Real analyticity of periodic layer potentials upon perturbation of the periodicity parameters and of the support

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Abstract. We prove that the periodic layer potentials for the Laplace operator depend real analytically on the density function, on the supporting hypersurface, and on the periodicity parameters.

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1. Introduction

This paper is devoted to the study of the dependence of the periodic simple and double layer potentials upon perturbation of the periodicity cell and of the support of the integration. A periodic version of potential theory has revealed to be a powerful tool to analyze boundary value problems for elliptic differential equations in spatially periodic domains. If one is interested into studying the behavior of the solutions of boundary value problems for the Laplacian in a periodic domain upon perturbation of the periodicity cell and of the shape of the domain, then one faces the problem of studying the behavior of the corresponding layer potentials upon the same perturbations.

In view of such an application, several authors have studied the dependence of the layer potentials upon domain perturbations. Potthast [15, 16] has obtained Fréchet differentiability results for the dependence of the layer potentials. Costabel and Le Louër [5] have analyzed the Fréchet differentiability of a class of boundary integral operators with pseudohomogeneous hypersingular and weakly singular kernels in the framework of Sobolev spaces. For elastic obstacle scattering, we mention Le Louër [13]. Also, Lanza de Cristoforis and collaborators have developed a method based on potential theory with the aim of proving real analyticity results in the framework of Schauder spaces for the dependence of the solutions of boundary value problems upon domain perturbations. In order to apply such a method, one has to verify the real analytic dependence of the layer potentials on both variation of the support of integration and on data. In [11, 12], Lanza de Cristoforis and Rossi have considered the layer potentials associated with the Laplace and the Helmholtz operators. In [4], instead, Dalla Riva and Lanza de Cristoforis have studied the case of layer potentials associated to a family of second order differential operators with constant coefficients. In Dalla Riva [2, 3] the author has considered the single layer potential corresponding to the fundamental solution of a given elliptic partial differential operator of order $2k$ with constant coefficients.

In order to introduce the problem, we fix $n \in \mathbb{N}\setminus\{0,1\}$. If $(q_{11},\dots,q_{nn}) \in]0,+\infty[^n$ we introduce a periodicity cell Q and a matrix q by setting

$$Q \equiv \prod_{j=1}^{n}]0, q_{jj}[\,, \qquad q \equiv \begin{pmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_{nn} \end{pmatrix}.$$

We also denote by $|Q|_n$ the n-dimensional measure of the fundamental cell Q, by ν_Q the outward unit normal to ∂Q, where it exists, and by q^{-1} the inverse matrix of q. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q. In order to construct periodic layer potentials, we replace the fundamental solution of the Laplace operator by a q-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n}\,,$$

where δ_{qz} denotes the Dirac measure with mass in qz (see *e.g.*, [8, p. 84]). The distribution $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|Q|_n 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x}$$

in the sense of distributions in $\mathbb{R}^n$ (see *e.g.*, Ammari and Kang [1, p. 53], [8, §3]). Moreover, $S_{q,n}$ is even, real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (see *e.g.*, [8, §3]). We now introduce the periodic layer potentials. We take a bounded open subset Ω_Q of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ such that $\overline{\Omega_Q} \subseteq Q$. For the definition of sets and functions of the Schauder class $C^{k,\alpha}$ ($k \in \mathbb{N}$) we refer, *e.g.*, to Gilbarg and Trudinger [7]. We set

$$\begin{aligned} v_q[\partial\Omega_Q,\mu](x) &\equiv \int_{\partial\Omega_Q} S_{q,n}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,, \\ w_q[\partial\Omega_Q,\mu](x) &\equiv -\int_{\partial\Omega_Q} DS_{q,n}(x-y)\cdot\nu_{\Omega_Q}(y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,, \end{aligned}$$

$$w_{q,*}[\partial\Omega_Q,\mu](x) \equiv \int_{\partial\Omega_Q} DS_{q,n}(x-y)\cdot\nu_{\Omega_Q}(x)\mu(y)\,d\sigma_y \qquad \forall x\in\partial\Omega_Q\,,$$

for all $\mu\in L^2(\partial\Omega_Q)$. Here above, the symbol ν_{Ω_Q} denotes the outward unit normal field to $\partial\Omega_Q$, $d\sigma$ denotes the area element on $\partial\Omega_Q$, and $DS_{q,n}(\xi)$ denotes the gradient of $S_{q,n}$ computed at the point $\xi\in\mathbb{R}^n\setminus q\mathbb{Z}^n$. The functions $v_q[\partial\Omega_Q,\mu]$ and $w_q[\partial\Omega_Q,\mu]$ are called the q-periodic simple (or single) and double layer potentials, respectively.

