Decisions in Economics and Finance
https://doi.org/10.1007/s10203-023-00420-2

Generally acceptable principles for financial amortization: a modest proposal

Francesca Beccacece[1] · Marco LiCalzi[2]

Received: 25 April 2023 / Accepted: 15 November 2023

Abstract
We propose a minimal set of commonly acceptable principles to consistently formulate amortization schedules in accordance with different contractual clauses. Our goal is bringing to the fore premises that are sometimes left implicit, and yet seem to draw a wide consensus in practice. We demonstrate by means of examples how these principles may be used to deal with risk or financial innovations, and to fill gaps arising from unforeseen contingencies.

Keywords Loan reimbursement · Flexible payments · Risk · Unforeseen contingencies

Mathematics Subject Classification 91G05 · 91G99

JEL Codes G21 · G53

1 Introduction

The repayment of a loan has two goals: returning the borrowed sum (the principal) and providing an interest payment to the lender. There exists a variety of reimbursement schemes, attesting both human ingenuity and a kaleidoscope of practical needs. When

We wish to thank Margherita Cigola, Gino Favero, and Laura Ziani for their comments.

✉ Francesca Beccacece
francesca.beccacece@unibocconi.it

Marco LiCalzi
licalzi@unive.it

1 Department of Decision Sciences, Università Bocconi, Via Röntgen 1, Milano 20136, Italy

2 Department of Management, Università Ca' Foscari Venezia, Venice 30121, Italy

the amount of the principal is sizable, it is often convenient to arrange its repayment over multiple installments that allow a gradual reimbursement of the debt.

Amortization is a time-honored and well-known method to spread the reimbursement of a loan over a sequence of payments. The typical example is the fixed-rate constant-payment 30-year mortgage, cemented in U.S. as the default option for finance housing after the Great Depression (Green 2013). In British English, amortization has been understood as "The action or an act of paying off a debt, liability, etc., gradually by making regular repayments over a period of time" at least since the early XIX century (OED Online 2022).

From a technical point of view, the amortization method derives a schedule of installments and separate each of them into a principal component and an interest component. A well-formed amortization plan ensures that the sum of the principal components paid over time matches the borrowed sum, and thus extinguishes the debt. The etymology of amortization, from the Vulgar Latin *admortire* (to extinguish), relates to the outstanding debt being progressively decremented to zero.

Amortization is not the only repayment scheme where the principal is gradually reimbursed and the debt is fully repaid. The defining feature of amortization as "a method of extinguishing a debt" is that "any payment over what is needed to pay interest on the principal [is] to be applied at once toward liquidation of the debt. As the debt is being paid off, a smaller amount goes toward the payment of interest." (Richardson and Miller 1946, pp. 111–112).

Besides the standard textbook example of a fixed-rate constant-payment amortization plan, there is a huge variety of clauses that may affect a schedule. For instance, the interest rate may be variable (or adjustable) according to predetermined rules, whose effect is not known at the time of drawing the payment schedule: this may prevent constant payments. Variable-rate amortization plans cover a significant amount of loans in many countries. Other common occurrences include, e.g., a stoppage in payment; a capped rate kicking in; a switch from fixed-rate to variable-rate; a discount on either principal or interest payments; adjustable payment dates.

We argue that there is a widespread consensus on the proper amortization schedule for each of these (and many other) variants. This paper modestly set forth a minimal set of generally acceptable principles to consistently formulate amortization schedules in accordance with many different contractual clauses, including possible future innovations.

We emphasize that our proposal for a set of generally acceptable principles concerns only amortization schedules. One may design gradual repayment schemes based on alternative assumptions, or even devise alternative interpretations for a series of multiple payments matching an amortization plan. These formulations are outside the scope of this paper; see for instance (Pressacco et al. 2022) for issues concerning the interpretation of repayment schemes in legal controversies.

However, we do claim that our proposal matches a general consensus about what characterizes an amortization plan. We offer a set of principles that is compact, general, and simple. It is compact because it contains four principles. It is general because, as the paper demonstrates, it covers many different occurrences. It is simple because it is formulated in a plain language easily understood by non-mathematicians such as customers or courts. Indeed, using plain language contributes to generality by glossing over notation and details. The paper turns to a technical language when it exemplifies the application of the principles in specific cases.

From now on, we speak of generally acceptable principles (GAP) for financial amortization.[1] We state the GAP in Sect. 2, and demonstrate them by revisiting two standard fixed-rate amortization plans. Section 3 applies GAP to situations where the amortization schedule may change because of future events mentioned in the contractual conditions; e.g., when the rate is variable or when flexible payments are allowed. Section 4 expands the approach to situations where unexpected occurrences may induce the parties to restructure the amortization plan; e.g., after a temporary stoppage in payments. Section 5 exemplifies the use of GAP to draw an amortization plan tailored to specific arrangements; finally, we test the boundaries of GAP and exhibit for each of the four principles an amortization contract that violates it—our implicit claim is that these counterexamples are contractually feasible but extremely unlikely to occur in practice.

2 Generally acceptable principles for amortization

Financial amortization refers to a contract that spreads the reimbursement of a loan over a sequence of payments at different time points. The underlying contract is often exemplified by an amortization plan, that lists for each time point the due payment and its subdivision into a principal and an interest component, along with the outstanding debt. Typically, the amortization contract presumes at least two payments with (strictly) positive principal components: consequently, the loan is reimbursed gradually.

