RESULTS IN ECONOMIC MODELS WITH EXTERNALITIES

PhD Thesis in Applied Mathematics

Vincenzo Platino

Thesis Committee:

Thesis Director:	Prof. Jean-Marc Bonnisseau	(University Paris 1 Panthéon-Sorbonne)
Rapporteur:	Prof. Paolo Siconolfi	(Columbia Business School)
Rapporteur:	Prof. Abderrahim Jourani	(University of Bourgogne)
Examinateur	Prof. Francis Bloch	(University Paris 1 Panthéon-Sorbonne)
Co-directeur	Prof. Agar Brugiavini	(University Ca Foscari)
Co-directeur	MCF Elena L. del Mercato	(University Paris 1 Panthéon-Sorbonne)

UNIVERSITÉ PARIS I – PANTHÉON SORBONNE UNIVERSITÉ CA FOSCARI DE VENISE
106 - 112 BOULEVARD DE L'HÔPITAL 75647 PARIS CEDEX 13
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Jean-Marc Bonnisseau

Composition du Jury

Jean-Marc Bonnisseau	Professeur à l'Université Paris I Panthéon Sorbonne	Directeur de Thése
Abderrahim Jourani	Professeur à l'Université de Bourgogne	Rapporteur
Paolo Siconolfi	Professeur à Columbia Business School	Rapporteur
Francis Bloch	Professeur à Université Paris 1 Panthéon-Sorbonne	Examinateur
Agar Brugiavini	Professeur à Université Ca'Foscari di Venezia	Co-directeur
Elena L. del Mercato	MCF à Université Paris 1 Panthéon-Sorbonne	Co-directeur

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Contributions sur les économies avec externalités**" by **Vincenzo Platino** of the requirements for the degree of **Docteur en sciences**.

Dated: January 2014

Supervisor: ______________________
Jean-Marc Bonnisseau

Readers: ______________________

____

UNIVERSITÉ PARIS I – PANTHÉON SORBONNE
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Author: **Vincenzo Platino**

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Signature of Author

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Introduction générale

La thèse porte sur des modèles économiques en présence d'*externalités*. En suivant Laffont (1988), nous donnons la définition suivante d'externalité.

> *"Une externalité est chaque "effet indirect" d'une activité de consommation ou d'une activité de production sur les préféerences individuelles, sur les possibilités de consommation ou les possibilités de production."*

"Effet indirect" signifie que l'effet est créé par un agent économique différent de celui qui est affecté, et que l'effet n'est pas produit par l'intermédiaire du système de prix. Par conséquence, le système des prix ne joue que le rôle d'égaler à l'équilibre l'offre globale et la demande globale.[1] La définition ci-dessus montre que la présence d'effets externes nécessite une nouvelle description des caractéristiques des agents, c'est-à-dire des préférences individuelles, des ensembles de consommation et des ensembles de production des producteurs.

La thèse se compose de trois chapitres. Le premier chapitre étudie les restrictions de testabilité d'un modèle spécifique avec des externalités et des biens publics. Dans le deuxième chapitre et le troisième chapitre, nous considérons un modèle d'équilibre général avec des externalités au niveau des préférences individuelles et des ensembles de production des producteurs. Dans le deuxième chapitre nous traitons l'existence d'un équilibre concurrentiel en utilisant un approche différentiable, et dans le troisième chapitre nous donnons un résultat de régularité.

Chapitre 1 - La consommation privée par rapport à la consommation publique dans les groupes : Tester la nature des biens à partir des données agrégées.

Les restrictions testables dans des modèles d'équilibre générale ont été étudiés par Brown and Matzkin (1996) et Chiappori et al. (2004). Les premiers résultats de testabilité dans un modèle

[1]Les externalités qui passent directement par le système de prix sont appelées externalités pécuniaires.

qui prend en compte des externalités et des biens publics, sont donnés par Browning and Chiappori (1998) pour un modèle des choix collectifs. En particulier, les auteurs considèrent un modèle de ménage (c'est-à-dire non-unitaire) dans lequel les décisions de consommation sont prises par deux membres du ménage et elles sont Pareto efficaces.

Récemment, en ce qui concerne l'analyse des décisions de consommation des ménages, le modèle des choix collectifs est devenu très populaire. La raison de cet intérêt est basé sur le fait que les individus qui font partie d'un ménage sont hétérogènes (c'est-à-dire ils ont des préférences différentes) et un processus de décision prend place dans le ménage. Dans le modèle unitaire, un ménage est considéré comme un seul décideur qui maximise ses préférences sous sa contrainte budgétaire. Cependant, il y a des preuves empiriques qui montrent que le modèle unitaire n'est pas suffisant pour modéliser les décisions d'un ménage. Le modèle unitaire est évidemment trop restrictive, car il est implicitement supposé que le ménage ait une seule préférence sur les biens de consommation, plutôt que supposer que les individus qui font partie du ménage aient des préférences individuelles.

Dans Browning and Chiappori (1998), on n'observe pas quels biens sont consommés en privé ou quels biens sont consommés publiquement dans le ménage. Les auteurs supposent que seulement les prix des biens et la demande agrégée, qui est générée par une distrubution de pouvoir entre les deux membres du ménage, sont observés publiquement. En utilisant une "approche" basée sur des techniques différentielles, les auteurs montrent que la demande globale est compatible avec un choix optimal au sens de Pareto s'elle répond à certaines restrictions sur une matrice "pseudo-Slutsky".

Ensuite, Chiappori and Ekeland (2006) généralisent le modèle précédent. Les auteurs montrent qu'en utilisant une approche "paramétrique" il n'est pas possible de tester le caractère privé ou publique de la consommation. Plus précisément, le modèle de consommation collective a exactement les mêmes implications testables que deux modèles spécifiques de consommation collective. Dans le premier modèle spécifique de consommation collective, tous les biens sont consommés publiquement au sein du ménage, et dans le deuxième modèle spécifique de consommation collective tous les biens sont consommés en privé par les membres du ménage.

Dans le Chapitre 1, en utilisant des restrictions "non-param'etriques" dérivée par Cherchye et al. (2007), nous donnons des exemples qui montrent que la nature publique ou privée de la consommation est testable. Ainsi, contrairement à la littérature précédente, nous constatons que l'approche "non-paramétrique" implique la testabilité du caractère privé ou publique de la consommation, même si on observe seulement la consommation agrégée du ménage. En outre, nous constatons que le modèle de ménage dans lequel tous les biens sont consommés publiquement est distinct du modèle dans lequel tous les biens sont consommés en privé par les membres du ménage. Plus précisément, un ensemble de données qui satisfait les restrictions testables pour le premier modèle de ménage, il ne satisfait pas nécessairement les restrictions testables pour le deuxième modèle de ménage, et vice versa.

Chapitre 2 - Economies de propriété privée avec externalités et l'existence de l'équilibre concurrentiel : une approche différentiable.

Dans le Chapitre 2, nous considérons un modèle d'économie de propriété privée avec des externalités de consommation et de production. En utilisant une approche différentiable, nous prouvons que l'ensemble des équilibres concurrentiels avec des consommations et des prix strictement positifs est non vide et compact.

Notre modèle d'externalités est basé sur les travaux de Laffont and Laroque (1972), Laffont (1977, 1978, 1988), dans lequel les préférences des individus et les technologies des entreprises dépendent des choix des autres individus et des choix des autres entreprises. Nous étudions une économie de propriété privée où la technologie de chaque entreprise est décrite par une fonction différentiable appelée fonction de transformation. Les préférences individuelles sont représentées par une fonction d'utilité. Les fonctions d'utilité et les fonctions de transformation sont affectés par la consommation des autres individus et par les choix de production des autres entreprises.

Les agents économiques (individus et entreprises) prennent le système de prix et les choix des autres comme donnés dans leur programme de maximisation. Le concept d'équilibre concurrentiel est un concept d'équilibre à la Nash et l'allocation d'équilibre doit être compatible avec les ressources initiales des agents. Ce concept couvre la notion classique d'équilibre en absence d'externalités. Le résultat principal du Chapitre 2 est le Théorème 2.8 qui énonce que pour toutes les dotations initiales, l'ensemble des équilibres concurrentiels avec des consommations et des prix strictement positifs est non vide et compact.

En suivant les travaux de Smale (1974, 1981) et les travaux plus récents de Villanacci and Zenginobuz (2005) et de Bonnisseau and del Mercato (2008), nous démontrons le Théorème 8 en utilisant : l'approche élargie (*extended approach*) de Smale, un démonstration par homotopie et la théorie du degré topologique *modulo* 2. L'approche élargie de Smale décrit les équilibres en termes d'équations en utilisant les conditions du premier ordre associées aux problèmes d'optimisation des agents économiques et les conditions d'équilibre sur les marché. L'idée de l'homotopie est que chaque économie avec des externalités est reliée par un arc à une économie sans externalités, et les equilibres bougent de manière continue tout au long de l'arc sans quitter la frontière.

Chapitre 3 - Régularité des économies de propriété privée avec externalités.

Dans le Chapitre 3, nous considérons des économies de propriété privée avec externalités de consommation et de production. Nous étudions des conditions suffisantes pour la régularité générique de ces économies.

Nous rappelons qu'une économie est régulière s'elle a un nombre fini d'équilibres et chaque équilibre dépend localement de manière continue ou différentiable des paramètres qui décrivent l'économie. Par conséquence, dans une économie régulière il est possible d'effectuer une analyse de statique comparative.
L'importance des économies régulières et les questions liées à l'approche globale de l'analyse des équilibres peuvent être retrouvés dans Smale (1981), Mas-Colell (1985), Balasko (1988).

En présence d'externalités, les économies régulières sont également importantes pour étudier des politiques d'amélioration au sens de Pareto et les restrictions testables qui sont encore des questions ouvertes et importantes.

Comme il a été démontré dans Bonnisseau and del Mercato (2010), dans le cas des externalités de consommation, la régularité peut échouer si les effets externes du second ordre sont trop forts. Donc, nous introduisons également une hypothèse supplémentaire sur les effets externes du second ordre sur les fonctions de transformation.

Dans le Chapitre 3, nous donnons deux exemples d'économie avec des externalités de production et une infinité d'équilibres pour toutes les dotations initiales. Dans les deux exemples, les fonctions de transformation satisfont nos hypothse sur les effets externes du second ordre. Ainsi, l'hypothèses de base et hypothèse supplémentaire sur les effets externes du second ordre mentionnées ci-dessus peuvent ne pas être suffisantes pour garantir le résultat de régularité. Par conséquent, nous introduisons des déplacements des frontières des ensembles de production, c'est-à-dire de simples perturbations des fonctions de transformation. Le résultat principal est le Théorème 3.19 qui énonce que presque toutes les économies perturbées sont régulières.

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Chapter 1

Private versus public consumption within groups: testing the nature of goods from aggregate data [1]

Abstract

We study the testability implications of public versus private consumption in collective models of group consumption. The distinguishing feature of our approach is that we start from a revealed preference characterization of collectively rational behavior. Remarkably, we find that assumptions regarding the public or private nature of specific goods do have testability implications, even if one only observes the aggregate group consumption. In fact, these testability implications apply as soon as the analysis includes three goods and four observations. This stands in sharp contrast with existing results that start from a differential characterization of collectively rational behavior. In our opinion, our revealed preference approach obtains stronger testability conclusions because it focuses on a global characterization of collective rationality, whereas the differential approach starts from a local characterization.

JEL classification: D11, D12, D13, C14.

Keywords: multi-person group consumption, collective model, revealed preferences, public goods, private goods, consumption externalities.

1.1 Introduction

Testable restrictions on the classical general equilibrium model have been widely studied in literature, see for example the seminal paper of Brown and Matzkin (1996), and Chiappori et al. (2004).

[1] **This Chapter is based on Cherchye et al. (2012) which has been presented at the "Dauphine Workshop in Economic Theory – Recent Advances in Revealed Preference Theory: testable restrictions in markets and games", Université Paris-Dauphine, France. So, Chapter 1 has also benefited from the comments of these audiences. We thank Georg Kirchsteiger and Paola Conconi for useful comments.**

The first testable restrictions in a model that involves externalities and public goods are provided by Browning and Chiappori (1998a) for a *collective consumption model*. More precisely, the authors consider a non-unitary household model in which the decisions taken by the two intra-household members are Pareto efficient.

In the last decades, the collective consumption model for the analysis of household decisions has become increasingly popular. The reasons for this interest stand in that individuals within a household are heterogeneous (i.e. they have different preferences) and an intra-household decision process takes place within a household. The standard *unitary model* considers a household as a single decision maker who maximizes his preferences subject to his budget constraint. The unitary model is obviously too restrictive, since it implicitly endows households, rather than individuals, with preferences over consumption goods. There exists empirical evidence showing that the unitary model does not hold for household decisions. In particular, the well-known properties of the classical demand function and especially the symmetry of the Slutsky matrix are often rejected.[2]

In Browning and Chiappori (1998a), one does not observe what goods are privately consumed and what goods are publicly consumed within the household. The authors assume that only prices and aggregate demand with respect to some power distribution between the two intra-household members are observed. Using a "parametric" approach based on differentiable techniques, they establish that for a two-person household, collectively rational group behavior requires a pseudo-Slutsky matrix that can be written as the sum of a symmetric negative semi-definite matrix and a rank one matrix. The symmetric negative semi-definite matrix is the classical Slutsky matrix, which measures the change in demand induced by the variation of prices and income. The rank one matrix measures the change in demand induced by the variation of power distribution. Furthermore, the authors show that a collective model with two intra-household members can be rejected if at least five goods are present in the economy.

Building further on the original work of Browning and Chiappori (1998a), Chiappori and Ekeland (2006) particularly focused on the testability conclusions regarding the private and public nature of group consumption. Their main conclusion is that, following a differential approach, **the private and public nature of consumption is not testable**. More precisely, the authors show that the collective consumption model has exactly the same testability implications as two more specific collective (benchmark) models. In the first benchmark model, all goods are publicly consumed within the household and in the second benchmark model, all goods are privately consumed within the

[2]See for example, Browning and Chiappori (1998b) and Browning and Chiappori (1998a).

household.

Differently from Browning and Chiappori (1998a), Cherchye et al. (2007) provide a "non-parametric" characterization of the collective consumption model in the tradition of Afriat (1967) and Varian (1982).[3] This approach does not rely on any functional specification regarding the group consumption process, and it typically focuses on revealed preference axioms (i.e. GARP or related axioms). In Cherchye et al. (2007), assuming positive externalities the authors derive necessary and sufficient conditions for a rationalization of a data set consistent with the collective consumption model. Furthermore, the authors show that it is sufficient to have a data set with three observations and three goods to reject collective rationality for a household with two members.

In Chapter 1, using the "non-parametric" restrictions found by Cherchye et al. (2007), we complement the results of Chiappori and Ekeland (2006). In particular, we provide examples showing that **the private and public nature of consumption have testable implications**. So, in contrast to the findings for the differential approach, we will conclude that our revealed preference approach does imply testability of privateness versus publicness of consumption, even if one only observes the aggregate group consumption. In addition, we will obtain that the model with all consumption public is independent from (or non-nested with) the model that assumes all consumption is private and preferences are egoistic: a data set that satisfies the revealed preference conditions for the first model does not necessarily satisfy the conditions for the second model, and vice versa.

How can we interpret this difference between the testability conclusions of our approach and the ones of the differential approach? Our explanation is that Chiappori and Ekeland's differential approach focuses on 'local' conditions for collective rationality (which apply in a sufficiently small neighborhood of a given point). By contrast, the revealed preference conditions on which we focus are 'global' by construction.[4] In this interpretation, the global nature of the revealed preference conditions implies stronger testability conclusions. In fact, we believe our results may have interesting implications from the viewpoint of practical applications. For example, they suggest that a practitioner may usefully apply the revealed preference characterization to verify if the data satisfies a particular specification the collective model (in terms of publicly and/or privately consumed goods), prior to the actual empirical analysis.

Following a similar revealed preference approach, Cherchye et al. (2010) also considered testability

[3] See also Samuelson (1938), Houthakker (1950) and Diewert (1973) for seminal contributions on the revealed preference approach to analyzing consumption behavior.

[4] See for example Hurwicz (1971) and Pollak (1990) for discussions on the difference between the global revealed preference approach and the local differential approach.

of the private versus public nature of consumption within groups. A specific feature of their analysis is that it allowed for non-convex preferences of the individual group members. These authors obtain the same nontestability conclusion as Chiappori and Ekeland (2006). Our following analysis differs from the one of Cherchye et al. (2010) in that we assume that individual preferences are convex (and represented by concave utility functions). This assumption of convex preferences follows the original analysis of Chiappori and Ekeland. As indicated above, we now do obtain different testability implications under alternative assumptions on the (public or private) nature of goods. When comparing this to the findings of Cherchye et al. (2010), we conclude that the assumption of convex preferences is crucial for obtaining our testability conclusions.

The remainder of the paper unfolds as follows. To set the stage, Section 1.2 defines collectively rational group consumption behavior in terms of the (general and specific) collective models that we will consider. Section 1.3 discusses the revealed preference characterization of such rational behavior. Section 1.4 shows our testability results on public versus private consumption in the group. Section 1.5 summarizes our main conclusions. Finally, in Appendix we provide some technical details.

1.2 Collective rationality

Following Chiappori and Ekeland (2006, 2009), we will concentrate on three collective consumption models in what follows. We will consider the general collective model (general-CR) of Browning and Chiappori (1998a) as well as two specific benchmark models, i.e. the collective model with all goods public (public-CR) and the collective model with all consumption private and egoistic preferences (egoistic-CR). In this section we introduce the necessary concepts to study these three collective models.

Throughout, we consider groups (or households) that consist of two members.[5] We assume a group that purchases the (non-zero) N-vector of quantities $\mathbf{q} \in \mathbb{R}^N_+$ with corresponding prices $\mathbf{p} \in \mathbb{R}^N_{++}$. All quantities can be consumed privately, publicly, or both. For the general collective model, we will assume that the empirical analyst has no information on the decomposition of the observed $\mathbf{q}$ into the bundles of private quantities $\mathbf{q}^1, \mathbf{q}^2$ and the bundle of public quantities $\mathbf{q}^h$. Therefore, we need to introduce (unobserved) *feasible personalized quantities* $\widehat{\mathbf{q}}$ that comply with the (observed) aggregate quantities $\mathbf{q}$. More formally, we define

[5] The results below can be generalized towards the setting of M members, with $M \geq 2$. However, we believe that the core arguments underlying our results are better articulated for this simple case.

$$\widehat{\mathbf{q}} = \left(\mathfrak{q}^1, \mathfrak{q}^2, \mathfrak{q}^h\right) \text{ with } \mathfrak{q}^1, \mathfrak{q}^2, \mathfrak{q}^h \in \mathbb{R}^N_+ \text{ and } \mathfrak{q}^1 + \mathfrak{q}^2 + \mathfrak{q}^h = \mathbf{q}.$$

Each $\widehat{\mathbf{q}}$ captures a feasible decomposition of the aggregate quantities $\mathbf{q}$ into private quantities and public quantities. This will be useful for modeling general preferences that depend on private consumption as well as public consumption. In the following, we consider feasible personalized quantities because we assume the minimalistic prior that only the aggregate quantity bundle $\mathbf{q}$ and not the 'true' personalized quantities are observed. Throughout, we will use that each $\widehat{\mathbf{q}}$ defines a unique $\mathbf{q}$.

The collective model explicitly recognizes the individual (convex) preferences of the group members. For the general model, these preferences may depend not only on the own private quantities and the public quantities, but also on the other individual's private quantities. This allows for externalities between the group members. Formally, this means that the preferences of each group member m ($m = 1, 2$) can be represented by a well-behaved utility function of the form $U^m(\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^h)$, with $\mathbf{q} = \mathbf{q}^1 + \mathbf{q}^2 + \mathbf{q}^h$ and $m = 1, 2$.[6]

Suppose then that we observe T choices of N-valued bundles. For each observation t the vector $\mathbf{q}_t \in \mathbb{R}^N_+$ records the quantities chosen by the group under the prices $\mathbf{p}_t \in \mathbb{R}^N_{++}$ (with strictly positive components). We let $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ be the corresponding set of T observations.[7] A collective rationalization of a set of observations S requires the existence of utility functions U^1 and U^2 such that each observed quantity bundle can be characterized as Pareto efficient. Thus, we get the following definition.

Definition 1.1 (general-CR) *Let $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ be a set of observations. A pair of utility functions U^1 and U^2 provides a general-CR of S (i.e. a collective rationalization in terms of the general collective model), if for each observation t there exist feasible personalized quantities $\widehat{\mathbf{q}}_t$ such that $U^m(\widehat{\mathbf{z}}) > U^m(\widehat{\mathbf{q}}_t)$ implies $U^l(\widehat{\mathbf{z}}) < U^l(\widehat{\mathbf{q}}_t)$ ($m \neq l$) for all feasible personalized quantities $\widehat{\mathbf{z}}$ with $\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{z}$.*

The two benchmark cases considered below involve restrictions on the individual preferences and the nature of the goods. In the first case we assume that all consumption is public. We formalize this by assuming individuals preferences that are represented by a well-behaved utility function $U^m_{pub}(\mathbf{q}^h)$. Clearly, in this case we have $\mathfrak{q}^h = \mathbf{q}$ (or $\mathfrak{q}^1 + \mathfrak{q}^2 = \mathbf{0}$), i.e. the true personalized quantities are effectively observed. Given all this, Definition 1.1 directly leads to the following definition.

[6] As in the differential approach, we say that a function is well-behaved if it is concave, differentiable and monotonically increasing.

[7] For ease of exposition, the scalar product $\mathbf{p}_t'\mathbf{q}_t$ is written as $\mathbf{p}_t\mathbf{q}_t$.

