10. *Let $y \in Y$ and $h \in G_Y(y)$. Then, $G_Y(hy) = G_Y(y)h^{-1}$.*

Proof. The first step of the proof is the inclusion $G_Y(hy) \subseteq G_Y(y)h^{-1}$: if $f \in G_Y(hy)$, then $f(hy) = (fh)y \in Y$. This means that $fh = g$ for some $g \in G_Y(y)$ and, hence, $f = gh^{-1}$, so closing the first step. Conversely, if $f \in G_Y(y)h^{-1}$, then $f = gh^{-1}$ for some $g \in G_Y(y)$. Thus, $f(hy) = gh^{-1}(hy) = (gh^{-1}h)y = gy$ and the claim now follows from $g \in G_Y(y)$. □

Proposition 2.11. *Let A be a non-empty subset of G. Suppose that A satisfies the following condition:*

$$a^{-1} \notin A \quad \text{for every } a \in A \text{ such that } a \neq e. \tag{2.2}$$

Then, for any $y \in Y$ there exists at most one $z \in \mathcal{O}_Y(y)$ such that $G_Y(z) = A$.

Proof. Suppose *ab absurdo* that there exist $z, w \in \mathcal{O}_Y(y)$ such that $z \neq w$ and $G_Y(z) = G_Y(w) = A$. By Lemma 2.9, $z = gy$ and $w = hy$ for some $g, h \in G_Y(y)$: further, $z \neq w$ implies $g \neq h$. Therefore, the previous proposition leads to $A = G_Y(y)g^{-1} = G_Y(y)h^{-1}$ or, equivalently $G_Y(y) = Ag = Ah$. Since $g \in G_Y(y)$, $G_Y(y) = Ah$ implies $g = ah$ for some $a \in A$: observe that $g \neq h$ entails $a \neq e$. Analogously, $h = bg$ for some $b \in A$. Then, $g = abg$ and consequently $ab = e$, so $a \in A \setminus \{e\}$ and $a^{-1} = b \in A$, which contradicts eq. (2.2), so closing the proof. □

If $G_Y(y)$ coincides with a particular subset of G, we have an interesting result concerning the stabilizer of y.

Proposition 2.12. *Let A be a non-empty subset of G. Assume that $G_Y(y) = A$ for some $y \in Y$. Suppose that eq. (2.2) holds. Then, the stabilizer of y is trivial.*

Proof. Suppose *ab absurdo* that there exists an $h \in St(y) \setminus \{e\}$. By eq. (2.1) and the assumption $G_Y(y) = A$, we infer that $h \in A \setminus \{e\}$. Since $St(y)$ is a subgroup, also $h^{-1} \in St(y) \setminus \{e\}$, hence, for the above reasons, $h^{-1} \in A \setminus \{e\}$. Consequently, both h and h^{-1} belong to $A \setminus \{e\}$ in contradiction to eq. (2.2). □

3. Distortion of semi–copulas

The main purpose of this section is to show that the distortion of semi–copulas may be considered as a particular group action. We start with some suitable notation and basic definitions. Let $\mathbb{I} = [0,1]$: the symbol $\mathbb{I}^d$ denotes the Cartesian product of d copies of $\mathbb{I}$, for any $d \in \mathbb{N}$ such that $d \geq 2$. Let $\mathbb{N}_d := \{1, \dots, d\}$. We will follow the vector notation for any point in $\mathbb{I}^d$, e.g. $\mathbf{u} = (u_1, \dots, u_d)$. Let $\mathbf{u}_{-j} := (u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_d)$ for any $\mathbf{u} \in \mathbb{R}^d$ and any $j \in \mathbb{N}_d$: clearly, $\mathbf{u}_{-j} \in \mathbb{R}^{d-1}$. For brevity, the notation $\mathbf{u}_j(t) := (u_1, \dots, u_{j-1}, t, u_{j+1}, \dots, u_d)$ will be exclusively adopted for $t \in \mathbb{I}$ and for $\mathbf{u}_{-j} \in \mathbb{I}^{d-1}$. If F is any real function defined on $\mathbb{I}^d$, let $D_k F(\mathbf{u})$ be, whenever and wherever it exists, the partial derivative of F with respect to the k-th variable at $\mathbf{u} \in \mathbb{I}^d$. The Lebesgue measure on the real line will be denoted by λ; it is intended that a zero measure set (or null set) $E \subset \mathbb{R}$ verifies $\lambda(E) = 0$. Throughout the paper, h is an automorphism of the real unit interval, i.e. an increasing bijection from $\mathbb{I}$ to $\mathbb{I}$, and denote by h^{-1} its inverse. Given any h and any $\mathbf{u} \in \mathbb{I}^d$, let $h(\mathbf{u}) := (h(u_1), \dots, h(u_d))$. Let Θ be the set of automorphisms of $\mathbb{I}$: notice that Θ, equipped with the composition operator, is a group. For the sake of simplicity, the composition $g \circ h$ of two elements $g, h \in \Theta$ will be simply denoted by gh, where, as usual, h is acting first: further, e is the identity element of Θ. Let Θ_c be the subset of Θ given by the convex automorphisms. Let Θ_{ac} be the subset of Θ given by the absolutely continuous automorphisms. Finally, let Θ^* be the subset of Θ such that $h \in \Theta^*$ if, and only if, both h and h^{-1} belong to Θ_{ac}.

Let us recall the notion of semi–copula.

Definition 3.1. A function $S : \mathbb{I}^d \to \mathbb{I}$ is called a (d*-dimensional*) *semi–copula* if it satisfies the following conditions:

($S1$) $S(1, \dots, 1, u_j, 1. \dots, 1) = u_j$ for every $j \in \mathbb{N}_d$ and $u_j \in \mathbb{I}$;
($S2$) S is increasing in each place.

It is easy to see that (S1) and (S2) imply

(S3) $S(\mathbf{u}) = 0$, if $u_j = 0$ for some $j \in \mathbb{N}_d$.

Let $\mathcal{S}_d$ be the class of semi-copulas of dimension d.

Definition 3.2. Let $h \in \Theta$. Let $S \in \mathcal{S}_d$. We call *distortion* of S (by means of h) the mapping $S_h : \mathbb{I}^d \to \mathbb{I}$ defined by

$$S_h(\boldsymbol{u}) := h(S(h^{-1}(u_1), \dots, h^{-1}(u_d))).$$

It is not difficult to see that the distortion preserves both (S1) and (S2) (see, for instance, [7, Theorem 2.2]), hence S_h is always a semi–copula. Further, if S is continuous, so is S_h.

Now, we show that the distortion of semi–copulas is a particular case of a left action. From now on, unless otherwise stated, we identify the group G and the set X of Definition 2.1 with Θ and $\mathcal{S}_d$, respectively.

Proposition 3.3. *The mapping $\alpha : \Theta \times \mathcal{S}_d \to \mathcal{S}_d$ such that $\alpha(h, S) = S_h$ is a left action.*

Proof. We have to show that the two properties (i)-(ii) given in Definition 2.1 are satisfied by the function α above defined. The first property is trivial. The second one amounts to $(S_h)_g = S_{gh}$. According to Definition 3.2, we have that

$$(S_h)_g(\mathbf{u}) = g(S_h(g^{-1}(u_1), \dots, g^{-1}(u_d))) = gh(S(h^{-1}g^{-1}(u_1), \dots, h^{-1}g^{-1}(u_d)))$$

and the claim now follows from the fact that $h^{-1}g^{-1} = (gh)^{-1}$. □

To the best of my knowledge, in [7] Durante and Sempi were the first to show that the distortion of semi–copulas is a special group action: in that case, since the authors adopted a different representation of distortion where the role of the automorphism h and its inverse is interchanged, they dealt with a right action.

4. Distortion of copulas

In the previous section, we showed that any distorted semi–copula is still a semi–copula. The same does not occur with copulas. First of all, let us recall the notion of d-copula.

Definition 4.1. Let R be the cartesian product $[a_1, b_1] \times \ldots \times [a_d, b_d]$ of d real intervals contained in $\mathbb{I}$. Let $F : \mathbb{I}^d \to \mathbb{R}$. We say that the *volume* of R (with respect to F) is given by

$$V_F(R) = \sum_{\boldsymbol{c}} (-1)^{S(\boldsymbol{c})} F(\boldsymbol{c}),$$

where $\boldsymbol{c} = (c_1, \dots, c_d)$ is such that each c_j is equal to either a_j or b_j and $S(\boldsymbol{c})$ is the cardinality of the set $\{j \in \mathbb{N}_d : c_j = a_j\}$.