When it is available, the amortization plan summarizes the relevant financial consequences. However, amortization involves a large set of contractual conditions to determine parties' obligations. Some are explicit and easily understood: e.g., the lender usually sets periods of equal length or quotes an interest rate. The parties may bargain over the time intervals between payments or the size of each interest component, but similar conventional clauses are usually adopted for the sake of simplicity.

A second set of conditions are also explicit, albeit potentially less transparent to the borrower: e.g., the constant-payment amortization plan associated with a variable-rate loan is subject to future revisions, whenever the underlying rate changes. The lender may provide a few schedules under alternative scenarios as illustrative examples, but for practical purposes the original amortization plan is unlikely to stay unchanged *ex post*.

A contractual condition may often be viewed as a function that is known at the time of signing the contract. It is convenient to distinguish three cases. The first, and simplest, is the case when the values of the function inputs are all known and, therefore, its output is also known. The standard constant-payment schedule fits this first case.

The second case is when at least one key input will be known in the future, thus the output is not (yet) known. Amortization plans with variable rates that are subject to future events fit the second case. Under additional restrictive assumptions, one

[1] There is no connection with the Generally Accepted Accounting Principles (GAAP), usually associated with the Accounting Standards Codification (ASC) published by the Financial Accounting Standards Board (FASB).

might reconfigure the first case as the class of deterministic amortization schedules and the second case as the class of aleatory schedules. This paper need not pursue this approach.

The third case is residual, and covers the possibility of (contractually) unforeseen contingencies that may affect amortization: e.g., how to deal with a stoppage in payments or, more generally, with circumstances leading to a restructuring of the amortization plan.

In general, unforeseen contingencies lead to incomplete contracts. There is a sizable economic literature studying incomplete contracts, viewed as agreements that are renegotiated whenever a contingency (unforeseen by at least one party) arises (Tirole 2009). The incompleteness may stem from different causes, ranging from the cost of writing or tracking contingencies (Dye 1985) to the differential awareness of parties over all possibilities (Filiz-Obay 2012; Zhao 2011).

This paper focuses on identifying generally acceptable principles that are used to fill the gaps in incomplete amortization contracts. Following the legal literature, we emphasize that the parties are free to contract over which default rules they wish to use (Ayres and Gertner 1989). The generally acceptable principles distill a convenient checklist of fallbacks for a great variety of gaps in the contractual provisions.[2] Financial innovations may over time lead to revise these principles or change the consensus supporting them. We demonstrate a few principles that suffice to revise or complete amortization plans affected by unforeseen contingencies.

We restrict attention to (financial) amortization, defined as gradual loan reimbursement where at least two payments are associated with (strictly positive) principal components; that is, the original debt undergoes a partial reduction before extinction. In particular, we do not consider lump-sum repayments or perpetuities.

We assume that the contract is clearly formulated and well understood by both parties; that is, the key values or the rules by which they can be derived are mutually known. For example, if the rate of interest i is fixed, then its value is given; if it is variable, the rules by which the interest is computed in each period are specified. Similarly, either the maturity (the time at which the debt is fully repaid) or the circumstances to achieve it are given. The assumption that the contract is not ambiguous is a minimal requirement for its legal validity. Nonetheless, we imagine that the contract may be incomplete, and unforeseen contingencies (not covered by its provisions) may occur.

We propose the following minimal set of Generally Acceptable Principles (GAP) for financial amortization as a compact checklist: (a) to recognize when a loan repayment can be framed as amortization, and (b) to complete the amortization plan, filling gaps in the explicit contractual clauses.[3]

- A1. The maturity is finite.
- A2. The sum of the principal components equals the initial outstanding debt.
- A3. The interest component matures at the time of payment, and is computed over the outstanding debt for the last elapsed period between consecutive maturities.

[2] That is, the GAP are meant to fill some gaps—pun intended.

[3] We are aware that the norms of a specific country may impose stronger restrictions, but the legalities of different countries are out of the scope of this paper. Yet, to the best of our knowledge, the GAP are compatible with the legal frameworks currently in force across the European Union.

A4. Each payment is first attributed to the interest component; the residual is applied to the outstanding debt.

We opt for a verbal format over a mathematical formulation for two reasons. First, it is shorter and more widely accessible to users. Second, because a full description of all the unforeseen contingencies is by definition unattainable, it is preferable to compress information and deliver principles in a form that is open to interpretation or revision in face of the unexpected. This implies that we cannot claim that GAP cover any possible eventuality; more modestly, we believe that they provide effective guidelines toward drawing proper amortization plans, although they may not be sufficient to pinpoint a unique solution.

For instance, consider A1. Apparently trivial, it covers the cases where debt restructuring include a change in the number of payments or, more generally, in its maturity, possibly subject to random events. It states that any gap filling must ensure that the maturity remains finite: f.i., if the number of payments is a random variable, its support must be finite. (We stay clear from theoretical subtleties such as "almost surely" when they carry no practical import.) A similar interpretation applies for A2: a debt restructuring may change the number of principal payments from what was originally planned, but it must ensure that *ex post* the debt is fully repaid,[4]

Principles A3 and A4 state how payments are accounted for. The interest is contractually determined, and comes due at the time of payment so that it is possible to decompose the payment into two components. Using mathematical notation, a payment P is decomposed into an interest I and a principal C so that $P = I + C$. At the time of payment, there must be enough information to determine two of the three values for P, I, C. Moreover, the defining feature of amortization is usually understood as involving the preliminary deduction of interest from the current payment, and the imputation of the difference toward reducing the outstanding debt. This applies even to the case of negative amortization, when the current payment does not cover the interest: therefore, the principal component is negative and the outstanding debt becomes higher.