Definition 1.2 (public-CR) *Let* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ *be a set of observations. A pair of utility functions* U^1_{pub} *and* U^2_{pub} *provides a public-CR of* S *(i.e. a collective rationalization in terms of the collective model with only public consumption), if for each observation* t *we have that* $U^m_{pub}(\mathbf{z}) > U^m_{pub}(\mathbf{q}_t)$ *implies* $U^l_{pub}(\mathbf{z}) < U^l_{pub}(\mathbf{q}_t)$ $(m \neq l)$ *for all* $\mathbf{z}$ *with* $\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{z}$*.*

The second benchmark case assumes that all consumption is private, i.e. $\mathfrak{q}^1 + \mathfrak{q}^2 = \mathbf{q}$ (or $\mathbf{q}^h = \mathbf{0}$). In addition, the individuals have egoistic preferences, which implies that they only care for their own consumption (i.e. no consumption externalities). We formalize this by assuming individual preferences that are represented by a well-behaved utility function $U^m_{ego}(\mathbf{q}^m)$, with $m = 1, 2$. The corresponding concept of collective rationality is as follows.

Definition 1.3 (egoistic-CR) *Let* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ *be a set of observations. A pair of utility functions* U^1_{ego} *and* U^2_{ego} *provides an egoistic-CR of* S *(i.e. a collective rationalization in terms of the collective model with all consumption private and egoistic preferences), if for each observation* t *there exist feasible personalized quantities* $\widehat{\mathbf{q}}_t$*, with* $\mathfrak{q}^h = \mathbf{0}$*, such that* $U^m_{ego}(\widehat{\mathbf{z}}) > U^m_{ego}(\widehat{\mathbf{q}}_t)$ *implies* $U^l_{ego}(\widehat{\mathbf{z}}) < U^l_{ego}(\widehat{\mathbf{q}}_t)$ $(m \neq l)$ *for all feasible personalized quantities* $\widehat{\mathbf{z}}$ *with* $\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{z}$ *and* $\mathfrak{z}^h = \mathbf{0}$*.*

1.3 Revealed preference characterization

Cherchye et al. (2007, 2011) derived the revealed preference characterizations for the three models discussed in the previous section. To formally define these revealed preference conditions, we will use the concept of *feasible personalized prices* $\widehat{\mathbf{p}}^1$ and $\widehat{\mathbf{p}}^2$.

$$
\begin{aligned}
\widehat{\mathbf{p}}^1 &= \left(\mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^h\right) \text{ and } \widehat{\mathbf{p}}^2 = \left(\mathbf{p} - \mathfrak{p}^1, \mathbf{p} - \mathfrak{p}^2, \mathbf{p} - \mathfrak{p}^h\right) \text{ with} \\
\mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^h &\in \mathbb{R}^N_+ \text{ and } \mathfrak{p}^c \leq \mathbf{p} \text{ for } c = 1, 2, h.
\end{aligned}
$$

This concept complements the concept of feasible personalized quantities defined above: $\widehat{\mathbf{p}}^1$ and $\widehat{\mathbf{p}}^2$ capture the fraction of the price for the personalized quantities $\widehat{\mathbf{q}}$ that is borne by the respective members. $\mathfrak{p}^1$ and $\mathfrak{p}^2$ refer to private quantities and are used to express the willingness to pay for the externalities related to these private quantities; $\mathfrak{p}^h$ refers to the public quantities and are similarly used to express the willingness to pay for the public quantities.

The revealed preference conditions make use of the *Generalized Axiom of Revealed Preference* (*GARP*). Varian (1982) introduced the *GARP* condition for individually rational behavior for observed prices and quantities; i.e. he showed that it is a necessary and sufficient condition for the observed quantity choices to maximize a single utility function under the given budget constraint.

We focus on the same condition in terms of feasible personalized prices and quantities; the next Proposition 1.5 will establish that collective rationality as defined in the above definitions requires *GARP* consistency for each individual member.

Definition 1.4 *Consider feasible personalized prices and quantities for a set of observations* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$*. For* $m = 1, 2$*, the set* $\{(\widehat{\mathbf{p}}_t^m, \widehat{\mathbf{q}}_t^m)\, ;\, t = 1, ..., T\}$ *satisfies* GARP *if there exist relations* R_0^m*,* R^m *that meet:*

(i) if $\widehat{\mathbf{p}}_s^m \widehat{\mathbf{q}}_s \geq \widehat{\mathbf{p}}_s^m \widehat{\mathbf{q}}_t$ *then* $\widehat{\mathbf{q}}_s\ R_0^m\ \widehat{\mathbf{q}}_t$*;*

(ii) if $\widehat{\mathbf{q}}_s\ R_0^m\ \widehat{\mathbf{q}}_u$*,* $\widehat{\mathbf{q}}_u\ R_0^m\ \widehat{\mathbf{q}}_v$*,* $\ldots$*,* $\widehat{\mathbf{q}}_z\ R_0^m\ \widehat{\mathbf{q}}_t$ *for some (possibly empty) sequence* $(u,\ v,\ ...,\ z)$ *then* $\widehat{\mathbf{q}}_s\ R^m\ \widehat{\mathbf{q}}_t$*;*

(iii) if $\widehat{\mathbf{q}}_s\ R^m\ \widehat{\mathbf{q}}_t$*, then* $\widehat{\mathbf{p}}_t \widehat{\mathbf{q}}_t \leq \widehat{\mathbf{p}}_t \widehat{\mathbf{q}}_s$*.*

We can now state the revealed preference characterization of the general collective model (i.e. general-CR) that is derived in Cherchye et al. (2007).

Proposition 1.5 *Let* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of well-behaved utility functions U^1 *and* U^2 *that provide a* general-CR *of* S;

(ii) there exist feasible personalized prices and quantities such that for each member $m = 1, 2$*, the set* $\{(\widehat{\mathbf{p}}_t^m, \widehat{\mathbf{q}}_t)\, ;\, t = 1, ..., T\}$ *satisfies* GARP*.*

Essentially, condition (ii) states that collective rationality requires individual rationality (i.e. *GARP* consistency) of each member in terms of personalized prices and quantities. In general, however, the *true* personalized prices and quantities are unobserved. Therefore, it is only required that there must exist at least one set of *feasible* personalized prices and quantities that satisfies the condition.

The characterization in Proposition 1.5 is easily adapted to the two benchmark cases considered in the previous section; see also Cherchye et al. (2011) for more discussion. For a public-CR of the data we need to include that all consumption is public. The implication is that only the willingness to pay for the public consumption will be relevant for the *GARP* test. This is contained in the following result.

Proposition 1.6 *Let* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of well-behaved utility functions U_{pub}^1 *and* U_{pub}^2 *that provide a* public-CR *of* S;

(ii) there exist feasible personalized prices and quantities, with $\mathfrak{q}_t^1 = \mathfrak{q}_t^2 = \mathbf{0}$*, such that for each member* $m = 1, 2$*, the set* $\{(\widehat{\mathbf{p}}_t^m, \widehat{\mathbf{q}}_t)\,;\, t = 1, ..., T\}$ *satisfies* GARP.

Similarly, for an egoistic-CR of the data we need to add to the second condition that all consumption is private (i.e. $\mathfrak{q}_t^h = \mathbf{0}$) and that the preferences are egoistic, implying that the willingness to pay for externalities is zero (i.e. $\mathfrak{p}_t^1 = \mathbf{p}_t$ and $\mathfrak{p}_t^2 = \mathbf{0}$).

Proposition 1.7 *Let* $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of well-behaved utility functions U_{ego}^1 *and* U_{ego}^2 *that provide an* egoistic-CR *of* S;

(ii) there exist feasible personalized prices, with $\mathfrak{p}_t^1 = \mathbf{p}_t$ *and* $\mathfrak{p}_t^2 = \mathbf{0}$*, and feasible personalized quantities, with* $\mathfrak{q}_t^h = \mathbf{0}$*, such that for each member* $m = 1, 2$*, the set* $\{(\widehat{\mathbf{p}}_t^m, \widehat{\mathbf{q}}_t)\,;\, t = 1, ..., T\}$ *satisfies* GARP.

1.4 Testing the nature of goods

We next show that the nature of goods is testable, even if we only observe the aggregate group behavior. More specifically, we will prove two main results by means of example data sets. Firstly, we provide data sets for which there exists a general-CR but not a public-CR or, respectively, an egoistic-CR. This implies that consistency with the general model does not necessarily imply consistency with any of the specific benchmark models. Putting it differently, rejection of the specific benchmark models in these examples is caused by the corresponding assumptions on the nature of the goods and not by the Pareto efficiency assumption as such. Secondly, our example data sets will show that the two benchmark models are independent from (or non-nested with) each other, i.e. data consistency with one benchmark model does not necessarily imply data consistency with the other benchmark model.

1.4.1 General-CR does not imply public-CR

The following example contains a data set for which there exists a general-CR but not a public-CR. The Appendix proves our claims in the examples.

Example 1 *Suppose that the dataset S contains the following 3 observations of bundles consisting of 3 quantities:*

$$\mathbf{q}_1 = (5,2,2)',\ \mathbf{q}_2 = (2,5,2)',\ \mathbf{q}_3 = (2,2,5)';$$
$$\mathbf{p}_1 = (4,1,1)',\ \mathbf{p}_2 = (1,4,1)',\ \mathbf{p}_3 = (1,1,4)'.$$

This dataset S satisfies the conditions in Proposition 1.5 (i.e. there exists a general-CR), but it rejects the conditions in Proposition 1.6 (i.e. there does not exist a public-CR).

This example has two important implications. Firstly, as discussed in the introduction, it contrasts with the results of Chiappori and Ekeland (2006): following a (local) differential approach, these authors show that the general collective model and the collective model with only public consumption are indistinguishable if one only observes aggregate group behavior. Example 1 illustrates that this is no longer the case if one adopts the (global) revealed preference approach.

Secondly, the example demonstrates that we need only three goods and three observations to obtain our conclusion. In fact, these numbers provide absolute lower bounds on the number of goods and observations for the collective models to have testable implications. Indeed, it can be verified that the conditions in Propositions 1.5 and 1.6 cannot be rejected if the number of observations or the number of goods is smaller than three.[8] Thus, as soon as collective rationality can be rejected, we can distinguish the specific model with all consumption public from the general collective consumption model. In this respect, it is also worth noting that the differential approach needs at least five goods for verifying the testable implications of the collective consumption model characterized in Propositions 1.6; see Browning and Chiappori (1998a) and Chiappori and Ekeland (2006). The fact that our revealed preference approach requires a smaller number of goods illustrates once more that the (global) revealed preference approach can yield stronger testability conclusions than the (local) differential approach.

1.4.2 General-CR does not imply egoistic-CR

We next provide an example with a data set for which there exists a general-CR but not an egoistic-CR.

[8] If $T = 2$, one can easily verify that $\mathfrak{p}_1^h = \mathbf{p}_1$ and $\mathfrak{p}_2^h = \mathbf{0}$ is a solution for the *GARP* conditions in Proposition 1.6 (and thus a fortiori also for the *GARP* conditions in Proposition 1.5). Next, if $N = 2$, one can again verify that member 1 paying for the first good (i.e. $(\mathfrak{p}_t^h)_1 = (\mathbf{p}_1)_1$) for all observations t and, similarly, member 2 paying for the second good (i.e. $(\mathfrak{p}_t^h)_2 = 0$) for all observations t obtains a solution for the *GARP* conditions in Proposition 1.6.

Example 2 *Suppose that the dataset S contains the following 4 observations of bundles consisting of 4 quantities:*

$$\mathbf{q}_1 = (1,0,0,0)',\ \mathbf{q}_2 = (0,1,0,0)',\ \mathbf{q}_3 = (0,0,1,0)',\ \mathbf{q}_4 = (0,0,0,1)';$$
$$\mathbf{p}_1 = (7,4,4,4)',\ \mathbf{p}_2 = (4,7,4,4)',\ \mathbf{p}_3 = (4,4,7,4)',\ \mathbf{p}_4 = (4,4,4,7)'.$$

This dataset S satisfies the conditions in Proposition 1.5 (i.e. there exists a general-CR), but it rejects the conditions in Proposition 1.7 (i.e. there does not exist an egoistic-CR).

Two remarks are in order. Similar to before, we conclude that the general collective model and the model with only private consumption and egoistic preferences are distinguishable from each other. Inter alia, this implies that the private nature of the goods is testable. Again, this conclusion contrasts with the one for the differential approach. Next, for mathematical elegance we have used four goods in Example 2.[9] Similar (but less elegant) examples exist for data sets that only consider three goods.

A final observation applies to the number of observations in Example 2. We have now used four observations, which contrasts with Example 1. In fact, in general we need minimally four observations for the collective model with private goods and egoistic preferences to be distinguishable from the general collective model. This result is formalized in the following proposition, which we prove in the Appendix.

Proposition 1.8 *Let $S = \{(\mathbf{p}_t, \mathbf{q}_t); t = 1,2,3\}$ be a set of three observations. Suppose that there exists a general-CR of S, then there also exists an egoistic-CR of S.*

1.4.3 Independence of egoistic-CR and public-CR

So far, we have shown that the general collective model is distinguishable from the two specific benchmark models. In the Appendix we argue that a similar conclusion also holds for the two benchmark cases. More precisely, we show that there exists an egoistic-CR for the data set considered in Example 1 and a public-CR for the data set considered in Example 2. Generally, this obtains that data consistency with one benchmark model does not necessarily imply data consistency with the other benchmark model.

Another interesting implication of this result is that we need no more than four observations and three goods to distinguish between the three collective consumption models under study. This conclusion directly carries over to 'intermediate' collective models that are situated between the two

[9] A similar qualification applies to the use of zeroes in Example 2.

benchmark cases, i.e models which assume that part of the goods is privately consumed (without externalities) while all other goods are publicly consumed. See Cherchye et al. (2011) for a detailed discussion (including revealed preference characterizations) of these intermediate models.

1.5 Conclusions

We have shown that the revealed preference approach implies different testability conclusions for collective consumption models with alternative assumptions on the (public or private) nature of goods. In particular, we obtain different testable implications as soon as we have three goods and four observations. Interestingly, these conclusions stand in sharp contrast with the existing results for the differential approach. As indicated before, our explanation is that we focus on revealed conditions that are global in nature, whereas the differential approach focuses on local testability conditions. As for practical applications, our results suggest that the practitioner may fruitfully apply revealed preference conditions to verify if the data satisfies a particular specification of the collective model that (s)he wants to use in the empirical analysis.

1.6 Appendix

1.6.1 Example 1

There exists a general-CR of S. Consider the following personalized quantities and prices:

$$\widehat{\mathbf{q}}_1 = (\mathbf{q}_1, \mathbf{0}, \mathbf{0}), \widehat{\mathbf{q}}_2 = (\frac{1}{2}\mathbf{q}_2, \frac{1}{2}\mathbf{q}_2, \mathbf{0}), \widehat{\mathbf{q}}_3 = (\mathbf{0}, \mathbf{q}_3, \mathbf{0});$$
$$\mathfrak{p}_t^1 = \mathbf{p}_1, \mathfrak{p}_t^2 = \mathbf{0} \text{ for } t = 1, 2, 3.$$

Then one can easily verify that the *GARP* conditions in Proposition 1.5 are satisfied for both members. This implies that there exists a general-CR of S.

There exists an egoistic-CR of S. By Proposition 1.7 we can conclude that the above construction also shows that there exists an egoistic-CR of S.

There does not exist a public-CR of S. Let us prove this ad absurdum and assume that we have a construction of feasible prices that satisfy condition (ii) in Proposition 1.6.

Observe that for the given set of observations we have for any $t,s \in \{1,2,3\}$, with $t \neq s$, that $\mathbf{p}_t\mathbf{q}_t > \mathbf{p}_t\mathbf{q}_s$. Therefore we must have for our solution of feasible prices that either $\mathfrak{p}_t^h\mathbf{q}_t > \mathfrak{p}_t^h\mathbf{q}_s$ or $(\mathbf{p}_t - \mathfrak{p}_t^h)\mathbf{q}_t > (\mathbf{p}_t - \mathfrak{p}_t^h)\mathbf{q}_s$. As a result the *GARP* conditions in Proposition 1.6 require that if $\mathfrak{p}_t^h\mathbf{q}_t \geq \mathfrak{p}_t^h\mathbf{q}_s$, we must have that $\mathfrak{p}_s^h\mathbf{q}_s \leq \mathfrak{p}_s^h\mathbf{q}_t$ and thus $(\mathbf{p}_s - \mathfrak{p}_s^h)\mathbf{q}_s > (\mathbf{p}_s - \mathfrak{p}_s^h)\mathbf{q}_t$. Or, alternatively, if $\widehat{\mathbf{q}}_t R_0^1 \widehat{\mathbf{q}}_s$, then we must have $\widehat{\mathbf{q}}_s R_0^2 \widehat{\mathbf{q}}_t$. Given that this holds for any $t,s \in \{1,2,3\}$, with $t \neq s$, we may therefore conclude that, without losing generality, the solution of feasible prices leads to (i) $\widehat{\mathbf{q}}_1 R_0^1 \widehat{\mathbf{q}}_2$ and $\widehat{\mathbf{q}}_2 R_0^1 \widehat{\mathbf{q}}_3$ for member 1; and (ii) $\widehat{\mathbf{q}}_3 R_0^2 \widehat{\mathbf{q}}_2$ and $\widehat{\mathbf{q}}_2 R_0^2 \widehat{\mathbf{q}}_1$ for member 2.

Assume that $\mathfrak{p}_2^h = (\pi_1, \pi_2, \pi_3)'$. The *GARP* condition for member 1 in Proposition 1.6 requires that

$$\begin{aligned}\mathfrak{p}_2^h\mathbf{q}_2 \leq \mathfrak{p}_2^h\mathbf{q}_1 \quad &\Leftrightarrow \quad 2\pi_1 + 5\pi_2 + 2\pi_3 \leq 5\pi_1 + 2\pi_2 + 2\pi_3 \\ &\Leftrightarrow \quad 0 \leq \pi_1 - \pi_2.\end{aligned}$$

The *GARP* condition for member 2 in Proposition 1.6 requires that

$$\begin{aligned}(\mathbf{p}_2 - \mathfrak{p}_2^h)\mathbf{q}_2 \leq (\mathbf{p}_2 - \mathfrak{p}_2^h)\mathbf{q}_3 \quad &\Leftrightarrow \quad 2(1-\pi_1) + 5(4-\pi_2) + 2(1-\pi_3) \\ &\qquad \leq 2(1-\pi_1) + 2(4-\pi_2) + 5(1-\pi_3) \\ &\Leftrightarrow \quad 3 \leq \pi_2 - \pi_3.\end{aligned}$$

Together this implies that $3 \leq \pi_2 \leq \pi_1$, which gives us the wanted contradiction since by construction $\pi_1 \leq 1$. We thus conclude that there cannot exists a public-CR of the data set in Example 1.

1.6.2 Example 2

There exists a general-CR of S. Consider the following personalized quantities and prices:

$$\begin{aligned}&\widehat{\mathbf{q}}_1 = (\mathbf{0},\mathbf{0},\mathbf{q}_1), \widehat{\mathbf{q}}_2 = (\mathbf{0},\mathbf{0},\mathbf{q}_2), \widehat{\mathbf{q}}_3 = (\mathbf{0},\mathbf{0},\mathbf{q}_3), \widehat{\mathbf{q}}_4 = (\mathbf{0},\mathbf{0},\mathbf{q}_4); \\ &\mathfrak{p}_1^h = (6,2,2,2)', \mathfrak{p}_2^h = (4,3.5,0,0)', \mathfrak{p}_3^h = (4,4,3.5,0)', \mathfrak{p}_4^h = (2,2,2,1)'.\end{aligned}$$

Then one can easily verify that the *GARP* conditions in Proposition 1.5 are satisfied for both members. This implies that there exists a general-CR of S.

There exists a public-CR of S. By Proposition 1.6 we can conclude that the above construction also shows that there exists a public-CR of S.

There does not exist an egoistic-CR of S. Let us prove this ad absurdum and assume that we have a construction of feasible prices that satisfy condition (ii) in Proposition 1.7.

Again we observe that for the given set of observations we have for any $t, s \in \{1, 2, 3, 4\}$, with $t \neq s$, that $\mathbf{p}_t\mathbf{q}_t > \mathbf{p}_t\mathbf{q}_s$. Therefore, without losing generality, we can as before assume that the solution of feasible prices leads to (i) $\widehat{\mathbf{q}}_1 R_0^1 \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_2 R_0^1 \widehat{\mathbf{q}}_3$ and $\widehat{\mathbf{q}}_3 R_0^1 \widehat{\mathbf{q}}_4$ for member 1; and (ii) $\widehat{\mathbf{q}}_4 R_0^2 \widehat{\mathbf{q}}_3, \widehat{\mathbf{q}}_3 R_0^1 \widehat{\mathbf{q}}_2$ and $\widehat{\mathbf{q}}_2 R_0^2 \widehat{\mathbf{q}}_1$ for member 2.

Assume that $\mathfrak{q}_2^1 = (0, \alpha, 0, 0)$ and $\mathfrak{q}_3^1 = (0, 0, \beta, 0)$. The *GARP* conditions for the two members in Proposition 1.7 require that the following holds:

$$\begin{aligned}
\widehat{\mathbf{p}}_2^1 \widehat{\mathbf{q}}_2 \leq \widehat{\mathbf{p}}_2^1 \widehat{\mathbf{q}}_1 \quad &\Leftrightarrow \quad 7\alpha \leq 4; \\
\widehat{\mathbf{p}}_3^1 \widehat{\mathbf{q}}_3 \leq \widehat{\mathbf{p}}_3^1 \widehat{\mathbf{q}}_2 \quad &\Leftrightarrow \quad 7\beta \leq 4\alpha \leq 4; \\
\widehat{\mathbf{p}}_2^2 \widehat{\mathbf{q}}_2 \leq \widehat{\mathbf{p}}_2^2 \widehat{\mathbf{q}}_3 \quad &\Leftrightarrow \quad 7(1-\alpha) \leq 4(1-\beta) \leq 4; \\
\widehat{\mathbf{p}}_3^2 \widehat{\mathbf{q}}_3 \leq \widehat{\mathbf{p}}_3^2 \widehat{\mathbf{q}}_4 \quad &\Leftrightarrow \quad 7(1-\beta) \leq 4.
\end{aligned}$$

This implies that $\frac{3}{7} \leq \alpha \leq \frac{4}{7}$, $\frac{3}{7} \leq \beta \leq \frac{4}{7}$ and $\frac{7\beta}{4} \leq \alpha$ and thus also that $\alpha \geq \frac{3}{4}$. As such we obtain the wanted contradiction and we conclude that there cannot exist an egoistic-CR of the data set in Example 2.