Definition 4.2. A function $C : \mathbb{I}^d \to \mathbb{I}$ is d-increasing if, and only if, $V_C(R) \geq 0$ for every cartesian product R of d real intervals in $\mathbb{I}$.

Definition 4.3. A function $C : \mathbb{I}^d \to \mathbb{I}$ is called a (d*-dimensional*) *copula* if it is d-increasing and fulfills the border conditions ($S1$) and ($S3$).

Let $\mathcal{C}_d$ be the class of d-copulas: it is well-known that $\mathcal{C}_d \subset \mathcal{S}_d$. It is quite easy to see that a distorted copula C_h is still a copula if, and only if, the distortion by h preserves the d-increasingness property.

Definition 4.4. Let $h \in \Theta$. We say that h is d*-copula preserving* if

$$C_h \in \mathcal{C}_d \quad \textit{for any } C \in \mathcal{C}_d.$$

In general, the class of d-copula preserving automorphisms is very restricted. If we consider, for instance, bivariate copulas, we have the following result.

Theorem 4.5. (Klement *et al.* [18]) *The class of* 2*-copula preserving automorphisms is given by* Θ_c.

However, this does not exclude that, for a *fixed* copula, its distortion is still a copula even if we use a non–convex automorphism, as we will show in a later example. This crucial remark leads us to weaken the notion of copula preserving automorphism in the following way.

Definition 4.6. Let $C \in \mathcal{C}_d$. We say that h is C-preserving if $C_h \in \mathcal{C}_d$.

Let C be a fixed copula. A natural question arises: what is the subset $\Theta(C)$ of Θ, depending on C, given by the C-preserving automorphisms? This was exactly the open problem posed by Durante and Sempi in [7], even if limited to bivariate copulas. What does the literature tell us about this problem? First of all, it is trivial to see that every h is M-preserving, where M is the minimum. With regard to bivariate copulas, we have the following results:

(i) h is W-preserving if, and only if, $h \in \Theta_c$, where $W(u, v) = max\{0, u + v - 1\}$ (see the proof of Theorem 2.4 in [18]);
(ii) h is Π-preserving if, and only if, $\log h^{-1}$ is concave (see [3, p. 318]);
(iii) the class of C-preserving functions includes the convex ones for every copula C (see Theorem 4.5).

Before proceeding with the multivariate case, let us introduce two types of monotonicity for any $f : J \to \mathbb{R}$, where $J \subseteq \mathbb{R}$ is an open interval.

Definition 4.7. We say that f is *d-monotone* if

($D1$) it is differentiable up to order $d - 2$;
($D2$) its derivatives satisfy $(-1)^k \cdot f^{(k)}(x) \geq 0$ for all $k = 0, \ldots, d - 2$ and every $x \in J$;
($D3$) $(-1)^{d-2} \cdot f^{(d-2)}$ is decreasing and convex.

Definition 4.8. We say that f is *d-absolutely monotone* if

($A1$) it is differentiable up to order d;
($A2$) its derivatives satisfy $f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}_d \cup \{0\}$ and every $x \in J$.

With regard to multivariate copulas, we have the following results:

(iv) h is Π-preserving if, and only, the function $h(e^{-t})$ is d-monotone on $]0, +\infty[$; (see [20, Theorem 2.2]);
(v) the class of C-preserving functions includes the d-absolutely monotone ones on $]0, 1[$ for every copula C (see [21, Theorem 4.7]).

An easy consequence of Definition 4.7 and Definition 4.8 is that if a function f is d-monotone (resp. d-absolutely monotone) for some $d \geq 3$, it is also k-monotone (resp. k-absolutely monotone) for any $2 \leq k \leq d$.

Remark 4.9. In the bivariate case, *(iv)* means that $h(e^{-t})$ is convex on $]0, +\infty[$ and, hence, *(iv)* is equivalent to the log-concavity of h^{-1}. Moreover, observe that a 2-absolutely monotone function is convex, hence any d-absolutely monotone function is necessarily convex.

Now, we are ready to transpose notions and results of the second part of Section 2 to our setting. To this purpose, from now on we will identify the set Y with C_d. Let us begin with Definition 2.6.

Definition 4.10. Let C be a copula. The C_d-orbit of C, simply denoted by $\mathcal{O}(C)$, is the subset of C_d given by

$$\{C_h : h \in \Theta\} \cap C_d.$$

In the sequel, we will refer to the C_d-orbit of C simply as orbit of C. According to Remark 2.7, any copula belongs to one and only one orbit. Based on Definition 2.2, the stabilizer of a copula C is the subset of $\Theta(C)$ given by $St(C) = \{h \in \Theta(C) : C_h = C\}$. In other terms, $h \in St(C)$ is equivalent to saying that h satisfies the following functional equation in d variables:

$$C(h^{-1}(\mathbf{u})) = h^{-1}(C(\mathbf{u})) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d.$$

Following Definition 2.8, it is not difficult to see that the generalized stabilizer of a copula C is nothing but $\Theta(C)$. By Lemma 2.9, for any $C \in C_d$ we have that

$$\mathcal{O}(C) = \{C_h : h \in \Theta(C)\}. \tag{4.1}$$

Now, let us focus on a single orbit and suppose that there exists a representative C with the property that we have partial (resp. full) knowledge of $\Theta(C)$. By means of Proposition 2.10, we can show that we also have partial (resp. full) knowledge of $\Theta(T)$ for any copula T belonging to the orbit of C.

Proposition 4.11. *Let C be a copula. Let T be a copula belonging to the orbit of C. Then, there exists $h \in \Theta(C)$ such that*

$$\Theta(T) = \{gh^{-1} : g \in \Theta(C)\}.$$

Proof. Let T be a copula such that $T \in \mathcal{O}(C)$. Due to eq. (4.1), there exists $h \in \Theta(C)$ such that $T = C_h$. The claim is now a straightforward consequence of Proposition 2.10. □

Example 4.12. Let $T : \mathbb{I}^2 \to \mathbb{I}$ be the function defined as

$$T(u,v) = \max\{0, (\sqrt{u} + \sqrt{v} - 1)^2\}$$

It is easy to see that T is the distortion of W by means of $h(t) = t^2$, hence it is a copula belonging to the orbit of W by (*iii*). Since $\Theta(W) = \Theta_c$, owing to *Proposition 4.11* we conclude that every T-preserving f is of the type $f(t) = g(\sqrt{t})$ for any $g \in \Theta_c$.

Example 4.13. Let $T : \mathbb{I}^3 \to \mathbb{I}$ be the function defined as

$$T(u,v,w) = \frac{8uvw}{2 + 2u + 2v + 2w + 2uv + 2uw + 2vw - 6uvw}$$

It is only a matter of calculations to check that T is the distortion of Π by means of the automorphism $h(t) = t/(2-t)$. Set $\varphi(t) := h(e^{-t})$ as $t > 0$. Clearly φ is infinitely differentiable: moreover, after easy computations, one finds that $\varphi'(t) = -2e^{-t}/(2 - e^{-t})^2$, $\varphi''(t) = 2e^{-t}(2 + e^{-t})/(2 - e^{-t})^3$ and finally $\varphi'''(t) = -2e^{-t}(e^{-2t} + 8e^{-t} + 4)/(2 - e^{-t})^4$. Consequently, according to Definition 4.7, φ is 3-monotone and, hence, T is a copula belonging to the orbit of Π. Therefore, if we apply Proposition 4.11, we conclude that every T-preserving f is of the type

$$f(t) = g\left(\frac{2t}{1+t}\right)$$

for any $g \in \Theta$ such that $g(e^{-t})$ is 3-monotone on $]0, +\infty[$.

The next definition aims to formalize the notion of indistinguishability between copulas from the point of view of their sets of preserving automorphisms.

Definition 4.14. Let $C, T \in \mathcal{C}_d$. We say that C and T are D-equivalent if $\Theta(C) = \Theta(T)$.

