Dependence of effective properties upon regular perturbations

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Abstract: In this survey, we present some results on the behavior of effective properties in presence of perturbations of the geometric and physical parameters. We first consider the case of a Newtonian fluid flowing at low Reynolds numbers around a periodic array of cylinders. We show the results of [43], where it is proven that the average longitudinal flow depends real analytically upon perturbations of the periodicity structure and the cross section of the cylinders. Next, we turn to the effective conductivity of a periodic two-phase composite with ideal contact at the interface. The composite is obtained by introducing a periodic set of inclusions into an infinite homogeneous matrix made of a different material. We show a result of [41] on the real analytic dependence of the effective conductivity upon perturbations of the shape of the inclusions, the periodicity structure, and the conductivity of each material. In the last part of the paper, we extend the result of [41] to the case of a periodic two-phase composite with imperfect contact at the interface.

Keywords: longitudinal flow; periodic composite; effective conductivity; transmission problem; perturbed domain; integral equations

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1 Introduction

In this paper, we review some of our recent results on the dependence of effective properties upon perturbations of the geometry and physical parameters. We first consider the case of a Newtonian fluid that flows at low Reynolds numbers around a periodic array of cylinders. By the results of [42, 43], we can see that the average of the longitudinal component of the flow velocity depends real analytically on perturbations of the periodicity structure and the cross section of the cylinders. Then we turn our attention to the thermal properties of two-phase composites that are obtained by introducing a periodic set of inclusions in an infinite homogeneous matrix made of a different material. Our aim is to prove that the effective conductivity depends real analytically on perturbations of the shape of the inclusions, the periodicity structure, and the conductivity of each material. First, we present a result of [41] on the case where we have an ideal contact at the interface. Then we show that the result of [41] can be extended to the case of imperfect contact conditions.

The average longitudinal flow and the effective conductivity are defined as specific functionals of the solutions of underlying periodic boundary value problems. In our work on domain perturbations, these problems are set in domains whose shape depends on certain perturbation parameters. Then we adopt a method based on a periodic

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version of the standard potential theory to transform the boundary value problems into systems of integral equations, which will be defined on the boundary of parameter-dependent domains. Next, with a suitable change of the functional variables, we obtain new systems of integral equations that depend on the geometry and the parameters under consideration but are defined on the boundary of fixed sets. These last systems can be studied by means of the implicit function theorem for analytic maps in Banach spaces. In particular, we can derive analytic dependence results for the solutions, which eventually yield the desired results for the effective properties. We note that here and throughout the paper the word 'analytic' always means 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [17, §15].

It is also worth noting that many existing methods in the literature are applied to periodic structures with specific shapes, *e.g.* two/three-dimensional periodic arrays of circles/spheres or ellipses/ellipsoids (see the references in the next sections). Our method, instead, can be used with arbitrary shapes, provided that they satisfy some reasonable regularity assumption. Moreover, the real analyticity results that we obtain surely imply the differentiability with respect to the parameters. Then one may want to compute the corresponding differentials, with the final goal of characterizing critical configurations. Since our approach is based on periodic potential theory, a preliminary step would be to compute the differentials of the periodic layer potentials. The computation of such differentials can be performed by following the lines of those of classic layer potentials as it is done in [37, Proposition 3.14].

The paper is organized as follows. In Section 2 we introduce the geometric setting of the considered periodic structures. Section 3 contains the result on the average longitudinal flow along a periodic array of cylinders. In Section 4 we present the result on the effective conductivity of a two-phase periodic composite with ideal contact condition. In Section 5 we state a new result on the effective conductivity of a composite with nonideal contact condition. Finally, in Section 6 we prove the result of Section 5.

2 The geometric setting

Throughout the paper

$$n \in \{2, 3\}$$

plays the role of the space dimension. If $q_{11}, \ldots, q_{nn} \in]0, +\infty[$, we use the following notation:

$$q = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \quad \text{if } n = 2, \qquad q = \begin{pmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{pmatrix} \quad \text{if } n = 3, \tag{1}$$

and

$$Q \equiv \prod_{j=1}^{n}]0, q_{jj}[\subseteq \mathbb{R}^n. \tag{2}$$

The set Q is the periodicity cell, while q is a diagonal matrix incorporating the information on the periodicity. Clearly, $|Q|_n \equiv \prod_{j=1}^{n} q_{jj}$ is the measure of the cell Q and $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the cell Q. We denote by q^{-1} the inverse matrix of q. We denote by $\mathbb{D}_n(\mathbb{R})$ the space of $n \times n$ diagonal matrices with real entries and by $\mathbb{D}_n^+(\mathbb{R})$ the set of elements of $\mathbb{D}_n(\mathbb{R})$ with diagonal entries in $]0, +\infty[$. Moreover, we find convenient to set

$$\widetilde{Q} \equiv]0, 1[^n .$$

If Ω_Q is a subset of $\mathbb{R}^n$ such that $\overline{\Omega_Q} \subseteq Q$, we define the following two periodic domains:

$$\mathbb{S}_q[\Omega_Q] \equiv \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega_Q), \qquad \mathbb{S}_q[\Omega_Q]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}_q[\Omega_Q]}.$$

The symbol '$\bar{\cdot}$' denotes the closure of a set. If u is a real valued function defined on $\mathbb{S}_q[\Omega_Q]$ or $\mathbb{S}_q[\Omega_Q]^-$, we say that u is q-periodic provided that $u(x + qz) = u(x)$ for all $z \in \mathbb{Z}^n$ and for all x in the domain of definition of u. If $k \in \mathbb{N}$, we set

$$C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \left\{ u \in C^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ s. t. } |\gamma| \leq k \right\}.$$

On $C^k_b(\overline{\mathbb{S}_q[\Omega_Q]^-})$ we consider the usual norm

$$\|u\|_{C^k_b(\overline{\mathbb{S}_q[\Omega_Q]^-})} \equiv \sum_{|\gamma|\le k} \sup_{x\in\overline{\mathbb{S}_q[\Omega_Q]^-}} |D^\gamma u(x)| \qquad \forall u \in C^k_b(\overline{\mathbb{S}_q[\Omega_Q]^-}),$$

where $|\gamma| \equiv \sum_{i=1}^n \gamma_i$ denotes the length of the multi-index $\gamma \equiv (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$. Moreover, if $\beta \in]0,1]$, then we set

$$C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \Big\{u \in C^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ s. t. } |\gamma| \le k\Big\}$$

and on $C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-})$ we consider the usual norm

$$\|u\|_{C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-})} \equiv \sum_{|\gamma|\le k} \sup_{x\in\overline{\mathbb{S}_q[\Omega_Q]^-}} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \overline{\mathbb{S}_q[\Omega_Q]^-}|_\beta \qquad \forall u \in C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-}),$$

where $|D^\gamma u : \overline{\mathbb{S}_q[\Omega_Q]^-}|_\beta$ denotes the β-Hölder constant of $D^\gamma u$ (see, *e.g.*, Gilbarg and Trudinger [24] for the definition of sets and functions of the Schauder class $C^{k,\beta}$). Then $C^k_q(\overline{\mathbb{S}_q[\Omega_Q]^-})$ denotes the Banach subspace of $C^k_b(\overline{\mathbb{S}_q[\Omega_Q]^-})$ defined by

$$C^k_q(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \Big\{u \in C^k_b(\overline{\mathbb{S}_q[\Omega_Q]^-}) : u \text{ is } q\text{-periodic}\Big\}$$

and $C^{k,\beta}_q(\overline{\mathbb{S}_q[\Omega_Q]^-})$ denotes the Banach subspace of $C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-})$ defined by

$$C^{k,\beta}_q(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \Big\{u \in C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]^-}) : u \text{ is } q\text{-periodic}\Big\}.$$

The spaces $C^k_b(\overline{\mathbb{S}_q[\Omega_Q]})$, $C^{k,\beta}_b(\overline{\mathbb{S}_q[\Omega_Q]})$, $C^k_q(\overline{\mathbb{S}_q[\Omega_Q]})$, and $C^{k,\beta}_q(\overline{\mathbb{S}_q[\Omega_Q]})$ can be defined in a similar way.

We denote by ν_{Ω_Q} the outward unit normal to $\partial\Omega_Q$ and by $d\sigma$ the area element on $\partial\Omega_Q$. We retain the standard notation for the Lebesgue space $L^1(\partial\Omega_Q)$ of Lebesgue integrable functions. We denote by $|\partial\Omega_Q|_{n-1}$ the $(n-1)$-dimensional measure of $\partial\Omega_Q$. To shorten our notation, we denote by $\fint_{\partial\Omega_Q} f\,d\sigma$ the integral mean $\frac{1}{|\partial\Omega_Q|_{n-1}}\int_{\partial\Omega_Q} f\,d\sigma$ for all $f \in L^1(\partial\Omega_Q)$. Also, if $\mathcal{X}$ is a vector subspace of $L^1(\partial\Omega_Q)$ then we set $\mathcal{X}_0 \equiv \Big\{f \in \mathcal{X} : \int_{\partial\Omega_Q} f\,d\sigma = 0\Big\}$.