1.6.3 Proof of Proposition 1.8

Example 1 of Cherchye et al. (2007) shows that there cannot exist a general-CR of S if we observe that $\mathbf{p}_1\mathbf{q}_1 \geq \mathbf{p}_1(\mathbf{q}_2+\mathbf{q}_3)$, $\mathbf{p}_2\mathbf{q}_2 \geq \mathbf{p}_2(\mathbf{q}_1+\mathbf{q}_3)$ and $\mathbf{p}_3\mathbf{q}_3 \geq \mathbf{p}_3(\mathbf{q}_1+\mathbf{q}_2)$ holds simultaneously. Without losing generality, we assume that $\mathbf{p}_2\mathbf{q}_2 < \mathbf{p}_2(\mathbf{q}_1+\mathbf{q}_3)$.

Consider the following personalized quantities and prices for an $\alpha \in [0, 1]$:

$$\begin{aligned}
&\widehat{\mathbf{q}}_1 = (\mathbf{q}_1, \mathbf{0}, \mathbf{0}), \widehat{\mathbf{q}}_2 = (\alpha\mathbf{q}_2, (1-\alpha)\mathbf{q}_2, \mathbf{0}), \widehat{\mathbf{q}}_3 = (\mathbf{0}, \mathbf{q}_3, \mathbf{0}); \\
&\mathfrak{p}_t^1 = \mathbf{p}_1, \mathfrak{p}_t^2 = \mathbf{0} \text{ for } t = 1, 2, 3.
\end{aligned}$$

These feasible prices and quantities are consistent with the collective model with only private goods (i.e. $\mathfrak{q}_t^h = \mathbf{0}$) and egoistic preferences (i.e. $\mathfrak{p}_t^1 = \mathbf{p}_t$ and $\mathfrak{p}_t^2 = \mathbf{0}$).

Given that $\mathbf{p}_2\mathbf{q}_2 < \mathbf{p}_2(\mathbf{q}_1+\mathbf{q}_3)$, there must exist an $\alpha \in [0, 1]$ such that $\alpha\mathbf{p}_2\mathbf{q}_2 < \mathbf{p}_2\mathbf{q}_1$ and $(1-\alpha)\mathbf{p}_2\mathbf{q}_2 < \mathbf{p}_2\mathbf{q}_3$. One can then easily verify that for such an α the *GARP* conditions in Proposition 1.7 are satisfied for both members. This implies that there exists an egoistic-CR of S. ■

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Chapter 2

Private ownership economy with externalities and existence of competitive equilibria: A differentiable approach[1]

Abstract

We consider a general model of private ownership economies with consumption and production externalities. Each firm is characterized by a technology described by a transformation function. Each household is characterized by a utility function, the shares on firms' profit and an initial endowment of commodities. Describing equlibria in terms of first order conditions and market clearing conditions, and using a homotopy approach based on the seminal work by Smale (1974), under differentiability and boundary conditions, we prove the non-emptiness and the compactness of the set of competitive equilibria with consumptions and prices strictly positive.

JEL classification: C62, D50, D62.

Keywords: externalities, production economies, competitive equilibrium, homotopy approach.

2.1 Introduction

We consider a general model of private ownership economies with consumption and production externalities. In a differentiable framework, our purpose is to prove the non-emptiness and the compactness of the set of competitive equilibria with consumptions and prices strictly positive.

[1]**This Chapter is based on del Mercato and Platino (2013b). We are indebted and very grateful to the participants of the "Public Economic Theory (PET 10) and Public Goods, Public Projects, Externalities (PGPPE) Closing Conference" in Bogazici University (2010), and of the "Fifth Economic Behavior and Interaction Models (EBIM) Doctoral Workshop on Economic Theory", in Bielefeld University (2010), for useful comments.**

Why do we care about the existence of equilibria from a differentiable viewpoint? The starting point of studying the set of *regular* economies is the non-empty and compact set of equilibria in a differentiable setting. The relevance of *regular* economies and issues related to the global approach of the equilibrium analysis can be found in Smale (1981), Mas-Colell (1985), Balasko (1988), Villanacci et al. (2002). So, because of the differentiable approach, this paper is a first step to study regular economies in production economies with externalities from a global viewpoint.

Our model of externalities is based on the seminal works by Laffont and Laroque (1972), Laffont (1977, 1978, 1988), where the choices of all households and firms affect individual preferences and production technologies. We consider a private ownership economy with a finite number of commodities, households and firms. Each firm is characterized by a technology described by an inequality on a differentiable function called the transformation function. Firms are owned by households. Each household is characterized by a consumption set which coincides with the strictly positive orthant of the commodity space, preferences, shares on firms' profit and an initial endowment of commodities. Individual preferences are represented by a utility function. Utility and transformation functions depend on the consumption of all households and on the production activity of all firms.

Facing a price, each firm chooses in his production set a production plan which solves his profit maximization problem taking as given the choices of the others, i.e., given the level of externality created by the other firms and households. Facing a price, each household chooses a consumption bundle which solves his utility maximization problem under the budget constraint taking as given the choices of the others, i.e., given the level of externality created by the other households and firms. The associated concept of competitive equilibrium is nothing else than an equilibrium *à la Nash*, the resulting allocation being feasible with the initial resources of agents. This notion includes as a particular case the classical equilibrium definition without externalities at all.

Our main result is Theorem 2.8 (Section 2.4) which states that for all strictly positive initial endowments, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. Following the seminal work by Smale (1974), and more recent contributions by Villanacci and Zenginobuz (2005) and Bonnisseau and del Mercato (2008), we prove Theorem 2.8 using:

(1) Smale's extended approach.
(2) Homotopy arguments.
(3) The topological *degree modulo* 2.

Smale's extended approach describes equilibria in terms of equations using the first order conditions associated to the individual optimization problems and market clearing conditions. This approach is used is different settings, such as incomplete markets, public goods and externalities, see for example Cass et al. (2001), Villanacci and Zenginobuz (2005), and del Mercato (2006). In the presence of externalities, this approach overcomes the following difficulty: the individual demand and supply functions depend on the individual demand and supply functions of the others, which depend on the individual demand and supply functions of the others, and so on. So, it would be problematic to define an aggregate supply and an aggregate excess demand which depend only on prices and initial endowments.

The homotopy idea is that any economy with externalities is connected by an arc to some economy without externalities at all. Along this arc, equilibria move in a continuous way without sliding off the boundary. In different settings, the homotopy approach is used, for instance, in Villanacci and Zenginobuz (2005), del Mercato (2006), Mandel (2008), Bonnisseau and del Mercato (2008) and Kung (2008).

Our homotopy approach is based on the topological *degree modulo* 2.[2]The degree modulo 2 is simpler than the *Brouwer degree* used in Mas-Colell (1985), which requires the concepts of *oriented manifold*.[3] The reader can find a brief review of the degree theory, for example, in Geanakoplos and Shafer (1990). In Section 2.6, we recall the definition and the fundamental properties of the degree modulo 2.

We now compare our contribution with previous works. The existence results by Laffont and Laroque (1972), Bonnisseau and Médecin (2001), and Mandel (2008) are more general than ours since in these works individual consumption sets or/and firms technologies are represented by correspondences. The contributions by Laffont and Laroque (1972), and Bonnisseau and Médecin (2001) are based on fixed point arguments. Furthermore, in Bonnisseau and Médecin (2001) and Mandel (2008), non-convexities are allowed on the production side. For that reason, their existence results involve the concept of *pricing rule* and more sophisticated techniques than those we use. In Mandel (2008), the author uses a homotopy approach which differs from ours for two main reasons, the author uses an excess demand approach and the Brouwer degree. In order to use an excess demand approach, the author has to enlarge the commodity space treating externalities as additional variables. In our mild context, we provide an existence proof simpler than the ones provided in Bonnisseau and Médecin (2001), and Mandel (2008).

[2]See Milnor (1965), Chapter 4.
[3]See also Milnor (1965), Chapter 5.

The paper is organized as follows. In Section 2.2, we present the model and the assumptions. In Section 2.3, the concept of competitive equilibrium is adapted to our economy. Then, we focus on the equilibrium function which is built on first order conditions associated with households and firms maximization problems, and market clearing conditions. In Section 2.4, we first present our main result Theorem 2.8 which states that for all initial endowments, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. Second, we provide the general homotopy theorem, namely Theorem 2.9, which is used to prove Theorem 2.8. In order to apply Theorem 2.9, in Subsection 2.4.1, fixing the externalities, we construct an appropriate private ownership economy that has a unique equilibrium and that is a regular economy. In Subsection 2.4.2, we provide our homotopy and its properties. All the lemmas are proved in Section 2.5. Finally, in Section 2.6, the reader can find a brief review on the degree *modulo* 2.

2.2 The model and the assumptions

There is a finite number C of physical commodities labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$ and a finite number H of households labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$. Each firm is owned by the households and it is characterized by a technology described by a transformation function. Each household is characterized by preferences described by a utility function, the shares on firms' profit and an endowment of commodities. Utility and transformation functions may be affected by the consumption choices of all households and by the production activities of all firms. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j, as usual output components are positive and input components are negative, $y_{-j} := (y_f)_{f \neq j}$ denotes the production plan of firms other than j, $y := (y_j)_{j \in \mathcal{J}}$.
- $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption, $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than h, $x := (x_h)_{h \in \mathcal{H}}$.
- Following Mas-Colell et al. (1995), the production set of firm j is described by an inequality on a function t_j called the *transformation function*. The transformation function is a convenient way to represent a production set using a function. We remind that, in the case of a single-output technology, the production set is commonly described by a *production function* f_j. That is, if $c(j) \in \mathcal{C}$ denotes the output of firm j, then the production function f_j gives the maximum amount of output that can be produced using a bundle of inputs $(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C)$. The

transformation function is the counterpart of the production function in the case of production processes which involve several outputs.

The main innovation of this paper comes from the dependency of the transformation function t_j with respect to the production activities of other firms and households consumption. So, we assume that t_j describes both the technology of firm j and the way firm j's technology is affected by the actions of the other agents. More precisely, given y_{-j} and x, the production set of the firm j is given by the following set,

$$Y_j(y_{-j}, x) := \left\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \leq 0 \right\}$$

where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$, $t := (t_j)_{j \in \mathcal{J}}$. In the particular case of a single-output technology, the transformation function of firm j is given by

$$t_j(y_j, y_{-j}, x) := y_j^{c(j)} - f_j(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C, y_{-j}, x) \tag{2.1}$$

where the dependency of the production function f_j with respect to the input amounts $(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C)$ has the usual meaning whereas the dependency with respect to (y_{-j}, x) simply means that the production function of firm j is affected by the actions of the other agents.

- Household h has preferences described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

  $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). So, u_h describes the way household h's preferences are affected by the actions of the other agents, $u := (u_h)_{h \in \mathcal{H}}$.
- $s_{jh} \in [0,1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j \in \mathcal{J}} \in [0,1]^J$ denotes the vector of the shares owed by household h; $s := (s_h)_{h \in \mathcal{H}} \in [0,1]^{JH}$. $S := \{s \in [0,1]^{JH} : \sum_{h \in \mathcal{H}} s_{jh} = 1, \forall j \in \mathcal{J}\}$ denotes the set of shares.
- $e_h := (e_h^1, .., e_h^c, .., e_h^C)$ denotes household h's endowment, $e := (e_h)_{h \in \mathcal{H}}$.
- $E := ((u, e, s), t)$ is a private ownership economy with externalities.
- p^c is the price of one unit of commodity c, $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote $w^{\backslash} := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

We make the following assumptions on the transformation functions.

Assumption 2.1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^1 function.

(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) \leq 0$.

(3) There is at least one commodity $c(j) \in \mathcal{C}$ *such that for every* $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $D_{y_j^{c(j)}} t_j(y'_j, y_{-j}, x) > 0$ *for all* $y'_j \in \mathbb{R}^C$.

(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, *the function* $t_j(\cdot, y_{-j}, x)$ *is* C^2 *and it is differentiably strictly quasi-convex, i.e., for all* $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ *is positive definite on* $\operatorname{Ker} D_{y_j} t_j(y'_j, y_{-j}, x)$.[4]

We remark that, fixing the externalities, the assumptions on t_j are standard in "smooth" general equilibrium models. Indeed, from Points 1 and 3 of Assumption 2.1 the production set is closed and smooth, from Point 4 of Assumption 2.1 it is convex. Point 2 of Assumption 2.1 states that inaction is possible. Point 3 of Assumption 2.1 means that t_j is strictly increasing with respect to some commodity $c(j)$. That is, it represents the "free disposal" property with respect to at least one commodity. In the particular case of a single-output technology, since the transformation function is given by (3.1), Point 3 of Assumption 2.1 is consistent with the fact that commodity $c(j)$ is the output of firm j. We also remark that we do not require any *strong convexity assumption* on the production sets, i.e., t_j is not required to be quasi-convex with respect to the externalities.

Let $e = (e_h)_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_{++}$ and $r = \sum_{h\in\mathcal{H}} e_h$, consider the set of feasible allocations

$$\mathcal{F}(r) := \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \mid t_j(y_j, y_{-j}, x) \leq 0, \forall j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r\}$$

and notice that $\mathcal{F}(r)$ is obviously non-empty by Point 2 of Assumption 2.1. However, Point 2 of Assumption 2.1 does not guarantee the non-emptiness of the following set

$$\mathcal{Z}(r) := \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \mid t_j(y_j, y_{-j}, x) = 0, \forall j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j = r\}$$

which is a necessary condition for the non-emptiness of the set of equilibrium allocations that belong to the boundary of all production sets and satisfy market clearing conditions. So, we make the following assumption.

Assumption 2.2 *For every* $r \in \mathbb{R}^C_{++}$, *the set* $\mathcal{Z}(r)$ *is non-empty.*

We remark that the assumption above is obviously satisfied if the production allocation $y = 0$ belongs the boundary of all production sets whatever is the consumption externality $x \in \mathbb{R}^{CH}_{++}$, which is an assumption of possibility of inaction stronger than Point 2 of Assumption 2.1.

[4] *Let* v *and* v' *be two vectors in* $\mathbb{R}^n$, $v \cdot v'$ *denotes the scalar product of* v *and* v'. *Let* A *be a real matrix with* m *rows and* n *columns, and* B *be a real matrix with* n *rows and* l *columns,* AB *denotes the matrix product of* A *and* B. *Without loss of generality, vectors are treated as row matrices and* A *denotes both the matrix and the following linear application* $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ *where* v^T *denotes the transpose of* v *and* $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. *When* $m = 1$, $A(v)$ *coincides with the scalar product* $A \cdot v$, *treating* A *and* v *as vectors in* $\mathbb{R}^n$.

For any given externality $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, we define the set of the production plans which belong to the production sets,

$$Y(x, y) := \{y' \in \mathbb{R}^{CJ} : t_j(y'_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J}\} \tag{2.2}$$

We make the following assumption which is in the same spirit as Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001), and Assumption P(3) in Mandel (2008).

Assumption 2.3 (Uniform Boundedness) *For every $r \in \mathbb{R}^{C}_{++}$, there exists a bounded set $C(r) \subseteq \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,*

$$Y(x, y) \cap \{y' \in \mathbb{R}^{CJ} : \textstyle\sum_{j \in \mathcal{J}} y'_j + r \gg 0\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 2.3.

Lemma 2.4

(1) For every $r \in \mathbb{R}^{C}_{++}$, there exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, the following set is included in $K(r)$.

$$A(x, y; r) := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : y' \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \leq r\}$$

(2) For every $r \in \mathbb{R}^{C}_{++}$, the set of feasible allocations $\mathcal{F}(r)$ is bounded.

It is well known that the boundedness of the set of feasible allocations is a crucial condition for the non-emptiness and the compactness of the equilibrium set. Fixing the externalities, from Assumption 2.3, one easily deduces that the set of feasible allocations is bounded. So, in this sense, Assumption 2.3 is standard. Assumption 2.3 also guarantees that the set of feasible allocations $A(x, y; r)$ is *uniformly bounded* with respect to any possible externality. In particular, it implies that the set of feasible allocations $\mathcal{F}(r)$ is bounded. However, for the non-emptiness of the equilibrium set it would not be sufficient to only assume the boundedness of the set of feasible allocations.[5] Lemma 2.4 is used to prove Steps 1.1 and 2.1 in the proof of Proposition 2.15, Section 2.5.

We make the following assumptions on the utility functions.

Assumption 2.5 *For all $h \in \mathcal{H}$,*

(1) The function u_h is continuous in its domain and it is C^1 in the interior of its domain.

(2) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiably strictly increasing, i.e., $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ for all $x'_h \in \mathbb{R}^{C}_{++}$.

[5] See also Bonnisseau and Médecin (2001) and Mandel (2008), where the authors need uniform boundedness assumptions in order to prove the non-emptiness of the equilibrium set.

(3) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$*, the function* $u_h(\cdot, x_{-h}, y)$ *is* C^2 *and it is differentiably strictly quasi-concave, i.e., for all* $x'_h \in \mathbb{R}^C_{++}$*,* $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ *is negative definite on* $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$*.*

(4) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ *and for every* $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$*,* $\mathrm{cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}^C_{++}$

Fixing the externalities, the assumptions on u_h are standard in "smooth" general equilibrium models. In particular, Point 4 of Assumption 2.5 is the classical Boundary Condition (BC) which means that u_h has upper counter sets with closure in $\mathbb{R}^C_{++}$. We notice that in Points 1 and 4 of Assumption 2.5, we consider consumption externalities x_{-h} on the boundary of the set $\mathbb{R}^{C(H-1)}_{++}$ in order to look at the limit of the behavior of u_h with respect to the consumption externalities. It means that BC is still valid whenever consumption externalities converge to zero for some commodity.[6] We also remark that we do not require any *strong convexity assumption* on the preferences, i.e., u_h is not required to be quasi-concave with respect to the externalities.

$\mathcal{T}$ denotes the set of t satisfying Assumptions 2.1, 2.2 and 2.3, $\mathcal{U}$ denotes the set of u satisfying Assumption 2.5, and $\mathcal{E} := \mathcal{U} \times \mathbb{R}^{CH}_{++} \times \mathcal{S} \times \mathcal{T}$ denotes the set of economies. From now on, $E = ((u, e, s), t)$ is any economy belonging to the set $\mathcal{E}$.

2.3 Competitive equilibrium and equilibrium function

In this section, we provide the definition of competitive equilibrium à la Nash and the notion of equilibrium function.[7]

Without loss of generality, commodity C is the "numeraire good". So, given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation, we denote $p := (p^{\setminus}, 1) \in \mathbb{R}^C_{++}$.

Definition 2.6 (Competitive equilibrium) $(x^*, y^*, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for the economy* E *if for all* $j \in \mathcal{J}$*,* y^*_j *solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y^*_{-j}, x^*) \leq 0 \end{aligned} \tag{2.3}$$

[6] See Step 2.2 in the proof of Proposition 2.15, Section 2.5.

[7] See Debreu (1952) for a game theoretical framework in which the preferences and the strategy set of an agent are affected by the choices of the others.

for all $h \in \mathcal{H}$, x_h^* solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ u_h(x_h, x^*_{-h}, y^*) \\ &\text{subject to } \ p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y_j^*) \end{aligned} \tag{2.4}$$

and (x^*, y^*) satisfies market clearing conditions, that is

$$\sum_{h \in \mathcal{H}} x_h^* = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y_j^* \tag{2.5}$$

Using the first order conditions, one easily characterizes the solutions of firms and households maximization problems. The proof of the following proposition is standard since in problems (2.3) and (2.4), each agent takes as given the price system and the actions of the other agents.

Proposition 2.7

(1) From Assumption 2.1, if y_j^* is a solution to problem (2.3), then it is unique and it is completely characterized by KKT conditions.[8]

(2) From Assumption 2.5, if x_h^* is a solution to problem (2.4), then it is unique and it is completely characterized by KKT conditions.

(3) As usual, from Point 2 of Assumption 2.5, household h's budget constraint holds with an equality. Thus, at equilibrium, due to the Walras law, the market clearing condition for commodity C is "redundant". So, one replaces condition (2.5) by $\sum_{h \in \mathcal{H}} x_h^{*\backslash} = \sum_{h \in \mathcal{H}} e_h^{\backslash} + \sum_{j \in \mathcal{J}} y_j^{*\backslash}$.

Let $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}^{C-1}_{++}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^{\backslash}) := ((x_h, \lambda_h)_{h \in \mathcal{H}}, (y_j, \alpha_j)_{j \in \mathcal{J}}, p^{\backslash})$ where λ_h denotes the Lagrange multiplier associated with household h's budget constraint, and α_j denotes the Lagrange multiplier associated with firm j's production constraint. We can now describe the competitive equilibria associated with the economy E using the *equilibrium function* $F : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F(\xi) := ((F^{h.1}(\xi), F^{h.2}(\xi))_{h \in \mathcal{H}}, (F^{j.1}(\xi), F^{j.2}(\xi))_{j \in \mathcal{J}}, F^M(\xi)) \tag{2.6}$$

where $F^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F^{h.2}(\xi) := -p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$, $F^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F^M(\xi) := \sum_{h \in \mathcal{H}} x_h^{\backslash} - \sum_{j \in \mathcal{J}} y_j^{\backslash} - \sum_{h \in \mathcal{H}} e_h^{\backslash}$.