This notion is quite intriguing, because it opens up a series of non-trivial problems. The first one is that, in general, we do not know whether a certain copula C admits D-equivalent copulas. For example, a bivariate copula C is D-equivalent to W if $\Theta(C) = \Theta_c$. Is there such a copula? The next theorem provides an affirmative answer: the crucial step of the proof is based upon a characterization of convexity for real continuous functions of one variable which is of independent interest.

Proposition 4.15. *Let $f :]a,b[\to \mathbb{R}$ be a continuous function, where $-\infty \le a < b \le \infty$. Then, f is convex if, and only if, for any $x_1, x_2 \in]a,b[$ such that $x_1 < x_2$, there exists a $\delta = \delta(x_1, x_2) > 0$ such that*

$$f(x_2) - f(x_1) \le f(x_2 + t) - f(x_1 + t) \quad \text{for all } t \in]0, \delta[.$$

Proof is in Appendix A

Now, we are ready to formulate the main theorem. Preliminarily, observe that the function $C_{\frac{1}{2},\frac{1}{2}} : \mathbb{I}^2 \to \mathbb{I}$ defined as

$$C_{\frac{1}{2},\frac{1}{2}}(u,v) = \frac{M(u,v) + W(u,v)}{2}$$

belongs to a two-parameter family of copulas due to Fréchet (see, e.g., [23, Exercise 2.4]).

Theorem 4.16. *The copula $C_{\frac{1}{2},\frac{1}{2}}$ is D-equivalent to W.*

Proof is in Appendix B

The notion of D-equivalence, being clearly an equivalence relation on the set of copulas, induces a partition of $\mathcal{C}_d$. An interesting problem concerns the relationship between such partition and the one given by the orbits (see Remark 2.7). Particularly, the question is: provided that C and T are D-equivalent, do they necessarily belong to different orbits? This is still an open problem. In some cases, the answer is in the affirmative.

Corollary 4.17. *Let $C \ne W$ be a copula D-equivalent to W. Then, C does not belong to $\mathcal{O}(W)$.*

Proof. Since $\Theta_c = \Theta(W) = \Theta(C)$, the claim is a direct consequence of Proposition 2.11, with $A = \Theta_c$. □

The next result is a straightforward consequence of Proposition 4.11, hence the proof will be omitted.

Corollary 4.18. *Let C and T be D-equivalent copulas. Then, C_h is D-equivalent to T_h for every $h \in \Theta(C)$.*

5. Absolute continuity of sections of distorted copulas

Let C_h be a distorted copula such that the function $t \mapsto C_h(\mathbf{u}_j(t))$ is invertible on a subdomain of the kind $[a, 1]$ for some $a \in [0, 1[$: in this section, we are interested in studying the assumptions on h and C which ensure that $t \mapsto C_h(\mathbf{u}_j(t))$ and its inverse are absolutely continuous. To this purpose, we need to recall a classical result and a few basic facts on absolutely continuous functions. The next theorem is due to M. A. Zarecki (see, e.g., [22]).

Theorem 5.1. *Let $f : [a, b] \to [c, d]$ be an increasing bijection that maps $[a, b]$ onto $[c, d]$. Then, f^{-1} is absolutely continuous if, and only if, $\lambda(\{x \in [a, b] : f'(x) = 0\}) = 0$.*

Remark 5.2. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function and $g : [c, d] \to \mathbb{R}$ be an absolutely continuous function, with $f([a, b]) \subseteq [c, d]$. If f is monotone, then the composite function $g \circ f$ is absolutely continuous (see, e.g., [27, Problem 13.12]).

Remark 5.3. An absolutely continuous function maps null sets into null sets (see, e.g., [27, Theorem 13.8]).

Lemma 5.4. *Let $f : [a, b] \to [c, d]$ be an increasing bijection that maps $[a, b]$ onto $[c, d]$ and suppose that f^{-1} is absolutely continuous. Let $g : [c, d] \to [g(c), g(d)]$ be an increasing bijection. Then, the chain rule*

$$(g \circ f)'(t) = g'(f(t)) \cdot f'(t)$$

holds for almost all $t \in [a, b]$. If g^{-1} is absolutely continuous, then $(g \circ f)' > 0$ almost everywhere.

Proof. First of all, observe that both f and g have finite derivative almost everywhere. Let Z be the subset of $[c, d]$ where g' does not exist or it is equal to ∞: as above remarked, Z is a null set. Observe that $f(t) \in Z$ if, and only if, $t \in f^{-1}(Z)$: according to Remark 5.3, we have $\lambda(f^{-1}(Z)) = 0$. This immediately implies that the chain rule for the derivative of the composite function $g \circ f$ may be applied. Finally, under the assumption that g^{-1} is absolutely continuous, we obtain that $f^{-1} \circ g^{-1}$ is absolutely continuous (see Remark 5.2) and, hence, by Theorem 5.1 we obtain that $(g \circ f)' > 0$ almost everywhere. □

For the sake of convenience, we adopt the alternative notation $\mu_{S,j,\mathbf{u}}(t) = S(\mathbf{u}_j(t))$, where S is any semi–copula. First of all, $\mu_{S,j,\mathbf{u}}(0) = 0$ by (S3); secondly, $\mu_{S,j,\mathbf{u}}$ is increasing as a consequence of (S2), whence its derivative $\mu'_{S,j,\mathbf{u}}(t)$ exists for almost all $t \in \mathbb{I}$ and it coincides with $D_j S(\mathbf{u}_j(t))$. Henceforth, to avoid a cumbersome notation, we shall simply write $\mu_{S,\mathbf{u}}$ instead of $\mu_{S,j,\mathbf{u}}$ unless ambiguity arises.

Remark 5.5. Let $C \in \mathcal{C}_d$. Then, we emphasize that $\mu_{C,\boldsymbol{u}}$ is 1-Lipschitz and, hence, absolutely continuous for every $j \in \mathbb{N}_d$ and every $\boldsymbol{u}_{-j} \in \mathbb{I}^{d-1}$ (see, for instance, [9]).

Given any automorphism h, from Definition 3.2 it follows that

$$\mu_{S_h,\mathbf{u}} = h \circ \mu_{S,h^{-1}(\mathbf{u})} \circ h^{-1}. \tag{5.1}$$

By the above remark, it is clear that a necessary condition for h to be C-preserving is that $\mu_{C_h,\mathbf{u}}$ is absolutely continuous for every $j \in \mathbb{N}_d$ and every $\mathbf{u}_{-j} \in \mathbb{I}^{d-1}$. This is easy to show when both h and h^{-1} are absolutely continuous ($h \in \Theta^*$, in symbols).

Lemma 5.6. *Let $C \in \mathcal{C}_d$. Assume that $h \in \Theta^*$. Then, $\mu_{C_h,\mathbf{u}}$ is absolutely continuous for every $j \in \mathbb{N}_d$ and every $\mathbf{u}_{-j} \in \mathbb{I}^{d-1}$.*

Proof. In view of Remark 5.5, taking into account Remark 5.2, we infer that the composite function $\mu_{C,h^{-1}(\mathbf{u})} \circ h^{-1}$ is absolutely continuous. Since $\mu_{C,h^{-1}(\mathbf{u})} \circ h^{-1}$ is clearly increasing, by virtue of eq. (5.1) we may repeat the above argument with $h \circ (\mu_{C,h^{-1}(\mathbf{u})} \circ h^{-1})$, hence the claim is established. □

Let us introduce a special form of strict monotonicity for semi–copulas.

Definition 5.7. A semi–copula S is said to be *j-strictly monotone* if, for any $\boldsymbol{u}_{-j} \in \mathbb{I}^{d-1}$ and any $t \in]0, 1[$ such that $S(\boldsymbol{u}_j(t)) > 0$, there holds $S(\boldsymbol{u}_j(t)) < S(\boldsymbol{u}_j(t'))$ for every $t' \in]t, 1]$. A semi–copula S is said to be *jointly strictly monotone* if it is j-strictly monotone for every $j \in \mathbb{N}_d$.