We now introduce the shape perturbations. In order to consider variable domains, we fix a set and consider a class of diffeomorphisms acting on its boundary. Then a perturbation of the diffeomorphism can be seen as a perturbation of the domain. To this aim, we fix

$$\begin{gathered}\alpha \in]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^n \text{ of class } C^{1,\alpha} \\ \text{such that } \mathbb{R}^n \setminus \overline{\Omega} \text{ is connected.}\end{gathered} \tag{3}$$

We denote by $\mathcal{A}_{\partial\Omega}$ the set of functions of class $C^1(\partial\Omega, \mathbb{R}^n)$ which are injective and whose differential is injective at all points of $\partial\Omega$. One can verify that $\mathcal{A}_{\partial\Omega}$ is open in $C^1(\partial\Omega, \mathbb{R}^n)$ (see, *e.g.*, Lanza de Cristoforis and Rossi [38, Lem. 2.2, p. 197] and [37, Lem. 2.5, p. 143]). Then we find it convenient to set

$$\mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \equiv \{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq \widetilde{Q}\},$$

that is the set of diffeomorphisms in $\mathcal{A}_{\partial\Omega}$ whose image is contained in $\widetilde{Q}$ (see Figure 1). If $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components and we denote by $\mathbb{I}[\phi]$ the bounded one (see, *e.g*, Deimling [17, Thm. 5.2, p. 26]). Clearly, the set $q\mathbb{I}[\phi] = \{qx : x \in \mathbb{I}[\phi]\}$ is contained in the periodicity cell Q (see Figure 2). Then

$$\mathbb{S}_q[q\mathbb{I}[\phi]] \quad \text{and} \quad \mathbb{S}_q[q\mathbb{I}[\phi]]^-$$

are two unbounded and periodic (q, ϕ)-dependent sets which model the periodic structure of the objects considered in this paper (see Figure 3). If we modify the entries of q, this will result in a modification of the periodicity of the sets. Instead, perturbing ϕ causes a change in the shape of the periodic inclusions.

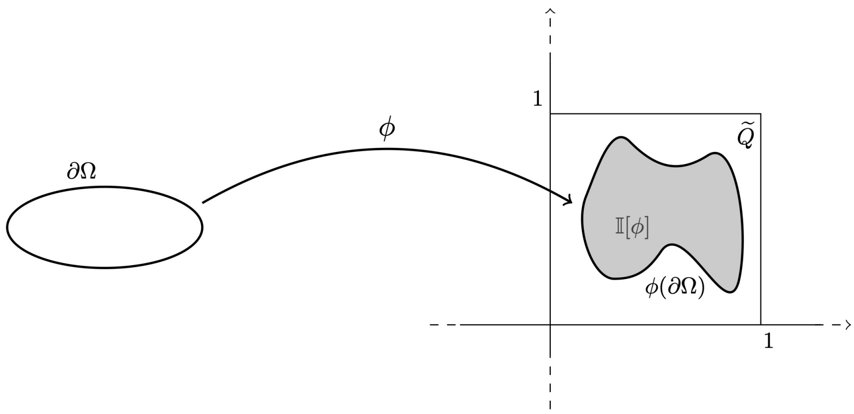

Figure 1: *A diffeomorphism* $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$ *in* $\mathbb{R}^2$.

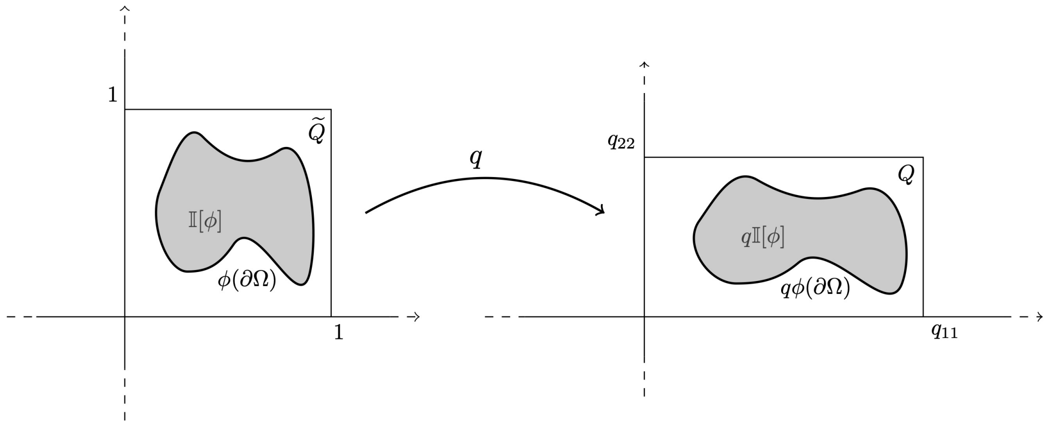

Figure 2: *The transformation induced by* q *in* $\mathbb{R}^2$.

3 The average longitudinal flow along a periodic array of cylinders

This section is devoted to the longitudinal flow of a Newtonian fluid flowing at low Reynolds numbers along a periodic array of cylinders. We study the effect of perturbations of the periodicity structure and the shape of the cross section of the cylinders. Since the cylinder's cross section is two-dimensional, in this section we set

$$n = 2\,.$$

As introduced in the previous section, the shape of the cross section of the cylinders is determined by the image of a fixed domain through a diffeomorphism ϕ and the periodicity cell is a rectangle of sides of length q_{11} and q_{22}, associated with the matrix

$$q = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \in \mathbb{D}_2^+(\mathbb{R})\,.$$

We assume that the pressure gradient is parallel to the cylinders. Under these assumptions, the velocity field has only one non-zero component which, by the Stokes equations, satisfies the Poisson equation (see problem (4)). Then, by integrating the longitudinal component of the velocity field over the fundamental cell, for each pair (q, ϕ), we define the average of the longitudinal component of the flow velocity $\Sigma[q, \phi]$ (see (5)). We note that $\Sigma[q, \phi]$ is a measure of the quantity of fluid flowing through the single periodicity cell and is sometimes referred as the longitudinal permeability of the array of cylinders (or its opposite, see, *e.g.*, Mityushev and Adler [46, 47]). Here, we are interested in the dependence of $\Sigma[q, \phi]$ upon the pair (q, ϕ).

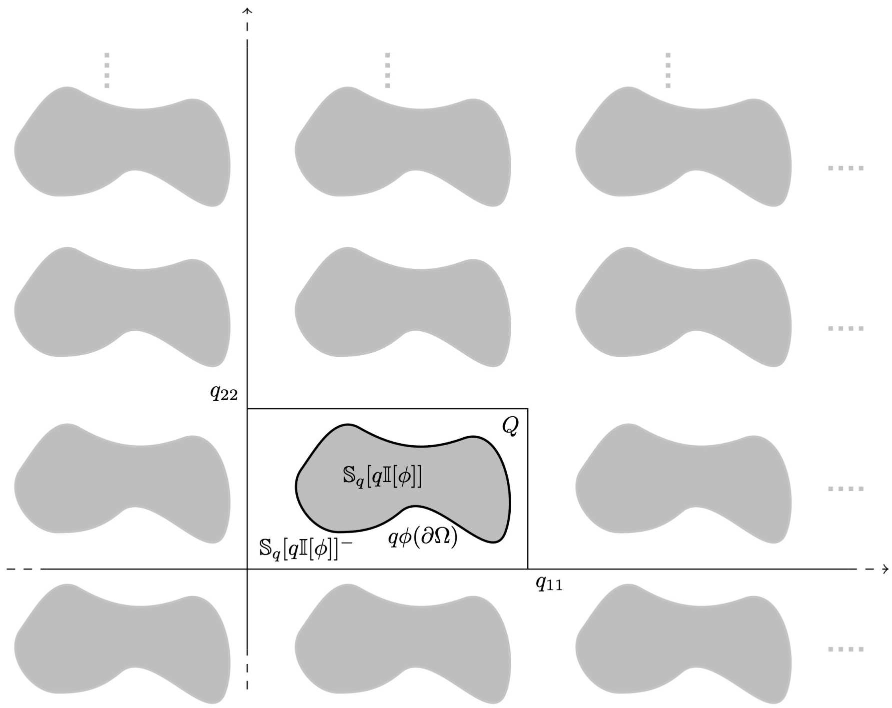

Figure 3: *The sets* $\mathbb{S}_q[q\mathbb{I}[\phi]]^-$, $\mathbb{S}_q[q\mathbb{I}[\phi]]$, *and* $q\phi(\partial\Omega)$ *in* $\mathbb{R}^2$.

The mathematical aspects of fluids in periodic structures have been studied by several authors and with a variety of different methods. With no expectation of being exhaustive, we mention some contributions. Hasimoto [29] has investigated the viscous flow past a cubic array of spheres and he has applied his results to the two-dimensional flow past a square array of circular cylinders. His techniques are based on the construction of a spatially periodic fundamental solution for the Stokes' system and are applied to specific shapes (circular/spherical obstacles and square/cubic arrays). Schmid [59] has investigated the longitudinal laminar flow in an infinite square array of circular cylinders. Sangani and Yao [57, 58] have studied the permeability of random arrays of infinitely long cylinders. Mityushev and Adler [46, 47] have considered the longitudinal permeability of periodic rectangular arrays of circular cylinders. Finally, the paper [53] with Mityushev deals with the asymptotic behavior of the longitudinal permeability of thin cylinders of arbitrary shape.

Here, instead, we are interested in the dependence of the (average) longitudinal velocity upon the length of the sides of the rectangular array and the shape of the cross section of the cylinders without restricting ourselves to particular shapes, as circles or ellipses.