Example Consider a standard fixed-rate constant-payment amortization plan for an amount of 1000 over four equal-length periods, at a constant interest rate $i = 5\%$. (It is understood that i is the effective interest rate per period.) The constant payment is $P = 1000/a_{\overline{n}|i}$ with $n = 4$ and $i = 0.05$; that is, $P = 282.01$ after rounding to the second digit.[5] The amortization plan is given in Table 1, where the columns respectively carry the (end of) period t, the payment P_t, the interest component I_t, the principal component C_t, and the outstanding debt $D_t = D_{t-1} - P_t$ (that is, after the payment P_t has taken place).

This example has no unforeseen contingencies. The amortization plan satisfies the four GAP: A1 holds because there are four payments; A2 holds because $\sum_{t=1}^{4} C_t = D_0$,

[4] In the literature, A2 is known as the *elementary equivalence* whereas the *financial equivalence* states the equality between the initial outstanding debt and the present value of all payments. We argue that the elementary equivalence is a simpler and thus better candidate for a generally acceptable principle.

[5] Throughout the paper, we carry out exact computations but show numbers rounded to the second digit. This explains seeming incongruities such as $D_3 - C_4 \neq D_4$ in Table 1.

Table 1 A constant-payment amortization plan

t	P_t	I_t	C_t	D_t
0	–	–	–	1000.00
1	282.01	50.00	232.01	767.99
2	282.01	38.40	243.61	524.38
3	282.01	26.22	255.79	268.59
4	282.01	13.43	268.58	0.00

Table 2 A constant-principal amortization plan

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	300.00	50.00	250.00	750.00
2	287.50	37.50	250.00	500.00
3	275.00	25.00	250.00	250.00
4	262.50	12.50	250.00	0

up to inconsequential rounding; A3 holds because $I_t = i D_{t-1}$; and A4 holds because $C_t = P_t - I_t$.

The key point goes in the opposite direction: the mere contractual clauses need to be complemented by the GAP to fully identify the amortization plan. For instance, A3 justifies the numbers in the third column and A4 those in the fourth column. The GAP summarize the implicit rules that, along with the contractual clauses, are used to draw up the amortization plan. The GAP are necessary to fill the gap between the contractual clauses ($n = 4$ and $i = 0.05$) and the full amortization schedule.

Similarly, the GAP allow an immediate derivation of the amortization plan assuming constant principal components. Given $n = 4$, A2 implies $C_t = D_0/n = 1000/4 = 250$ at the end of each period t. Then, using A3 and A4 as before, the amortization plan shown in Table 2 obtains.

We claim that the GAP complement the contractual conditions (e.g. i, n and the sequence of payments $\{P_t\}$) in order to derive the amortization plan. While the contractual conditions are by necessity explicit, the GAP are often implicitly understood. Their validity and applicability rest on general consensus. For standard amortization plans, their usage is widespread and so ingrained in common professional practice that there is little need to make them explicit.

This is no longer true when the amortization contract is subject to random events or, more generally, to unforeseen contingencies that call for gap-filling rules. For convenience, we distinguish two cases. We speak of *risks* when the contractual provisions include a variety of (possibly, random) events that are fully accounted for; for example, when the interest rate is variable because it is linked to some market rate. We speak of (unforeseen) *contingencies* when some actual occurrence is not covered by the contractual provisions; for example, when the possibility of a stoppage in payments is not explicitly mentioned. The next two sections demonstrate the reach of GAP in the case of risks and contingencies, respectively.

3 GAP in face of risks

The fixed-rate constant-payment amortization plan is a time-honored benchmark, but financial innovation has introduced a variety of amortization contracts. They are usually spurred by ingenious efforts to encourage the debtor's solvency without jeopardizing the lender's return. The balance between these motives is affected by the (exogenous) market conditions and the (endogenous) parties' obligations.

A common theme is to embed the amortization contract with an adequate flexibility to face risks; that is, to deal with future events that are contractually described but not yet known at the time of signing the contract. For instance, the parties may agree that the rate is variable and face the risk that the payments are not constant. In general, dealing with risks imply that the original amortization plan is subject to changes associated to specific (possibly, aleatory) events.

Because financial innovation accommodates emerging needs, it is often the case that new provisions are grafted onto the fixed-rate constant-payment standard without a full and explicit description of all possible consequences. We claim that the GAP are usually (and implicitly) invoked to fill the gaps, and help revise the amortization plan whenever a risk occurs. We demonstrate this over two common amortization contracts: First, the case of flexible payments where the borrower is given some latitude in choosing when and how much to pay back; second, two variants for the case of variable rates.

3.1 Flexible payments

Consider an amount of 1000 to be repaid over 60 monthly payments, at a nominal interest rate $i^{(12)} = 6\%$. Assuming a flexible-payment provision, the borrower is allowed to amortize the full debt under the following agreement: pay back 10% of the principal within 1 year, 60% of it within 3 years, and all of it within the maturity of 5 years. Interest payments are due monthly.

In a constant-payment amortization plan, given an effective monthly interest rate $i = 0.5\%$, the monthly payment is $P = 1000/a_{\overline{60}|0.005} = 19.33$ and a standard amortization schedule may be easily drawn *ex ante* and attached to the contract; see Table 3.