Fix a semi–copula S and let $\Gamma_j(S) = \{\mathbf{u}_{-j} \in \mathbb{I}^{d-1} : S(\mathbf{u}_j(1)) > 0\}$. Evidently, owing to (S1), $\Gamma_j(S)$ is not empty. In general, $\Gamma_j(S)$ is a strict subset of $]0, 1]^{d-1}$ even if S is a copula: consider for example the copula $C(u, v, w) = W(u, v) \cdot w$ (see, e.g., [9, Remark 1.5.3]). In this case, any $(u, v) \in \mathbb{I}^2$ such that $u + v \leq 1$ does not belong to $\Gamma_3(C)$. Given any $\mathbf{u}_{-j} \in \Gamma_j(S)$, let

$$a_S(\mathbf{u}_{-j}) := \inf\{t \leq 1 : S(\mathbf{u}_j(t)) > 0\}.$$

It is trivial to see that $a_S(\mathbf{u}_{-j}) \in [0,1[$ when S is continuous. Moreover, $S(\mathbf{u}_j(t)) = 0$ for every $t \in [0, a_S(\mathbf{u}_{-j})]$.

Remark 5.8. Observe that a continuous semi–copula S is j-strictly monotone if, and only if, for any $\boldsymbol{u}_{-j} \in \Gamma_j(S)$ we have that $\mu_{S,\boldsymbol{u}}$ is an increasing bijection from $[a_S(\boldsymbol{u}_{-j}), 1]$ to $[0, \mu_{S,\boldsymbol{u}}(1)]$. In this case, we denote its inverse from $[0, \mu_{S,\boldsymbol{u}}(1)]$ to $[a_S(\boldsymbol{u}_{-j}), 1]$ by $\mu^{-1}_{S,\boldsymbol{u}}$.

Let S be a continuous, j-strictly monotone semi–copula and fix any $\mathbf{u}_{-j} \in \Gamma_j(S)$: in the sequel, when we say that $\mu_{S,\mathbf{u}}$ is absolutely continuous, it is intended that the domain of $\mu_{S,\mathbf{u}}$ is $[a_S(\mathbf{u}_{-j}), 1]$. Let C be a j-strictly monotone copula: in view of Remark 5.5, we know that $\mu_{C,\mathbf{u}}$ is absolutely continuous for any $\mathbf{u}_{-j} \in \Gamma_j(C)$. Now, the question is: when does the same property hold for $\mu^{-1}_{C,\mathbf{u}}$? The next result is completely based upon Theorem 5.1, so we can omit the proof.

Corollary 5.9. *Let C be a j-strictly monotone copula. Then, for every $\mathbf{u}_{-j} \in \Gamma_j(C)$, $\mu^{-1}_{C,\mathbf{u}}$ is absolutely continuous if, and only if, $\mu'_{C,\mathbf{u}} > 0$ almost everywhere.*

Let C be a j-strictly monotone copula and consider the distorted copula C_h: due to eq. (5.1), it is easy to see that $\mathbf{u}_{-j} \in \Gamma_j(C_h)$ implies $h^{-1}(\mathbf{u}_{-j}) \in \Gamma_j(C)$. This leads to the following lemma.

Lemma 5.10. *Let C be a j-strictly monotone copula. Then, C_h is j-strictly monotone and $a_{C_h}(\mathbf{u}_{-j}) = h(a_C(h^{-1}(\mathbf{u}_{-j})))$ for every $\mathbf{u}_{-j} \in \Gamma_j(C_h)$*

Proof. Due to eq. (5.1), we deduce that $\mu_{C_h,\mathbf{u}}(t) > 0$ if, and only if, $h^{-1}(t) > a_C(h^{-1}(\mathbf{u}_{-j}))$ and this easily leads to $a_{C_h}(\mathbf{u}_{-j}) = h(a_C(h^{-1}(\mathbf{u}_{-j})))$. As a consequence, if we consider the restriction of h^{-1} to the subdomain $[a_{C_h}(\mathbf{u}_{-j}), 1]$, we infer that the composite function $\nu : [a_{C_h}(\mathbf{u}_{-j}), 1] \to \mathbb{R}$ given by $\nu = \mu_{C,h^{-1}(\mathbf{u})} \circ h^{-1}$ is well defined and strictly increasing, hence the claim is established. □

The main result of this section shows that, provided that C is j-strictly monotone, the absolute continuity of $\mu^{-1}_{C,\mathbf{u}}$ ensures the absolute continuity of $\mu^{-1}_{C_h,\mathbf{u}}$. A crucial step of the proof is the fact that the chain rule for differentiation may be applied in eq. (5.1) or, in other terms, that the following property holds for some $j \in \mathbb{N}_d$, for every $\mathbf{u}_{-j} \in \Gamma_j(C_h)$ and for almost all $t \in [a_{C_h}(\mathbf{u}_{-j}), 1]$:

$$\mu'_{C_h,\mathbf{u}}(t) = h'(\mu_{C,h^{-1}(\mathbf{u})}(h^{-1}(t))) \cdot \mu'_{C,h^{-1}(\mathbf{u})}(h^{-1}(t)) \cdot (h^{-1})'(t). \tag{5.2}$$

Proposition 5.11. *Let C be a j-strictly monotone copula. Assume that $\mu^{-1}_{C,\mathbf{u}}$ is absolutely continuous for any $\mathbf{u}_{-j} \in \Gamma_j(C)$. Let $h \in \Theta^*$. Then, $\mu^{-1}_{C_h,\mathbf{u}}$ is absolutely continuous for every $\mathbf{u}_{-j} \in \Gamma_j(C_h)$.*

Proof. Let $\mathbf{u}_{-j} \in \Gamma_j(C_h)$. If we adopt the notation introduced in the proof of Lemma 5.10, let ν be the increasing bijection from $[a_{C_h}(\mathbf{u}_{-j}), 1]$ to $[0, \mu_{C,h^{-1}(\mathbf{u})}(1)]$ given by $\nu = \mu_{C,h^{-1}(\mathbf{u})} \circ h^{-1}$. Note that both $\mu_{C,h^{-1}(\mathbf{u})}$ and h^{-1} are increasing bijections and, by assumption, their inverse functions are absolutely continuous. Thus, Lemma 5.4 applies and one finds that

$$\nu'(t) = \mu'_{C,h^{-1}(\mathbf{u})}(h^{-1}(t)) \cdot (h^{-1})'(t) > 0$$

almost everywhere on $[a_{C_h}(\mathbf{u}_{-j}), 1]$. It is easy to see that $\nu^{-1} = h \circ \mu^{-1}_{C,h^{-1}(\mathbf{u})}$ is absolutely continuous, hence eq. (5.2) is a straightforward consequence of Lemma 5.4 applied to $h \circ \nu$, which is nothing but $\mu_{C_h,\mathbf{u}}$ by eq. (5.1). Moreover, since $h \in \Theta^*$, Lemma 5.4 also implies that $\mu'_{C_h,\mathbf{u}} > 0$ almost everywhere on $[a_{C_h}(\mathbf{u}_{-j}), 1]$. Therefore, the claim is a direct consequence of Corollary 5.9. □

6. The bivariate case

The main purpose of this section is to establish a general method able to enlarge the class of bivariate copulas whose associated set of preserving automorphisms is (at least partially) known. Throughout the section, S is a bivariate continuous semi–copula and C is a bivariate copula. Given any $u \in \mathbb{I}$, we have that $\mu_{S,1,u}(t) = S(t,u)$ and $\mu'_{S,1,u}(t) = D_1 S(t,u)$ (and analogously for $\mu_{S,2,u}$). For brevity, without loss of generality, in this section we will exclusively deal with $\mu_{S,1,u}$, simply denoted by $\mu_{S,u}$ in the sequel. Note that for any $u \in \mathbb{I}$ we shall write $a_S(u)$ instead of $a_S(\mathbf{u}_{-1})$; recall that $a_S(u) = \inf\{t \leq 1 : S(t,u) > 0\}$ for any $u > 0$ and $a_S(0) = 1$. Moreover, $\mu_{S,u}$ is a bijection from $[a_S(u), 1]$ to $[0, u]$. Eventually, note that $\Gamma_1(S) =]0,1]$ independently of S.

Definition 6.1. Let F be a real function defined on some nonempty subset $D \subseteq \mathbb{I}^2$. We say that F is *increasing in the second place almost everywhere* if for any $u_1, u_2 \in \mathbb{I}$, $u_1 < u_2$, and any interval $J \subseteq \mathbb{I}$ such that $(t, u_i) \in D$ for almost all $t \in J$ and for $i = 1, 2$, there holds $F(t, u_1) \leq F(t, u_2)$ for almost all $t \in J$.