If $q \in \mathbb{D}_2^+(\mathbb{R})$ and $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the set $\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]} \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}_q[q\mathbb{I}[\phi]]^- \times \mathbb{R}$ is the region where a Newtonian fluid is flowing at low Reynolds numbers. We assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (cf., *e.g.*, Adler [1, Ch. 4], Sangani and Yao [58], and Mityushev and Adler [46, 47]), we can transform the Stokes system into a Poisson equation for the non-zero component of the velocity field. Without loss of generality, we may assume that the viscosity of the fluid and the non-zero component of the pressure gradient are both set equal to one. Accordingly, if $q \in \mathbb{D}_2^+(\mathbb{R})$ and $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the problem is reduced to the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\ u(x + qe_i) = u(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}\,, \forall i \in \{1,2\}\,, \\ u(x) = 0 & \forall x \in \partial\mathbb{S}_q[q\mathbb{I}[\phi]]^-\,. \end{cases} \tag{4}$$

Here $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. We can show that problem (4) has a unique solution in the space $C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$ of $C^{1,\alpha}$ q-periodic functions on $\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}$, and we denote it by $u[q,\phi]$. From the physical point of view, the function $u[q,\phi]$ represents the non-zero component of the velocity field (see Mityushev and Adler [46, §2]). Then we can define $\Sigma[q,\phi]$ as the integral of the flow velocity $u[q,\phi]$ over the periodicity cell (see Adler [1], Mityushev and Adler [46, §3]), *i.e.*,

$$\Sigma[q,\phi] \equiv \frac{1}{|Q|_2}\int_{Q\setminus q\mathbb{I}[\phi]} u[q,\phi](x)\,dx \qquad \forall (q,\phi)\in \mathbb{D}_2^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right). \quad (5)$$

In [42, 43] we have studied the regularity properties of $\Sigma[q,\phi]$ as a function of (q,ϕ). Among other results, we have proven that the map $(q,\phi)\mapsto\Sigma[q,\phi]$ is real analytic, as we state in the following theorem.

Theorem 3.1. *Let α, Ω be as in* (3)*. Then the map from*

$$\mathbb{D}_2^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$$

to $\mathbb{R}$ that takes a pair (q,ϕ) to $\Sigma[q,\phi]$ is real analytic.

4 The effective conductivity of a two-phase periodic composite with ideal contact condition

In this section, we recall the results of [41] about the effective conductivity of an n-dimensional periodic two-phase composite ($n\in\{2,3\}$) with ideal contact at the interface. The composite is obtained by introducing into a homogeneous matrix a periodic set of inclusions of sufficiently smooth shapes. Both the matrix and the set of inclusions are filled with two different homogeneous and isotropic heat conductive materials of conductivity λ^- and λ^+, respectively, with

$$(\lambda^+,\lambda^-)\in[0,+\infty[^2_*\equiv[0,+\infty[^2\setminus\{(0,0)\}.$$

The limit case of a material with zero conductivity corresponds to a thermal insulator. So here we are assuming that the two materials are not both insulators. On the other hand, if the conductivity tends to $+\infty$, the material is a perfect conductor. Similarly to what we have done in the previous section, the inclusions' shape is determined by the image of a fixed domain through a diffeomorphism ϕ, and the periodicity cell is a 'cuboid' of edges of lengths $q_{11},\dots,q_{nn}$. As it is known, it is possible to define the composite's effective conductivity matrix $\lambda^{\mathrm{eff,id}}$ by means of the solution of a transmission problem for the Laplace equation (see Definition 4.1, cf. Mityushev, Obnosov, Pesetskaya, and Rogosin [49, §5]). The effective conductivity can be thought as the conductivity of a homogeneous material whose global behavior as a conductor is 'equivalent' to the composite. Then we may want to understand the dependence of $\lambda^{\mathrm{eff,id}}$ upon the 'triple' $((q_{11},\dots,q_{nn}),\phi,(\lambda^+,\lambda^-))$, *i.e.*, upon perturbations of the periodicity structure of the composite, the inclusions' shape, and the conductivity parameters of each material.

The mathematical literature on the properties of composite materials is too vast to attempt a complete list of references. We confine ourselves to mention some contributions that are more focused on perturbation analysis of the effective properties. For example, in Ammari, Kang, and Touibi [5] the authors have exploited a potential theoretic approach in order to investigate the asymptotic behavior of the effective properties of a periodic dilute composite. Then Ammari, Kang, and Kim [3] and Ammari, Kang, and Lim [4] have studied anisotropic composite materials and elastic composites, respectively. The method of Functional Equations, first proposed in Mityushev [45], has been used to study the dependence on the radius of the inclusions for a wide class of 2D composites. For ideal composites, we mention here, for example, the works of Mityushev, Obnosov, Pesetskaya, and Rogosin [49], Gryshchuk and Rogosin [28], Kapanadze, Mishuris, and Pesetskaya [30]. Berlyand, Golovaty, Movchan, and Phillips [8] have analyzed the transport properties of fluid/solid and solid/solid composites and have investigated how the curvature of the inclusions affects such properties. Berlyand and Mityushev [9] have studied the dependence of the effective conductivity of two-phase composites upon the polydispersity parameter. Gorb and Berlyand [26] considered the asymptotic behavior of the effective properties of composites with close inclusions of optimal shape. For two-dimensional composites, the recent work by Mityushev, Nawalaniec, Nosov, and Pesetskaya [48] studies the effective conductivity of two-phase random composites with non-overlapping inclusions whose boundaries are $C^{1,\alpha}$ curves. In Lee and Lee [39], the authors have studied how the effective elasticity of dilute periodic elastic composites is affected by its periodic structure. In connection with doubly periodic problems for composite materials, we mention the monograph of Grigolyuk and Fil'shtinskij [27], where the authors have proposed a method of integral equations for planar periodic problems in the frame of elasticity (see also Fil'shtinskij [22] and the more recent work Filshtinsky and Mityushev [23]). Finally, in [55] the fourth named author has explicitly computed the effective conductivity of a periodic dilute composite with perfect contact as a power series in the size of the inclusions (see also [54]).

With the aim of introducing the definition of the effective conductivity, we first have to introduce a family of boundary value problems for the Laplace equation. If $q \in \mathbb{D}_n^+(\mathbb{R})$, $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, and $(\lambda^+, \lambda^-) \in [0, +\infty[^2_*$, for each $j \in \{1, \dots, n\}$ we consider the following transmission problem for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$:

$$
\begin{cases}
\Delta u_j^+ = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]\,, \\
\Delta u_j^- = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\
u_j^+(x + q e_h) = u_j^+(x) + \delta_{hj} q_{jj} & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}\,,\ \forall h \in \{1, \dots, n\}\,, \\
u_j^-(x + q e_h) = u_j^-(x) + \delta_{hj} q_{jj} & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}\,,\ \forall h \in \{1, \dots, n\}\,, \\
\lambda^+ \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u_j^+ - \lambda^- \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u_j^- = 0 & \text{on } \partial q\mathbb{I}[\phi]\,, \\
u_j^+ - u_j^- = 0 & \text{on } \partial q\mathbb{I}[\phi]\,, \\
\int_{\partial q\mathbb{I}[\phi]} u_j^+ \, d\sigma = 0,
\end{cases}
\tag{6}
$$

where $\nu_{q\mathbb{I}[\phi]}$ is the outward unit normal to $\partial q\mathbb{I}[\phi]$ and $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. We recall that here above for $(h, j) \in \{1, \dots, n\}^2$ the symbol δ_{hj} denotes the Kronecker delta symbol, so that $\delta_{hj} = 1$ for $h = j$ and $\delta_{hj} = 0$ otherwise. Problem (6) admits a unique solution (u_j^+, u_j^-) in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$, which we denote by $(u_j^+[q, \phi, (\lambda^+, \lambda^-)], u_j^-[q, \phi, (\lambda^+, \lambda^-)])$. This solution is used to define the effective conductivity as follows (cf., *e.g.*, Mityushev, Obnosov, Pesetskaya, and Rogosin [49, §5]).

Definition 4.1. *Let $q \in \mathbb{D}_n^+(\mathbb{R})$, $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, and $(\lambda^+, \lambda^-) \in [0, +\infty[^2_*$. Then the effective conductivity*

$$
\lambda^{\mathrm{eff,id}}[q, \phi, (\lambda^+, \lambda^-)] \equiv (\lambda^{\mathrm{eff,id}}_{ij}[q, \phi, (\lambda^+, \lambda^-)])_{i,j=1,\dots,n}
$$

is the $n \times n$ matrix with (i, j)-entry defined by

$$
\begin{aligned}
\lambda^{\mathrm{eff,id}}_{ij}[q, \phi, (\lambda^+, \lambda^-)] \equiv \frac{1}{|Q|_n} \Bigg\{ & \lambda^+ \int_{q\mathbb{I}[\phi]} \frac{\partial}{\partial x_i} u_j^+[q, \phi, (\lambda^+, \lambda^-)](x)\, dx \\
& + \lambda^- \int_{Q \setminus \overline{q\mathbb{I}[\phi]}} \frac{\partial}{\partial x_i} u_j^-[q, \phi, (\lambda^+, \lambda^-)](x)\, dx \Bigg\} \\
& \forall i, j \in \{1, \dots, n\}.
\end{aligned}
$$

As for the average longitudinal flow in Section 3, we are interested in the function

$$
(q, \phi, (\lambda^+, \lambda^-)) \mapsto \lambda^{\mathrm{eff,id}}_{ij}[q, \phi, (\lambda^+, \lambda^-)]\,.
$$

The following result of [41, Thm. 5.1] describes the regularity of the effective conductivity matrix $\lambda^{\mathrm{eff,id}}[q, \phi, (\lambda^+, \lambda^-)]$ of the ideal composite upon the triple '*periodicity-shape-conductivity*'. More in details, it shows that the (i, j)-entry $\lambda^{\mathrm{eff,id}}_{ij}[q, \phi, (\lambda^+, \lambda^-)]$ can be expressed in terms of the conductivity λ^- of the matrix, the conductivity λ^+ of the inclusions, and an analytic map of the periodicity q, of the inclusions' shape ϕ, and the ratio $\frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-}$, which is sometimes referred to as the contrast parameter.