The flexible-payment provision relaxes the obligation to deliver constant payments, requiring only that a given amount E_t of the principal be reimbursed by the end of month t. (The letter E_t is a mnemonic for the extinguished debt at time t.) Continuing with our example, the amount E_t for three specific months ($t = 12, 36, 60$) is

Table 3 Another constant-payment amortization plan

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
...	...	...	...	...
11	19.33	4.27	15.07	838.34
12	19.33	4.19	15.14	823.20
13	19.33	4.12	15.22	807.98
14	19.33	4.04	15.29	792.69
...	...	...	...	...
36	19.33	2.27	17.07	436.20
37	19.33	2.18	17.15	419.05
...	...	...	...	...
60	19.33	0.10	19.24	0.00

$$
\begin{aligned}
E_{12} &= \sum_{t=1}^{12} C_t = 100, \\
E_{36} &= E_{12} + \sum_{t=13}^{36} C_t = 600, \\
E_{60} &= E_{36} + \sum_{t=37}^{60} C_t = 1000.
\end{aligned}
\tag{1}
$$

It may be checked that the constant-payment plan would reimburse $E'_{12} = 176.80$ by $t = 12$ and $E'_{36} = 563.80$ by $t = 36$. In our example, therefore, the flexible-payment provision is less burdensome because it allows the borrower a partial right to postpone the principal repayment. This comparison is irrelevant in the following discussion.

The borrower has the option of repaying the initial debt using any sequence of principal components that satisfies (1) and can adjust the monthly payments as he[6] sees fit until before the three contractually specified due dates. This borrower's option adds a risk in the contract, and precludes drawing the *ex ante* amortization plan at $t = 0$. Nonetheless, the GAP can be used to derive the amortization plan as time proceeds and the borrower delivers his payments.

For instance, suppose that the borrower avails himself of the option at its fullest and delays his principal payments as much as possible. Therefore, $C_{12} = 100$, $C_{36} = E_{36} - E_{12} = 500$ and $C_{60} = E_{60} - E_{36} = 400$ are the only principal payments made to the lender, while $C_t = 0$, for $t \neq 12, 36, 60$.

Let us review how the GAP can be used to deliver the amortization plan, and the total payments due by the lender. First, note that A1 holds because there are at most sixty payments, even though their exact number is not known at the time of the contract. Similarly, A2 is satisfied because of (1).

[6] We assume a male borrower and a female lender.

Table 4 Ex post amortization plan with flexible payments

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	5.00	5.00	0	1000
...	...	...	...	...
12	105.00	5.00	100	900
13	4.50	4.50	0	900
...	...	...	...	...
36	504.50	4.50	500	400
37	2.00	2.00	0	400
...	...	...	...	...
60	402.00	2.00	400	0

Because the three nonzero principal payments are $C_{12} = 100$, $C_{36} = 500$ and $C_{60} = 400$, A4 implies that the outstanding debt sequence decreases only at $t = 12, 36$ and 60. Therefore, $D_0 = \cdots = D_{11} = 1000$; $D_{12} = \cdots = D_{35} = 900$; $D_{36} = \cdots = D_{59} = 400$; and $D_{60} = 0$. By A3, the corresponding interest components are $I_t = 1000 \cdot 0.005 = 5$ for $t = 1, \cdots, 12$; $I_t = 900 \cdot 0.005 = 4.5$ for $t = 13, \cdots, 36$; and $I_t = 400 \cdot 0.005 = 2$ for $t = 37, \cdots, 60$.

Ex post, given the borrower's choice, the amortization schedule is as given in Table 4. (We use ellipsis when the rows are identical except for t.) Clearly, $P_t = I_t + C_t$ by A4.

Analogously, one may use the GAP to deal with different payment at different times. For instance, if at $t = 1$ the borrower pays $P_1 = 7$, using A4 we decompose P_1 into $I_1 = 5$ and $C_1 = P_1 - I_1 = 2$. Then the third line of the amortization plan would carry $D_2 = D_1 - C_1 = 1000 - 2 = 998$.

We have discussed one specific example, but there are many variants that confer different degrees of flexibility in the arrangement of the payments, for either principal or interest components. A common case is to restructure the amortization by increasing the frequency of payments while reducing their amounts, for example by splitting monthly dues into two semi-monthly payments.

3.2 Variable rates

An alternative arrangement calls for a loan to be repaid using a variable rate, where the interest rate is computed monthly as a function of some contractually given index. For example, a typical provision for euro-denominated loans is to add a contractually fixed spread over the EURIBOR (Euro Interbank Offered Rate) for a given maturity.

Consider again an amount of 1000 to be repaid over 60 monthly payments, and suppose that at $t = 0$ the nominal interest rate is $i^{(12)} = 6\%$. Changes in the variable rate lead to changes in the payment schedule. This constitutes a risk that usually prevents the *ex ante* firm amortization schedule drawn at $t = 0$ from staying the same until the end.

As mere guidelines, the GAP allow for different arrangements after a change in the interest rate. This latitude conforms to the precept that the generally acceptable principles should govern financial innovation by reducing controversies or misunderstandings, while leaving the parties space to seek whatever solutions fit them best. For this reason, it is necessary that the amortization contract specifies which convention applies to remove an ambiguity that may impair its validity.