$\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$ is an *extended equilibrium* for the E if and only if $F(\xi^*) = 0$. We remark that, by Proposition 3.7, $(x^*, y^*, p^{*\backslash})$ is a competitive equilibrium for E if and only if there exists (λ^*, α^*) such that ξ^* is an *extended equilibrium* for E. We simply call ξ^* an equilibrium.

[8] *From now on, "KKT conditions" means Karush–Kuhn–Tucker conditions.*

2.4 Existence and compactness

In this section, we show that the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. The result is provided by the following theorem.

Theorem 2.8 *The equilibrium set $F^{-1}(0)$ is non-empty and compact.*

In order to prove Theorem 2.8, following the seminal paper by Smale (1974) we use a homotopy approach, that is, the following theorem which is a consequence of the homotopy invariance of the topological degree. Following Chapter 4 in Milnor (1965), and more recent contributions by Villanacci and Zenginobuz (2005), and Bonnisseau and del Mercato (2008), our homotopy approach is based on the theory of degree *modulo 2*. The theory of degree *modulo 2* is simpler than the one used in Mas-Colell (1985) which requires the concepts of *oriented manifold* and the associated topological degree – the *Brouwer degree*.[9] The reader can find a brief review of the degree theory, for example, in Geanakoplos and Shafer (1990). In Section 2.6, we recall the definition and the fundamental properties of the degree *modulo 2*.[10]

Theorem 2.9 (Homotopy Theorem) *Let M and N be C^2 manifolds of the same dimension contained in euclidean spaces, $y \in N$ and $f, g : M \to N$ be such that f is a continuous function, g is a C^1 function, y is a regular value for g and $\#g^{-1}(y)$ is odd, there exists a continuous homotopy L from g to f such that $L^{-1}(y)$ is compact. Then,*

(1) $g^{-1}(y)$ *is compact and* $\deg_2(g, y) = 1$,
(2) $f^{-1}(y)$ *is compact and* $\deg_2(f, y) = 1$.

To apply Theorem 2.9, the equilibrium function F plays the role of the function f. In order to construct the function playing the role of the function g, we proceed as follows. We fix the externalities and we construct the so called "test economy". The test economy is an appropriate private ownership economy à la Arrow–Debreu without externalities at all. G is the equilibrium function associated with the test economy and it plays the role of g. In Subsection 2.4.1, we construct the test economy and the function G, and we provide the main properties of G in Proposition 2.12, i.e., $\#G^{-1}(0) = 1$ and 0 is a regular value of G. In Subsection 2.4.2, we provide the required homotopy H from G to F playing the role of the homotopy L, and Proposition 2.15 which states the compactness of $H^{-1}(0)$. Using Propositions 2.12 and 2.15, all the assumptions of Theorem 2.9 are satisfied, and so one gets the following lemma.

[9] Also see Milnor (1965), Chapters 4 and 5.

[10] The reader can find a survey of the theory of degree *modulo 2* in Villanacci et al. (2002).

Lemma 2.10 *$F^{-1}(0)$ is compact and* $\deg_2(F,0) = 1$.

Using Lemma 2.10 and the non-triviality property of the topological degree one gets $F^{-1}(0) \neq \emptyset$, and so Theorem 2.8 is completely proved.

Finally, we remark that if E is a *regular economy* (i.e., the equilibrium function F is C^1 and 0 is a regular value of F), then using Lemma 2.10 and the computation of the degree *modulo 2*, one obviously finds that, at a regular economy, the number of equilibria is finite and odd.[11] However, this paper does not address regularity issues. In the presence of externalities, the analysis of regular economies is a sensitive topics, see Bonnisseau (2003), Kung (2008), Mandel (2008), and Bonnisseau and del Mercato (2010). With regard to our model, in the context of the extended approach, it deserves a separate analysis, see del Mercato and Platino (2013a).

2.4.1 The "test economy" and its properties

We construct the test economy in two steps. First, fixing the externalities, we consider a standard production economy $\overline{E}$ and a Pareto optimal allocation of $\overline{E}$. Second, using the Second Welfare Theorem, we construct an appropriate private ownership economy $\widetilde{E}$ that has a unique equilibrium. $\widetilde{E}$ is the test economy and it is an economy without externalities at all.

Fix $(\overline{x}, \overline{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. Define $\overline{u}_h(x_h) := u_h(x_h, \overline{x}_{-h}, \overline{y})$ for all $h \in \mathcal{H}$, $\overline{t}_j(y_j) := t_j(y_j, \overline{y}_{-j}, \overline{x})$ for all $j \in \mathcal{J}$, and the production economy $\overline{E} := (\overline{u}, \overline{t}, \sum_{h\in\mathcal{H}} e_h)$ which is a standard production economy without externalities at all.

As stated in the following proposition, it is well known that, under Assumptions 2.1, 2.3 and 2.5, there exists a Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ of the economy $\overline{E}$ and Lagrange multipliers $(\widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ is completely characterized by the first order conditions for Pareto optimality.

Proposition 2.11 *There exists a Pareto optimal allocation $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ of the economy $\overline{E}$ and $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) = ((\widetilde{\beta}_j)_{j\in\mathcal{J}}, (\widetilde{\theta}_h)_{h\neq 1}, \widetilde{\gamma}) \in \mathbb{R}^{J}_{++} \times \mathbb{R}^{H-1}_{++} \times \mathbb{R}^{C}_{++}$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ is the unique solution to the following system.*

$$\left\{ \begin{array}{l} (1)\ D_{x_1}\overline{u}_1(x_1) = \gamma,\ \forall h \neq 1:\ (2)\ \theta_h D_{x_h}\overline{u}_h(x_h) = \gamma,\ (3)\ \overline{u}_h(x_h) = \overline{u}_h(\widetilde{x}_h) \\ \forall j \in \mathcal{J}:\ (4)\ \gamma = \beta_j D_{y_j}\overline{t}_j(y_j),\ (5)\ -\overline{t}_j(y_j) = 0,\ (6)\ \displaystyle\sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j = \sum_{h\in\mathcal{H}} e_h \end{array} \right. \tag{2.7}$$

[11]The computation of the degree *modulo 2* for C^1 functions and regular values is provided by Proposition 2.17, Section 2.6.

Also, it is well known that the Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ can be *supported* by some price system $\widetilde{p}$.[12] From system (2.7), one easily deduces a *supporting price* $\widetilde{p}$, a redistribution of initial endowments $\widetilde{e} = (\widetilde{e}_h)_{h\in\mathcal{H}}$ and the equilibrium equations satisfied by $(\widetilde{x}, \widetilde{y})$ for appropriate Lagrange multipliers $(\widetilde{\lambda}, \widetilde{\alpha}) \in \mathbb{R}^H_{++} \times \mathbb{R}^J_{++}$. More precisely, define $\widetilde{e}_h := \widetilde{x}_h - \sum_{j\in\mathcal{J}} s_{jh}\widetilde{y}_j$ and the function $G : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$G(\xi) := ((G^{h.1}(\xi), G^{h.2}(\xi))_{h\in\mathcal{H}}, (G^{j.1}(\xi), G^{j.2}(\xi))_{j\in\mathcal{J}}, G^M(\xi)) \tag{2.8}$$

where $G^{h.1}(\xi) := D_{x_h}u_h(x_h, \overline{x}_{-h}, \overline{y}) - \lambda_h p$, $G^{h.2}(\xi) := -p \cdot (x_h - \widetilde{e}_h - \sum_{j\in\mathcal{J}} s_{jh}y_j)$, $G^{j.1}(\xi) := p - \alpha_j D_{y_j}t_j(y_j, \overline{y}_{-j}, \overline{x})$, $G^{j.2}(\xi) := -t_j(y_j, \overline{y}_{-j}, \overline{x})$ and $G^M(\xi) := \sum_{h\in\mathcal{H}} x_h^{\backslash} - \sum_{j\in\mathcal{J}} y_j^{\backslash} - \sum_{h\in\mathcal{H}} \widetilde{e}_h^{\backslash}$.

G is nothing else than the equilibrium function associated with the private ownership economy $\widetilde{E} := ((\overline{u}, \widetilde{e}, s), \overline{t})$. Now, define $\widetilde{\xi} := (\widetilde{x}, \widetilde{\lambda}, \widetilde{y}, \widetilde{\alpha}, \widetilde{p}^{\backslash})$ with $\widetilde{p} := \frac{\widetilde{\gamma}}{\widetilde{\gamma}^C}$, $\widetilde{\lambda}_1 := \widetilde{\gamma}^C$, $\widetilde{\lambda}_h := \frac{\widetilde{\gamma}^C}{\widetilde{\theta}_h}$ for all $h \neq 1$ and $\widetilde{\alpha}_j := \frac{\widetilde{\beta}_j}{\widetilde{\gamma}^C}$ for all $j \in \mathcal{J}$. Using system (2.7), it is an easy matter to check that $G(\widetilde{\xi}) = 0$. As stated in the following lemma, $\widetilde{\xi}$ is the unique equilibrium for the economy $\widetilde{E}$ and $\widetilde{E}$ is a regular economy.

Proposition 2.12 *$G^{-1}(0) = \{\widetilde{\xi}\}$, G is C^1 and 0 is a regular value for G.*

Remark 2.13 *We remark that the economy $\widetilde{E}$ does not necessarily belong to the set of economies $\mathcal{E}$ since the initial endowment $\widetilde{e}_h$ is not necessarily strictly positive. However, at equilibrium, the individual wealth is equal to $\widetilde{p} \cdot \widetilde{x}_h$ which is strictly positive. One might consider different redistributions which also involve the shares and give rise to positive endowments.*[13] *The redistributions we consider do not involve the shares. So, we do not need to homotopize the shares (see the next subsection).*

2.4.2 The homotopy and its properties

The basic idea is to homotopize the endowments and the externalities by an arc from the equilibrium conditions associated with the test economy $\widetilde{E}$ to the ones associated with our economy E. But, one finds the following difficulty. At equilibrium, the individual wealth is positive at the beginning as well as at the end of the homotopy arc.[14] However, since the production sets are not required

[12] Using Debreu's vocabulary, $(\widetilde{x}, \widetilde{y})$ is an *equilibrium relative to some price system* $\widetilde{p}$, see Section 6.4 of Debreu (1959).

[13] *For example,* $\widetilde{s}_{jh} := \frac{\widetilde{p}\cdot\widetilde{x}_h}{\widetilde{p}\cdot\sum_{h\in\mathcal{H}}\widetilde{x}_h}$ *and* $\widetilde{e}_h := \widetilde{s}_{jh}\sum_{h\in\mathcal{H}} e_h$.

[14] At the economy E, the individual wealth is positive because of the possibility of inaction (Point 2 of Assumption 2.1) and standard arguments from profit maximization.

to be convex with respect to the externalities, the individual wealth may not be positive along the homotopy arc. Consequently, the individual budget constraint may be empty. We illustrate the details below.

Homotopize the endowments by a segment. Then, for every $\tau \in [0,1]$ the individual wealth is given by $p \cdot [\tau e_h + (1-\tau)\widetilde{e}_h] + p \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j$ which is equal to

$$p \cdot [\tau e_h + (1-\tau)\widetilde{x}_h] + p \cdot \sum_{j \in \mathcal{J}} s_{jh}[y_j - (1-\tau)\widetilde{y}_j]$$

So, the individual wealth is positive if $p \cdot y_j \geq p \cdot (1-\tau)\widetilde{y}_j$ for all $j \in \mathcal{J}$. Using standard arguments from profit maximization, at equilibrium, this condition is satisfied if $(1-\tau)\widetilde{y}_j$ belongs to the production set of firm j. On the other hand, if at same time, one homotopizes the externalities by a segment, the production set becomes

$$Y_j(\tau y_{-j} + (1-\tau)\overline{y}_{-j}, \tau x + (1-\tau)\overline{x})$$

But, one does not know whether or not the production plan $(1-\tau)\widetilde{y}_j$ belongs to the production set given above unless it satisfies *strong convexity assumptions*, i.e., the production set is also convex with respect to the externalities.

To overcome the difficulty described above, we define the homotopy H in two times using two homotopies Φ and Γ. Namely,

- in the first homotopy Φ, we homotopize the endowments **without** homotopizing the externalities,
- in the second homotopy Γ, we homotopize the externalities in preferences and production sets **without** homotopizing the endowments.

Remark 2.14 *We remark that,*

(1) Under strong convexity assumptions on the production side, endowments and externalities can be obviously homotopized at the same time.

(2) If the externalities are fixed, then only one homotopy is needed, namely the homotopy Φ. So, our homotopy proof covers the case in which the economy E is a standard private ownership economy without externalities at all.[15]

[15] *In the absence of externalities, in Chapter 9 of Villanacci et al. (2002), one finds a homotopy proof for classical private ownership economies. Our proof is simpler than the latter one since we do not homotopize the shares.*

Formally, define the following convex combinations

$$e_h(\tau) := \tau e_h + (1-\tau)\widetilde{e}_h,\ x(\tau) := \tau x + (1-\tau)\overline{x},\ y(\tau) := \tau y + (1-\tau)\overline{y} \tag{2.9}$$

and the homotopies $\Phi, \Gamma : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$,

$$\Phi(\xi,\tau) := ((\Phi^{h.1}(\xi,\tau), \Phi^{h.2}(\xi,\tau))_{h\in\mathcal{H}}, (\Phi^{j.1}(\xi,\tau), \Phi^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Phi^M(\xi,\tau)) \tag{2.10}$$

where $\Phi^{h.1}(\xi,\tau) = D_{x_h}u_h(x_h, \overline{y}_{-j}, \overline{x}) - \lambda_h p$, $\Phi^{h.2}(\xi,\tau) = -p\cdot[x_h - e_h(\tau) - \sum_{j\in\mathcal{J}} s_{jh}y_j]$, $\Phi^{j.1}(\xi,\tau) = p - \alpha_j D_{y_j}t_j(y_j, \overline{y}_{-j}, \overline{x})$, $\Phi^{j.2}(\xi,\tau) = -t_j(y_j, \overline{y}_{-j}, \overline{x})$, $\Phi^M(\xi,\tau) = \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h(\tau)^{\setminus}$.

$$\Gamma(\xi,\tau) := ((\Gamma^{h.1}(\xi,\tau), \Gamma^{h.2}(\xi,\tau))_{h\in\mathcal{H}}, (\Gamma^{j.2}(\xi,\tau), \Gamma^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Gamma^M(\xi,\tau)) \tag{2.11}$$

where $\Gamma^{h.1}(\xi,\tau) = D_{x_h}u_h(x_h, x_{-h}(\tau), y(\tau)) - \lambda_h p$, $\Gamma^{h.2}(\xi,\tau) = -p\cdot[x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh}y_j]$, $\Gamma^{j.1}(\xi,\tau) = p - \alpha_j D_{y_j}t_j(y_j, y_{-j}(\tau), x(\tau))$, $\Gamma^{j.2}(\xi,\tau) = -t_j(y_j, y_{-j}(\tau), x(\tau))$, $\Gamma^M(\xi,\tau) = \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\setminus}$.
Finally, define the homotopy $H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$,

$$H(\xi,\psi) := \begin{cases} \Phi(\xi, 2\psi) \text{ if } 0 \le \psi \le \frac{1}{2} \\ \Gamma(\xi, 2\psi - 1) \text{ if } \frac{1}{2} \le \psi \le 1 \end{cases}$$

The homotopy H is continuous since Φ and Γ are composed by continuous functions. Importantly, $H\left(\xi, \frac{1}{2}\right)$ is well defined since

$$\Phi(\xi,1) = \Gamma(\xi,0)$$

Furthermore, $H(\xi,0) = \Phi(\xi,0) = G(\xi)$ and $H(\xi,1) = \Gamma(\xi,1) = F(\xi)$. We conclude providing the following lemma.

Proposition 2.15 $H^{-1}(0)$ *is compact.*

2.5 Proofs

Proof of Lemma 2.4. Let $(x', y') \in A(x,y;r)$. Since $\sum_{h\in\mathcal{H}} x'_h \gg 0$, y' belongs to the bounded set $C(r)$ given by Assumption 2.3. Furthermore, for every $h \in \mathcal{H}$, $0 \ll x'_h \ll \sum_{h\in\mathcal{H}} x'_h \le \sum_{j\in\mathcal{J}} y'_j + r$. Thus, there exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $A(x,y;r) \subseteq K(r)$. Let $(x,y) \in \mathcal{F}(r)$. By definition, the allocation (x,y) belongs to the set $A(x,y;r)$. So, Point 1 of Lemma 2.4 implies that $\mathcal{F}(r) \subseteq K(r)$.

■

Proof of Proposition 2.11. Let $\overline{E}$ be the production economy defined in Subsection 2.4.1. Denote $r := \sum_{h\in\mathcal{H}} e_h$ and remind that $A(\overline{x},\overline{y};r) := \{(x',y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : \overline{t}_j(y'_j) \leq 0,\ \forall j \in \mathcal{J}$ and $\sum_{h\in\mathcal{H}} x'_h - \sum_{j\in\mathcal{J}} y'_j \leq r\}$. Consider the set $U(r) := \{(u'_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Im}\overline{u}_h \mid \exists (x',y') \in A(\overline{x},\overline{y};r)\colon \overline{u}_h(x'_h) \geq u'_h,\ \forall h \in \mathcal{H}\}$

By Point 2 of Assumption 2.1, the set U_r is non-empty. Fix $(u'_h)_{h\in\mathcal{H}} \in U(r)$ and consider the following maximization problem

$$\begin{array}{ll} \max\limits_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} & \overline{u}_1(x_1) \\ \text{subject to} & \overline{t}_j(y_j) \leq 0,\ \forall j \in \mathcal{J} \\ & \overline{u}_h(x_h) \geq u'_h,\ \forall h \in \mathcal{H} \\ & \sum\limits_{h\in\mathcal{H}} x_h - \sum\limits_{j\in\mathcal{J}} y_j \leq r \end{array} \tag{2.12}$$

Step 1. *Problem (2.12) has at least a solution.* Let K be the set determined by the constraints of problem (2.12). K is non-empty since $(u'_h)_{h\in\mathcal{H}} \in U_r$. We claim that K is compact. Define the set $N := \{(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}\colon \overline{u}_h(x_h) \geq u'_h,\ \forall h \in \mathcal{H}\}$ and notice that $K = N \cap A(\overline{x},\overline{y};r)$. So, K is bounded by Lemma 2.4. Furthermore, K is a closed set included in $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. Indeed, take a sequence $(x^\nu,y^\nu)_{\nu\in\mathbb{N}}$ in K converging to some (x,y). Since $(x^\nu,y^\nu)_{\nu\in\mathbb{N}} \subseteq N$, (x,y) belongs to the closure of N which is included in $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ by Point 4 of Assumption 2.5. So, $(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. Since the functions $\overline{u}_h$ and $\overline{t}_j$ are continuous, $(x,y) \in K$ which completes the proof of the claim. By Weierstrass' Theorem, there exists a solution to problem (2.12).

Step 2. *Let $(\widetilde{x},\widetilde{y})$ be a solution to problem (2.12). Then, $(\widetilde{x},\widetilde{y})$ solves the following problem and it is a Pareto optimal allocation of the economy $\overline{E}$.*

$$\begin{array}{ll} \max\limits_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} & \overline{u}_1(x_1) \\ \text{subject to} & -\overline{t}_j(y_j) \geq 0,\ \forall j \in \mathcal{J} \\ & \overline{u}_h(x_h) - \overline{u}_h(\widetilde{x}_h) \geq 0,\ \forall h \neq 1 \\ & r - \sum\limits_{h\in\mathcal{H}} x_h + \sum\limits_{j\in\mathcal{J}} y_j \geq 0 \end{array} \tag{2.13}$$

Let K_1 be the set determined by the constraints of problem (2.13), $(\widetilde{x},\widetilde{y})$ obviously belongs to K_1. Consider now $(x,y) \in K_1$ and remind that K is the set determined by the constraints of problem (2.12). If $\overline{u}_1(x_1) \geq u'_1$, then $(x,y) \in K$ and so $\overline{u}_1(\widetilde{x}_1) \geq \overline{u}_1(x_1)$. If $\overline{u}_1(x_1) < u'_1$, then $\overline{u}_1(\widetilde{x}_1) > \overline{u}_1(x_1)$ since $\overline{u}_1(\widetilde{x}_1) \geq u'_1$. Thus, $(\widetilde{x},\widetilde{y})$ solves problem (2.13). Now, suppose by contradiction that $(\widetilde{x},\widetilde{y})$ is not a Pareto optimal allocation of $\overline{E}$. Then, there is another allocation $(\widehat{x},\widehat{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $\bar{t}_j(\widehat{y}_j) \leq 0$ for all j, $\sum_{h\in\mathcal{H}} \widehat{x}_h \leq r + \sum_{j\in\mathcal{J}} \widehat{y}_j$, $\overline{u}_h(\widehat{x}_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all h, and $\overline{u}_k(\widehat{x}_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$. If $k = 1$, then we get a contradiction since $(\widetilde{x}, \widetilde{y})$ solves problem (2.13). If $k \neq 1$, using the continuity of $\overline{u}_k$, there exists $\varepsilon > 0$ such that $\overline{u}_k(\widehat{x}_k - \varepsilon\mathbf{1}^c) > \overline{u}_k(\widetilde{x}_k)$ where the vector $\mathbf{1}^c \in \mathbb{R}^C_+$ has all the components equal to 0 except the component c which is equal to 1. Consider the allocation (x, y) defined by $x_1 := \widehat{x}_1 + \varepsilon\mathbf{1}^c$, $x_k := \widehat{x}_k - \varepsilon\mathbf{1}^c$, $x_h := \widehat{x}_h$ for all $h \neq 1$, $h \neq k$, and $y_j := \widehat{y}_j$ for all j. $(x, y) \in K_1$ and $\overline{u}_1(x_1) > \overline{u}_1(\widetilde{x}_1)$ since $\overline{u}_1$ is strictly increasing. So, once again we get a contradiction since $(\widetilde{x}, \widetilde{y})$ solves problem (2.13).