In order to consider the dependence of periodic layer potentials under shape perturbations, we need to introduce some notation. First, we find convenient to set $\widetilde{Q}\equiv\,]0,1[^n$ and to set $\tilde{q}$ equal to the $n\times n$ identity matrix. Then we take

$$\begin{aligned}&\alpha\in\,]0,1[\text{ and a bounded open connected subset }\Omega\text{ of }\mathbb{R}^n\text{ of class }C^{1,\alpha}\\&\text{such that }\mathbb{R}^n\setminus\overline{\Omega}\text{ is connected.}\end{aligned}\tag{1.1}$$

The symbol '$\overline{\cdot}$' denotes the closure. Then we consider a class of diffeomorphisms $\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$ from $\partial\Omega$ into their images contained in $\widetilde{Q}$ (see (1.2)). To define such a class, we take Ω as in (1.1) and a bounded open connected subset Ω' of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We denote by $\mathcal{A}_{\partial\Omega}$ and by $\mathcal{A}_{\overline{\Omega'}}$ the sets of functions of class $C^1(\partial\Omega,\mathbb{R}^n)$ and of class $C^1(\overline{\Omega'},\mathbb{R}^n)$ which are injective and whose differential is injective at all points of $\partial\Omega$ and of $\overline{\Omega'}$, respectively. One can verify that $\mathcal{A}_{\partial\Omega}$ and $\mathcal{A}_{\overline{\Omega'}}$ are open in $C^1(\partial\Omega,\mathbb{R}^n)$ and $C^1(\overline{\Omega'},\mathbb{R}^n)$, respectively (see, *e.g.*, Lanza de Cristoforis and Rossi [12, Lem. 2.2, p. 197] and [11, Lem. 2.5, p. 143]). Then we set

$$\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\equiv\{\phi\in\mathcal{A}_{\partial\Omega}:\phi(\partial\Omega)\subseteq\widetilde{Q}\},\quad \mathcal{A}^{\widetilde{Q}}_{\overline{\Omega'}}\equiv\{\Phi\in\mathcal{A}_{\overline{\Omega'}}:\Phi(\overline{\Omega'})\subseteq\widetilde{Q}\}.\tag{1.2}$$

If $\phi\in\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n\setminus\phi(\partial\Omega)$ has exactly two open connected components (see, *e.g*, Deimling [6, Thm. 5.2, p. 26]), and we denote by $\mathbb{I}[\phi]$ the bounded open connected component of $\mathbb{R}^n\setminus\phi(\partial\Omega)$. We denote by $\mathbb{D}_n(\mathbb{R})$ the space of $n\times n$ diagonal matrices with real entries and by $\mathbb{D}_n^+(\mathbb{R})$ the set of elements of $\mathbb{D}_n(\mathbb{R})$ with diagonal entries in $]0,+\infty[$.

Then for each triple (q,ϕ,θ) in $\mathbb{D}_n^+(\mathbb{R})\times(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega})\times C^{0,\alpha}(\partial\Omega)$ we denote by $V[q,\phi,\theta]$ the function in $C^{1,\alpha}(\partial\Omega)$ defined by

$$V[q,\phi,\theta](x)\equiv\int_{q\phi(\partial\Omega)}S_{q,n}(q\phi(x)-s)(\theta\circ\phi^{(-1)})(q^{-1}s)d\sigma_s\quad\forall x\in\partial\Omega\,,$$

and by $W_*[q,\phi,\theta]$ the function in $C^{0,\alpha}(\partial\Omega)$ defined by

$$W_*[q,\phi,\theta](x)\equiv\int_{q\phi(\partial\Omega)}DS_{q,n}(q\phi(x)-s)\cdot\nu_{q\mathbb{I}[\phi]}(q\phi(x))(\theta\circ\phi^{(-1)})(q^{-1}s)d\sigma_s \qquad \forall x\in\partial\Omega\,.$$

Similarly, for each triple (q,ϕ,θ) in $\mathbb{D}_n^+(\mathbb{R})\times(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}})\times C^{1,\alpha}(\partial\Omega)$ we denote by $W[q,\phi,\theta]$ the function in $C^{1,\alpha}(\partial\Omega)$ defined by

$$W[q,\phi,\theta](x) \equiv -\int_{q\phi(\partial\Omega)} DS_{q,n}(q\phi(x)-s)\cdot\nu_{q\mathbb{I}[\phi]}(s)(\theta\circ\phi^{(-1)})(q^{-1}s)d\sigma_s \qquad \forall x\in\partial\Omega\,.$$