Theorem 4.2. *Let α, Ω be as in* (3). *Let $i, j \in \{1, \dots, n\}$. Then there exist an open neighborhood $\mathcal{U}$ of $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times [-1, 1]$ in the space $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times \mathbb{R}$ and a real analytic map Λ_{ij} from $\mathcal{U}$ to $\mathbb{R}$ such that*

$$\lambda_{ij}^{\mathrm{eff,id}}[q, \phi, (\lambda^+, \lambda^-)] \equiv \delta_{ij}\lambda^- + (\lambda^+ + \lambda^-)\Lambda_{ij}\left[q, \phi, \frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-}\right]$$

for all $(q, \phi, (\lambda^+, \lambda^-)) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times [0, +\infty[^2_*$.

5 The effective conductivity of a two-phase periodic composite with nonideal contact condition

We now turn our attention to the study of the effective conductivity of an n-dimensional periodic two-phase composite ($n \in \{2, 3\}$) with imperfect (or nonideal) contact at the interface.

As in the previous section, the composite consists of a matrix and a periodic set of inclusions. The matrix and the inclusions are filled with two (possibly different) homogeneous and isotropic heat conductive materials. The normal component of the heat flux is assumed to be continuous at the two-phase interface, while we impose that the temperature field displays a jump proportional to the normal heat flux by means of a parameter r. In physics, the appearance of such a discontinuity in the temperature field is a well-known phenomenon and has been largely investigated since 1941, when Kapitza carried out the first systematic study of thermal interface behavior in liquid helium (see, *e.g.*, Swartz and Pohl [61], Lipton [40] and references therein). As in the ideal case, our aim is to study the behavior of the effective conductivity of the nonideal composite upon perturbation of the geometry and the parameters of the problem. The expression defining the effective conductivity of a composite with imperfect contact conditions was introduced by Benveniste and Miloh in [7] by generalizing the dual theory of the effective behavior of composites with perfect contact (see also Benveniste [6] and for a review Drygaś and Mityushev [20]). By the argument of Benveniste and Miloh, in order to evaluate the effective conductivity, one has to study the thermal distribution of the composite when so-called 'homogeneous conditions' are prescribed.

We first introduce the parameters of the problem. Both the matrix and the set of inclusions are filled with two different homogeneous and isotropic heat conductive materials of conductivity λ^- and λ^+, respectively, with

$$(\lambda^+, \lambda^-) \in]0, +\infty[^2.$$

The normal component of the heat flux is assumed to be continuous at the two-phase interface, while we impose that the temperature field displays a jump proportional to the normal heat flux by means of a parameter

$$r \in [0, +\infty[\,.$$

We find it convenient to set

$$\mathcal{P} \equiv]0, +\infty[^2 \times [0, +\infty[.$$

As in the ideal case, the set $\mathbb{S}_q[q\mathbb{I}[\phi]]^-$ represents the homogeneous matrix made of a material with conductivity λ^- where the periodic set of inclusions $\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}$ with conductivity λ^+ is inserted. The two-phase composite consists of the union of the matrix and the inclusions.

Let $q \in \mathbb{D}_n^+(\mathbb{R})$, $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, $(\lambda^+, \lambda^-, r) \in \mathcal{P}$. To define the effective conductivity in the nonideal case we introduce the following boundary value problem:

$$\begin{cases} \Delta u_j^+ = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]\,, \\ \Delta u_j^- = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\ u_j^+(x+qe_h) = u_j^+(x) + \delta_{hj}q_{jj} & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}\,,\ \forall h \in \{1,\dots,n\}\,, \\ u_j^-(x+qe_h) = u_j^-(x) + \delta_{hj}q_{jj} & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}\,,\ \forall h \in \{1,\dots,n\}\,, \\ \lambda^+ \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u_j^+ - \lambda^- \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u_j^- = 0 & \text{on } \partial q\mathbb{I}[\phi]\,, \\ \lambda^+ \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u_j^+ + r\left(u_j^+ - u_j^-\right) = 0 & \text{on } \partial q\mathbb{I}[\phi]\,, \\ \int_{\partial q\mathbb{I}[\phi]} u_j^+ \, d\sigma = 0\,, \end{cases} \tag{7}$$

with $j \in \{1,\dots,n\}$. As we will see, problem (7) admits a unique solution (u_j^+, u_j^-) in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$, which we denote by

$$(u_j^+[q,\phi,\lambda^+,\lambda^-,r], u_j^-[q,\phi,\lambda^+,\lambda^-,r])\,.$$

Then, with this family of solutions, we can define the effective conductivity as follows. (The reader might note the similarity with Definition 4.1 given in the case of an ideal contact.)

Definition 5.1. *Let $q \in \mathbb{D}_n^+(\mathbb{R})$, $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, and $(\lambda^+,\lambda^-,r) \in \mathcal{P}$. Then the effective conductivity*

$$\lambda^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r] \equiv (\lambda_{ij}^{\mathrm{eff}}[q,\phi,\lambda^+,\lambda^-,r])_{i,j=1,\dots,n}$$

is the $n \times n$ matrix with (i,j)-entry defined by

$$\lambda_{ij}^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r] \equiv \frac{1}{|Q|_n}\Bigg\{\lambda^+ \int_{q\mathbb{I}[\phi]} \frac{\partial}{\partial x_i} u_j^+[q,\phi,\lambda^+,\lambda^-,r](x)\,dx \\ + \lambda^- \int_{Q\setminus\overline{q\mathbb{I}[\phi]}} \frac{\partial}{\partial x_i} u_j^-[q,\phi,\lambda^+,\lambda^-,r](x)\,dx\Bigg\} \\ \forall i,j \in \{1,\dots,n\}.$$

Before describing the main result of this section, we mention that composites with contact conditions different from the ideal ones are studied, for example, in Drygaś and Mityushev [20], in Castro, Kapanadze, and Pesetskaya [10, 11] (about non-ideal composites), and in Castro and Pesetskaya [12] (about composites with inextensible-membrane-type interface). We also mention that the asymptotic behavior of the effective conductivity of a periodic dilute composite with imperfect contact has been studied in [15, 16]. In addition, we note that effective properties of heat conductors with interfacial contact resistance have been studied via homogenization theory (cf. Donato and Monsurrò [18], Faella, Monsurrò, and Perugia [21], Monsurrò [51, 52]).

The main goal of the rest of our paper is to study the regularity of the map

$$(q,\phi,\lambda^+,\lambda^-,r) \mapsto \lambda^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r]\,.$$

We will prove the following theorem.

Theorem 5.2. *Let α, Ω be as in* (3). *Let $i,j \in \{1,\dots,n\}$. Then there exist an open neighborhood $\mathcal{V}$ of $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times \mathcal{P}$ in the space $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times \mathbb{R}^3$ and a real analytic map Λ_{ij} from $\mathcal{V}$ to $\mathbb{R}$ such that*

$$\lambda_{ij}^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r] \equiv \delta_{ij}\lambda^- + \Lambda_{ij}\left[q,\phi,\lambda^+,\lambda^-,r\right]$$

for all $(q,\phi,\lambda^+,\lambda^-,r) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times \mathcal{P}$.

The approach that we use to prove Theorem 5.2 was introduced by Lanza de Cristoforis in [33] and then extended to a large variety of singular and regular perturbation problems (cf., *e.g.*, Lanza de Cristoforis [34], Lanza de Cristoforis and the first named author [14], and [13]).

In particular, in the present paper, we follow the strategy of [43] where we have studied the behavior of the longitudinal flow along a periodic array of cylinders upon perturbations of the shape of the cross section of the cylinders and the periodicity structure (see also Section 3), and of [41] where we have considered the effective conductivity of an ideal composite (see also Section 4). More precisely, we transform the problem into a set of integral equations defined on a fixed domain and depending on the set of variables $(q, \phi, \lambda^+, \lambda^-, r)$. We study the dependence of the solution of the integral equations upon $(q, \phi, \lambda^+, \lambda^-, r)$ and then we deduce the result on the behavior of $\lambda_{ij}^{\mathrm{eff,nonid}}[q, \phi, \lambda^+, \lambda^-, r]$. In this paper, the integral equations are derived by a potential theoretic approach. However, integral equations could also be deduced by the generalized alternating method of Schwarz (cf. Gluzman, Mityushev, and Nawalaniec [25] and Drygaś, Gluzman, Mityushev, and Nawalaniec [19]).

Incidentally, we observe that there are several contributions concerning the optimization of effective parameters from many different points of view. For example, one can look for *optimal lattices* without confining to rectangular distributions. In this direction, Kozlov [31] and Mityushev and Rylko [50] have discussed extremal properties of hexagonal lattices of disks. In Rylko [56], the author has studied the influence of perturbations of the shape of the circular inclusion on the macroscopic conductivity properties of 2D dilute composites. For an experimental work concerning the analysis of particle reinforced composites we mention Kurtyka and Rylko [32].

Finally, we note that we do not consider the case where $r \to +\infty$. The asymptotic analysis of such case in a (non-periodic) transmission problem can be found in Schmidt and Hiptmair [60].

6 Proof of Theorem 5.2

6.1 Preliminaries

Our method is based on a periodic version of the classical potential theory. Periodic layer potentials are constructed by replacing the fundamental solution of the Laplace operator with a q-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n} ,$$

where δ_{qz} denotes the Dirac measure with mass at the point $qz \in \mathbb{R}^n$ (see, *e.g.*, Lanza de Cristoforis and the third named author [36, p. 84]). The distribution $S_{q,n}$ is determined up to an additive constant, and we have

$$S_{q,n}(x) = - \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|Q|_n 4\pi^2 |q^{-1}z|^2} e^{2\pi \mathbf{i}(q^{-1}z)\cdot x} ,$$

where the generalized sum is defined in the sense of distributions in $\mathbb{R}^n$ (see, *e.g.*, Ammari and Kang [2, p. 53], Lanza de Cristoforis and the third named author [36, §3]). It is known that $S_{q,n}$ is real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$ and locally integrable in $\mathbb{R}^n$ (see, *e.g.*, Lanza de Cristoforis and the third named author [36, §3]).