Remark 6.2. Observe that, for any $u_1, u_2 \in \mathbb{I}$ with $u_1 < u_2$, the increasing monotonicity of the mapping $t \mapsto S(t, u_i)$ on $\mathbb{I}$ for $i = 1, 2$ entails the existence of both $D_1 S(t, u_1)$ and $D_1 S(t, u_2)$ for almost all $t \in \mathbb{I}$. Therefore, the increasingness of $D_1 S$ in the second place almost everywhere amounts to the following property: for any $0 \le u_1 < u_2 \le 1$ the inequality $D_1 S(t, u_1) \le D_1 S(t, u_2)$ holds for almost all $t \in \mathbb{I}$. Note that we can always assume $u_1 > 0$, because $D_1 S(t, 0) = 0$ for every $t \in \mathbb{I}$ as a consequence of $(S3)$. Moreover, it is easy to see that $a_S(u_1) \ge a_S(u_2)$. Finally, we can limit ourselves to consider only $t > a_S(u_1)$, because $D_1 S(t, u) = 0$ when $a_S(u_1) > 0$ and $0 \le t < a_S(u_1)$.

The next result is a simplified version of a characterization theorem of the d-increasingness property ([16, Theorem 2.14]) for real bivariate functions defined on $\mathbb{I}^2$, hence we can omit the proof.

Theorem 6.3. *Let $F : \mathbb{I}^2 \to \mathbb{R}$ be such that $t \mapsto F(t, v)$ is absolutely continuous for every fixed $v \in \mathbb{I}$. Assume that F satisfies* (S1) *and* (S3). *Then, F is a copula if, and only if, $D_1 F$ is increasing in the second place almost everywhere.*

Proposition 6.4. *Assume that $h \in \Theta^*$. Suppose that $D_1 C_h$ is increasing in the second place almost everywhere. Then, $h \in \Theta(C)$.*

Proof. As remarked in Section 4, the distortion preserves (S1) and (S3): moreover, by Lemma 5.6, we have that $t \mapsto C_h(t, v)$ is absolutely continuous for every fixed $v \in \mathbb{I}$. Then, the claim directly follows from Theorem 6.3. □

Now, our purpose is to exploit the previous result in order to find a sufficient condition for an automorphism h to be C-preserving. We shall act on two fronts by introducing two crucial functions associated with C and h, respectively. The first one is $\psi_C := \frac{D_1 C}{C}$. We emphasize that, by the celebrated Rademacher's theorem (see [24]), the domain of ψ_C is the set $L_C = \{(u, v) \in \mathbb{I}^2 : C(u, v) > 0\}$ up to a (possibly empty) subset $\Omega \subset L_C$ such that $\lambda_2(\Omega) = 0$, where λ_2 stands for the two-dimensional Lebesgue measure.

Remark 6.5. Observe that for any C we have $L_C \subseteq L^*$, where $L^* := \mathbb{I}^2 \setminus (\{(x, 0) : x \in \mathbb{I}\} \cup \{(0, x) : x \in \mathbb{I}\})$.

Let σ_h be the function defined as $\sigma_h = \frac{h^{-1}}{(h^{-1})'}$. In the sequel, it is implicitly intended that σ_h is defined almost everywhere on $\mathbb{I}$: this is equivalent to saying that the associated automorphism h is absolutely continuous (see Theorem 5.1).

The stage is now set for the main result of the section. Let $\mathcal{F}_1$ denote the subclass of (bivariate) 1-strictly monotone copulas such that $\mu_{C,u}^{-1}$ is absolutely continuous for any $u > 0$ and ψ_C is increasing in the second place almost everywhere. Observe that, in view of Remark 6.2, the increasingness of ψ_C in the second place almost everywhere amounts to the following property: for any $0 < u_1 < u_2 \le 1$, the inequality $\psi_C(t, u_1) \le \psi_C(t, u_2)$ holds for almost all $t > a_C(u_1)$.

Theorem 6.6. *Let $C \in \mathcal{F}_1$. Let $h \in \Theta^*$. Assume that σ_h is increasing. Then, $h \in \Theta(C)$.*

Proof is in Appendix C

Theorem 6.6 allows us to address the problem of determining a family of automorphisms preserving a certain class of copulas by separating the subclass $\mathcal{F}_1$ of considered copulas and the required kind of automorphisms. Let us begin with some distinguished examples belonging to $\mathcal{F}_1$. Preliminarily, recall that C is *exchangeable* when $C(u, v) = C(v, u)$ for all $(u, v) \in \mathbb{I}^2$: obviously, an exchangeable copula is 1-strictly monotone if, and only if, it is jointly strictly monotone.

Example 6.7. Given any $f : \mathbb{I} \to \mathbb{R}$, let C be defined as

$$C(u, v) = M(u, v) \cdot f(\max\{u, v\}). \tag{6.1}$$

According to the results presented in [13], C is an exchangeable copula, called *Marshall copula* or *semilinear copula*, (see [19,10]), if, and only if, f satisfies the following properties:

$(P1)$ $f(1) = 1$;
$(P2)$ f is increasing;
$(P3)$ f is absolutely continuous;
$(P4)$ $f(t) \ge t f'(t)$ for almost all $t \in \mathbb{I}$.

We shall call f a *generator* if, and only if, it fulfills $(P1) - (P4)$ and C as in eq. (6.7) is the generated copula. Note that $(P3)$ and $(P4)$ imply that the mapping $f(t)/t$ is decreasing on $]0, 1]$, hence a generator f satisfies the condition $t \le f(t) \le 1$ for all $t \in \mathbb{I}$. It is easy to see that a generated copula C is jointly strictly monotone if, and only if, the generator f is strictly increasing. In this case, f admits an inverse function $f^{-1} : [f(0), 1] \to \mathbb{I}$, with $f(0) \in [0, 1[$: moreover, for any $u > 0$ we have

$$\mu_{C,u}^{-1}(t) = \begin{cases} t/f(u), & \text{if } t \in [0, u f(u)]; \\ f^{-1}(t/u), & \text{if } t \in [u f(u), u]. \end{cases}$$

Consequently, it is not difficult to show that $\mu_{C,u}^{-1}$ is absolutely continuous for any $u > 0$ if, and only if, f^{-1} is absolutely continuous. A strictly increasing generator f whose inverse is absolutely continuous will be called a *strong generator* and a copula generated by a strong generator will be called a *strong semilinear* copula. Let C be a strong semilinear copula: after some computations, for any $(u, v) \in L^*$ one finds that

$$\psi_C(u,v) = \begin{cases} 1/u, & \text{if } u < v; \\ f'(u)/f(u), & \text{for almost all } u > v. \end{cases}$$

Given any $0 < v_1 < v_2 \leq 1$, it is immediate to see that $\psi_C(u, v_1) = \psi_C(u, v_2)$ for all $0 < u < v_1$ and for almost all $u > v_2$, while the inequality $\psi_C(u, v_1) \leq \psi_C(u, v_2)$ reduces to (P4) when $v_1 < u < v_2$. In conclusion, any strong semilinear copula C belongs to $\mathcal{F}_1$.

Example 6.8. The two parameters family of copulas given by

$$C_{\alpha,\beta}(u,v) = \begin{cases} u^{1-\alpha} v, & \textit{if } u^\alpha \geq v^\beta; \\ u v^{1-\beta}, & \textit{otherwise}, \end{cases}$$

where $0 < \alpha, \beta < 1$, is known as the *Marshall-Olkin* family (see, e.g., [23]). Note that $C_{\alpha,\beta}$ is generally not exchangeable. It is easy to see that $C_{\alpha,\beta}$ is jointly strictly monotone. Moreover, for any $u > 0$ we have

$$\mu_{C_{\alpha,\beta},u}^{-1}(t) = \begin{cases} t \cdot u^{\beta-1}, & \text{if } t \in [0, u^\rho]; \\ (t/u)^{1/(1-\alpha)}, & \text{if } t \in [u^\rho, u], \end{cases}$$

where $\rho = \rho(\alpha, \beta) = 1 - \beta + \beta/\alpha$, hence $\mu_{C_{\alpha,\beta},u}^{-1}$ is evidently absolutely continuous for any $u > 0$. After some computations, for any $(u, v) \in L^*$ one finds that

$$\psi_{C_{\alpha,\beta}}(u,v) = \begin{cases} 1/u, & \text{if } u < v^{\beta/\alpha}; \\ (1-\alpha)/u, & \text{if } u > v^{\beta/\alpha}. \end{cases}$$

Given any $0 < v_1 < v_2 \leq 1$, it is immediate to see that $\psi_{C_{\alpha,\beta}}(u, v_1) = \psi_{C_{\alpha,\beta}}(u, v_2)$ for all $0 < u < v_1^{\beta/\alpha}$ and for all $u > v_2^{\beta/\alpha}$, while the assumption $\alpha > 0$ forces $\psi_{C_{\alpha,\beta}}(u, v_1) < \psi_{C_{\alpha,\beta}}(u, v_2)$ for all $v_1^{\beta/\alpha} < u < v_2^{\beta/\alpha}$. In conclusion, any copula of this family belongs to $\mathcal{F}_1$.