The functions $V[q,\phi,\theta]$ and $W[q,\phi,\theta]$ are associated with the $q\phi$-pull backs on $\partial\Omega$ of the periodic simple layer potential and of the periodic double layer potential, respectively. The function $W_*[q,\phi,\theta]$, instead, is associated with the $q\phi$-pull back on $\partial\Omega$ of the normal derivative of the periodic simple layer potential. These functions are well known to intervene in the integral equations associated with periodic boundary value problems. We are interested in understanding the dependence of $V[q,\phi,\theta]$, $W[q,\phi,\theta]$, and $W_*[q,\phi,\theta]$ upon perturbation of (q,ϕ,θ), *i.e.*, of the periodicity matrix, the support of integration, and the density function. Hence, we pose the following question:

What can be said on the regularity of the maps $(q,\phi,\theta)\mapsto V[q,\phi,\theta]$, $(q,\phi,\theta)\mapsto W[q,\phi,\theta]$, and $(q,\phi,\theta)\mapsto W_*[q,\phi,\theta]$? (1.3)

Our work stems from that of Lanza de Cristoforis and Preciso [10] for the Cauchy integral operator, from that of Lanza de Cristoforis and Rossi [11, 12] for the Laplace and for the Helmholtz operator, and from that of Dalla Riva and Lanza de Cristoforis [4] for second order elliptic operators. Moreover this work can be seen as complement of [8], where it has been shown that periodic layer potentials associated with parameter dependent analytic families of fundamental solutions of second order differential operators with constant coefficients depend real analytically upon the density function and on a suitable parametrization of the supporting hypersurface and on the parameter. Furthermore, it generalizes a part of [14] where the authors have proven analyticity results for the double layer potential in dimension two for a specific perturbation of the periodicity cell, in order to study the longitudinal flow through a periodic array of cylinders.

In this paper, we answer to the question (1.3) by proving that the maps in (1.3) are real analytic (see Theorem 3.2).

2. Preliminary technical results

To prove the analyticity of the operators $V[\cdot,\cdot,\cdot]$, $W[\cdot,\cdot,\cdot]$, and $W_*[\cdot,\cdot,\cdot]$, we need the following results from Lanza de Cristoforis and Rossi [12, §2].

Lemma 2.1. *Let α, Ω be as in* (1.1)*. Then there exists $\beta\in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)$ such that $|\beta(x)|=1$ and $\beta(x)\cdot\nu_\Omega(x)>1/2$ for all $x\in\partial\Omega$.*

Lemma 2.2. *Let α, Ω be as in* (1.1)*. Let β be as in Lemma* 2.1*. Then the following statements hold.*

(i) *There exists $\delta_\Omega\in\,]0,+\infty[$ such that the sets*

$$\Omega_{\beta,\delta}\equiv\{x+t\beta(x) : x\in\partial\Omega,\ t\in\,]-\delta,\delta[\},$$

$$\Omega^+_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\ t \in]-\delta, 0[\},$$

are connected and of class $C^{1,\alpha}$, *and*

$$\partial\Omega_{\beta,\delta} = \{x + t\beta(x) : x \in \partial\Omega,\ t \in \{-\delta, \delta\}\},$$
$$\partial\Omega^+_{\beta,\delta} = \{x + t\beta(x) : x \in \partial\Omega,\ t \in \{-\delta, 0\}\},$$

and $\Omega^+_{\beta,\delta} \subseteq \Omega$ *for all* $\delta \in]0, \delta_\Omega[$.

(ii) *Let* $\delta \in]0, \delta_\Omega[$. *If* $\Phi \in \mathcal{A}_{\overline{\Omega_{\beta,\delta}}}$, *then* $\Phi_{|\partial\Omega} \in \mathcal{A}_{\partial\Omega}$.

(iii) *If* $\delta \in]0, \delta_\Omega[$, *then the set*

$$\mathcal{A}'_{\overline{\Omega_{\beta,\delta}}} \equiv \{\Phi \in \mathcal{A}_{\overline{\Omega_{\beta,\delta}}} : \Phi(\Omega^+_{\beta,\delta}) \subseteq \mathbb{I}[\Phi_{|\partial\Omega}]\}$$

is open in $\mathcal{A}_{\overline{\Omega_{\beta,\delta}}}$.