We now introduce periodic layer potentials. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0, 1[$ such that $\overline{\Omega_Q} \subseteq Q$. We set

$$\begin{aligned} v_q[\partial\Omega_Q, \mu](x) &\equiv \int_{\partial\Omega_Q} S_{q,n}(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n , \\ w_{q,*}[\partial\Omega_Q, \mu](x) &\equiv \int_{\partial\Omega_Q} \nu_{\Omega_Q}(x) \cdot DS_{q,n}(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega_Q , \end{aligned}$$

for all $\mu \in C^0(\partial\Omega_Q)$. Here above, $DS_{q,n}(\xi)$ denotes the gradient of $S_{q,n}$ computed at the point $\xi \in \mathbb{R}^n \setminus q\mathbb{Z}^n$. The function $v_q[\partial\Omega_Q, \mu]$ is called the q-periodic single layer potential, and $w_{q,*}[\partial\Omega_Q, \mu]$ is a function related to the normal derivative of the single layer potential. As is well known, if $\mu \in C^0(\partial\Omega_Q)$, then $v_q[\partial\Omega_Q, \mu]$ is continuous in $\mathbb{R}^n$ and q-periodic. We set

$$v_q^+[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\overline{\mathbb{S}_q[\Omega_Q]}} \qquad \text{and} \qquad v_q^-[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\overline{\mathbb{S}_q[\Omega_Q]^-}} .$$

In the following theorem, we collect some properties of $v_q^\pm[\partial\Omega_Q, \cdot]$ and $w_{q,*}[\partial\Omega_Q, \cdot]$ that are the periodic analog of classical regularity results and jump formulas for the single layer potential. For a proof of statements (i)–(iii) we refer to Lanza de Cristoforis and the third named author [36, Thm. 3.7] and to [15, Lem. 4.2]. For a proof of statement (iv) we refer to [15, Lem. 4.2 (i), (iii)].

Theorem 6.1. *Let q, Q be as in (1) and (2). Let $\alpha \in]0, 1[$. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\overline{\Omega_Q} \subseteq Q$. Then the following statements hold.*

(i) The map from $C^{0,\alpha}(\partial\Omega_Q)$ to $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\Omega_Q]})$ that takes μ to $v_q^+[\partial\Omega_Q, \mu]$ is linear and continuous. The map from $C^{0,\alpha}(\partial\Omega_Q)$ to $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\Omega_Q]^-})$ that takes μ to $v_q^-[\partial\Omega_Q, \mu]$ is linear and continuous.

(ii) Let $\mu \in C^{0,\alpha}(\partial\Omega_Q)$. Then

$$\frac{\partial}{\partial\nu_{\Omega_Q}} v_q^\pm[\partial\Omega_Q, \mu] = \mp\frac{1}{2}\mu + w_{q,*}[\partial\Omega_Q, \mu] \qquad \text{on } \partial\Omega_Q.$$

Moreover,

$$\int_{\partial\Omega_Q} w_{q,*}[\partial\Omega_Q, \mu]\, d\sigma = \left(\frac{1}{2} - \frac{|\Omega_Q|_n}{|Q|_n}\right) \int_{\partial\Omega_Q} \mu\, d\sigma .$$

(iii) Let $\mu \in C^{0,\alpha}(\partial\Omega_Q)_0$. Then

$$\Delta v_q[\partial\Omega_Q, \mu] = 0 \qquad \text{in } \mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega_Q].$$

(iv) The operator $w_{q,*}[\partial\Omega_Q, \cdot]$ is compact in $C^{0,\alpha}(\partial\Omega_Q)$ and in $C^{0,\alpha}(\partial\Omega_Q)_0$.

Next we turn to problem (7). By means of the following proposition, whose proof is of immediate verification, we can transform problem (7) into a q-periodic transmission problem for the Laplace equation.

Proposition 6.2. *Let q be as in (1), Q be as in (2), and α, Ω be as in (3). Let $(\lambda^+, \lambda^-, r) \in \mathcal{P}$. Let $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$. Let $j \in \{1, \dots, n\}$. A pair*

$$(u_j^+, u_j^-) \in C^{1,\alpha}_{\text{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\text{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$$

solves problem (7) if and only if the pair

$$(\tilde{u}_j^+, \tilde{u}_j^-) \in C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$$

defined by

$$\begin{aligned} \tilde{u}_j^+(x) &\equiv u_j^+(x) - x_j \quad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}, \\ \tilde{u}_j^-(x) &\equiv u_j^-(x) - x_j \quad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}, \end{aligned}$$

solves

$$\begin{cases} \Delta\tilde{u}_j^+ = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]\,, \\ \Delta\tilde{u}_j^- = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\ \tilde{u}_j^+(x + qe_h) = \tilde{u}_j^+(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]},\ \forall h \in \{1, \dots, n\}\,, \\ \tilde{u}_j^-(x + qe_h) = \tilde{u}_j^-(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-},\ \forall h \in \{1, \dots, n\}\,, \\ \lambda^+ \frac{\partial}{\partial\nu_{q\mathbb{I}[\phi]}}\tilde{u}_j^+ - \lambda^- \frac{\partial}{\partial\nu_{q\mathbb{I}[\phi]}}\tilde{u}_j^- = (\lambda^- - \lambda^+)(\nu_{q\mathbb{I}[\phi]})_j & \text{on } \partial q\mathbb{I}[\phi]\,, \\ \lambda^+ \frac{\partial}{\partial\nu_{q\mathbb{I}[\phi]}}\tilde{u}_j^+ + r\left(\tilde{u}_j^+ - \tilde{u}_j^-\right) = -\lambda^+(\nu_{q\mathbb{I}[\phi]})_j & \text{on } \partial q\mathbb{I}[\phi]\,, \\ \int_{\partial q\mathbb{I}[\phi]} \tilde{u}_j^+\, d\sigma = -\int_{\partial q\mathbb{I}[\phi]} y_j\, d\sigma_y. \end{cases} \tag{8}$$

By [15, Prop. 5.1, Thm. 5.3], we deduce the validity of the following proposition, stating that (the equivalent) problems (7) and (8) have unique solution.

Proposition 6.3. *Let q be as in (1), Q be as in (2), and α, Ω be as in (3). Let $(\lambda^+,\lambda^-,r)\in\mathcal{P}$. Let $\phi\in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let $j\in\{1,\dots,n\}$. Then the following statements hold.*

(i) Problem (7) has a unique solution (u^+_j,u^-_j) in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]})\times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$.

(ii) Problem (8) has a unique solution $(\tilde{u}^+_j,\tilde{u}^-_j)$ in $C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]})\times C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$.

6.2 An integral equation formulation of problem (7)

In this section, we convert problem (7) into a system of integral equations. As done in [43] for the longitudinal flow and in [41] for the ideal contact, we do so by representing the solution in terms of single layer potentials with densities that solve certain integral equations. We first start with the following proposition on the invertibility of an integral operator that will appear in the integral formulation of problem (7).

Proposition 6.4. *Let q be as in (1), Q be as in (2), and α, Ω be as in (3). Let $(\phi,\lambda^+,\lambda^-,r)\in(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega})\times\mathcal{P}$. Let $J\equiv(J_1,J_2)$ be the operator from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ defined by*

$$\begin{aligned}
J_1[\mu^+,\mu^-]\equiv&\lambda^+\Big(-\frac{1}{2}\mu^++w_{q,*}[\partial q\mathbb{I}[\phi],\mu^+]\Big)-\lambda^-\Big(\frac{1}{2}\mu^-+w_{q,*}[\partial q\mathbb{I}[\phi],\mu^-]\Big)\,,\\
J_2[\mu^+,\mu^-]\equiv&\lambda^+\Big(-\frac{1}{2}\mu^++w_{q,*}[\partial q\mathbb{I}[\phi],\mu^+]\Big)\\
&+r\Big(v^+_q[\partial q\mathbb{I}[\phi],\mu^+]_{|\partial q\mathbb{I}[\phi]}-\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]}v^+_q[\partial q\mathbb{I}[\phi],\mu^+]\,d\sigma\\
&-v^-_q[\partial q\mathbb{I}[\phi],\mu^-]_{|\partial q\mathbb{I}[\phi]}+\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]}v^-_q[\partial q\mathbb{I}[\phi],\mu^-]\,d\sigma\Big)\,,
\end{aligned}$$

for all $(\mu^+,\mu^-)\in(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$, where $|\partial q\mathbb{I}[\phi]|_{n-1}$ denotes the $(n-1)$-dimensional measure of $\partial q\mathbb{I}[\phi]$. Then the following statements hold.

(i) The operator J restricts to a homeomorphism from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0)^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0)^2$.

(ii) The operator J is a homeomorphism from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$.