Example 6.9. The one parameter family of exchangeable copulas given by

$$C_a(u,v) = uv + auv(1-u)(1-v),$$

where $-1 \leq a \leq 1$, is known as the *Farlie-Gumbel-Morgenstern* family (see, e.g., [23]). A simple calculation shows that $D_1 C_a(u, v) = v(1 + a(1 - v)(1 - 2u))$ for all $(u, v) \in \mathbb{I}^2$: this ensures at the same time that C_a is jointly strictly monotone and that $\mu_{C_a,u}^{-1}$ is absolutely continuous for all $u > 0$ (the last conclusion is due to Theorem 5.1). After some computations, for any $(u, v) \in L^*$ one finds that

$$\psi_{C_a}(u,v) = \frac{1}{u} - a\frac{1-v}{1+a(1-u)(1-v)}.$$

An elementary calculation shows that $\partial\psi_{C_a}/\partial v = a/(1 + a(1 - u)(1 - v))^2$, hence ψ_{C_a} is increasing in the second place for every $u > 0$ when $a \geq 0$. In conclusion, any copula of this family belongs to $\mathcal{F}_1$ for $a \geq 0$.

Note that in Example 6.7 and in Example 6.8, the function ψ_C does not depend on the second variable when the first variable belongs to a certain interval. This strong form of increasingness in the second variable almost everywhere of ψ_C forces σ_h to be increasing in order for h to be C-preserving. Before proceeding to the next theorem, we need some preliminary results concerning the set of 1-strictly monotone copulas.

Let C be 1-strictly monotone. Given any $b \in]0, 1[$, define $\zeta_b : [b, 1] \to \mathbb{I}$ as $\zeta_b(x) = \mu_{C,x}^{-1}(b)$. Observe that ζ_b is well-defined, since the domain of $\mu_{C,x}^{-1}$ is $[0, x]$ for any $x > 0$. Moreover, it is easy to see that

$$\mu_{C,x}(\zeta_b(x)) = b = C(\zeta_b(x), x) \tag{6.2}$$

for any $x \in [b, 1]$. We will show that the mapping ζ_b fulfills the following properties:

(E1) $\zeta_b(b) = 1$;
(E2) ζ_b is decreasing;
(E3) ζ_b is continuous.

The first property follows from $b = C(\zeta_b(b), b) = C(1, b)$ and the 1-strict monotonicity of C. With regard to the second property, given any $x < x'$ suppose *ab absurdo* that $\zeta_b(x) < \zeta_b(x')$: this implies $b = C(\zeta_b(x), x) \leq C(\zeta_b(x), x') < C(\zeta_b(x'), x') = b$, which is a contradiction. Finally, for any sequence $\{x_n\} \subset [b, 1]$ such that $x_n \to x$, by the monotonicity of ζ_b there holds $\zeta_b(x_n) \to l$ for some

$l \in \mathbb{I}$. Consequently, we get $b = C(\zeta_b(x_n), x_n) \to C(l, x)$, hence $l = \zeta_b(x)$ by definition, so showing (E3). Evidently, ζ_b is a decreasing bijection from $[b, 1]$ to $[b, 1]$ under the assumption that C is jointly strictly monotone: in this case, we will denote by ζ_b^{-1} its inverse.

Finally, we need a simplified version of a necessary condition required by the d-increasingness property ([16, Proposition 2.2], [15, Proposition 2.2]) for real 2-increasing functions defined on $\mathbb{I}^2$ (the proof is omitted).

Theorem 6.10. *Let $F : \mathbb{I}^2 \to \mathbb{R}$ be 2-increasing. Let $t \in \mathbb{I}$ and $0 \le u_1 < u_2 \le 1$ be such that both $D_1F(t, u_1)$ and $D_1F(t, u_2)$ exist. Then, $D_1F(t, u_1) \le D_1F(t, u_2)$.*

Theorem 6.11. *Let C be jointly strictly monotone and such that $\mu_{C,u}^{-1}$ is absolutely continuous for any $u > 0$. Assume that there exists a null set $E \subset \mathbb{I}$, a mapping $f :]0, 1] \to]0, 1]$ continuous at 1, with $f(1) = 1$, and a function $\kappa :]0, 1] \to]0, \infty[$ such that*

$$\psi_C(u, v) = \kappa(u) \quad \text{for any } (u, v) \in]0, 1] \setminus E \times]0, f(u)[. \tag{6.3}$$

Suppose that ζ_b^{-1} is absolutely continuous for any $b > 0$. Let $h \in \Theta^*$. If $h \in \Theta(C)$, then σ_h is increasing.

Proof is in Appendix D

Let us show that Theorem 6.11 applies to any strong semilinear copula. Indeed, in this case, after some computations, it is possible to see that

$$\zeta_b^{-1}(x) = \begin{cases} (f^{-1} \circ \gamma_b)(x), & \text{if } x \in [b, \varphi^{-1}(b)]; \\ (\gamma_b \circ f)(x), & \text{if } x \in [\varphi^{-1}(b), 1], \end{cases}$$

where $\gamma_b(x) := b/x$ and $\varphi(x) := x f(x)$, hence ζ_b^{-1} is absolutely continuous for any $b \in]0, 1[$, because it is clearly absolutely continuous on both the subdomains $[b, \varphi^{-1}(b)]$ and $[\varphi^{-1}(b), 1]$ (see Remark 5.2 and [27, Problem 13.2]). Let E be the subset of $\mathbb{I}$ where f' does not exist or it is ∞ or zero: by Theorem 5.1, we know that E is a null set and we get $\psi_C(u, v) = \kappa(u)$, where $\kappa(x) = f'(x)/f(x)$, for all $(u, v) \in]0, 1] \setminus E \times]0, f(u)[$, with $f(x) = x$. The same conclusion may be drawn for any Marshall-Olkin copula: in fact, in this case, a simple calculation shows that

$$\zeta_b(x) = \begin{cases} (b/x)^{1/(1-\alpha)}, & \text{if } x \in [b, b^{1/\rho}]; \\ b x^{\beta - 1}, & \text{if } x \in [b^{1/\rho}, 1], \end{cases}$$

and the absolute continuity of ζ_b^{-1} follows from the evident fact that $\zeta_b' < 0$ almost everywhere. Finally, in this case E is the empty set and $\psi_C(u, v) = \kappa(u)$, where $\kappa(x) = (1 - \alpha)/x$, for all $(u, v) \in]0, 1] \times]0, f(u)[$, with $f(x) = x^{\alpha/\beta}$.