Step 3. *Let $(\widetilde{x}, \widetilde{y})$ be a solution of problem (2.12). Then, there exists $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) := ((\widetilde{\beta}_j)_{j\in\mathcal{J}}, (\widetilde{\theta}_h)_{h\neq 1}, \widetilde{\gamma}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{H-1}_{++} \times \mathbb{R}^C_{++}$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ is the unique solution to system (2.7).* We first prove the existence of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$, afterwards we show the uniqueness of $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$.

Existence of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) \gg 0$. By Step 2, $(\widetilde{x}, \widetilde{y})$ solves problem (2.13). The KKT conditions associated with problem (2.13) are given by

$$\begin{cases} D_{x_1}\overline{u}_1(x_1) = \gamma, \ \forall h \neq 1 : \ \theta_h D_{x_h}\overline{u}_h(x_h) = \gamma, \ \theta_h(\overline{u}_h(x_h) - \overline{u}_h(\widetilde{x}_h)) = 0 \\ \forall j \in \mathcal{J} : \ \gamma = \beta_j D_{y_j}\bar{t}_j(y_j), \ \beta_j(-\bar{t}_j(y_j)) = 0, \ \forall c \in \mathcal{C} : \ \gamma^c(r^c - \sum_{h\in\mathcal{H}} x^c_h + \sum_{j\in\mathcal{J}} y^c_j) = 0 \end{cases} \tag{2.14}$$

where $(\beta, \theta, \gamma) := ((\beta_j)_{j\in\mathcal{J}}, (\theta_h)_{h\neq 1}, (\gamma^c)_{c\in\mathcal{C}}) \in \mathbb{R}^J_+ \times \mathbb{R}^{H-1}_+ \times \mathbb{R}^C_+$ are the Lagrange multipliers associated with the constraint functions of problem (2.13). We first claim that KKT conditions are necessary conditions to solve problem (2.13). It is enough to verify that the Jacobian matrix associated with the constraint functions of problem (2.13) has full row rank equal to $N := J + (H-1) + C$. The matrix given below is the $N \times N$ square sub-matrix which is obtained considering the partial derivatives of the constraint functions with respect to $((y^{c(j)}_j)_{j\in\mathcal{J}}, (x^1_h)_{h\neq 1}, x_1)$, where for every $j \in \mathcal{J}$, $c(j)$ denotes the commodity given by Point 3 of Assumption 2.1.[16] Point 3 of Assumption 2.1 and Point 2 of Assumption 2.5 imply that the determinant of this square sub-matrix is different from zero, which complete the proof of the claim.

$$\begin{bmatrix} -D_{y_1^{c(1)}}\bar{t}_1(y_1) & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -D_{y_J^{c(J)}}\bar{t}_J(y_J) & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & D_{x_2^1}\overline{u}_2(x_2) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & D_{x_H^1}\overline{u}_H(x_H) & 0 \\ \left[\mathbf{1}^{c(1)}\right]^T & \cdots & \left[\mathbf{1}^{c(J)}\right]^T & -\left[\mathbf{1}^1\right]^T & \cdots & -\left[\mathbf{1}^1\right]^T & -I_C \end{bmatrix}$$

Therefore, there exists $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) \geq 0$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ solves system (2.14). Furthermore, Point 3 of Assumption 2.1 and Point 2 of Assumption 2.5 imply that all the Lagrange multipliers $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ must be strictly positive. Consequently, all the constraints of problem (2.13) are binding, and so $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ is a

[16] We remind that for every commodity c, the vector $\mathbf{1}^c \in \mathbb{R}^C_+$ has all the components equal to 0 except the component c which is equal to 1.

solution to system (2.7).

Uniqueness of $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$. Define $\widetilde{\theta}_1 := 1$, by equations (1) and (2) of system (2.7), for all h one gets $D_{x_h}\overline{u}_h(\widetilde{x}_h) = \frac{\widetilde{\gamma}}{\widetilde{\theta}_h}$. So, for every h, $\widetilde{x}_h$ solves the maximization problem: $\max_{x_h \in \mathbb{R}^C_{++}} \overline{u}_h(x_h)$ subject to $\frac{\widetilde{\gamma}}{\widetilde{\theta}_h} \cdot x_h \leq \frac{\widetilde{\gamma}}{\widetilde{\theta}_h} \cdot \widetilde{x}_h$ because KKT are sufficient conditions to solve this problem. Thus, the uniqueness of $\widetilde{x}_h$ obviously follows from the strict quasi-concavity of $\overline{u}_h$. Analogously, by equations (4) and (5) of system (2.7), $\widetilde{y}_j$ solves the maximization problem: $\max_{y_j \in \mathbb{R}^C} \frac{\widetilde{\gamma}}{\widetilde{\beta}_j} \cdot y_j$ subject to $\overline{t}_j(y_j) \leq 0$ for every j. Thus, the uniqueness of $\widetilde{y}_j$ follows from the continuity and the strict quasi-convexity of $\overline{t}_j$. Therefore, $(\widetilde{x}, \widetilde{y})$ is unique, and consequently, the uniqueness of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ obviously follows by equations (1), (2) and (4) of system (2.7).

■

Proof of Proposition 2.12. For the proof we use the functions $\overline{u}_h$ and $\overline{t}_j$ defined in Subsection 2.4.1. We have already pointed out that $G(\widetilde{\xi}) = 0$. Let $\xi' = (x', \lambda', y', \alpha', p'^{\setminus}) \in \Xi$ be such that $G(\xi') = 0$, we show that $\widetilde{\xi} = \xi'$.

First, notice that

$$\sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j = \sum_{h \in \mathcal{H}} e_h \tag{2.15}$$

Indeed, summing $G^{h.2}(\xi') = 0$ over h, by $G^M(\xi') = 0$, one gets $\sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j = \sum_{h \in \mathcal{H}} \widetilde{e}_h$. Using the definition of $\widetilde{e}_h$ and Proposition 2.11, one easily deduces (2.15).

Second, we show that

$$\overline{u}_h(x'_h) = \overline{u}_h(\widetilde{x}_h), \forall h \in \mathcal{H} \tag{2.16}$$

From $G^{h.1}(\xi') = G^{h.2}(\xi') = 0$, x'_h solves the following maximization problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ \overline{u}_h(x_h) \\ &\text{subject to } \ p' \cdot x_h \leq p' \cdot \widetilde{x}_h + \sum_{j \in \mathcal{J}} s_{jh} \ p' \cdot (y'_j - \widetilde{y}_j) \end{aligned} \tag{2.17}$$

because KKT are sufficient conditions to solve this problem. Analogously, from $G^{j.1}(\xi') = G^{j.2}(\xi') = 0$, y'_j solves the maximization problem: $\max_{y_j \in \mathbb{R}^C} p' \cdot y_j$ subject to $\overline{t}_j(y_j) \leq 0$. Notice that $\widetilde{y}_j$ satisfies the constraint of this problem because $G^{j.2}(\widetilde{\xi}) = 0$. Thus, $p' \cdot (y'_j - \widetilde{y}_j) \geq 0$ for all j, and consequently, $\widetilde{x}_h$ belongs to the budget constraint of problem (2.17). So, $\overline{u}_h(x'_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all h. Now, suppose that $\overline{u}_k(x'_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$. From (2.15) and $G^{j.2}(\xi') = 0$ for all j, one deduces that (x', y') is a feasible allocation of the production economy $\overline{E}$, and so one gets a contradiction since $(\widetilde{x}, \widetilde{y})$ is a Pareto optimal allocation of $\overline{E}$ by Proposition 2.11. Thus, (2.16) is completely proved.

Now, define $\beta' := (\beta'_j)_{j \in \mathcal{J}}$ where $\beta'_j := \lambda'_1 \alpha'_j$ for all j, $\theta' := (\theta'_h)_{h \neq 1}$ where $\theta'_h := \frac{\lambda'_1}{\lambda'_h}$ for all $h \neq 1$ and $\gamma' := \lambda'_1 p'$. From $G^{h.1}(\xi') = 0$ for all h, $G^{j.1}(\xi') = G^{j.2}(\xi') = 0$ for all j, (2.15) and (2.16), it is an easy matter to check that $(x', y', \beta', \theta', \gamma')$ solves system (2.7). So, Proposition 2.11 implies that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) = (x', y', \beta', \theta', \gamma')$, and consequently, one obviously deduces that $\widetilde{\xi} = \xi'$.

We remark that G is C^1 by Point 4 of Assumption 2.1 and by Point 3 of Assumption 2.5. Finally, in order to show that 0 is a regular value for G, one proves that $D_\xi G(\widetilde{\xi})$ has full row rank. In this regard, we show that if $\Delta D_\xi G(\widetilde{\xi}) = 0$, then $\Delta = 0$ where $\Delta := \Big((\Delta x_h, \Delta\lambda_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^\setminus\Big) \in \mathbb{R}^{\dim \Xi}$. The system $\Delta D_\xi G(\widetilde{\xi}) = 0$ is given below.

$$\begin{cases} (h.1) \;\; \Delta x_h D^2_{x_h}\overline{u}_h(\widetilde{x}_h) - \Delta\lambda_h \widetilde{p} + \Delta p^\setminus [I_{C-1}|0] = 0, \;\forall\, h \in \mathcal{H} \\ (h.2) \;\; -\Delta x_h \cdot \widetilde{p} = 0, \;\forall\, h \in \mathcal{H} \\ (j.1) \;\; \sum\limits_{h\in\mathcal{H}} \Delta\lambda_h s_{jh}\widetilde{p} - \widetilde{\alpha}_j \Delta y_j D^2_{y_j}\overline{t}_j(\widetilde{y}_j) - \Delta\alpha_j D_{y_j}\overline{t}_j(\widetilde{y}_j) - \Delta p^\setminus [I_{C-1}|0] = 0, \;\forall\, j \in \mathcal{J} \\ (j.2) \;\; -\Delta y_j \cdot D_{y_j}\overline{t}_j(\widetilde{y}_j) = 0, \;\forall\, j \in \mathcal{J} \\ (M) \;\; -\sum\limits_{h\in\mathcal{H}} \widetilde{\lambda}_h \Delta x_h^\setminus + \sum\limits_{j\in\mathcal{J}} \Delta y_j^\setminus = 0 \end{cases}$$

We first prove that $\Delta x_h = 0$ for all $h \in \mathcal{H}$. Otherwise, suppose that there is $\overline{h} \in \mathcal{H}$ such that $\Delta x_{\overline{h}} \neq 0$. The proof goes through the two following claims that contradict each others.

We first claim that $\Delta p^\setminus \cdot (\sum_{h\in\mathcal{H}} \widetilde{\lambda}_h \Delta x_h^\setminus) > 0$. Multiplying $(h.1)$ by $\widetilde{\lambda}_h \Delta x_h$ and summing over h, from $(h.2)$ we get $\sum_{h\in\mathcal{H}} \widetilde{\lambda}_h \Delta x_h D^2_{x_h}\overline{u}_h(\widetilde{x}_h)(\Delta x_h) = -\Delta p^\setminus \cdot (\sum_{h\in\mathcal{H}} \widetilde{\lambda}_h \Delta x_h^\setminus)$. From $(h.2)$, multiplying $G^{h.1}(\widetilde{\xi}) = 0$ by Δx_h, we get $\Delta x_h \cdot D_{x_h}\overline{u}_h(\widetilde{x}_h) = 0$ for all h. Therefore, Point 3 of Assumption 2.5 completes the proof of the claim since $\widetilde{\lambda}_h > 0$ for all h and $\Delta x_{\overline{h}} \neq 0$.

Second, we claim that $\Delta p^\setminus \cdot (\sum_{h\in\mathcal{H}} \widetilde{\lambda}_h \Delta x_h^\setminus) \leq 0$. From $(j.2)$, multiplying both sides of $G^{j.1}(\widetilde{\xi}) = 0$ by Δy_j, we get $\Delta y_j \cdot \widetilde{p} = 0$. So, multiplying $(j.1)$ by Δy_j and summing over j, from $(j.2)$ we get $-\sum_{j\in\mathcal{J}} \widetilde{\alpha}_j \Delta y_j D^2_{y_j}\overline{t}_j(\widetilde{y}_j)(\Delta y_j) = \Delta p^\setminus \cdot \sum_{j\in\mathcal{J}} \Delta y_j^\setminus$. Since, $\widetilde{\alpha}_j > 0$ for all j, Point 4 of Assumption 2.1 and $(j.2)$ imply that $\Delta p^\setminus \cdot \sum_{j\in\mathcal{J}} \Delta y_j^\setminus \leq 0$. Using (M), the claim is completely proved.

Since $\widetilde{p}^C = 1$ and $\Delta x_h = 0$ for all $h \in \mathcal{H}$, from $(h.1)$ we get $\Delta\lambda_h = 0$ for all h, and so $\Delta p^\setminus = 0$. Thus, multiplying $(j.1)$ by Δy_j, Point 4 of Assumption 2.1 and $(j.2)$ imply that $\Delta y_j = 0$. So, using once again $(j.1)$, we get $\Delta\alpha_j = 0$ by Point 3 of Assumption 2.1. Therefore, $\Delta = 0$.

■

Proof of Proposition 2.15. Observe that $H^{-1}(0) = \Phi^{-1}(0) \cup \Gamma^{-1}(0)$. Since the union of a finite number of compact sets is compact, it is enough to show that $\Phi^{-1}(0)$ and $\Gamma^{-1}(0)$ are compact.

Claim 1. $\Phi^{-1}(0)$ *is compact.*

We prove that, up to a subsequence, every sequence $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}} \subseteq \Phi^{-1}(0)$ converges to an element of $\Phi^{-1}(0)$, where $\xi^\nu := (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^{\nu\,\backslash})_{\nu\in\mathbb{N}}$. Since $\{\tau^\nu : \nu \in \mathbb{N}\} \subseteq [0,1]$, up to a subsequence, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to some $\tau^* \in [0,1]$. From Steps 1.1, 1.2, 1.3 and 1.4 below, up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to some $\xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^{*\,\backslash}) \in \Xi$. Since Φ is continuous, taking the limit, one gets $(\xi^*, \tau^*) \in \Phi^{-1}(0)$.

We remind that for every $\tau \in [0,1]$, $e_h(\tau)$ is given by (2.9).

Step 1.1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.* We first show that for some $r = (r^c)_{c\in\mathcal{C}} \gg 0$, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ is included in the bounded set $K(r)$ given by Point 1 of Lemma 2.4. By $\Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j we have that,

$$t_j(y_j^\nu, \overline{y}_{-j}, \overline{x}) = 0, \ \forall \nu \in \mathbb{N}$$

Thus, the sequence $(y^\nu)_{\nu\in\mathbb{N}}$ is included in the set $Y(\overline{x}, \overline{y})$ given by (2.2). Now, for every h and for every commodity c, define the set $E_c := \{e_h^c(\tau^\nu) \colon \nu \in \mathbb{N}\} \cup \{e_h^c(\tau^*)\}$ which is a compact set. Then, there exists $r^c > 0$ such that $\max\limits_{e_h^c \in E_c} \sum\limits_{h\in\mathcal{H}} e_h^c \leq r^c$. Summing $\Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Phi^M(\xi^\nu, \tau^\nu) = 0$, we get $\sum\limits_{h\in\mathcal{H}} x_h^\nu - \sum\limits_{j\in\mathcal{J}} y_j^\nu = \sum\limits_{h\in\mathcal{H}} e_h(\tau^\nu) \leq r, \ \ \forall\, \nu \in \mathbb{N}$. So, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq A(\overline{x}, \overline{y}; r) \subseteq K(r)$.

Consequently, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ is included in $\operatorname{cl} K(r)$ which is a compact set. Then, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} K(r) \subseteq \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$, and so $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.

Step 1.2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* By $\Phi^{h.1}(\xi^\nu, \tau^\nu) = \Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$ and KKT sufficient conditions, x_h^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ u_h(x_h, \overline{x}_{-h}, \overline{y}) \\ &\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot [\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h] + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh}(y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \end{aligned} \tag{2.18}$$

We first claim that for every $\nu \in \mathbb{N}$, the bundle $\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h$ belongs to the budget constraint of the problem above. By $\Phi^{j.1}(\xi^\nu, \tau^\nu) = \Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$ and KKT sufficient conditions, y_j^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ &\text{subject to } t_j(y_j, \overline{y}_{-j}, \overline{x}) \leq 0 \end{aligned} \tag{2.19}$$

$t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0$ since $G^{j.2}(\widetilde{\xi}) = 0$, see (2.8). By Point 2 of Assumption 2.1, $t_j(0, \overline{y}_{-j}, \overline{x}) \leq 0$. So, we get $t_j((1-\tau^\nu)\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) < 0$ since $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-convex, that is, the production plan $(1-\tau^\nu)\widetilde{y}_j$ belongs to the constraint set of problem (2.19). Thus, $p^\nu \cdot (y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$ for every j, and so $p^\nu \cdot \sum\limits_{j\in\mathcal{J}} s_{jh}(y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$ which completes the proof of the claim.

Therefore, for every $\nu \in \mathbb{N}$, $u_h(x_h^\nu, \overline{x}_{-h}, \overline{y}) \geq u_h(\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h, \overline{x}_{-h}, \overline{y})$. By Point 2 of Assumption 2.5, for every $\varepsilon > 0$ we get $u_h(x_h^\nu + \varepsilon\mathbf{1}, \overline{x}_{-h}, \overline{y}) > u_h(\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h, \overline{x}_{-h}, \overline{y})$ where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$.

Taking the limit over ν and using Point 1 of Assumption 2.5, $x_h^* \gg 0$ since it belongs to the closure of the upper counter set associated with $u_h(\tau^* e_h + (1-\tau^*)\widetilde{x}_h, \overline{x}_{-h}, \overline{y})$ which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 2.5. Thus, $x^* \gg 0$.

Step 1.3. *Up to a subsequence, $(\lambda^\nu, p^{\nu\,\backslash})_{\nu\in\mathbb{N}}$ converges to some $\lambda^* \in \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++}$.* The proof is similar to the proof of Step 2.3 in Claim 2.

Step 1.4. *Up to a subsequence, $(\alpha^\nu)_{\nu\in\mathbb{N}}$ converges to some $\alpha^* \in \mathbb{R}^J_{++}$.* The proof is similar to the proof of Step 2.4 in Claim 2.

Claim 2. *$\Gamma^{-1}(0)$ is compact.*

Let $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}}$ be a sequences in $\Gamma^{-1}(0)$. As in Claim 1, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to $\tau^* \in [0,1]$. From Seps 2.1, 2.2, 2.3 and 2.4 below, up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to an element $\xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$. Since Γ is a continuous function, taking limit, we get $(\xi^*, \tau^*) \in \Gamma^{-1}(0)$.

We remind that for every $\tau \in [0,1]$, $x(\tau)$ and $y(\tau)$ are given by (2.9).

Step 2.1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}^{CH}_{+} \times \mathbb{R}^{CJ}$.* We show that for $r := \sum_{h\in\mathcal{H}} e_h$, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ is included in the bounded set $K(r)$ given by Lemma 2.4. Then, one completes the proof as in Step 1.1 of Claim 1. By $\Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j we have that

$$t_j(y_j^\nu, y_{-j}^\nu(\tau^\nu), x^\nu(\tau^\nu)) = 0,\ \forall \nu \in \mathbb{N}$$

Thus, for every $\nu \in \mathbb{N}$, the production allocation y^ν belongs to the set $Y(x^\nu(\tau^\nu), y^\nu(\tau^\nu))$ given by (2.2). Now, summing $\Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Gamma^M(\xi^\nu, \tau^\nu) = 0$, we get $\sum_{h\in\mathcal{H}} x_h^\nu - \sum_{j\in\mathcal{J}} y_j^\nu = r$. So, for every $\nu \in \mathbb{N}$, the allocation (x^ν, y^ν) belongs to the set $A(x^\nu(\tau^\nu), y^\nu(\tau^\nu); r) \subseteq K(r)$, and consequently, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq K(r)$.

Step 2.2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* The argument is similar to the one used in Step 1.2 of Claim 1 except for the last part which is quite different due to the presence of consumption externalities in the utility functions.

According to $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = \Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$, replace problem (2.18) with the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) \\ &\text{subject to } p^\nu \cdot x_h \le p^\nu \cdot e_h + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y_j^\nu \end{aligned} \tag{2.20}$$

According to $\Gamma^{j.1}(\xi^\nu,\tau^\nu)=\Gamma^{j.2}(\xi^\nu,\tau^\nu)=0$, replace problem (2.19) with the following problem

$$\begin{aligned}&\max_{y_j\in\mathbb{R}^C} p^\nu\cdot y_j\\ &\text{subject to } t_j(y_j,y^\nu_{-j}(\tau^\nu),x^\nu(\tau^\nu))\leq 0\end{aligned}$$

In order to prove that $x^*_h\gg 0$ for every h, we show that x^*_h belongs to the closure of the upper contour set associated to $(e_h,x^*_{-h}(\tau^*),y^*(\tau^*))$. One should notice that if $\tau^*=1$, then $x^*_{-h}(\tau^*)=x^*_{-h}$ which *a priori* is not necessarily strictly positive. For this reason, in Points 1 and 4 of Assumption 2.5 we allow for consumption externalities on the boundary of $\mathbb{R}^{C(H-1)}_{++}$.