Proof. We first notice that the validity of statement (i) follows by [15, Prop. 5.2]. We now consider statement (ii) and we follow the lines of the proof of [15, Prop. 5.2]. Let $\hat{J}\equiv(\hat{J}_1,\hat{J}_2)$ be the linear operator from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ defined by

$$\hat{J}_1[\mu^+,\mu^-]\equiv-(\lambda^-/2)\mu^--(\lambda^+/2)\mu^+\,,\quad \hat{J}_2[\mu^+,\mu^-]\equiv-(\lambda^+/2)\mu^+$$

for all $(\mu^+,\mu^-)\in(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$. Clearly, $\hat{J}$ is a linear homeomorphism from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$. Then let

$$\tilde{J}\equiv(\tilde{J}_1,\tilde{J}_2)$$

be the operator from $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ to $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ defined by

$$\begin{aligned}
\tilde{J}_1[\mu^+,\mu^-]\equiv&\lambda^+w_{q,*}[\partial q\mathbb{I}[\phi],\mu^+]-\lambda^-w_{q,*}[\partial q\mathbb{I}[\phi],\mu^-]\,,\\
\tilde{J}_2[\mu^+,\mu^-]\equiv&\lambda^+w_{q,*}[\partial q\mathbb{I}[\phi],\mu^+]\\
&+r\Big(v^+_q[\partial q\mathbb{I}[\phi],\mu^+]_{|\partial q\mathbb{I}[\phi]}-\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]}v^+_q[\partial q\mathbb{I}[\phi],\mu^+]\,d\sigma\\
&-v^-_q[\partial q\mathbb{I}[\phi],\mu^-]_{|\partial q\mathbb{I}[\phi]}+\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]}v^-_q[\partial q\mathbb{I}[\phi],\mu^-]\,d\sigma\Big)
\end{aligned}$$

for all $(\mu^+,\mu^-)\in(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$. Then, by Theorem 6.1 (i), the operator from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])$ to $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$ that takes μ to

$$v_q[\partial q\mathbb{I}[\phi],\mu]_{|\partial q\mathbb{I}[\phi]}-\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]}v_q[\partial q\mathbb{I}[\phi],\mu]\,d\sigma\,,$$

is bounded and by Theorem 6.1 (iv) the map $w_{q,*}[\partial q\mathbb{I}[\phi],\cdot]$ is compact. Then the compactness of the imbedding of $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$ into $C^{0,\alpha}(\partial q\mathbb{I}[\phi])$ implies that $\tilde{J}$ is a compact operator. Now, since $J = \hat{J} + \tilde{J}$ and since compact perturbations of isomorphisms are Fredholm operators of index 0, we deduce that J is a Fredholm operator of index 0. Thus, to show that J is a linear homeomorphism, it suffices to show that it is injective. So, let $(\mu^+,\mu^-) \in (C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ be such that

$$J[\mu^+,\mu^-] = (0,0)\,. \tag{9}$$

Clearly,

$$\begin{aligned}\int_{\partial q\mathbb{I}[\phi]} r\Big(& v_q^+[\partial q\mathbb{I}[\phi],\mu^+]_{|\partial q\mathbb{I}[\phi]} - \frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]} v_q^+[\partial q\mathbb{I}[\phi],\mu^+]\,d\sigma \\ &- v_q^-[\partial q\mathbb{I}[\phi],\mu^-]_{|\partial q\mathbb{I}[\phi]} + \frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu^-]\,d\sigma\Big)\,d\sigma = 0\,.\end{aligned}$$

Then Theorem 6.1 (ii) and the second component of equality (9) imply that

$$\int_{\partial q\mathbb{I}[\phi]} \mu^+\,d\sigma = 0\,,$$

i.e., $\mu^+ \in C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$. Then again Theorem 6.1 (ii) and the first component of equality (9) imply that

$$\int_{\partial q\mathbb{I}[\phi]} \mu^-\,d\sigma = 0\,,$$

i.e., $\mu^- \in C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$. In other words, we have shown that if $(\mu^+,\mu^-) \in (C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ is such that $J[\mu^+,\mu^-] = (0,0)$, then we have $(\mu^+,\mu^-) \in (C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0)^2$. As a consequence, statement (i) implies that $(\mu^+,\mu^-) = (0,0)$, and, therefore, the validity of statement (ii). □

We are now ready to show that problem (7) can be reformulated in terms of a system of integral equations which admits a unique solution.

Theorem 6.5. *Let q be as in* (1)*, Q be as in* (2)*, and α, Ω be as in* (3)*. Let $(\lambda^+,\lambda^-,r) \in \mathcal{P}$. Let $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let $j \in \{1,\dots,n\}$. Then the unique solution*

$$(u_j^+[q,\phi,\lambda^+,\lambda^-,r], u_j^-[q,\phi,\lambda^+,\lambda^-,r]) \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$$

of problem (7) *is delivered by*

$$\begin{aligned} u_j^+[q,\phi,\lambda^+,\lambda^-,r](x) =& v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](x) - \fint_{\partial q\mathbb{I}[\phi]} v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](y)\,d\sigma_y \\ &- \fint_{\partial q\mathbb{I}[\phi]} y_j\,d\sigma_y + x_j \qquad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}, \\ u_j^-[q,\phi,\lambda^+,\lambda^-,r](x) =& v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](x) - \fint_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](y)\,d\sigma_y \\ &- \fint_{\partial q\mathbb{I}[\phi]} y_j\,d\sigma_y + x_j \qquad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}, \end{aligned} \tag{10}$$

where (μ_j^+,μ_j^-) is the unique solution in $(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ of the system of integral equations

$$J[\mu_j^+,\mu_j^-] = \Big((\lambda^- - \lambda^+)(\nu_{q\mathbb{I}[\phi]})_j, -\lambda^+(\nu_{q\mathbb{I}[\phi]})_j\Big)\,, \tag{11}$$

where J is as in Proposition 6.4.

Proof. Proposition 6.3 (i) implies that problem (7) has a unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$. Accordingly, we only need to prove that the pair of functions defined by (10) solves problem (7). Since

$$(\nu_{q\mathbb{I}[\phi]})_j \in C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0,$$

Proposition 6.4 (i) implies that there exists a unique pair $(\mu_j^+, \mu_j^-) \in (C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0)^2$ that solves the integral equation (11). Accordingly, a straightforward computation based on the properties of the single layer potential (see Theorem 6.1) together with Proposition 6.2 implies that the pair of functions defined by (10) solves problem (7). □

Remark 6.6. *The previous theorem provides an integral equation formulation of problem (7) and a representation formula for its solution. Plugging this representation formula into Definition 5.1, we can rewrite the effective conductivity in terms of the densities μ_j^+ and μ_j^- solving equation (11). Let the assumptions of Theorem 6.5 hold and let $u_j^+[q,\phi,\lambda^+,\lambda^-,r]$, $u_j^-[q,\phi,\lambda^+,\lambda^-,r]$, and μ_j^+, μ_j^- be as in Theorem 6.5. Then the divergence theorem implies that*

$$\begin{aligned}
&\int_{q\mathbb{I}[\phi]} \frac{\partial}{\partial x_i} u_j^+[q,\phi,\lambda^+,\lambda^-,r](x)\,dx \\
&\quad = \int_{\partial q\mathbb{I}[\phi]} u_j^+[q,\phi,\lambda^+,\lambda^-,r](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \\
&\quad = \int_{\partial q\mathbb{I}[\phi]} \Bigg(v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](y) - \fint_{\partial q\mathbb{I}[\phi]} v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](z)\,d\sigma_z \\
&\qquad\qquad - \fint_{\partial q\mathbb{I}[\phi]} z_j\,d\sigma_z + y_j \Bigg)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \\
&\quad = \int_{\partial q\mathbb{I}[\phi]} v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \\
&\qquad - \int_{\partial q\mathbb{I}[\phi]} (\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \fint_{\partial q\mathbb{I}[\phi]} v_q^+[\partial q\mathbb{I}[\phi],\mu_j^+](z)\,d\sigma_z \\
&\qquad - \int_{\partial q\mathbb{I}[\phi]} (\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \fint_{\partial q\mathbb{I}[\phi]} z_j\,d\sigma_z + \delta_{ij}|q\mathbb{I}[\phi]|_n.
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\int_{Q\setminus\overline{q\mathbb{I}[\phi]}} \frac{\partial}{\partial x_i} u_j^-[q,\phi,\lambda^+,\lambda^-,r](x)\,dx \\
&\quad = \int_{\partial Q} u_j^-[q,\phi,\lambda^+,\lambda^-,r](y)(\nu_Q(y))_i\,d\sigma_y \\
&\qquad - \int_{\partial q\mathbb{I}[\phi]} u_j^-[q,\phi,\lambda^+,\lambda^-,r](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \\
&\quad = \delta_{ij}|Q|_n - \int_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \\
&\qquad + \int_{\partial q\mathbb{I}[\phi]} (\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \fint_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](z)\,d\sigma_z \\
&\qquad + \int_{\partial q\mathbb{I}[\phi]} (\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y \fint_{\partial q\mathbb{I}[\phi]} z_j\,d\sigma_z - \delta_{ij}|q\mathbb{I}[\phi]|_n.
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\int_{\partial Q} \Bigg(v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](y) - \fint_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu_j^-](z)\,d\sigma_z \\
&\qquad\qquad - \fint_{\partial q\mathbb{I}[\phi]} z_j\,d\sigma_z + y_j \Bigg)(\nu_Q(y))_i\,d\sigma_y \\
&\quad = \int_{\partial Q} y_j(\nu_Q(y))_i\,d\sigma_y = \delta_{ij}|Q|_n.
\end{aligned}$$

Also, by the divergence theorem, we have

$$\int_{\partial q\mathbb{I}[\phi]} (\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y = 0 \qquad \forall i \in \{1,\dots,n\}\,.$$

Accordingly, by a straightforward computation we have

$$
\begin{aligned}
&\lambda_{ij}^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r] \qquad (12)\\
&= \frac{1}{|Q|_n}\left\{\lambda^+\int_{q\mathbb{I}[\phi]}\frac{\partial}{\partial x_i}u_j^+[q,\phi,\lambda^+,\lambda^-,r](x)\,dx+\lambda^-\int_{Q\setminus\overline{q\mathbb{I}[\phi]}}\frac{\partial}{\partial x_i}u_j^-[q,\phi,\lambda^+,\lambda^-,r](x)\,dx\right\}\\
&= \frac{1}{|Q|_n}\left\{\delta_{ij}\lambda^-|Q|_n+(\lambda^+-\lambda^-)\delta_{ij}|q\mathbb{I}[\phi]|_n+\lambda^+\int_{\partial q\mathbb{I}[\phi]}v_q[\partial q\mathbb{I}[\phi],\mu_j^+](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y\right.\\
&\qquad\left.-\lambda^-\int_{\partial q\mathbb{I}[\phi]}v_q[\partial q\mathbb{I}[\phi],\mu_j^-](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y\right\}.
\end{aligned}
$$

6.3 Analyticity of the solution of the integral equation

Equality (12) suggests that the next step in order to study the dependence of the effective conductivity $\lambda_{ij}^{\mathrm{eff,nonid}}[q,\phi,\lambda^+,\lambda^-,r]$ upon the quintuple $(q,\phi,\lambda^+,\lambda^-,r)$ is to analyze the dependence of the solutions μ_j^+,μ_j^- of equation (11). Before starting with this plan, we note that equation (11) is defined on the (q,ϕ)-dependent domain $\partial q\mathbb{I}[\phi]$, while a formulation on a fixed domain would be easier to analyze. Thus, we first provide a reformulation on a fixed domain.