We conclude this section with some consideration regarding the subclass of Θ given by $\Theta_\sigma = \{h \in \Theta_{ac} : \sigma_h \text{ is increasing}\}$. We assert that if $\log h^{-1}$ is concave then $h \in \Theta_\sigma$. First of all, we state that the concavity of $\log h^{-1}$ entails $h \in \Theta^*$. Indeed, given any $a \in]0, 1[$, the absolute continuity of $\log h^{-1}$ on $[a, 1]$ follows from its concavity (see, e.g., [27, Theorem 14.13]) and, hence, h^{-1} is absolutely continuous on $[a, 1]$ since $h^{-1} = \exp \circ \log h^{-1}$ (see Lemma 5.2). This implies that $h^{-1} \in \Theta_{ac}$ by the arbitrariness of a (see, e.g., [27, Problem 13.8]). By the same token, it may be proven that $h \in \Theta_{ac}$, in consideration of the fact that the inverse of $\log h^{-1}$ from $[a, 1]$ to $[\log h^{-1}(a), 0]$, given by $h \circ \exp$, is convex for any $a \in]0, 1[$, so closing the statement. If we combine the statement with the log-concavity of h^{-1}, we infer that $(\log h^{-1})' = 1/\sigma_h$ is strictly positive almost everywhere and is decreasing, so closing the assertion. The converse is easy when $h \in \Theta^*$ (the key point of the proof is based upon [27, Theorem 14.14]), but it is generally not true for an arbitrary automorphism, even if it is absolutely continuous. Indeed, consider $h_\tau = \varphi^{-1}$, where $\varphi := (\mathrm{e} + \tau)/2$ and τ denotes the Cantor ternary function (see, for instance, [6]). Note that $\varphi' = 1/2$ almost everywhere, hence h_τ is absolutely continuous by Theorem 5.1. At the same time, we have $\int_0^1 \varphi'(t)\, dt = 1/2 < 1 = \varphi(1) - \varphi(0)$, so that h_τ^{-1} is not absolutely continuous and, hence, h_τ does not belong to Θ^*. It is immediate to see that $\sigma_{h_\tau} = \mathrm{e} + \tau$, so that $h_\tau \in \Theta_\sigma$. However, $\log h_\tau^{-1}$ cannot be concave, because otherwise we would get $h_\tau \in \Theta^*$. Consequently, h_τ does not preserve the product (see (ii) in Section 4) even if σ_{h_τ} is increasing: the crucial point is just that $h_\tau \notin \Theta^*$. In general, for any copula $C \in \mathcal{C}_d$, it seems hard that an automorphism h not belonging to Θ^* may be C-preserving: we note in passing that all the preserving automorphisms listed in (i)-(v) in Section 4 belong to Θ^*. Thus, the idea is to restrict the family of considered automorphisms to Θ^*: in other terms, the left action given by the distortion of semi–copulas is referred to the group $G = \Theta^*$. This little loss of generality is motivated by the fact that the excluded automorphisms are somewhat negligible, in the sense that any $h \in \Theta \setminus \Theta^*$ is essentially a pathological and hardly tractable function (see, for instance, [28] and the references therein). Set $\Theta^*(C) := \Theta(C) \cap \Theta^*$: as a straightforward consequence of Theorem 6.6 and Theorem 6.11, we obtain a characterization of the C-preserving automorphisms when C is a strong semilinear copula or belongs to the Marshall-Olkin family.

Theorem 6.12. *Let C be a strong semilinear copula or a Marshall-Olkin copula. Then, $h \in \Theta^*(C)$ if, and only if, $\log h^{-1}$ is concave or, equivalently, $\Theta^*(C) = \Theta(\Pi)$.*

7. Conclusions

In this work, we have focused mainly on the problem of determining the automorphisms which preserve the copula property under a distortion. Firstly, we have analyzed this issue within the algebraic theoretical framework of the group actions whose basic notions have allowed us to achieve an important result about the connection between the preserving automorphisms of a d-copula C and every copula belonging to the same orbit of C, regardless of d. We have also introduced the notion of D-equivalence between two copulas in order to formalize the concept of indistinguishability from the point of view of their distortion and we have proved that the lower Fréchet-Hoeffding bound admits a D-equivalent copula. Secondly, we have shown the crucial role of the property of absolute continuity of the one-dimensional sections of a distorted copula. Finally, we have established a sufficient condition in order for an automorphism to preserve a large family of bivariate copulas under distortion. In particular, we have shown that if we refer the distortion to the group $G = \Theta^*$, any strong semilinear (or Marshll-Olkin) copula is indistinguishable from the product. Recall that the (strict) Archimedean copulas, widely used in the applications, are nothing but distortions of the product, hence they all are associative, but so far we have not found any convincing statistical interpretation of associativity (see, for instance, [1, Problem 15]). If we replace a (strict) Archimedean copula Π_h with the distortion C_h of a strong semilinear (or Marshll-Olkin) copula C, we are sure that C_h is still a copula with the undoubted advantage that C_h does not fulfill the (mysterious) property of associativity.

CRediT authorship contribution statement

Roberto Ghiselli Ricci: Writing – review & editing, Writing – original draft, Resources, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Roberto Ghiselli Ricci has patent pending to Licensee. RGR: No relationship If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Proposition 4.15

Proof. Suppose first that f is convex. Then, for any sufficiently small $t > 0$, we have that $x_1 < x_1 + t < x_2$, thus, according to [27, Proposition 14.4], we obtain

$$\frac{f(x_1+t)-f(x_1)}{t} \le \frac{f(x_2)-f(x_1+t)}{x_2-x_1-t} \tag{A.1}$$

By the same token, we get

$$\frac{f(x_2)-f(x_1+t)}{x_2-x_1-t} \le \frac{f(x_2+t)-f(x_2)}{t} \tag{A.2}$$

The combination of eq. (A.1) with eq. (A.2) leads to the claim. Conversely, by the continuity of f, it suffices to show that for any $u, v \in]a, b[$ we have $f(\xi) \le (f(u) + f(v))/2$, where $\xi := (u+v)/2$. Without loss of generality, let $u < v$: by assumption, there exists a $\delta = \delta(u, \xi) > 0$ such that

$$f(\xi) - f(u) \le f(\xi+t) - f(u+t) \quad \text{for all } t \in]0, \delta[. \tag{A.3}$$

Now, set $\varphi(t) := f(\xi+t) - f(u+t)$ and $E := \{t > 0 : f(\xi) - f(u) \le \varphi(\tau) \text{ for all } 0 \le \tau \le t\}$. Note that eq. (A.3) implies $t^* \ge \delta$, where $t^* := \sup E$. We assert that either $t^* = \infty$ if $b = \infty$ or $\xi + t^* = b$ if b is finite. Suppose that the assertion is false: this means that, regardless of whether b is finite or not, t^* is finite and $\xi + t^* < b$. In this case, obviously, t^* belongs to the domain of φ. Remark that the continuity of φ and the fact that $t^* = \sup E$ entail $f(\xi) - f(u) = \varphi(t^*)$. Again, by assumption, there exists a $\delta_1 = \delta_1(u+t^*, \xi+t^*) > 0$ such that

$$f(\xi+t^*) - f(u+t^*) \le f(\xi+t^*+t) - f(u+t^*+t) \quad \text{for all } t \in]0, \delta_1[.$$

Since $f(\xi+t^*) - f(u+t^*) = \varphi(t^*)$, due to the above remark the previous equation leads to

$$f(\xi) - f(u) \le f(\xi+s) - f(u+s) \quad \text{for all } s \in]0, t^*+\delta_1[,$$

so contradicting the fact that $t^* = \sup E$ and showing the assertion. Accordingly, since $\xi + (v-u)/2 = v < b$, we have that $(v-u)/2 < t^*$ and the claim is achieved just applying eq. (A.3) with $t = (v-u)/2$. □

Appendix B. Proof of Theorem 4.16

Proof. For simplicity of notation, throughout this proof we will denote the copula $C_{\frac{1}{2},\frac{1}{2}}$ by C^*. Suppose *ab absurdo* that there exists a non-convex $\varphi \in \Theta(C^*)$. Due to Proposition 4.15, there exist $x_1, x_2 \in]0,1[$, with $x_1 < x_2$, and a sequence $\{\delta_n\}$ that is decreasing to zero such that

$$\varphi(x_2) - \varphi(x_1) > \varphi(x_2 + \delta_n) - \varphi(x_1 + \delta_n) \quad \text{for all } n \in \mathbb{N}. \tag{B.1}$$

Now, given any $u_1 \in]0,1[$, set $v_j := x_j + (1 - u_1)/2$ for $j = 1,2$. Assume that $u_1 > \max\{1 - 2x_1, (1 + 2x_2)/3\}$: in this case, it is easy to check that $0 < v_1 < v_2 < u_1$ and $v_1 + u_1 > 1$. Fix a natural number n and set $u_2 := u_1 + 2\delta_n$: if n is sufficiently large, one finds that $u_1 < u_2 < 1$. In summary, the following properties are satisfied:

$$0 < v_1 < v_2 < u_1 < u_2 < 1, \; v_j + u_i > 1 \quad \text{for } i,j = 1,2.$$

Moreover, a simple computation shows that for any $j \in \mathbb{N}_2$ we have

$$v_j + \frac{u_2 - 1}{2} = x_j + \delta_n \quad \text{and} \quad v_j + \frac{u_1 - 1}{2} = x_j. \tag{B.2}$$