(iv) *If* $\delta \in]0, \delta_\Omega[$ *and* $\Phi \in C^{1,\alpha}(\overline{\Omega_{\beta,\delta}}, \mathbb{R}^n) \cap \mathcal{A}'_{\overline{\Omega_{\beta,\delta}}}$, *then* $\Phi(\Omega^+_{\beta,\delta})$ *is an open set of class* $C^{1,\alpha}$ *and* $\partial\Phi(\Omega^+_{\beta,\delta}) = \Phi(\partial\Omega^+_{\beta,\delta})$.

3. Analyticity of the integral operators associated with layer potentials

In this section, we prove our main result on the analyticity of the maps in (1.3). We first need to prove the following lemma which represents an intermediate step.

Lemma 3.1. *Let* α, Ω *be as in* (1.1). *Let* β *and* δ_Ω *be as in Lemma* 2.2. *Let*

$$\mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}} \equiv \mathcal{A}'_{\overline{\Omega_{\beta,\delta}}} \cap \mathcal{A}^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}} \quad \forall \delta \in]0, \delta_\Omega[.$$

Let $\eta \in]0, 1[$. *Then there exists* $\delta_\eta \in]0, \delta_\Omega[$ *such that for all* $\delta \in]0, \delta_\eta[$ *the map which takes*

$$(q, \Phi, \theta) \in \mathbb{D}^+_n(\mathbb{R}) \times \left(C^{1,\alpha}(\overline{\Omega_{\beta,\delta}}, \mathbb{R}^n) \cap \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}}\right) \times C^{0,\alpha}(\partial\Omega)$$

to the function $V^+[q, \Phi, \theta]$, *which is defined as*

$$V^+[q, \Phi, \mu](x) \equiv \int_{q\Phi(\partial\Omega)} S_{q,n}(q\Phi(x) - s)\left(\mu \circ \Phi^{(-1)}\right)(q^{-1}s)\,d\sigma_s \quad \forall x \in \overline{\Omega^+_{\beta,\delta}},$$

is real analytic from $\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{0,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$, *where*

$$\mathcal{O}(\eta) \equiv \left\{q \in \mathbb{D}^+_n(\mathbb{R}) :\ \max\{(q_{jj})^{-1} : j = 1, \dots, n\} < \eta^{-1}\right\},$$

$$\mathcal{U}_{\eta,\delta} \equiv \left\{\Phi \in \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}} \cap C^{1,\alpha}(\overline{\Omega_{\beta,\delta}}, \mathbb{R}^n) :\ \sup_{\overline{\Omega_{\beta,\delta}}} |\det(D\Phi)| < \eta^{-1}\right\}.$$

Proof. Let $\delta \in]0, \delta_\Omega[$. Next, we note that if

$$(q, \Phi) \in \mathbb{D}^+_n(\mathbb{R}) \times \left(C^{1,\alpha}(\overline{\Omega_{\beta,\delta}}, \mathbb{R}^n) \cap \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}}\right)$$

then

$$V^+[q, \Phi, \mu](x) = \int_{q\Phi(\partial\Omega)} S_{q,n}(q\Phi(x) - s)\left(\mu \circ \Phi^{(-1)}\right)(q^{-1}s)\,d\sigma_s$$

$$= \int_{\Phi(\partial\Omega)} S_{q,n}(q(\Phi(x)-y)) \left(\mu\circ\Phi^{(-1)}\right)(y)|\det q||q^{-1}\cdot \nu_{\mathbb{I}[\Phi_{|\partial\Omega}]}(y)|d\sigma_y$$

for all $\mu \in C^{0,\alpha}(\partial\Omega)$ and for all $x \in \Omega^+_{\beta,\delta}$. Then we set

$$\tilde{S}_n(q,x) \equiv S_{q,n}(qx) \qquad \forall x \in \mathbb{R}^n \setminus \mathbb{Z}^n .$$