Lemma 6.7. *Let q be as in* (1)*, Q be as in* (2)*, and α, Ω be as in* (3)*. Let $(\lambda^+,\lambda^-,r)\in\mathcal{P}$. Let $\phi\in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$. Let $j\in\{1,\dots,n\}$. Then the pair $(\theta_j^+,\theta_j^-)\in(C^{0,\alpha}(\partial\Omega))^2$ solves the system of equations*

$$
\begin{aligned}
&\lambda^+\Big(-\frac{1}{2}\theta_j^+(t)+\int_{q\phi(\partial\Omega)}DS_{q,n}(q\phi(t)-s)\cdot\nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta_j^+\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big) \qquad (13)\\
&-\lambda^-\Big(\frac{1}{2}\theta_j^-(t)+\int_{q\phi(\partial\Omega)}DS_{q,n}(q\phi(t)-s)\cdot\nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta_j^-\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big)\\
&\qquad=(\lambda^--\lambda^+)(\nu_{q\mathbb{I}[\phi]}(q\phi(t)))_j\quad\forall t\in\partial\Omega,\\
&\lambda^+\Big(-\frac{1}{2}\theta_j^+(t)+\int_{q\phi(\partial\Omega)}DS_{q,n}(q\phi(t)-s)\cdot\nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta_j^+\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big) \qquad (14)\\
&+r\Big(\int_{q\phi(\partial\Omega)}S_{q,n}(q\phi(t)-s)\left(\theta_j^+\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\\
&-\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{q\phi(\partial\Omega)}\int_{q\phi(\partial\Omega)}S_{q,n}(y-s)\left(\theta_j^+\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\,d\sigma_y\\
&-\int_{q\phi(\partial\Omega)}S_{q,n}(q\phi(t)-s)\left(\theta_j^-\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\\
&+\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{q\phi(\partial\Omega)}\int_{q\phi(\partial\Omega)}S_{q,n}(y-s)\left(\theta_j^-\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\,d\sigma_y\Big)\\
&\qquad=-\lambda^+(\nu_{q\mathbb{I}[\phi]}(q\phi(t)))_j\quad\forall t\in\partial\Omega,
\end{aligned}
$$

if and only if the pair $(\mu_j^+,\mu_j^-)\in(C^{0,\alpha}(\partial q\mathbb{I}[\phi]))^2$ defined by

$$
\mu_j^\pm(x)\equiv(\theta_j^\pm\circ\phi^{(-1)})(q^{-1}x)\qquad\forall x\in\partial q\mathbb{I}[\phi] \qquad (15)
$$

solves (11)*. Moreover, system* (13)–(14) *has a unique solution in* $(C^{0,\alpha}(\partial\Omega))^2$.

Proof. The equivalence of equations (13)–(14) in the unknown (θ_j^+,θ_j^-) and equation (11) in the unknown (μ_j^+,μ_j^-), with (μ_j^+,μ_j^-) delivered by (15), is a straightforward consequence of a change of variables. Then the existence and uniqueness of a solution of equations (13)–(14) in $(C^{0,\alpha}(\partial\Omega))^2$ follows from Theorem 6.5 and from the equivalence of equations (11) and (13)–(14). □

Inspired by Lemma 6.7, for all $j\in\{1,\dots,n\}$ we introduce the map

$$
M_j\equiv(M_{j,1},M_{j,2}):\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times\mathbb{R}^3\times(C^{0,\alpha}(\partial\Omega))^2\to(C^{0,\alpha}(\partial\Omega))^2
$$

by setting

$$
\begin{aligned}
&M_{j,1}[q,\phi,\lambda^+,\lambda^-,r,\theta^+,\theta^-](t)\\
&\equiv \lambda^+\Big(-\frac{1}{2}\theta^+(t)+\int_{q\phi(\partial\Omega)} DS_{q,n}(q\phi(t)-s)\cdot \nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta^+\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big)\\
&-\lambda^-\Big(\frac{1}{2}\theta^-(t)+\int_{q\phi(\partial\Omega)} DS_{q,n}(q\phi(t)-s)\cdot \nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta^-\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big)\\
&-(\lambda^--\lambda^+)(\nu_{q\mathbb{I}[\phi]}(q\phi(t)))_j \quad \forall t\in\partial\Omega,\\
&M_{j,2}[q,\phi,\lambda^+,\lambda^-,r,\theta^+,\theta^-](t)\\
&\equiv \lambda^+\Big(-\frac{1}{2}\theta^+(t)+\int_{q\phi(\partial\Omega)} DS_{q,n}(q\phi(t)-s)\cdot \nu_{q\mathbb{I}[\phi]}(q\phi(t))(\theta^+\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\Big)\\
&+r\Big(\int_{q\phi(\partial\Omega)} S_{q,n}(q\phi(t)-s)\left(\theta^+\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\\
&-\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{q\phi(\partial\Omega)}\int_{q\phi(\partial\Omega)} S_{q,n}(y-s)\left(\theta^+\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\,d\sigma_y\\
&-\int_{q\phi(\partial\Omega)} S_{q,n}(q\phi(t)-s)\left(\theta^-\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\\
&+\frac{1}{|\partial q\mathbb{I}[\phi]|_{n-1}}\int_{q\phi(\partial\Omega)}\int_{q\phi(\partial\Omega)} S_{q,n}(y-s)\left(\theta^-\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s\,d\sigma_y\Big)\\
&+\lambda^+(\nu_{q\mathbb{I}[\phi]}(q\phi(t)))_j \quad \forall t\in\partial\Omega,
\end{aligned}
$$

for all $(q,\phi,\lambda^+,\lambda^-,r,\theta^+,\theta^-)\in\mathbb{D}_n^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)\times\mathbb{R}^3\times(C^{0,\alpha}(\partial\Omega))^2$. As one can readily verify, under the assumptions of Lemma 6.7, the system (13)-(14) can be rewritten as

$$
M_j\left[q,\phi,\lambda^+,\lambda^-,r,\theta^+,\theta^-\right]=0. \tag{16}
$$

Our aim is to describe the dependence of the pair (θ^+,θ^-) that solves equation (16) on the periodicity matrix q, the inclusions' shape ϕ, and the parameters λ^+,λ^-,r. To do so, we plan to apply the implicit function theorem for real analytic maps in Banach spaces to equation (16). So, as a first step we need to prove that M_j is real analytic. We start with some technical results on the analyticity of certain operators involved in the definition of M_j. The first one concerns integral operators associated with the single layer potential and its normal derivative and shows their analytical dependence upon the periodicity matrix q and the shape ϕ. For a proof we refer to [44].

Lemma 6.8. *Let α, Ω be as in* (3)*. Then the following statements hold.*

(i) The map from $\mathbb{D}_n^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)\times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ that takes a triple (q,ϕ,θ) to the function $V[q,\phi,\theta]$ defined by

$$
V[q,\phi,\theta](t)\equiv\int_{q\phi(\partial\Omega)} S_{q,n}(q\phi(t)-s)\left(\theta\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s \qquad \forall t\in\partial\Omega,
$$

is real analytic.

(ii) The map from $\mathbb{D}_n^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)\times C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ that takes a triple (q,ϕ,θ) to the function $W_*[q,\phi,\theta]$ defined by

$$
W_*[q,\phi,\theta](t)\equiv\int_{q\phi(\partial\Omega)} DS_{q,n}(q\phi(t)-s)\cdot\nu_{q\mathbb{I}[\phi]}(q\phi(t))\left(\theta\circ\phi^{(-1)}\right)(q^{-1}s)d\sigma_s \qquad \forall t\in\partial\Omega,
$$

is real analytic.

Next, we need the following lemma about the real analytic dependence of certain maps related to the change of variables in integrals and to the pullback of the outer normal field. For a proof we refer to Lanza de Cristoforis and Rossi [37, p. 166] and to Lanza de Cristoforis [35, Prop. 1].

Lemma 6.9. *Let α, Ω be as in* (3). *Then the following statements hold.*

(i) For each $\psi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, there exists a unique $\tilde{\sigma}[\psi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\psi] > 0$ and

$$\int_{\psi(\partial\Omega)} w(s)\, d\sigma_s = \int_{\partial\Omega} w \circ \psi(y) \tilde{\sigma}[\psi](y)\, d\sigma_y, \qquad \forall w \in L^1(\psi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ from $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) The map from $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ that takes ψ to $\nu_{\mathbb{I}[\psi]} \circ \psi$ is real analytic.