After the assignments $U_j = \varphi(u_j)$ and $V_j = \varphi(v_j)$ for $j = 1,2$, owing to the above properties and eq. (B.2), it is not difficult to see that the volume of $R = [V_1, V_2] \times [U_1, U_2]$ with respect to C^*_φ is given by

$$V_{C^*_\varphi}(R) = \varphi(x_2 + \delta_n) - \varphi(x_1 + \delta_n) - \varphi(x_2) + \varphi(x_1).$$

The assumption $\varphi \in \Theta(C^*)$ implies $V_{C^*_\varphi}(R) \geq 0$ or, equivalently,

$$\varphi(x_2) - \varphi(x_1) \leq \varphi(x_2 + \delta_n) - \varphi(x_1 + \delta_n),$$

which clearly contradicts eq. (B.1), so concluding the proof. □

Appendix C. Proof of Theorem 6.6

Proof. Since $h \in \Theta^*$, by Proposition 6.4 the claim is true if we prove that $D_1 C_h$ is increasing in the second place almost everywhere. Due to Remark 6.2, we have to show that for any $0 < u_1 < u_2 \leq 1$ the following inequality holds for almost all $t > a_{C_h}(u_1)$:

$$D_1 C_h(t, u_1) \leq D_1 C_h(t, u_2). \tag{C.1}$$

Observe that both u_1 and u_2 belong to $\Gamma_1(C_h) =]0,1]$. Consequently, we may apply Proposition 5.11 to C both for $u = u_1$ and $u = u_2$ and we find that

$$D_1 C_h(t, u_i) = h'(C(h^{-1}(t), h^{-1}(u_i))) \cdot D_1 C(h^{-1}(t), h^{-1}(u_i)) \cdot (h^{-1})'(t) \tag{C.2}$$

and $D_1 C_h(t, u_i) > 0$ for almost all $t > a_{C_h}(u_1)$ and for $i = 1,2$. This obviously implies that

$$h'(C(h^{-1}(t), h^{-1}(u_i))) > 0, \; i = 1,2, \tag{C.3}$$

for almost all $t > a_{C_h}(u_1)$. For the sake of convenience, set $v_i(t) := C_h(t, u_i)$ for $i = 1,2$: note that $h^{-1}(v_i(t)) = C(h^{-1}(t), h^{-1}(u_i))$ for $i = 1,2$. It is easy to see that $0 < v_1(t) \leq v_2(t)$ for all $t > a_{C_h}(u_1)$ or, equivalently,

$$0 < h^{-1}(v_1(t)) \leq h^{-1}(v_2(t)) \quad \text{for all } t > a_{C_h}(u_1). \tag{C.4}$$

By Theorem 5.1, we know that $(h^{-1})' > 0$ almost everywhere on $\mathbb{I}$, hence, in view of eq. (C.2), eq. (C.1) is equivalent to

$$h'(h^{-1}(v_1(t))) D_1 C(h^{-1}(t), h^{-1}(u_1)) \leq h'(h^{-1}(v_2(t))) D_1 C(h^{-1}(t), h^{-1}(u_2)). \tag{C.5}$$

By virtue of eq. (C.3), for $i = 1,2$ we have that

$$0 < h'(C(h^{-1}(t), h^{-1}(u_i))) = h'(h^{-1}(v_i(t))) = \frac{1}{(h^{-1})'(v_i(t))}.$$

Accordingly, eq. (C.5) becomes

$$\frac{D_1 C(h^{-1}(t), h^{-1}(u_1))}{(h^{-1})'(v_1(t))} \leq \frac{D_1 C(h^{-1}(t), h^{-1}(u_2))}{(h^{-1})'(v_2(t))}. \tag{C.6}$$

Owing to eq. (C.4), we can multiply and divide the left hand-side and the right-hand side of eq. (C.6) by $h^{-1}(v_1(t))$ and $h^{-1}(v_2(t))$, respectively: recalling that $h^{-1}(v_i(t)) = C(h^{-1}(t), h^{-1}(u_i))$ for $i = 1,2$, we get

$$\psi_C(h^{-1}(t), h^{-1}(u_1))\, \sigma_h(v_1(t)) \leq \psi_C(h^{-1}(t), h^{-1}(u_2))\, \sigma_h(v_2(t)).$$

Since $\sigma_h > 0$ and $a_{C_h}(u_1) = h(a_C(h^{-1}(u_1)))$ (see Lemma 5.10), the validity of the above equation for almost all $t > a_{C_h}(u_1)$ follows from the assumption of increasingness of ψ_C in the second place almost everywhere and the assumption of increasingness of σ_h, so concluding the proof. □

Appendix D. Proof of Theorem 6.11

Proof. Suppose *ab absurdo* that there exist $a, b \in]0,1[$, with $a < b$, such that $\sigma_h(a) > \sigma_h(b)$. Remark that this implies

$$(h^{-1})'(t) > 0 \quad \text{both at } t = a \text{ and } t = b. \tag{D.1}$$

Now, set $q := h^{-1}(b)$ and let Z be the subset where h' is not strictly positive. By Theorem 5.1, we know that Z is a null set, thus the absolute continuity of ζ_q^{-1} entails $\lambda(\zeta_q^{-1}(E \cup Z)) = 0$. Therefore, given a sufficiently small $\delta > 0$, we can always fix a $w \in J_\delta$, where

$$J_\delta :=]q, q+\delta[\setminus \zeta_q^{-1}(E \cup Z).$$

We assert that $\psi_C(\zeta_q(w), v) = \kappa(\zeta_q(w))$ for any $v \in]0, w]$, provided that δ is sufficiently small. Indeed, first of all, $\zeta_q(w) \in]0,1] \setminus E$: moreover, owing to (E1), (E3) and the properties of f, it is not difficult to see that $w < f(\zeta_q(w))$ as $\delta \to 0$, thus eq. (6.3) holds with $u = \zeta_q(w)$ and $v \le w$, so closing the assertion. Observe that the jointly strict monotonicity of C ensures the existence of a unique solution of the equation $C(\zeta_q(w), x) = h^{-1}(a)$ in the variable x, here denoted by s_w. Obviously, $s_w > 0$: moreover, $s_w < w$ is a direct consequence of the fact that $h(C(\zeta_q(w), s_w)) = a < b = h(C(\zeta_q(w), w))$, where the last equality is due to eq. (6.2). Therefore, based upon the assertion, we have that $\psi_C(\zeta_q(w), s_w) = \kappa(\zeta_q(w))$. We emphasize that from the assertion it directly follows that both $D_1C(\zeta_q(w), w)$ and $D_1C(\zeta_q(w), s_w)$ exist and are strictly positive. Further, we deduce that

$$h'(C(\zeta_q(w), w)) = \frac{1}{(h^{-1})'(h(C(\zeta_q(w), w)))} = \frac{1}{(h^{-1})'(b)},$$

hence $h'(C(\zeta_q(w), w))$ exists and is strictly positive by eq. (D.1). The same conclusion may be similarly drawn for $h'(C(\zeta_q(w), s_w))$. Finally,

$$(h^{-1})'((h \circ \zeta_q)(w)) = \frac{1}{h'(\zeta_q(w))},$$

hence $(h^{-1})'((h \circ \zeta_q)(w))$ exists and is strictly positive due to the fact that $\zeta_q(w) \notin Z$. Thus, in summary, we can state that the chain rule holds for both $D_1C_h(t, u_1)$ and $D_1C_h(t, u_2)$ for $t = (h \circ \zeta_q)(w)$, $u_1 = h(s_w)$ and $u_2 = h(w)$. Since C_h is a copula, Theorem 6.10 applies and we conclude that $D_1C_h(t, u_1) \le D_1C_h(t, u_2)$. Now, if we employ the same algebraic manipulations illustrated in the proof of Theorem 6.6, taking into account the assertion, it is not difficult to see that the inequality $D_1C_h(t, u_1) \le D_1C_h(t, u_2)$ goes to

$$\kappa(\zeta_q(w)) \cdot \sigma_h(C_h(t, u_1)) \le \kappa(\zeta_q(w)) \cdot \sigma_h(C_h(t, u_2)).$$

Since $C_h(t, u_2) = b$ and $C_h(t, u_1) = a$, the previous inequality becomes $\sigma_h(a) \le \sigma_h(b)$. The contradiction shows the claim. □

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