We discuss the two most common ways to redraw the amortization schedule after a change in the interest rate. We respectively call them same-principle and same-principal, for reasons that will be clear momentarily. The two nicknames sound similar, but the two conventions carry some differences in the ensuing amortization plan.

Same-principle Under this convention, after a change in the interest rate, it is assumed that the interest will no longer change and a new constant-payment plan is recomputed over the current outstanding debt. The assumption of no future changes is the same made at the time the *ex ante* amortization plan was drawn: it is a mere convention used after any change; hence, the nickname "same-principle".

At $t = 0$, the nominal interest rate is $i^{(12)} = 6\%$; therefore, the first payment is $P_1 = 1000/a_{\overline{60}|0.005} = 19.33$. Suppose that, at $t = 1$, the nominal interest rate goes up to 6.6%. Under the same-principle convention, the second payment is computed on the outstanding debt $D_1 = 1000 - C_1 = 1000 - (19.33 - 5) = 1000 - 14.33 = 985.67$ using the (new) monthly interest rate of 0.0055; this yields $P_2 = 985.67/a_{\overline{59}|0.0055} = 19.61$.

The GAP kick in to complete (assuming no future changes) the rest of the new amortization plan. We have $I_2 = 985.67 \cdot 0.0055 = 5.42$ by A3 and $C_2 = P_2 - I_2 = 19.61 - 5.42 = 4.19$ by A4. If no future changes occur, all other lines in the new schedule are drawn as for a constant-payment amortization plan. If, at a future time t, the monthly interest changes again to i, then one recomputes $P_t = D_{t-1}/a_{\overline{60-t}|i}$ and draws the revised amortization plan from t onwards.

Same-principal Under this convention, after a change in the interest rate, the original constant-payment plan is ditched but the new schedule is drawn preserving the principal components of the original plan; hence, the nickname "same-principal". In simple words, the columns C_t and D_t from the original plan are cut-and-pasted in the revised schedule.

This convention preserves a substantial piece of the information provided in the amortization schedule at $t = 0$ at the cost of giving up the principle that payments should be (theoretically) constant. There is no settled consensus on which convention is preferable, but the same-principal convention shifts all the risk from the variability of the interest rate exclusively on the interest payments.

Returning to our example, the first payment is still $P_1 = 1000/a_{\overline{60}|0.005} = 19.33$ and the sequence of principal components from the original amortization plan verifies A2. Under the same-principal convention, this sequence is used as the starting point for the revised amortization schedule. At $t = 2$, when the monthly interest changes to 0.0055, this is used to recompute the interest component I_2 on the outstanding debt D_1. Then, the full payment expected from the borrower is $P_2 = I_2 + C_2$, where C_2 is the same as in the original amortization plan while I_2 and thus P_2 have changed. More generally, after a change in the interest rate at time t, the new interest is used to compute I_t on the outstanding debt D_{t-1} and $P_t = I_t + C_t$, where C_t is still the same as in the original amortization plan.

4 GAP in face of contingencies

The previous section covers cases where the amortization contract is subject to risks that are expected and covered by its provisions. This section moves on to deal with (unforeseen) contingencies, where the GAP may be used to configure the space of options open to the parties. We leave it understood that an unforeseen contingency may lead to resolution of the contract or to one party's default. We focus instead on examples where the contract is kept alive by revising the initial amortization schedule. We consider respectively the aftermath of a stoppage in payments, and the more general case of debt restructuring.

4.1 Stoppage

Suppose that the borrower misses one full payment at some time τ, so that $P_\tau = 0$. (We postpone the case where the borrower delivers only a partial payment and $P_\tau > 0$.) This omission is usually a breach of the amortization plan; we seek to remedy it by saving the scope of the contract and revising the original schedule. Our discussion is general, but we emphasize that the legal provisions in some countries may restrict the set of feasible revisions.

Returning to our simple case, given $P_\tau = C_\tau + I_\tau = 0$, the borrower needs to make up both the principal component C_τ and the interest component I_τ. There is a variety of possible arrangements conforming to GAP. For instance, a penalty such as a default interest rate $i' > i$ may apply to compensate the lender for the borrower's delay concerning P_τ.

We illustrate two of the most common options, amenable to the same distinction (same-principle vs. same-principal) introduced above. For simplicity, we assume that payments restart one period later, at $\tau + 1$, the underlying amortization is based on constant payments at a fixed interest rate i, and that the interest rate after the stoppage stays unchanged.

Same-principle The missing payment is capitalized and rolled over to the outstanding debt in the next period $\tau + 1$. This is consistent with A4, as discussed at the end of this subsection. Given $P_\tau = 0$, from the recursive formula for the outstanding debt

$$D_\tau = D_{\tau-1} \cdot (1 + i) - P_\tau \tag{2}$$

it follows that

$$D_\tau = D_{\tau-1}(1 + i).$$

The new amortization plan is recomputed for the outstanding debt D_τ from period τ to the same maturity n in the original contract. For $t = \tau + 1, \ldots, n$, the new constant payment is

Table 5 Ex post amortization plan, after a stoppage at $\tau = 12$

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
...	...	...	...	...
11	19.33	4.27	15.07	838.34
12	0	4.19	−4.19	842.53
13	19.79	4.21	15.57	826.96
14	19.79	4.13	15.65	811.30
...	...	...	...	...
36	19.79	2.32	17.47	446.45
37	19.79	2.23	17.55	428.89
...	...	...	...	...
60	19.79	0.10	19.69	0.00

$$P'_t = D_\tau / a_{\overline{n-\tau}|i}$$

For instance, consider an amount of 1000 to be repaid over 60 monthly payments at the nominal interest rate $i^{(12)} = 6\%$. The amortization plan is given in Table 3. After a stoppage at $\tau = 12$, with $P_\tau = 0$, the outstanding debt at $\tau = 12$ increases to $D_{12} = D_{11} \cdot (1 + i) = 838.34 \cdot 1.005 = 842.53$.