Since $t_j(0,y^\nu_{-j}(\tau^\nu),x^\nu(\tau^\nu))\leq 0$ (Point 2 of Assumption 2.1), one easily checks that e_h belongs to the budget constraint of problem (2.20). So, for every $\nu\in\mathbb{N}$, $u_h(x^\nu_h,x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))\geq u_h(e_h,x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))$. By Point 2 of Assumption 2.5, for every $\varepsilon>0$ we have that $u_h(x^\nu_h+\varepsilon\mathbf{1},x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))>u_h(e_h,x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))$. Taking the limit over ν and using the continuity of u_h (Point 1 of Assumption 2.5), we get $u_h(x^*_h+\varepsilon\mathbf{1},x^*_{-h}(\tau^*),y^*(\tau^*))\geq u_h(e_h,x^*_{-h}(\tau^*),y^*(\tau^*))$. That is, for every $\varepsilon>0$ the point $(x^*_h+\varepsilon\mathbf{1})$ belongs to the upper contour set

$$\{x_h\in\mathbb{R}^C_{++}:u_h(x_h,x^*_{-h}(\tau^*),y^*(\tau^*))\geq u_h(e_h,x^*_{-h}(\tau^*),y^*(\tau^*))\}$$

So, the point x^*_h belongs to the closure of set above which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 2.5.

Step 2.3. *Up to a subsequence,* $(\lambda^\nu,p^{\nu\,\setminus})_{\nu\in\mathbb{N}}$ *converges to some* $(\lambda^*,p^{*\,\setminus})\in\mathbb{R}^H_{++}\times\mathbb{R}^{C-1}_{++}$. By $\Gamma^{h.1}(\xi^\nu,\tau^\nu)=0$, fixing commodity C, for every $\nu\in\mathbb{N}$ we have $\lambda^\nu_h=D_{x^C_h}u_h(x^\nu_h,x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))$. Taking the limit, by Points 1 and 2 of Assumption 2.5, we get $\lambda^*_h:=D_{x^C_h}u_h(x^*_h,x^*_{-h}(\tau^*),y^*(\tau^*))>0$.

By $\Gamma^{h.1}(\xi^\nu,\tau^\nu)=0$, for all commodity $c\neq C$ and for all $\nu\in\mathbb{N}$ we have $p^{\nu\,c}=\dfrac{D_{x^c_h}u_h(x^\nu_h,x^\nu_{-h}(\tau^\nu),y^\nu(\tau^\nu))}{\lambda^\nu_h}$. Taking the limit, by Points 1 and 2 of Assumption 2.5, we get $p^{*\,c}:=\dfrac{D_{x^c_h}u_h(x^*_h,x^*_{-h}(\tau^*),y^*(\tau^*))}{\lambda^*_h}>0$. Therefore, $p^{*\,\setminus}\gg 0$.

Step 2.4. *Up to a subsequence,* $(\alpha^\nu)_{\nu\in\mathbb{N}}$ *converges to some* $\alpha^*\in\mathbb{R}^J_{++}$. By $\Gamma^{j.1}(\xi^\nu,\tau^\nu)=0$, for every $\nu\in\mathbb{N}$ we have that

$$\alpha^\nu_j=\frac{p^{\nu\,c(j)}}{D_{y^{c(j)}_j}t_j(y^\nu_j,y^\nu_{-j}(\tau^\nu),x^\nu(\tau^\nu))}$$

for some commodity $c(j)$ given by Point 3 of Assumption 2.1. Taking the limit, we get
$\alpha^*_j:=\dfrac{p^{*\,c(j)}}{D_{y^{c(j)}_j}t_j(y^*_j,y^*_{-j}(\tau^*),x^*(\tau^*))}>0$ by Points 1 and 3 of Assumption 2.1.

■

2.6 Appendix

We introduce a definition of the degree *modulo 2* of continuous functions, see Appendix B in Geanakoplos and Shafer (1990), and Chapter 7 in Villanacci et al. (2002).

Let M and N be two C^2 manifolds of the same dimension contained in euclidean spaces. Let $\mathcal{A}$ be the set of triples (f, M, y) where,
(1) $f : M \to N$ is a continuous function,
(2) $y \in N$ and $f^{-1}(y)$ is compact.

Theorem 2.16 *There exists a unique function, called degree modulo 2 and denoted by* $\deg_2 : \mathcal{A} \to \{0, 1\}$ *such that*

(1) (Normalisation) $\deg_2(id_M, M, y) = 1$
where $y \in M$ *and* id_M *denotes the identity of* M*.*

(2) (Non–triviality) If $(f, M, y) \in \mathcal{A}$ *and* $\deg_2(f, M, y) = 1$*, then* $f^{-1}(y) \neq \emptyset$*.*

(3) (Excision) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *such that* $f^{-1}(y) \subseteq U$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U, y)$$

(4) (Additivity) If $(f, M, y) \in \mathcal{A}$ *and* U_1 *and* U_2 *are open and disjoint subsets of* M *such that* $f^{-1}(y) \subseteq U_1 \cup U_2$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U_1, y) + deg_2(f, U_2, y)$$

(5) (Local constantness) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *with compact closure such that* $f^{-1}(y) \subseteq U$*, then there is an open neighborhood* V *of* y *in* N *such that for every* $y' \in V$*,*

$$\deg_2(f, U, y') = \deg_2(f, U, y)$$

(6) (Homotopy invariance) Let $L : (z, \tau) \in M \times [0, 1] \to L(z, \tau) \in N$ *be a continuous homotopy. If* $y \in N$ *and* $L^{-1}(y)$ *is compact, then*

$$\deg_2(L_0, U, y) = \deg_2(L_1, U, y)$$

where $L_0 := L(\cdot, 0) : M \to N$ *and* $L_1 := L(\cdot, 1) : M \to N$*.*

If there is no possible confusion, we denote by $\deg_2(f, y)$ the degree *modulo 2* of the triple (f, M, y).

As stated in the following proposition, in the case of C^1 functions and regular values, the degree *modulo 2* is computed using the *residue class modulo 2*.

Proposition 2.17 *If* $(g, M, y) \in \mathcal{A}$*,* g *is a* C^1 *function and* y *is a regular value of* g *(i.e., for all* $z^* \in g^{-1}(y)$*, the differential mapping* $Dg(z^*)$ *is onto), then* $g^{-1}(y)$ *is finite (possibly empty) and the degree modulo 2 of* *g is given by*

$$\deg_2(g, M, y) = [\#g^{-1}(y)]_2 = \begin{cases} 0 & \textit{if } \#g^{-1}(y) \textit{ is even} \\ 1 & \textit{if } \#g^{-1}(y) \textit{ is odd} \end{cases}$$

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Chapter 3

Regularity of competitive equilibria in a production economy with externalities[1]

Abstract

We consider a general equilibrium model of private ownership economy with consumption and production externalities. Each firm is owned by the households and it is characterized by a technology described by a transformation function. Each household is characterized by a utility function, the shares on the firms profits and an endowment of commodities. The choices of all agents (households and firms) affect utility functions and production technologies. Showing by two examples that basic assumptions are not enough to guarantee a regularity result in the space of the initial endowments, we provide sufficient conditions for the regularity in the space of endowments and perturbations of the transformation functions.

JEL classification: C62, D50, D62.

Keywords: externalities, production economies, competitive equilibrium, regular economies.

3.1 Introduction

We consider a general model of private ownership economy with consumption and production externalities. Our purpose is to provide sufficient conditions for the regularity of such economies.

Why do we care about regular economies? We recall that an economy is regular if it has a finite set of equilibria and if every equilibrium locally depends in a continuous or differentiable manner on the parameters describing the economy. Therefore, at a regular economy it is possible to perform comparative static

[1] **This Chapter is based on del Mercato and Platino (2013) which has been presented at the "37 Simposio de la Asociación Española de Economa (SAEe 2012)", in Vigo, "European Economic Association and the Econometric Society European meeting (EEA-ESEM)", in Malaga, "XX European Workshop on General Equilibrium Theory, 2011 (EWGET 2011)", in Vigo and "11th Society for the Advancement of Economic Theory (SAET 2011) Conference", in Faro. So, Chapter 3 has also benefited from the comments of theese audiences. We thank Paolo Siconolfi for useful comments.**

analysis. The relevance of regular economies and issues related to the global approach of the equilibrium analysis can be found in Smale (1981), Mas-Colell (1985), Balasko (1988).

Regular economies are also important for two key aspects listed below.

(1) *Pareto improving policies.* It is well known that several sources of market failures such as incomplete financial markets, externalities, public goods prevent competitive equilibrium allocations to be Pareto optimal. In recent works, the achievement of Pareto improving policies is based on the set of regular economies. In different settings, see for instance Geanakoplos and Polemarchakis (1986, 2008), Citanna et al. (1998), Citanna et al. (2006), Villanacci and Zenginobuz (2006, 2012).

(2) *Testable restrictions.* An economic model is testable if it generates restrictions that must be satisfied by the observable data. It is well known that there are two ways to construct testable restrictions. The "parametric" approach is based on differentiable techniques which give rise to conditions remindful Slutsky conditions. This approach focuses on the *local structure* of the equilibrium manifold, that is, on regular economies, see for instance Chiappori et al. (2004).

It is an important and still open issue to study Pareto improving policies and testable restrictions in the presence of externalities from a differentiable viewpoint.

We remark that, the model, the equilibrium concept and the approach are the same as in Chapter 2. Now we describe our contributions. We start our analysis by considering the case in which there are only production externalities among firms. As shown in Bonnisseau and del Mercato (2010), in the case of only consumption externalities, regularity may fail whenever the second order external effects are too strong. Thus, in the spirit of Bonnisseau and del Mercato (2010), in Subsection 3.3.1, we introduce an additional assumption on the second order external effects on the transformation functions.

Furthermore, we provide two examples of a private ownership economy with externalities and an infinite set of equilibria for all the initial endowments. In both examples, the transformation functions satisfy our assumption on the second order external effects. So, the additional assumption mentioned above may be not sufficient to guarantee a regularity result. Thus, we also introduce displacements of the boundaries of the production sets, that is, *simple perturbations* of the transformation functions.

Our main result is Theorem 3.19 which states that almost all *perturbed economies* are regular, where the term *almost all* means in a open and full measure.[2] We remark that in order to prove our results, we follow Smale's extended approach as in Chapter 2.

Finally, we compare our contribution with previous contributions. Concerning recent works on externalities and public goods, in Bonnisseau (2003), Villanacci and Zenginobuz (2005), Kung (2008) and Bonnisseau and del Mercato (2010), the authors use Smale's extended approach too. Villanacci and Zenginobuz (2005)

[2] See Smale (1981) for details.

focus on a specific kind of externalities, namely public goods. In Kung (2008), differently from our paper, there are no externalities on the production side. Furthermore, in order to get a regularity result, the author does not make any additional assumptions on the utility functions, but perturbations of the utility functions are also needed. In Bonnisseau and del Mercato (2010), only consumption externalities are considered. So, our model extends the latter one to the case of production economy.

In Mandel (2008), the contribution mainly concerns an existence result. At the end of the paper, the author just mentions an assumption on the demand and supply functions to get the classical regularity result in the space of the endowments, namely Assumption TR2. But, this assumption implicitly involves endogenous variables, that is, equilibrium prices and Lagrange multipliers.

The paper is organized as follows. To set the stage, Section 3.2 introduces the model. In Subsection 3.2.1, we present the basic assumptions. In Subsection 3.2.2, we briefly provide the definitions of competitive equilibria and equilibrium function. In Subsection 3.2.3, we remind the definition of a regular economy. Section 3.3 is devoted to the case in which there are only production externalities among firms. In Subsection 3.3.1, we introduce an assumption on the second order external effects on the transformation functions. In Subsection 3.3.2, we provide two examples of a private ownership economy with externalities, where for all endowments one gets infinitely many equilibria. Section 3.4 is devoted to the analysis of the general model. In Subsection 3.4.1, we introduce the perturbations of the transformation functions and we adapt the basic assumptions and the notion of equilibrium function to the case of the perturbed economies. In Subsection 3.4.2, we consider the second order external effects assumption made by Bonnisseau and del Mercato (2010) on the utility functions, and we adapt our second order external effects assumption to the case of the perturbed economies. In Subsections 3.4.3, we provide our main result, that is Theorem 3.19 which states that almost all perturbed economies are regular. All the lemmas are proved in Section 3.5. Finally, in Section 3.6, the reader can find classical results from differential topology used in our analysis.

3.2 The model

There is a finite number C of physical commodities labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$ and a finite number H of households labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$. Each firm is owned by the households and it is characterized by a technology described by a transformation function. Each household is characterized by preferences described by a utility function, the shares on firms' profit and an endowment of commodities. Utility and transformation functions may be affected by the consumption choices of all households and by the production activities of all firms. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j, as usual if $y_j^c > 0$ then commodity c is produced as an output, if $y_j^\ell < 0$ then commodity ℓ is used as an input, $y_{-j} := (y_f)_{f \neq j}$ denotes the production plan of firms other than j, $y := (y_j)_{j \in \mathcal{J}}$.

- $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption, $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than h, $x := (x_h)_{h \in \mathcal{H}}$.
- Following Mas-Colell et al. (1995), the production set of firm j is described by an inequality on a function t_j called the *transformation function*. The transformation function is a convenient way to represent a production set using a function. We remind that, in the case of a single-output technology, the production set is commonly described by a *production function* f_j. That is, if $c(j) \in \mathcal{C}$ denotes the output of firm j, then the production function f_j gives the maximum amount of output that can be produced using a bundle of inputs $(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C)$. The transformation function is the counterpart of the production function in the case of production processes which involve several outputs.

  The main innovation of this paper comes from the dependency of the transformation function t_j with respect to the production activities of other firms and households consumption. So, we assume that t_j describes both the technology of firm j and the way firm j's technology is affected by the actions of the other agents. More precisely, given y_{-j} and x, the production set of the firm j is given by the following set,
$$Y_j(y_{-j}, x) := \left\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \leq 0 \right\}$$
  where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$, $t := (t_j)_{j \in \mathcal{J}}$. In the particular case of a single-output technology, the transformation function of firm j is given by
$$t_j(y_j, y_{-j}, x) := y_j^{c(j)} - f_j(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C, y_{-j}, x) \tag{3.1}$$
  where the dependency of the production function f_j with respect to the input amounts $(y_j^1, \ldots, y_j^{c(j)-1}, y_j^{c(j)+1}, \ldots, y_j^C)$ has the usual meaning whereas the dependency with respect to (y_{-j}, x) simply means that the production function of firm j is affected by the actions of the other agents.
- Household h has preferences described by a utility function,
$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$
  $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). So, u_h describes the way household h's preferences are affected by the actions of the other agents, $u := (u_h)_{h \in \mathcal{H}}$.
- $s_{jh} \in [0,1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j \in \mathcal{J}} \in [0,1]^J$ denotes the vector of the shares owed by household h; $s := (s_h)_{h \in \mathcal{H}} \in [0,1]^{JH}$. $\mathcal{S} := \{s \in [0,1]^{JH} : \sum_{h \in \mathcal{H}} s_{jh} = 1, \ \forall j \in \mathcal{J}\}$ denotes the set of shares.
- $e_h := (e_h^1, .., e_h^c, .., e_h^C)$ denotes household h's endowment, $e := (e_h)_{h \in \mathcal{H}}$.
- $E := ((u, e, s), t)$ is a private ownership economy with externalities.
- p^c is the price of one unit of commodity c, $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote $w^{\setminus} := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

3.2.1 Basic assumptions

We make the following assumptions on the transformation functions $t = (t_j)_{j \in \mathcal{J}}$.

Assumption 3.1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^2 function.

(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) \le 0$.

(3) There is at least one commodity $c(j) \in \mathcal{C}$ such that for every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $D_{y_j^{c(j)}} t_j(y'_j, y_{-j}, x) > 0$ for all $y'_j \in \mathbb{R}^C$.

(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly quasi-convex, i.e., for all $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ is positive definite on $\operatorname{Ker} D_{y_j} t_j(y'_j, y_{-j}, x)$.[3]

Assumption 3.1 is identical to Assumption 2.1 of Chapter 2. We remark that, in order to study regularity properties, we also required t_j to be a C^2 function.

Let $e = (e_h)_{h \in \mathcal{H}} \in \mathbb{R}^{CH}_{++}$ and $r := \sum_{h \in \mathcal{H}} e_h$, and define the following sets

$$\mathcal{F}(r) := \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} | \; t_j(y_j, y_{-j}, x) \le 0, \forall j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j \le r\}$$

$$\mathcal{Z}(r) := \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \mid t_j(y_j, y_{-j}, x) = 0, \forall j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j = r\}$$

and for any given externality $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,

$$Y(x, y) := \{y' \in \mathbb{R}^{CJ} : t_j(y'_j, y_{-j}, x) \le 0, \; \forall \, j \in \mathcal{J}\}$$

$$A(x, y; r) := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : y' \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \le r\}$$

The following two assumptions are identical to Assumption 2.2 and 2.3 of Chapter 2.

Assumption 3.2 *For every $r \in \mathbb{R}^C_{++}$, the set $\mathcal{Z}(r)$ is non-empty.*

Assumption 3.3 (Uniform Boundedness) *For every $r \in \mathbb{R}^C_{++}$, there exists a bounded set $C(r) \subseteq \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,*

$$Y(x, y) \cap \{y' \in \mathbb{R}^{CJ} : \sum\nolimits_{j \in \mathcal{J}} y'_j + r \gg 0\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 3.3.[4]

Lemma 3.4

(1) For every $r \in \mathbb{R}^C_{++}$, there exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $A(x, y; r) \subseteq K(r)$.

(2) For every $r \in \mathbb{R}^C_{++}$, the set of feasible allocations $\mathcal{F}(r)$ is bounded.

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[3] *Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the inner product of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the matrix product of A and B. Without loss of generality, vectors are treated as row matrices and A denotes both the matrix and the following linear application $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T denotes the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the inner product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.*

[4] See for instance, the proof of Lemma 2.4 in Section 2.5 of Chapter 2.

We make the following assumptions on the utilities functions $u = (u_h)_{h \in \mathcal{H}}$.

Assumption 3.5 *For all $h \in \mathcal{H}$,*

(1) The function u_h is continuous in its domain and C^2 in the interior of its domain.

(2) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiably strictly increasing, i.e., $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ for all $x'_h \in \mathbb{R}^C_{++}$.

(3) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiably strictly quasi-concave, i.e., for all $x'_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ is negative definite on $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$.

(4) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ and for every $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\mathrm{cl}_{\mathbb{R}^C} \{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}^C_{++}$$

Assumption 3.5 is identical to Assumption 2.5 of Chapter 2. We remark that, in order to study regularity properties, we also required u_h to be a C^2 function on the interior of its domain.

$\mathcal{T}$ denotes the set of t satisfying Assumptions 3.1, 3.2 and 3.3, $\mathcal{U}$ denotes the set of u satisfying Assumption 3.5, and $\mathcal{E} := \mathcal{U} \times \mathbb{R}^{CH}_{++} \times \mathcal{S} \times \mathcal{T}$ denotes the set of economies. From now on, $E = ((u, e, s), t)$ is any economy belonging to the set $\mathcal{E}$.

3.2.2 Competitive equilibrium and equilibrium function

In this section, we remind the definitions of competitive equilibrium à la Nash and equilibrium functions provided in Chapter 2.

Without loss of generality, commodity C is the “numeraire good”. So, given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation, we denote $p := (p^{\setminus}, 1) \in \mathbb{R}^C_{++}$.

Definition 3.6 (Competitive equilibrium) *$(x^*, y^*, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ is a competitive equilibrium for the economy E if for all $j \in \mathcal{J}$, y^*_j solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y^*_{-j}, x^*) \leq 0 \end{aligned} \tag{3.2}$$

for all $h \in \mathcal{H}$, x^*_h solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^*_{-h}, y^*) \\ &\text{subject to } \; p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j) \end{aligned} \tag{3.3}$$

and (x^*, y^*) satisfies market clearing conditions, that is

$$\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y^*_j \tag{3.4}$$

Using the first order conditions, one easily characterizes the solutions of firms and households maximization problems. The proof of the following proposition is standard since in problems (3.2) and (3.3), each agent takes as given the price system and the actions of the other agents.

Proposition 3.7

(1) *From Assumption 3.1, if y_j^* is a solution to problem (3.2), then it is unique and it is completely characterized by KKT conditions.*[5]

(2) *From Assumption 3.5, if x_h^* is a solution to problem (3.3), then it is unique and it is completely characterized by KKT conditions.*

(3) *As usual, from Point* 2 *of Assumption 3.5, household h's budget constraint holds with an equality. Thus, at equilibrium, due to the Walras law, the market clearing condition for commodity C is "redundant". So, one replaces condition (3.4) by* $\sum_{h\in\mathcal{H}} x_h^{*\backslash} = \sum_{h\in\mathcal{H}} e_h^{\backslash} + \sum_{j\in\mathcal{J}} y_j^{*\backslash}$.

Let $\Xi := (\mathbb{R}_{++}^C \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}_{++}^{C-1}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^{\backslash}) := ((x_h, \lambda_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}}, p^{\backslash})$ where λ_h denotes the Lagrange multiplier associated with household h's budget constraint, and α_j denotes the Lagrange multiplier associated with firm j's production constraint. We can now describe the competitive equilibria associated with the economy E using the *equilibrium function* $F_E : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F_E(\xi) := ((F_E^{h.1}(\xi), F_E^{h.2}(\xi))_{h\in\mathcal{H}}, (F_E^{j.1}(\xi), F_E^{j.2}(\xi))_{j\in\mathcal{J}}, F_E^M(\xi)) \tag{3.5}$$

where $F_E^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F_E^{h.2}(\xi) := -p\cdot(x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh} y_j)$, $F_E^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F_E^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F_E^M(\xi) := \sum_{h\in\mathcal{H}} x_h^{\backslash} - \sum_{j\in\mathcal{J}} y_j^{\backslash} - \sum_{h\in\mathcal{H}} e_h^{\backslash}$.

$\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$ is an *extended equilibrium* for the E if and only if $F_E(\xi^*) = 0$. We remark that, by Proposition 3.7, $(x^*, y^*, p^{*\backslash})$ is a competitive equilibrium for E if and only if there exists (λ^*, α^*) such that ξ^* is an *extended equilibrium* for E. We simply call ξ^* an equilibrium.