We note that the $\tilde{q}$-periodic function $\tilde{S}_n(q,\cdot)$ is a $\tilde{q}$-periodic $\{0\}$-analog of the fundamental solution of the operator $\sum_{j=1}^n \frac{1}{q_{jj}}\frac{\partial^2}{\partial x_j^2}$, *i.e.*, a $\tilde{q}$-periodic tempered distribution such that

$$\sum_{j=1}^n \frac{1}{q_{jj}}\frac{\partial^2}{\partial x_j^2}\tilde{S}_n(q,\cdot) = \sum_{z\in\mathbb{Z}^n}\delta_z - 1\,,$$

in the sense of distributions (see [8, §1]). Then, if we set

$$\sigma_\#[q,\Phi](s) \equiv |\det q||q^{-1}\cdot(\nu_{\mathbb{I}[\Phi_{|\partial\Omega}]}\circ\Phi)(s)| \qquad \forall s\in\partial\Omega\,,$$

for all $(q,\Phi)\in\mathbb{D}^+_n(\mathbb{R})\times\mathcal{U}_{\eta,\delta}$, we can write

$$\begin{aligned}
&\int_{\Phi(\partial\Omega)} S_{q,n}(q(\Phi(x)-y)) \left(\mu\circ\Phi^{(-1)}\right)(y)|\det q||q^{-1}\cdot \nu_{\mathbb{I}[\Phi_{|\partial\Omega}]}(y)|d\sigma_y\\
&\quad = \int_{\Phi(\partial\Omega)} \tilde{S}_n(q,\Phi(x)-y)\left(\mu\circ\Phi^{(-1)}\right)(y)\sigma_\#[q,\Phi]\circ\Phi^{(-1)}(y)d\sigma_y\\
&\quad \equiv \widetilde{V}^+_{\tilde{q}}[q,\Phi,\mu](x) \qquad \forall x\in\overline{\Omega^+_{\beta,\delta}},
\end{aligned}$$

for all $(q,\Phi,\mu)\in\mathbb{D}^+_n(\mathbb{R})\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$. By Lanza de Cristoforis and Rossi [11, Lem. 3.3 and p. 166] and standard calculus in Banach spaces, we have that the map from $\mathbb{D}^+_n(\mathbb{R})\times\mathcal{U}_{\eta,\delta}$ to $C^{0,\alpha}(\partial\Omega)$ which takes a pair (q,Φ) to $\sigma_\#[q,\Phi]$ is real analytic. Now we note that by [9, Thm. 7] and [8, §3] the map from $\mathbb{D}^+_n(\mathbb{R})\times(\mathbb{R}^n\setminus\mathbb{Z}^n)$ to $\mathbb{R}$ which takes the pair (q,x) to $\tilde{S}_n(q,x)=S_{q,n}(qx)$ is real analytic. Moreover, as noted above, for all $q\in\mathbb{D}^+_n(\mathbb{R})$, the map $\tilde{S}_n(q,\cdot)$ is a $\tilde{q}$-periodic function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\sum_{j=1}^n\frac{1}{q_{jj}}\frac{\partial^2}{\partial x_j^2}\tilde{S}_n(q,\cdot)=\sum_{z\in\mathbb{Z}^n}\delta_z-1$ in the sense of distributions. Accordingly, one can readily verify that the assumptions (1.8) of [8, pp. 78, 79] are satisfied and thus we can apply the results of [8]. Hence, [8, Prop. 5.6, pp. 105, 106] implies that there exists $\delta_\eta\in]0,\delta_\Omega[$ such that for all $\delta\in]0,\delta_\eta[$ the map $\widetilde{V}^+_{\tilde{q}}[\cdot,\cdot,\cdot]$ is real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$, and thus the proof is complete. □

We can now deduce our main theorem on the analyticity of the periodic layer potentials upon the periodicity parameter, the shape, and the density. The proof follows the strategy exploited in Lanza de Cristoforis and Rossi [11, Thm. 3.12] and in [8, Thm. 5.10].

Theorem 3.2. *Let* α, Ω *be as in* (1.1). *Then the following statements hold.*

(i) *The map from* $\mathbb{D}^+_n(\mathbb{R})\times(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega})\times C^{0,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes a triple* (q,ϕ,θ) *to the function* $V[q,\phi,\theta]$ *is real analytic.*

(ii) *The map from* $\mathbb{D}_n^+(\mathbb{R}) \times (C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}) \times C^{1,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes a triple* (q, ϕ, θ) *to the function* $W[q, \phi, \theta]$ *is real analytic.*

(iii) *The map from* $\mathbb{D}_n^+(\mathbb{R}) \times (C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}) \times C^{0,\alpha}(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$ *which takes a triple* (q, ϕ, θ) *to the function* $W_*[q, \phi, \theta]$ *is real analytic.*

Proof. To prove statements (i) and (iii), it suffices to argue as in the proof of [8, Thm. 5.10] and to replace [8, Prop. 5.6] by Lemma 3.1. Instead, the proof of statement (ii), follows by the same argument as the one of the proof of [14, Lem. 4.2], with the combination of the proof of [14, Lem. 4.1] and Lemma 3.1. □

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