The new constant-payment amortization plan that recovers the missing payment starts at $\tau = 13$ and the new payment is $P' = 842.53/a_{\overline{60-12}|0.005} = 19.79$. Table 5 provides the ex post amortization plan, juxtaposing the original schedule until $t = 11$, the missed payment in $\tau = 12$, and the revised schedule from $t = 13$ onward. The stoppage causes the outstanding debt to increase at $\tau = 12$. Consistent with A4, because the (null) payment at $t = 12$ cannot cover the due interest, the residual applied to the principal component is negative. GAP still hold, as in the previous example about the same-principle convention.

Same-principal A different option is to return as fast as possible to the constant payments of the original amortization plan. The quickest route, after $P_\tau = 0$, is to set the new payment to $P'_{\tau+1} = P_\tau + P_{\tau+1}$; we use a prime to denote the values revised after the stoppage. Indeed, by (2), we have

$$D_{\tau+1} = D_{\tau-1} \cdot (1 + i)^2 - P_\tau \cdot (1 + i) - P_{\tau+1}.$$

Given $P_\tau = 0$, this can be rewritten as

$$D'_{\tau+1} = D_{\tau-1} \cdot (1 + i)^2 - P'_{\tau+1}.$$

Therefore, $D'_{\tau+1} = D_{\tau+1}$ if and only if $P'_{\tau+1} = P_\tau \cdot (1 + i) + P_{\tau+1}$. With obvious meaning, we call this the same-principal convention.

Returning to our example, suppose again a stoppage at $\tau = 12$ with $P_\tau = 0$. As before, the missing payment is capitalized and rolled over to the outstanding debt

Table 6 Another ex post amortization plan, after a stoppage at $\tau = 12$

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
...	...	...	...	...
11	19.33	4.27	15.07	838.34
12	0	4.19	−4.19	842.53
13	38.76	4.21	34.55	807.98
14	19.33	4.04	15.29	792.69
...	...	...	...	...
36	19.33	2.27	17.07	436.20
...	...	...	...	...
60	19.33	0.10	19.24	0.00

in the next period so that $D_{12} = 842.53$. Following the same-principal convention, the missing payment is recovered entirely at $\tau = 13$ setting $P'_{13} = 19.33 \cdot 1.005 + 19.33 = 38.76$. The new payment P'_{13} is decomposed into the interest component $I_{13} = 0.005 \cdot 842.53 = 4.21$ and the principal component $C_{13} = D_{12} - D_{13} = 842.53 - 807.98 = 34.55$. Consistent with A3 and A4, $P'_{13} = C_{13} + I_{13}$. Table 6 shows the amortization plan that, from $t = 14$ onward, repeats the same amounts as the original plan in Table 3.

Moving beyond the special case of a zero payment, there usually arises an unexpected contingency whenever the borrower pays any amount $0 \leq P < P_\tau$ in some period τ. The discussion above extends straightforwardly, with a distinction. By A4, the amount P is applied first to the interest component.

If $P < I_\tau$, then the payment cannot even cover the interest I_τ. Therefore, part of I_τ must be carried over to subsequent payments and thus the outstanding debt becomes higher: $D'_\tau > D_{\tau-1}$. We have seen this increase in both examples above, with a zero payment. This temporary increase in the outstanding debt is usually known as *negative amortizaton*.

If $P \geq I_\tau$, then P is large enough to cover the interest payment in period τ and thus the outstanding debt is still decreasing: $D_{\tau-1} > D'_\tau > D_\tau$. Therefore, negative amortization does not occur.

4.2 Restructuring

It is possible to view the revision of the amortization schedule after an (unexpected) stoppage as an instance of debt restructuring, but we conveniently draw the following distinction. A stoppage is a discontinuation in payments that comes without a forewarning; whereas a restructuring initiates after the borrower signals to the lender that he expects to face difficulties in meeting his future obligations, with the purpose of avoiding unforetold interruptions.

For simplicity, we assume that before the payment P_τ comes due, the borrower informs the lender that he cannot pay it in full at time τ. The parties seek to agree

Table 7 Longer maturity

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
…	…	…	…	…
11	19.33	4.27	15.07	838.34
12	15.98	4.19	11.79	826.55
13	15.98	4.13	11.85	814.70
…	…	…	…	…
60	15.98	1.00	14.98	185.67
…	…	…	…	…
72	15.98	0.08	15.90	0.00

on a different amortization schedule that is compatible with the borrower's distress. Assuming constant payments, two typical arrangements are the following. First, the parties may extend the maturity of the contract to a longer (finite) date, increasing the number of payments but preserving the rate of interest and the length of each period. Second, the parties may keep the same maturity and agree to postpone payments for a number of periods; when payments restart, a higher rate of interest applies.

Longer maturity The borrower informs the lender of his financial difficulties during the period between $\tau - 1$ and τ; then the outstanding debt to be restructured is $D_{\tau-1}$. If they agree to increase the number of payments from n to n', the new (constant) payment is $P' = D_{\tau-1}/a_{\overline{n'-(\tau-1)}|i}$.