Theorem 3.8 (Existence and compactness) *For every economy $E \in \mathcal{E}$, the equilibrium set $F_E^{-1}(0)$ is non-empty and compact.*

In Chapter 2, one can find a proof of Theorem 3.8 by homotopy arguments.

3.2.3 Regular economies

We recall below the formal notion of a regular economy.

Definition 3.9 (Regular economy) *E is a regular economy if F_E is a C^1 function and* 0 *is a regular value of F_E, i.e., for every $\xi^* \in F_E^{-1}(0)$, the differential $D_\xi F_E(\xi^*)$ is onto.*

[5] *From now on, "KKT conditions" means Karush–Kuhn–Tucker conditions.*

Using the extended approach, the definition of a regular economy becomes a very natural notion. The fact that $D_\xi F_E(\xi^*)$ is a nonsingular matrix simply means that the linear approximation at ξ^* of the equilibrium system $F_E(\xi) = 0$ has a unique solution. So, applying the Implicit Function Theorem, around ξ^*, the equilibrium system has a unique solution which is a continuous or differentiable function of the parameters describing the economy.[6] If the equilibrium set $F_E^{-1}(0)$ is also non-empty and compact, as a consequence of the Regular Value Theorem (see Corollary 3.25 in Section 3.6), one easily deduces that a regular economy has a finite number of equilibria.

In the presence of externalities, the possibility of infinitely many equilibria cannot be excluded by making the previous basic assumptions. Indeed, the equilibrium notion given in Definition 3.6 has the following characteristics. All the agents take as given the price and the choices of the others. So, given the price and the choices of the others, the individual optimal solutions are completely determined since the transformation functions and the utility functions are respectively strictly quasi-convex and strictly quasi-concave with respect to the individual choices. This is trivial. But, for a given price, the equilibrium allocation (x^*, y^*) has a feature of a Nash equilibrium, and the problem is that, under the previous basic assumptions, for a given price, one might get infinitely many Nash equilibria (x^*, y^*).

3.3 The analysis of only production externalities among firms

We first focus our analysis to the case in which there are only production externalities among firms and no externalities at all among consumers.

As shown in Bonnisseau and del Mercato (2010), in the case of pure exchange economies with only consumption externalities, regularity may fail because of the second order external effects on the utility functions, see the example given in Section 4 of Bonnisseau and del Mercato (2010), for which one gets infinitely many equilibria for all initial endowments. In order to guarantee a regularity result for almost all initial endowments, Bonnisseau and del Mercato introduce an assumption on the second order external effects on the utility functions.[7] So, in Subsection 3.3.1, we first introduce an assumption on the second order external effects on the transformation functions, namely Assumption 3.10, which is the counterpart of the assumption provided in Bonnisseau and del Mercato (2010) in the case of only production externalities among firms. Second, in Subsection 3.3.1, we provide three examples of transformation functions that satisfy Assumption 3.10, namely Examples 3, 4 and 5.

One can easily verify that, if one considers the transformation functions given in Example 3, for a given price, the Nash supply y^* is uniquely determined. Whereas, in Examples 4 and 5, for a given price, one gets infinitely many Nash supplies. In Subsection 3.3.2, using Examples 4 and 5, we provide two examples

[6] Continuity or differentiability depends on whether the space of economies is a finite dimensional space or a topological space.

[7] See also Assumption 3.17 in Subsection 3.4.2.

of private ownership economies where for all initial endowments one gets infinitely many equilibria, namely Examples A and B. Importantly, in Example B, the indeterminacy is "price relevant". That is, one has infinitely many equilibrium prices, and consequently, the indeterminacy has an impact on the welfare of the economy.

So, differently from the case of a pure exchange economy with externalities, Examples A and B show that, alone, an assumption on the second order external effects on the transformation functions is not enough to guarantee a regularity result. Why so? Because of the first order external effects on the transformation functions. Thus, we also introduce displacements of the boundaries of the production sets, that is, *simple perturbations* of the transformation functions. In the case of only production externalities among firms, under Assumption 3.10, the regularity result holds true for almost all *perturbed economies*. The perturbations of the transformation functions are introduced for the general model in Section 3.4. In Section 3.4, we also adapt the assumption of Bonnisseau and del Mercato (2010) and Assumption 3.10 to the general model. Under these assumptions, we provide our main result, namely Theorem 3.19, which states that almost all perturbed economies are regular. Finally, we remark that, under Assumption 3.10, the proof of Theorem 3.19 can be easily adapted to the case of only production externalities among firms.

3.3.1 Second order external effects: An additional assumption

Assumption 3.10 *Let $y \in \mathbb{R}^{CJ}$ such that $t_j(y_j, y_{-j}) = 0$ for every $j \in \mathcal{J}$ and the gradients $(D_{y_j} t_j(y_j, y_{-j}))_{j\in\mathcal{J}}$ are positively collinear. Let $z \in \mathbb{R}^{CJ}$ such that $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_{-j})$ for every $j \in \mathcal{J}$ and $\sum_{j\in\mathcal{J}} z_j = 0$. Then,*

$$z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j, y_{-j})(z_f) > 0 \text{ whenever } z_j \neq 0$$

In the absence of second order external effects, Assumption 3.10 is trivially satisfied because of the strict quasi-convexity assumption of the second order on the transformation functions. In the presence of second order external effects, Assumption 3.10 still is an assumption of the second order which in addition takes into account the first order external effects on the marginal transformation $D_{y_j} t_j(y_j, y_{-j})$ of firm j. It does not mean that the Hessian matrix of the transformation function t_j (with respect to all the variables) is positive definite on $\operatorname{Ker} D_y t_j(y_j, y_{-j})$. That is, we are not requiring t_j to be differentiably strictly quasi-convex with respect to all its variables. We are using positive forms that may induce to think of strict quasi-convexity, but actually we are taking into account only a partial block of rows of the Hessian matrix of t_j.

In the case of a single-output technology, Assumption 3.10 means that the changes in the marginal productivities of firm j, that result from changing the production plans $(y_f)_{f\neq j}$ of firms other than j, are "dominated" by the changes in the marginal productivities of firm j that result from changing its own production plan y_j.[8] Indeed, consider the single-output technology given by (3.1). Without loss of generality, for simplicity

[8] As usual, the partial derivatives of the production function of firm j with respect to the inputs of firm j are called the marginal productivities of firm j.

of exposition, assume that commodity C is the output of firm j, so that we can write the transformation function as

$$t_j(y_j, y_{-j}) = y_j^C - f_j(y_j^{\setminus}, y_{-j})$$

It is an easy matter to check that $z_j D^2_{y_j} t_j(y_j, y_{-j})(z_j) = -z_j^{\setminus} D^2_{y_j^{\setminus}} f_j(y_j^{\setminus}, y_{-j})(z_j^{\setminus})$ and $z_j D^2_{y_f y_j} t_j(y_j, y_{-j})(z_f) = -z_j^{\setminus}\left[D_{y_f}\left(D_{y_j^{\setminus}} f_j(y_j^{\setminus}, y_{-j})\right)\right](z_f)$ for every $f \neq j$. So, under Point 4 of Assumption 3.1, Assumption 3.10 states that the absolute value of $z_j^{\setminus} D^2_{y_j^{\setminus}} f_j(y_j^{\setminus}, y_{-j}, x)(z_j^{\setminus})$ is strictly greater than the remaining term, i.e.,

$$\left| z_j^{\setminus} D^2_{y_j^{\setminus}} f_j(y_j^{\setminus}, y_{-j})(z_j^{\setminus}) \right| > z_j^{\setminus} \sum_{f \neq j} \left[D_{y_f}\left(D_{y_j^{\setminus}} f_j(y_j^{\setminus}, y_{-j})\right)\right](z_f)$$

We provide below three examples of transformation functions that satisfy Assumption 3.10. In all the examples,

(1) there are two commodities and two firms, $y_j = (y_j^1, y_j^2)$ denotes the production plan of firm j,
(2) without loss of generality, for simplicity of exposition, the subscript f denotes the subscript $-j$, so that $y_f = (y_f^1, y_f^2)$ denotes the production plan of the firm other than j,
(3) both firms use commodity 2 to produce commodity 1.

Example 3 The production technology of firm j is affected by the amount of output y_f^1 of the other firm in the following way. Given y_f^1, the production set of firm j is $Y_j(y_f^1) = \{y_j \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } y_j^1 \leq f_j(y_j^2, y_f^1)\}$ where the production function is defined by $f_j(y_j^2, y_f^1) := 2y_f^1\sqrt{(-y_j^2)\rho_j y_f^1}$ with $\rho_j > 0$. So, for every firm j, one considers the transformation function

$$t_j(y_j, y_f) := y_j^1 - 2y_f^1\sqrt{(-y_j^2)\rho_j y_f^1} \quad \text{with } \rho_j > 0$$

Then, $D_{y_j} t_j(y_j, y_f) = \left(1, \dfrac{\rho_j (y_f^1)^2}{\sqrt{(-y_j^2)\rho_j y_f^1}}\right)$. Take $z = (z_1, z_2) \in \mathbb{R}^4$ such that

$$z_1 + z_2 = 0 \tag{3.6}$$

and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_f)$, that is,

$$z_j^1 = -\frac{\rho_j (y_f^1)^2}{\sqrt{(-y_j^2)\rho_j y_f^1}} z_j^2 \tag{3.7}$$

We provide below the two matrices involved in Assumption 3.10, i.e.,

$$D^2_{y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\rho_j (y_f^1)^2}{2(-y_j^2)\sqrt{(-y_j^2)\rho_j y_f^1}} \end{pmatrix}, \quad D^2_{y_f y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ \dfrac{3\sqrt{(-y_j^2)\rho_j y_f^1}}{2(-y_j^2)} & 0 \end{pmatrix}$$

Thus, $z_j D^2_{y_j} t_j(y_j, y_f)(z_j) + z_j D^2_{y_f y_j} t_j(y_j, y_f)(z_f)$ is equal to

$$\frac{\rho_j (y_f^1)^2}{2(-y_j^2)\sqrt{(-y_j^2)\rho_j y_f^1}} (z_j^2)^2 + \frac{3\sqrt{(-y_j^2)\rho_j y_f^1}}{2(-y_j^2)} z_j^2 z_f^1$$

By (3.7), if $z_j \neq 0$ then $z_j^2 \neq 0$. Thus, using (3.6) and (3.7), the quantity above is strictly positive. Indeed, z_j^2 and z_j^1 have opposite sign from (3.6), and z_j^1 and z_f^1 have opposite sign from (3.7). So, z_j^2 and z_f^1 have the same sign. Then, Assumption 3.10 holds true. Finally, we remark that $t_j(y_j, y_f)$ is not differentiably (strictly) quasi-convex with respect to all its variables. ■

Example 4 The production technology of firm j is affected by the amount of output y_f^1 of the other firm in the following way. Given y_f^1, the production set of firm j is $Y_j(y_f^1) = \{y_j \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } y_j^1 \leq f_j(y_j^2, y_f^1)\}$ where the production function is defined by $f_j(y_j^2, y_f^1) := 2\sqrt{-y_j^2} - y_f^1$. So, for every firm j, one considers the transformation function

$$t_j(y_j, y_f) := y_j^1 - 2\sqrt{-y_j^2} + y_f^1$$

Assumption 3.10 is obviously satisfied. Indeed, the transformation function t_j is strictly quasi-convex with respect to the production plans of firm j, and there are no second order external effects, since the partial derivatives of the marginal transformation $D_{y_j} t_j(y_j, y_f)$ of firm j with respect to the production plan of the other firm are equal to zero.

Example 5 The production technology of firm j is affected by the production plan y_f of the other firm in the following way. Given y_f, the production set of firm j is $Y_j(y_f) = \{y_j \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } y_j^1 \leq f_j(y_j^2, y_f)\}$ where the production function is defined by $f_j(y_j^2, y_f) := 2\phi_j(y_f)\sqrt{-y_j^2}$ with $\phi_j(y_f) := \dfrac{y_f^1}{2\sqrt{-y_f^2}}$. So, for every firm j, one considers the transformation function

$$t_j(y_j, y_f) := y_j^1 - 2\phi_j(y_f)\sqrt{-y_j^2}$$

Thus, $D_{y_j} t_j(y_j, y_f) = \left(1, \dfrac{\phi_j(y_f)}{\sqrt{-y_j^2}}\right)$. Take $z = (z_1, z_2) \in \mathbb{R}^4$ such that

$$z_1 + z_2 = 0 \tag{3.8}$$

and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_f)$, that is,

$$z_j^1 = -\frac{\phi_j(y_f)}{\sqrt{-y_j^2}} z_j^2 \tag{3.9}$$

We provide below the two matrices involved in Assumption 3.10, i.e.,

$$D^2_{y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\phi_j(y_f)}{2(-y_j^2)\sqrt{-y_j^2}} \end{pmatrix} \quad , \quad D^2_{y_f y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ \dfrac{\phi_j(y_f)}{y_f^1\sqrt{-y_j^2}} & \dfrac{\phi_j(y_f)}{2(-y_f^2)\sqrt{-y_j^2}} \end{pmatrix}$$

Using (3.8) and (3.9), it is an easy matter to compute $z_j D^2_{y_j} t_j(y_j, y_f)(z_j) + z_j D^2_{y_f y_j} t_j(y_j, y_f)(z_f)$ which is given by

$$\left[\frac{1}{(-y_j^2)} + \frac{1}{\sqrt{-y_j^2}\sqrt{-y_f^2}} - \frac{1}{(-y_f^2)}\right] \frac{\phi_j(y_f)}{2\sqrt{(-y_j^2)}} (z_j^2)^2 \tag{3.10}$$

By (3.9), if $z_j \neq 0$ then $z_j^2 \neq 0$. If $D_{y_1} t_1(y_1, y_2)$ and $D_{y_2} t_2(y_2, y_1)$ are positively collinear, then $y_1^1 = y_2^1$, and so, if in addition $t_j(y_j, y_f) = 0$ for every firm $j = 1, 2$, one gets $y_1^2 = y_2^2$. Thus, the quantity given by (3.10) is strictly positive for every j, and Assumption 3.10 is completely verified. Finally, we remark that $t_j(y_j, y_f)$ is not differentiably (strictly) quasi-convex with respect to y_f. ■

3.3.2 A continuum of competitive equilibria: Two examples

In this subsection, using the transformation functions given in Example 4 and Example 5, we provide two private ownership economies with production externalities among firms, where for all endowments one gets infinitely many equilibria since, at equilibrium, there are infinitely many Nash supplies (y_1^*, y_2^*). Importantly, in the second example, the indeterminacy is “price relevant”. That is, one has infinitely many equilibrium prices, and consequently, the indeterminacy has an impact on the welfare of the economy. In both examples, there are two commodities and one household, $x = (x^1, x^2)$ denotes the consumption of the household and $e = (e^1, e^2)$ is his initial endowment. The utility function of the household is given by $u(x^1, x^2) = x^1 x^2$. So, there are no externalities on the consumption side. The price of commodity 2 is normalized to 1. As in the previous subsection,

(1) there are two firms, $y_j = (y_j^1, y_j^2)$ denotes the production plan of firm j,
(2) without loss of generality, for simplicity of exposition, the subscript f denotes the subscript $-j$ so that $y_f = (y_f^1, y_f^2)$ denotes the production plan of the firm other than j,
(3) both firms use commodity 2 to produce commodity 1.

Example A Consider the two firms given in Example 4, Subsection 3.3.1. $(x^*, y_1^*, y_2^*, (p^*, 1))$ is a competitive equilibrium if for every j, y_j^* solves

$$\begin{aligned} &\max_{y_j^1>0,\ y_j^2<0} \quad p^* y_j^1 + y_j^2 \\ &\text{subject to } \ y_j^1 \leq 2\sqrt{-y_j^2} - y_f^1 \end{aligned}$$

For each firm $j = 1, 2$, $D_{y_j} t_j(y_j, y_f^2) = \left(1, \dfrac{1}{\sqrt{-y_j^2}}\right)$. So, the associated KKT conditions provide the following equilibrium equations, $p^* = \alpha_j$, $1 = \alpha_j \dfrac{1}{\sqrt{-y_j^2}}$, and $2\sqrt{-y_j^2} - y_f^{*1} - y_j^1 = 0$. Thus, at equilibrium, one gets

$$y_1^{*1} = 2p^* - y_2^{*1} \text{ and } y_1^{*2} = -(p^*)^2 \tag{3.11}$$

and

$$y_2^{*1} = 2p^* - y_1^{*1} \text{ and } y_2^{*2} = -(p^*)^2 \tag{3.12}$$

Consequently, at equilibrium, the aggregate profit is given by

$$\sum_{j=1}^{2} (p^* y_j^{*1} + y_j^{*2}) = p^*(2p^* - y_2^{*1}) - (p^*)^2 + p^* y_2^{*1} - (p^*)^2 = 0$$

So, household's maximization problem is given by

$$\max_{x\in\mathbb{R}^2_{++}} \; x^1x^2$$
$$\text{subject to } \; p^*x^1 + x^2 \le p^*e^1 + e^2$$

Thus, at equilibrium, the optimal solution of the household is given by

$$x^{*1} = \frac{1}{2p^*}(p^*e^1 + e^2) \text{ and } x^{*2} = \frac{1}{2}(p^*e^1 + e^2) \tag{3.13}$$

Using market clearing condition for commodity 1, one finds the equilibrium price

$$p^* = \frac{1}{8}\left(\sqrt{(e^1)^2 + 16e^2} - e^1\right) \tag{3.14}$$

Finally, using (3.11), (3.12), (3.13) and (3.14), any bundle

$$((p^*,1), x^*, y_1^*, y_2^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++} \times \mathbb{R}^2 \times \mathbb{R}^2 \text{ such that } y_2^{*1} \in [0, 2p^*]$$

is a competitive equilibrium. Thus, for all initial endowments we have a continuum of equilibria parametrized by $y_2^{*1} \in [0, 2p^*]$.

One should notice that without externalities at all, if the output price increases then the output supply of both firms increases too.[9] So, equilibria are completely determined. Whereas, in the previous example, for given y_2^{*1}, if the output price p^* increases by k units then the output supply y_1^{*1} of firm 1 increases by $2k$ units, and consequently the output supply y_2^{*1} of firm 2 does not change since the price increase is compensated by firm 1's output increase. Therefore, the output supply of firm 2 is indeterminate since the two effects offset each others. ■

Example B Consider the two firms given in Example 5, Subsection 3.3.1. $(x^*, y_1^*, y_2^*, (p^*,1))$ is a competitive equilibrium if for every j, y_j^* solves

$$\max_{y_j^1>0,\; y_j^2<0} \; p^*y_j^1 + y_j^2$$
$$\text{subject to } \; y_j^1 \le 2\phi_j(y_f^*)\sqrt{-y_j^2}$$

x^* solves the following problem

$$\max_{x\in\mathbb{R}^2_{++}} \; x^1x^2$$
$$\text{subject to } \; p^*x^1 + x^2 \le p^*e^1 + e^2 + \sum_{j=1}^{2}(p^*y_j^{*1} + y_j^{*2})$$

and markets clear.

[9] In that case, the transformation function of firm j is given for instance by $t_j(y_j^1, y_j^2) := 2\sqrt{-y_j^2} - y_j^1$.

For each firm $j = 1, 2$, $D_{y_j} t_j(y_j, y_f) = \Big(1, \phi_j(y_f)\frac{1}{\sqrt{-y_j^2}}\Big)$. So, the associated KKT conditions provide the following equilibrium equations, $p^* = \alpha_j$, $1 = \alpha_j \phi_j(y_f^*)\frac{1}{\sqrt{-y_j^2}}$ and $y_j^1 = \phi_j(y_f^*) 2\sqrt{-y_j^2}$. Consequently, at the optimal solution, one gets

$$y_j^{*2} = -(p^*)^2[\phi_j(y_f^*)]^2 \text{ and } y_j^{*1} = 2p^*[\phi_j(y_f^*)]^2$$

Thus, at equilibrium, one easily deduces that

$$y_1^{*1} = y_2^{*1} \text{ and } y_1^{*2} = y_2^{*2} = -\frac{1}{2}p^* y_2^{*1} \text{ for any } y_2^{*1} > 0 \tag{3.15}$$

So, at equilibrium, the aggregate profit is equal to $p^* y_2^{*1}$, and consequently, the optimal solution of the household is given by

$$x^{*1} = \frac{1}{2p^*}(p^* e^1 + e^2 + p^* y_2^{*1}) \text{ and } x^{*2} = \frac{1}{2}(p^* e^1 + e^2 + p^* y_2^{*1}) \tag{3.16}$$

Using market clearing condition for commodity 1, the equilibrium price is

$$p^* = \frac{e^2}{e^1 + 3y_2^{*1}} \tag{3.17}$$

Finally, using (3.15), (3.16) and (3.17), any bundle

$$((p^*, 1), x^*, y_1^*, y_2^*) \text{ with } y_2^{*1} > 0$$

is a competitive equilibrium. Thus, for all initial endowments we get a continuum of equilibria parametrized by $y_2^{*1} > 0$. ■

3.4 The regularity result

We now come back to the general model introduced in Section 3.2. The examples given in Subsection 3.3.2 suggest to introduce *displacements* of the boundaries of the production sets, that is, simple perturbations of the productions sets. So, in Subsection 3.4.1, we introduce the perturbations of the transformation functions and we adapt the basic assumptions and the notion of equilibrium function given in Subsections 3.2.1 and 3.2.2. Next, in Subsection 3.4.2, we consider the second order external effects assumption made by Bonnisseau and del Mercato (2010) on the utility functions, and we adapt our second order external effects assumption given in Subsection 3.3.1 to the case of the perturbed economies. Finally, we provide our main result, that is Theorem 3.19 which states that almost all perturbed economies are regular.