The last technical result that we need is about the analyticity of certain maps related to the measure of sets.

Lemma 6.10. *Let α, Ω be as in* (3). *Then the following statements hold.*

(i) The map from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$ to $\mathbb{R}$ that takes (q, ϕ) to $|q\mathbb{I}[\phi]|_n$ is real analytic.

(ii) The map from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$ to $\mathbb{R}$ that takes (q, ϕ) to $|\partial q\mathbb{I}[\phi]|_{n-1}$ is real analytic.

Proof. The validity of statement (i) follows by the proof of [41, Thm. 5.1]. In order to prove statement (ii) we notice that

$$|\partial q\mathbb{I}[\phi]|_{n-1} = \int_{q\phi(\partial\Omega)} d\sigma = \int_{\partial\Omega} \tilde{\sigma}[q\phi] d\sigma\,,$$

for all $(q, \phi) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$. By the analyticity of the map from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$ to $\left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}\right)$ that takes the pair (q, ϕ) to $q\phi$ and by Lemma 6.9 (i), we deduce that

$$\int_{q\phi(\partial\Omega)} d\sigma = \int_{\partial\Omega} \tilde{\sigma}[q\phi] d\sigma$$

depends real analytically on $(q, \phi) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right)$. As a consequence, the validity of statement (ii) follows. □

By Lemmas 6.8, 6.9, 6.10 and by standard calculus in Banach spaces, we immediately deduce the validity of the following.

Proposition 6.11. *Let α, Ω be as in* (3). *Let $j \in \{1, \ldots, n\}$. The map M_j is real analytic from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times \mathbb{R}^3 \times (C^{0,\alpha}(\partial\Omega))^2$ to $(C^{0,\alpha}(\partial\Omega))^2$.*

We are now ready to prove that the solution of (16) depends real analytically upon the quintuple $(q, \phi, \lambda^+, \lambda^-, r)$.

Proposition 6.12. *Let α, Ω be as in* (3). *Let $j \in \{1, \ldots, n\}$. Then the following statements hold.*

(i) There exists an open neighborhood $\mathcal{V}$ of $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times \mathcal{P}$ in $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times \mathbb{R}^3$ such that for each $(q, \phi, \lambda^+, \lambda^-, r) \in \mathcal{V}$ the operator

$$M_j[q, \phi, \lambda^+, \lambda^-, r, \cdot, \cdot]$$

is a linear homeomorphism from $(C^{0,\alpha}(\partial\Omega))^2$ onto $(C^{0,\alpha}(\partial\Omega))^2$. In particular, for each $(q, \phi, \lambda^+, \lambda^-, r) \in \mathcal{V}$ there exists a unique pair (θ_j^+, θ_j^-) in $(C^{0,\alpha}(\partial\Omega))^2$ such that

$$M_j[q, \phi, \lambda^+, \lambda^-, r, \theta_j^+, \theta_j^-] = 0.$$

We denote this pair by $(\theta_j^+[q, \phi, \lambda^+, \lambda^-, r], \theta_j^-[q, \phi, \lambda^+, \lambda^-, r])$.

(ii) The map from $\mathcal{V}$ to $(C^{0,\alpha}(\partial\Omega))^2$ that takes $(q,\phi,\lambda^+,\lambda^-,r)$ to the pair

$$(\theta_j^+[q,\phi,\lambda^+,\lambda^-,r],\theta_j^-[q,\phi,\lambda^+,\lambda^-,r])$$

is real analytic

Proof. Statement (i) follows by arguing as in the proof of Lemma 6.7 and by the fact that the set of linear homeomorphisms is open in the set of linear and continuous operators. To prove statement (ii) we first note that, by Proposition 6.11, M_j is a real analytic map from $\mathbb{D}_n^+(\mathbb{R})\times\left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right)\times\mathbb{R}^3\times(C^{0,\alpha}(\partial\Omega))^2$ to $(C^{0,\alpha}(\partial\Omega))^2$. Moreover, for all $(q,\phi,\lambda^+,\lambda^-,r)\in\mathcal{V}$, the partial differential

$$\partial_{(\theta^+,\theta^-)}M_j\left[q,\phi,\lambda^+,\lambda^-,r,\theta^+[q,\phi,\lambda^+,\lambda^-,r],\theta^-[q,\phi,\lambda^+,\lambda^-,r]\right]$$

of M_j at the point

$$(q,\phi,\lambda^+,\lambda^-,r,\theta^+[q,\phi,\lambda^+,\lambda^-,r],\theta^-[q,\phi,\lambda^+,\lambda^-,r])$$

with respect to the variable (θ^+,θ^-) is given by

$$\begin{aligned}\partial_{(\theta^+,\theta^-)}M_j\Big[q,\phi,\lambda^+,\lambda^-,r,\theta^+[q,\phi,\lambda^+,\lambda^-,r],\theta^-[q,\phi,\lambda^+,\lambda^-,r]\Big](\psi^+,\psi^-)\\ =M_j\left[q,\phi,\lambda^+,\lambda^-,r,\psi^+,\psi^-\right]\ ,\end{aligned}$$

for all $(\psi^+,\psi^-)\in(C^{0,\alpha}(\partial\Omega))^2$. Accordingly, by statement (i) and by the implicit function theorem for real analytic maps in Banach spaces (see, *e.g.*, Deimling [17, Thm. 15.3]), we deduce the analyticity of the map

$$(q,\phi,\lambda^+,\lambda^-,r)\mapsto(\theta_j^+[q,\phi,\lambda^+,\lambda^-,r],\theta_j^-[q,\phi,\lambda^+,\lambda^-,r])$$

as in the statement. □

6.4 Analyticity of the effective conductivity

We are now ready to prove our main Theorem 5.2 for the effective conductivity in the case of nonideal contact conditions. To this aim, we exploit formula (12) for $\lambda^{\mathrm{eff,nonid}}$ and the analyticity result of Proposition 6.12.

Proof of Theorem 5.2. Let (θ_j^+,θ_j^-) and $\mathcal{V}$ be as in Proposition 6.12. Then, we set Λ_{ij} to be the map from the $\mathcal{V}$ to $\mathbb{R}$ defined by

$$\begin{aligned}\Lambda_{ij}&[q,\phi,\lambda^+,\lambda^-,r]\\ &\equiv\frac{1}{|Q|_n}\Bigg\{\lambda^+\int_{\partial q\mathbb{I}[\phi]}v_q[\partial q\mathbb{I}[\phi],(\theta_j^+[q,\phi,\lambda^+,\lambda^-,r]\circ\phi^{(-1)})(q^{-1}\cdot)](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y\\ &\quad-\lambda^-\int_{\partial q\mathbb{I}[\phi]}v_q[\partial q\mathbb{I}[\phi],(\theta_j^-[q,\phi,\lambda^+,\lambda^-,r]\circ\phi^{(-1)})(q^{-1}\cdot)](y)(\nu_{q\mathbb{I}[\phi]}(y))_i\,d\sigma_y\\ &\quad+(\lambda^+-\lambda^-)\delta_{ij}|q\mathbb{I}[\phi]|_n\Bigg\}\end{aligned}$$

for all $(q,\phi,\lambda^+,\lambda^-,r)\in\mathcal{V}$. By formula (12) for the effective conductivity, by Proposition 6.12, by Lemma 6.7 and by Theorem 6.5, the only thing that remains in order to complete the proof is to show that the map Λ_{ij} is real analytic. Lemma 6.9 implies that

$$\begin{aligned}\Lambda_{ij}[q,\phi,\lambda^+,\lambda^-,r]=&\frac{1}{|Q|_n}\Bigg\{\lambda^+\int_{\partial\Omega}V[q,\phi,\theta_j^+[q,\phi,\lambda^+,\lambda^-,r]](y)(\nu_{q\mathbb{I}[\phi]}(q\phi(y)))_i\tilde{\sigma}[q\phi](y)\,d\sigma_y\\ &-\lambda^-\int_{\partial\Omega}V[q,\phi,\theta_j^-[q,\phi,\lambda^+,\lambda^-,r]](y)(\nu_{q\mathbb{I}[\phi]}(q\phi(y)))_i\tilde{\sigma}[q\phi](y)\,d\sigma_y\\ &+(\lambda^+-\lambda^-)\delta_{ij}|q\mathbb{I}[\phi]|_n\Bigg\}\end{aligned}$$

for all $(q, \phi, \lambda^+, \lambda^-, r) \in \mathcal{V}$. Since

$$|Q|_n = \prod_{l=1}^{n} q_{ll} \qquad \forall q \in \mathbb{D}_n^+(\mathbb{R}),$$

clearly $|Q|_n$ depends analytically on $q \in \mathbb{D}_n^+(\mathbb{R})$. Lemma 6.10 implies that the map from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right)$ to $\mathbb{R}$ that takes (q, ϕ) to $|q\mathbb{I}[\phi]|_n$ is real analytic. Thus, by Proposition 6.12, by Lemma 6.8 (i), by Lemma 6.9, together with the above considerations, we can conclude that the map Λ_{ij} is real analytic from $\mathcal{V}$ to $\mathbb{R}$. Accordingly, the statement of Theorem 5.2 holds true. □

7 Conclusions

We have presented some of our recent results about the dependence of effective properties upon regular perturbation of the geometric and physical parameters. We have considered the average flow velocity along a periodic array of cylinders and the effective conductivity of periodic composites, both with ideal and nonideal contact conditions. We have proven that these quantities depend real analytically upon the parameters involved. The method used is based on the so-called *Functional Analytic Approach* proposed by Lanza de Cristoforis for the analysis of regular and singular domain perturbations (cf. [33, 34, 35]).

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