Freeze in payments Alternatively, the parties may agree to skip $k < n-\tau$ payments and restart the amortization at the end of period $\tau + k$ until the maturity n, using a higher interest rate $i' > i$. Then the number of nonzero payments that remain to be made is $n-(\tau-1)-k$. Assuming that the parties agree to apply the former interest rate i until payments resume, the new constant payment is $P' = D_{\tau-1}(1+i)^k/a_{\overline{n-(\tau-1)-k}|i'}$. (It is a simple matter to replace i' for i if the new interest rate applies from τ on.)

We illustrate these two simple cases returning to the example where an amount of 1000 is to be repaid over $n = 60$ monthly payments at the nominal interest rate $i^{(12)} = 6\%$; see the amortization schedule in Table 3. Assume that the restructuring begins at $\tau = 12$, and that the borrower has made regular payments up to $\tau - 1 = 11$.

Suppose $n' = n + 12 = 72$: the original maturity is extended by 12 months and the number of payments still to be made is $n' - (\tau - 1) = 61$. We have $P' = 838.34/a_{\overline{61}|0.005} = 15.98$, for $t = 12, 13, \cdots, 72$. Table 7 shows the amortization schedule. Since the reimbursement of the loan is delayed, the outstanding debt and the interest components from $t = 12$ onward are higher than in the original plan of Table 3.

If instead the parties agree to skip $k = 12$ payments and keep the original maturity, then there remain $60 - (12 - 1) - 12 = 37$ payments to be made. Assuming that the nominal interest rate is raised to $i^{(12)\prime} = 6.6\%$, the new payment is $P' = 838.34 \cdot$

Table 8 Freeze in payments

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
...	...	...	...	...
11	19.33	4.27	15.07	838.34
12	0	4.64	−4.64	842.95
13	0	4.61	−4.61	847.59
...	...	...	...	...
23	0	4.90	−4.90	895.37
24	26.81	4.92	21.89	873.49
...	...	...	...	...
60	26.81	0.15	26.66	0.00

$1.0055^{12}/a_{\overline{37}|0.0055} = 26.81$, to be paid on $t = 24, 25, \cdots, 60$. Again, the delay in reimbursement induces higher interest components as shown in Table 8.

Two tranches In general, debt restructuring aims to balance a temporary reduction in the borrower's outflow with a straightforward amortization plan. The two previous examples illustrate especially simple options. Another possibility is to split the reimbursement still due into two *tranches* with constant payments, where each payment from the second tranche is proportionally higher than each payment in the first tranche.

As before, assume that the outstanding debt to be restructured is $D_{\tau-1}$ and that there are $n-(\tau-1)$ payments until maturity. Let m_1, m_2 be the number of payments in the first and second tranche, respectively; then $m_1 + m_2 = n - (\tau - 1)$. Given the constant payment P_s for tranche $s = 1, 2$, assume that $P_2 = \alpha P_1$, where $\alpha > 0$ is a given proportion. (For instance, if $\alpha = 2$, then P_2 is twice higher than P_1.)

Then the two payments P_1, P_2 satisfy the system

$$\begin{cases} P_2 = \dfrac{D_{\tau-1} \cdot (1+i)^{m_1} - P_1 s_{\overline{m_1}|i}}{a_{\overline{m_2}|i}} \\ P_2 = \alpha P_1 \end{cases}$$

which yields

$$P_1 = \frac{D_{\tau-1} \cdot (1+i)^{m_1} / a_{\overline{m_2}|i}}{\alpha + s_{\overline{m_1}|i} / a_{\overline{m_2}|i}} \tag{3}$$

Returning to the example where an amount of 1000 is to be repaid over $n = 60$ monthly payments at the nominal interest rate $i^{(12)} = 6\%$, assume again that the restructuring occurs at $\tau = 12$. The 49 future payments are split into a first tranche with $m_1 = 24$ constant payments P_1 and a second tranche with $m_2 = 25$ constant

Table 9 Two *tranches*

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	19.33	5.00	14.33	985.67
...	...	...	...	...
11	19.33	4.27	15.07	838.34
12	13.07	4.19	8.87	829.46
...	...	...	...	...
35	13.07	3.11	9.95	612.66
36	26.13	3.06	23.07	589.59
...	...	...	...	...
60	26.13	0.13	26.00	0.00

payments P_2. Given $\alpha = 2$, we obtain from (3) that

$$P_1 = \frac{D_{11} \cdot (1.005)^{24} / a_{\overline{25}|0.005}}{2 + s_{\overline{24}|0.005} / a_{\overline{25}|0.005}} = 13.07$$

and thus $P_2 = 26.13$. The amortization plan is drawn in Table 9.

5 Concluding remarks

This section ties two loose ends. First, we illustrate the use of GAP to draw an amortization plan tailored to a specific arrangement requested by the borrower. Second, for each of the four principles we exhibit an amortization contract that violates it; we argue that these (or other) counterexamples may be both theoretically and contractually feasible but that financial arrangements outside of GAP are definitely unusual.

5.1 GAP after tailored arrangements

Consider a borrower who uses the full loan for purchasing a long-lived asset and is required to depreciate its book value following a specific accounting rule. The borrower might ask the lender (e.g., for tax reasons) to draw an amortization plan where the principal components per period match the depreciation values.

Assume for simplicity that the purchase price of the asset is equal to the initial debt D_0, and the depreciation horizon covers n periods, after which the asset has a salvage value $D_n \geq 0$. Finally, we suppose that depreciation is carried out using the *declining balance method*, although the approach works with any another method; see f.i. Section 2.4.2 in Broverman (2017) for other depreciation formulas.

The declining balance method postulates that the depreciation amount per period is computed by applying a discount rate d to the current value D_{t-1}; that is, $C_t = dD_{t-1}$. Because $C_t = D_{t-1} - D_t$, this implies $D_t = (1 - d)D_{t-1}$ and, more generally,

Table 10 Amortization plan with settlement at n

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	250	50	200	800
2	200	40	160	640
3	160	32	128	512
...	...	...	...	...
9	41.94	8.39	33.55	134.22
10	140.93	6.71	134.22	0

Table 11 Amortization plan with settlement at $n + 1$

t	P_t	I_t	C_t	D_t
0	–	–	–	1000
1	250	50	200	800
2	200	40	160	640
3	160	32	128	512
...	...	...	...	...
9	41.94	8.39	33.55	134.22
10	33.55	6.71	26.84	107.37
11	112.74	5.37	107.37	0

$D_t = D_0(1-d)^t$. The salvage value at the end of the depreciation period is $D_n = D_0(1-d)^n > 0$.

The goal is to draw an amortization plan where the outstanding debt matches the depreciated value in the books. This goal obtains if we set the principal components equal to the depreciation amounts per period, and then use GAP to complete the schedule. Additional care is needed only for the salvage value $D_n > 0$.

If $D_n \approx 0$, the parties may conventionally agree that the debt is settled. Otherwise, the two simplest options are to settle the debt at the end of period n or $n+1$. Settlement in n requires a payment $P'_n = C'_n + I_n$, with $C'_n = C_n + D_n = D_{n-1}$ and $I_n = i\,D_{n-1}$. Settlement in period $n+1$ requires a payment $P_{n+1} = C_{n+1} + I_{n+1}$, with $C_{n+1} = D_n$ and $I_{n+1} = i\,D_n$. Either approach respects GAP.

For example, consider a loan (and a corresponding asset) with value 1000. The loan carries a yearly interest rate $i = 5\%$. The asset value is to depreciated in $n = 10$ years with annual discount rate $d = 20\%$, and salvage value $D_{10} = 1000 \cdot (1-0.20)^{10} = 107.37$.

Assuming full settlement at the end of period $n = 10$, we have $C'_{10} = D_9 = 1000 \cdot (1-0.20)^9 = 134.22$ and $I_{10} = 134.22 \cdot 0.05 = 6.71$, so that $P'_{10} = C_{10} + I_{10} = 134.22 + 6.71 = 140.93$. See Table 10.

Assuming full settlement at $n + 1 = 11$, we have $C_{11} = D_{10} = 107.37$ and $I_{11} = 107.37 \cdot 0,05 = 5.37$, so that $P_{11} = 107.37 + 5.37 = 112.74$. See Table 11.

5.2 Gaps in GAP

It is important to clarify that one can conceive legitimate amortization plans outside of the GAP perimeter. Our modest claim is that GAP distill a widespread consensus: In theory, the class of amortization schedules is large; in practice, only contracts that match GAP are unlikely to come under serious scrutiny. We test the boundaries of our claim by exhibiting four minimal counterexamples: each of them violates one GAP and can be completed preserving the remaining three GAP.

A1. At the end of each equal-length period t, the borrower is expected to pay back a fraction $(1/2)^t$ of the original debt. This simple provision implies that the original debt is repaid using an infinite sequence of payments, summing up to a converging series. Therefore, the maturity for this contract is not finite.

A2. The initial outstanding debt is divided into n equal payments $C = D_0/n$, but in each period $t = 1, \ldots, n$ the borrower pays back as principal component the real (inflation-adjusted) value of the original amount, computed as $C(1 + r_t)^t$ where r_t is the cumulative inflation rate over the period $[0, t]$. The sum of these principal components is (very likely) different from the outstanding debt. The plan can be completed preserving A1 and A3-A4.

It is worth emphasizing that GAP do not rule out inflation-adjusted amortization plans. A2 claims only that the sum of the principal components must equal the initial outstanding debt, which is usually interpreted in nominal terms. But one may draw an amortization schedule that keeps distinct the principal component (in nominal value), the inflation-adjusted correction component, and the interest component. It is reasonable to expect that future financial innovation may lead to amortization plans in real terms. When this happens, experience and practice may suggest a revision of GAP.

A3: the interest component at time t is computed over the principal component C_t for the whole period $[0, t]$; that is, assuming that the period interest rate is i, we have $I_t = C_t \cdot (it)$. This arrangement ignores the outstanding debt, in violation of A3. The plan can be completed using A1-A2 and A4. This theoretical construct has been suggested amidst legal controversies over anatocism in Italy, but to the best of our knowledge is not used in practice.

A4: every payment is initially attributed entirely to the principal component until the outstanding debt is zero, postponing the payment of the interest meanwhile matured; that is, $P_t = C_t$ until period τ with $\sum_{t=1}^{\tau} C_t = D_0$, followed by a final payment $P_{\tau+1} = I$ that settles the interest, regardless of how this is specifically calculated.

Declarations

Conflict of interest The authors did not receive support from any organization for the submitted work. They have no financial or proprietary interest to disclose. One of the authors is a member of the Editorial Board for *Decisions in Economics and Finance*.

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