3.4.1 Perturbations of the production sets, basic assumptions and equilibrium function for perturbed economies

Let t_j be the transformation function of firm j and b_j be a positive number, we consider the perturbation defined by

$$t_j(\cdot; b_j) := t_j(\cdot) - b_j$$

which generates the production set $Y_j(y_{-j}, x; b_j) := \{y_j \in \mathbb{R}^C : t_j(y_j, y_{-j}, x) \leq b_j\}$.

For every $b := (b_j)_{j\in\mathcal{J}} \in \mathbb{R}^J_+$, denote $t(\cdot; b) := (t_j(\cdot; b_j))_{j\in\mathcal{J}}$. The definition of a perturbed economy is provided below.

Definition 3.11 (Perturbed economies) *A perturbed production economy is given by $E(b) := ((u, e, s), t(\cdot; b))$ and it is parametrized by $(b, e) \in \mathbb{R}^J_+ \times \mathbb{R}^{CH}_{++}$.*

It is an easy matter to check that if $t = (t_j)_{j\in\mathcal{J}}$ satisfies Assumption 3.1, then $t(\cdot; b)$ satisfies Assumption 3.1 for all $b \in \mathbb{R}^J_+$.

For any given externality $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, define the set

$$Y(x, y; b) := \{y' \in \mathbb{R}^{CJ} : t_j(y'_j, y_{-j}, x; b_j) \leq 0, \ \forall\, j \in \mathcal{J}\}$$

Using the notation above, one defines the sets $\mathcal{Z}(b, r)$, $A(x, y; b, r)$ and $\mathcal{F}(b, r)$ as a natural adaptation of the sets $\mathcal{Z}(r)$, $A(x, y; r)$ and $\mathcal{F}(r)$ defined in Section 3.2.1. We introduce the following two assumptions which are the counterpart of Assumptions 3.2 and 3.3 in the case of perturbed economies.

Assumption 3.12 *For all $b \in \mathbb{R}^J_+$ and for every $r \in \mathbb{R}^C_{++}$, the set $\mathcal{Z}(b, r)$ is non-empty.*

Assumption 3.13 (Uniform Boundedness for perturbed economies) *For all $b \in \mathbb{R}^J_+$ and for every $r \in \mathbb{R}^C_{++}$, there exists a bounded set $C(b, r) \subseteq \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $Y(x, y; b) \cap \{y' \in \mathbb{R}^{CJ} : \sum_{j\in\mathcal{J}} y'_j + r \gg 0\} \subseteq C(b, r)$.*

The following lemma is an immediate consequence of Assumption 3.13.

Lemma 3.14

(1) For all $b \in \mathbb{R}^J_+$ and for every $r \in \mathbb{R}^C_{++}$, there exists a bounded set $K(b, r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $A(x, y; b, r) \subseteq K(b, r)$.

(2) For all $b \in \mathbb{R}^J_+$ and for every $r \in \mathbb{R}^C_{++}$, the set of feasible allocations $\mathcal{F}(b, r)$ is bounded.

Lemma 3.14 is used to prove Theorem 3.16 and Step 1 in the proof of Lemma 3.20.

Under the previous assumptions, for every $(b, e) \in \mathbb{R}^J_+ \times \mathbb{R}^{CH}_{++}$, $t(\cdot; b) \in \mathcal{T}$, and so, the perturbed economy $E(b) \in \mathcal{E}$. Consequently, all the notions and the results provided in Subsection 3.2.2 apply to the perturbed economy $E(b)$.

Remark 3.15 *With innocuous abuse of notation, from now on we simply call (b, e) a perturbed economy, and for every $(b, e) \in \mathbb{R}^J_+ \times \mathbb{R}^{CH}_{++}$, one defines in a natural way the equilibrium function $F_{b,e}$ associated with (b, e), which is nothing else than the equilibrium function associated with $E(b)$, i.e.,*

$$F_{b,e}(\xi) := F_{E(b)}(\xi)$$

Theorem 3.16 (Existence and compactness for perturbed economies) *For every perturbed economy $(b, e) \in \mathbb{R}^J_+ \times \mathbb{R}^{CH}_{++}$, the equilibrium set $F^{-1}_{b,e}(0)$ is non-empty and compact.*

3.4.2 Assumptions on the second order external effects for perturbed economies

We remind below the assumption on the utility functions made by Bonnisseau and del Mercato (2010) – Section 4, Assumption 9.(1) – where the reader can find its interpretation as well as an example of utility functions that satisfy this assumption. Notice that this assumption concerns the second order external effects due to the presence of consumption externalities on the utility functions.

Assumption 3.17 (Bonnisseau and del Mercato, 2010) *Let* $(x, v) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH}$ *such that* $v_h \in \operatorname{Ker} D_{x_h} u_h(x_h, x_{-h}, y)$ *for every* $h \in \mathcal{H}$ *and* $\sum_{h \in \mathcal{H}} v_h = 0$*. Then,*

$$v_h \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k) < 0 \textit{ whenever } v_h \neq 0$$

The following assumption is an adaptation of Assumption 3.10 to the case of the perturbed economies which takes into account the second order external effects due to the presence of the production externalities on the transformation and utility functions.

Assumption 3.18 *For every* $b = (b_j)_{j \in \mathcal{J}} \in \mathbb{R}^J_+$ *and for every* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ *such that* $t_j(y_j, y_{-j}, x; b_j) = 0$ *for every* $j \in \mathcal{J}$ *and the gradients* $(D_{y_j} t_j(y_j, y_{-j}, x; b_j))_{j \in \mathcal{J}}$ *are positively collinear. Let* $(v, z) \in \mathbb{R}^{CH} \times \mathbb{R}^{CJ}$ *such that* $v_h \in \operatorname{Ker} D_{x_h} u_h(x_h, x_{-h}, y)$ *for every* $h \in \mathcal{H}$*,* $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_{-j}, x; b_j)$ *for every* $j \in \mathcal{J}$*, and* $\sum_{h \in \mathcal{H}} v_h = \sum_{j \in \mathcal{J}} z_j$*. Then,*

(1) $z_j \sum_{f \in \mathcal{J}} D^2_{y_f y_j} t_j(y_j, y_{-j}, x; b_j)(z_f) > 0$ *whenever* $z_j \neq 0$*,*

(2) $v_h \sum_{f \in \mathcal{J}} D^2_{y_f x_h} u_h(x_h, x_{-h}, y)(z_f) \leq 0$*.*

From now on, $\widetilde{\mathcal{T}}$ denotes the set of $t = (t_j)_{j \in \mathcal{J}}$ satisfying Assumptions 3.1, 3.12, 3.13 and 3.18, and $\widetilde{\mathcal{U}}$ denotes the set of $u = (u_h)_{h \in \mathcal{H}}$ satisfying Assumptions 3.5, 3.17 and 3.18.

3.4.3 Regularity for almost all perturbed economies

In this section, we prove the following theorem, which is our main result. The utility functions, the shares and the transformation functions $(u, s, t) \in \widetilde{\mathcal{U}} \times \mathcal{S} \times \widetilde{\mathcal{T}}$ are fixed. We focus our analysis on the open set of perturbed economies defined by $\Lambda := \mathbb{R}^J_{++} \times \mathbb{R}^{CH}_{++}$.

Theorem 3.19 (Regularity for almost all perturbed economies) *The set* Λ^* *of perturbed economies* $(b, e) \in \Lambda$ *such that* (b, e) *is a regular economy is an open and full measure subset of* Λ*.*

In order to prove the theorem above, we introduce the following notations and we provide two auxiliary lemmas, namely Lemmas 3.20 and 3.21.

We remind that in Remark 3.15, we have defined the equilibrium function $F_{b,e}$ associated with any $(b, e) \in \Lambda$. By Point 1 of Assumptions 3.1 and 3.5 the equilibrium function $F_{b,e}$ is C^1 everywhere. So, by Definition 3.9, the economy (b, e) is regular if

$$\forall\, \xi^* \in F_{b,e}^{-1}(0),\ \operatorname{rank} D_\xi F_{b,e}(\xi^*) = \dim \Xi$$

Define the following set

$$B := \left\{(\xi, b, e) \in F^{-1}(0) :\ \operatorname{rank} D_\xi F(\xi, b, e) < \dim \Xi\right\}$$

where the function $F : \Xi \times \Lambda \to \mathbb{R}^{\dim \Xi}$ is defined by

$$F(\xi, b, e) := F_{b,e}(\xi)$$

and denote Π the restriction to $F^{-1}(0)$ of the projection of $\Xi \times \Lambda$ onto Λ, i.e.

$$\Pi : (\xi, b, e) \in F^{-1}(0) \to \Pi(\xi, b, e) := (b, e) \in \Lambda$$

We can now write the set Λ^* given in Theorem 3.19 as

$$\Lambda^* = \Lambda \setminus \Pi(B)$$

So, in order to prove Theorem 3.19, it is enough to show that $\Pi(B)$ is a closed set in Λ and $\Pi(B)$ is of measure zero.

We first claim that $\Pi(B)$ is a closed set in Λ. From Point 1 of Assumptions 3.1 and 3.5, F and $D_\xi F$ are continuous on $\Xi \times \Lambda$. The set B is characterized by the fact that the determinant of all the square submatrices of $D_\xi F(\xi, b, e)$ of dimension $\dim \Xi$ is equal to zero. Since the determinant is a continuous function and $D_\xi F$ is continuous on $F^{-1}(0)$, the set B is closed in $F^{-1}(0)$. Thus, $\Pi(B)$ is closed since the projection Π is proper.[10] The properness of the projection Π is provided in the following lemma.

Lemma 3.20 *The projection $\Pi : F^{-1}(0) \to \Lambda$ is a proper function.*

To complete the proof of Theorem 3.19, we claim that $\Pi(B)$ is of measure zero in Λ. The result follows by Lemma 3.21 given below and a consequence of Sard's Theorem (see Theorem 3.26 in Section 3.6). Indeed, Lemma 3.21 and Theorem 3.26 imply that there exists a full measure subset Ω of Λ such that for each $(b, e) \in \Omega$ and for each ξ^* such that $F(\xi^*, b, e) = 0$, $\operatorname{rank} D_\xi F(\xi^*, b, e) = \dim \Xi$. Now, let $(b, e) \in \Pi(B)$, then there exists $\xi \in \Xi$ such that $F(\xi, b, e) = 0$ and $\operatorname{rank} D_\xi F(\xi, b, e) < \dim\, \Xi$. So, $(b, e) \notin \Omega$. This prove that $\Pi(B)$ is included in the complementary of Ω, that is in $\Omega^C := \Lambda \setminus \Omega$. Since Ω^C has zero measure, so too does $\Pi(B)$. Thus, the set of regular perturbed economies Λ^* is of full measure since $\Omega \subseteq \Lambda^*$, which completes the proof of Theorem 3.19.

Lemma 3.21 *0 is a regular value for F.*

[10]See Definition 3.27 in Section 3.6.

Finally, one easily deduces the following proposition from Theorems 3.16 and 3.19, a consequence of the Regular Value Theorem (i.e., Corollary 3.25 in Section 3.6) and the Implicit Function Theorem.

Proposition 3.22 (Properties of a regular economy) *For each* $(b, e) \in \Lambda^*$,

(1) the equilibrium set associated with the economy (b, e) *is a non-empty finite set, i.e.*

$$\exists\, r \in \mathbb{N} \setminus \{0\} : \; F_{b,e}^{-1}(0) = \{\xi^1, ..., \xi^r\}$$

(2) there exists an open neighborhood I *of* (b, e) *in* Λ^**, and for each* $i = 1, \dots, r$ *there exist an open neighborhood* U_i *of* ξ^i *in* Ξ *and a* C^1 *function* $g_i : I \to U_i$ *such that*

(a) $U_i \cap U_k = \emptyset$ *if* $i \neq k$,

(b) $g_i(b, e) = \xi^i$ *and* $\xi' \in F_{b',e'}^{-1}(0)$ *holds for* $(\xi', b', e') \in U_i \times I$ *if and only if* $\xi' = g_i(b', e')$.

3.5 Proofs

In this section, we prove all the lemmas stated in Sections 3.4.1 and 3.4.3.

Proof of Lemma 3.14. See the proof of Lemma 2.4 in Section 2.5 of Chapter 2. ■

Proof of Theorem 3.16. See the proof of Theorem 2.8 in Section 2.4 of Chapter 2. ■

Proof of Lemma 3.20. We show that any sequence $(\xi^\nu, b^\nu, e^\nu)_{\nu \in \mathbb{N}} \subseteq F^{-1}(0)$, up to a subsequence, converges to an element of $F^{-1}(0)$, knowing that the sequence $\Pi(\xi^\nu, b^\nu, e^\nu)_{\nu\in\mathbb{N}} = (b^\nu, e^\nu)_{\nu\in\mathbb{N}} \subseteq \Lambda$ converges to some $(b^*, e^*) \in \Lambda$. We recall that $\xi^\nu = (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^{\nu\setminus})$. By the definition of $t_j(\cdot; b_j)$, $t_j(\cdot; b_j^\nu) = t_j(\cdot) - b_j^\nu$.

Step 1. *Up to a subsequence,* $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ *converges to* $(x^*, y^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$. We show that for an appropriate $(\bar{b}, \bar{r}) \in \mathbb{R}_{++}^J \times \mathbb{R}_{++}^C$, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the set $\mathcal{F}(\bar{b}, \bar{r})$ which is bounded by Lemma 3.14. Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the compact set $\mathrm{cl}\,\mathcal{F}(\bar{b}, \bar{r})$. Thus, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathrm{cl}\,\mathcal{F}(\bar{b}, \bar{r}) \subseteq \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$.

For every $j \in \mathcal{J}$, consider the following compact set $\{b_j^\nu : \nu \in \mathbb{N}\} \cup \{b_j^*\}$ and define

$$\bar{b}_j := \max\{b_j^\nu : \nu \in \mathbb{N}\} \cup \{b_j^*\} \;\forall\, j \in \mathcal{J} \text{ and } \bar{b} := (\bar{b}_j)_{j\in\mathcal{J}}$$

By definition, $t_j(y_j, y_{-j}, x; \bar{b}_j) \leq t_j(y_j, y_{-j}, x; b_j^\nu)$ for every $(y_j, y_{-j}, x) \in \mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$ and for every $\nu \in \mathbb{N}$. Since $F^{j.2}(\xi^\nu, b^\nu, e^\nu) = 0$, for every $\nu \in \mathbb{N}$ we get

$$t_j(y_j^\nu, y_{-j}^\nu, x^\nu; \bar{b}_j) \leq 0$$

Now, for every commodity c consider the following compact set $\{e^{\nu c} : \nu \in \mathbb{N}\} \cup \{e^{*c}\}$ and define

$$\bar{r}^c := \max_{e_h^c \in \{e^{\nu c}:\, \nu\in\mathbb{N}\}\cup\{e^{*c}\}} \sum_{h\in\mathcal{H}} e_h^c \quad \text{and} \quad \bar{r} := (\bar{r}^c)_{c\in\mathcal{C}}$$

Summing $F^{h.2}(\xi^\nu, b^\nu, e^\nu) = 0$ over h, by $F^M(\xi^\nu, b^\nu, e^\nu) = 0$ we have that $\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu = \sum_{h \in \mathcal{H}} e_h^\nu$ for all $\nu \in \mathbb{N}$. By definition of $\bar{r}$, $\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu \leq \bar{r}$ for all $\nu \in \mathbb{N}$. Thus, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}} \subseteq \mathcal{F}(\bar{b}, \bar{r})$.

Step 2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* We show that for every $h \in \mathcal{H}$, x_h^* belongs to the closure of the following set

$$\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x^*_{-h}, y^*) \geq u_h(e^*_h, x^*_{-h}, y^*)\} \tag{3.18}$$

which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 3.5. Thus, $x_h^* \gg 0$.

By $F^{h.1}(\xi^\nu, b^\nu, e^\nu) = F^{h.2}(\xi^\nu, b^\nu, e^\nu) = 0$ and KKT sufficient conditions, x_h^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^\nu_{-h}, y^\nu) \\ &\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot e^\nu_h + p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j^\nu \end{aligned}$$

We claim that the point e_h^ν belongs to the budget constraint of the problem above. By $F^{j.1}(\xi^\nu, b^\nu, e^\nu) = F^{j.2}(\xi^\nu, b^\nu, e^\nu) = 0$ and KKT sufficient conditions, y_j^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^\nu \cdot y_j \\ &\text{subject to } t_j(y_j, y^\nu_{-j}, x^\nu; b^\nu_j) \leq 0 \end{aligned}$$

By Point 2 of Assumption 3.1, $p^\nu \cdot y_j^\nu \geq p^\nu \cdot 0 = 0$. So, one gets $p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j^\nu \geq 0$ which completes the proof of the claim. Therefore, for every $\nu \in \mathbb{N}$

$$u_h(x^\nu_h, x^\nu_{-h}, y^\nu) \geq u_h(e^\nu_h, x^\nu_{-h}, y^\nu)$$

By Point 2 of Assumption 3.5, for every $\varepsilon > 0$ we have that $u_h(x^\nu_h + \varepsilon \mathbf{1}, x^\nu_{-h}, y^\nu) > u_h(e^\nu_h, x^\nu_{-h}, y^\nu)$ where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. So, taking the limit for $\nu \to +\infty$ and using the continuity of u_h, one gets

$$u_h(x^*_h + \varepsilon \mathbf{1}, x^*_{-h}, y^*) \geq u_h(e^*_h, x^*_{-h}, y^*)$$

since $(e^\nu_h)_{\nu \in \mathbb{N}}$ converges to e^*_h. Thus, for every $\varepsilon > 0$ the point $(x^*_h + \varepsilon \mathbf{1})$ belongs to the set defined in (3.18), which implies that x^*_h belongs to the closure of this set.

Step 3. *Up to a subsequence, $(\lambda^\nu, p^{\nu \setminus})_{\nu \in \mathbb{N}}$ converges to some $(\lambda^*, p^{* \setminus}) \in \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++}$.* By $F^{h.1}(\xi^\nu, b^\nu, e^\nu) = 0$, fixing commodity C, for every $\nu \in \mathbb{N}$ we have $\lambda^\nu_h = D_{x^C_h} u_h(x^\nu_h, x^\nu_{-h}, y^\nu)$. Taking the limit over ν, by Points 1 and 2 of Assumption 3.5, we get $\lambda^*_h := D_{x^C_h} u_h(x^*_h, x^*_{-h}, y^*) > 0$.

By $F^{h.1}(\xi^\nu, b^\nu, e^\nu) = 0$, for all commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have $p^{\nu\, c} = \dfrac{D_{x^c_h} u_h(x^\nu_h, x^\nu_{-h}, y^\nu)}{\lambda^\nu_h}$. Taking the limit over ν, by Points 1 and 2 of Assumption 3.5, we get $p^{*\, c} := \dfrac{D_{x^c_h} u_h(x^*_h, x^*_{-h}, y^*)}{\lambda^*_h} > 0$.

Therefore, $p^{*\setminus} \gg 0$.

Step 4. *Up to a subsequence, $(\alpha^\nu)_{\nu\in\mathbb{N}}$ converges to some $\alpha^* \in \mathbb{R}^J_{++}$.* By $F^{j.1}(\xi^\nu, b^\nu, e^\nu) = 0$, for every $\nu \in \mathbb{N}$ we have that $\alpha_j^\nu = \frac{p^{\nu\, c(j)}}{D_{y_j^{c(j)}} t_j(y_j^\nu, y_{-j}^\nu, x^\nu; b_j^\nu)}$ for some commodity $c(j)$ given by Point 3 of Assumption 3.1. Taking the limit, by Points 1 and 3 of Assumption 3.1, one gets $\alpha_j^* := \frac{p^{*\, c(j)}}{D_{y_j^{c(j)}} t_j(y_j^*, y_{-j}^*, x^*; b_j^*)} > 0$. ■

Proof of Lemma 3.21.

We show that for each $(\xi^*, b^*, e^*) \in F^{-1}(0)$, the Jacobian matrix $D_{\xi,b,e}F(\xi^*, b^*, e^*)$ has full row rank. It is enough to prove that $\Delta D_{\xi,b,e}F(\xi^*, b^*, e^*) = 0$ implies $\Delta = 0$, where

$$\Delta := ((\Delta x_h, \Delta\lambda_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^\setminus) \in \mathbb{R}^{H(C+1)} \times \mathbb{R}^{J(C+1)} \times \mathbb{R}^{C-1}$$

The computation of $D_{\xi,b,e}F(\xi^*, b^*, e^*)$ is described in Section 3.7 and the system $\Delta D_{\xi,b,e}F(\xi^*, b^*, e^*) = 0$ is written in detail below.

$$\begin{cases}
\sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*) - \Delta\lambda_k p^* - \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*) + \\
- \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{x_k} t_j(y_j^*, y_{-j}^*, x^*) + \Delta p^\setminus [I_{C-1}|0] = 0, \ \forall\, k \in \mathcal{H} \\
-\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\
\sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*) + \sum_{h\in\mathcal{H}} \Delta\lambda_h s_{fh} p^* - \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*) + \\
- \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{y_f} t_j(y_j^*, y_{-j}^*, x^*) - \Delta p^\setminus [I_{C-1}|0] = 0, \ \forall\, f \in \mathcal{J} \\
-\Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, j \in \mathcal{J} \\
\Delta\lambda_h p^* - \Delta p^\setminus [I_{C-1}|0] = 0, \ \forall\, h \in \mathcal{H} \\
-\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^\setminus - \sum_{h\in\mathcal{H}} \Delta\lambda_h (x_h^{*\setminus} - e_h^{*\setminus} - \sum_{j\in\mathcal{J}} s_{jh} y_j^{*\setminus}) + \sum_{j\in\mathcal{J}} \Delta y_j^\setminus = 0 \\
\Delta\alpha_j = 0, \ \forall\, j \in \mathcal{J}
\end{cases}$$

Since $p^{*C} = 1$, we get $\Delta\lambda_h = 0$ for all $h \in \mathcal{H}$ and $\Delta p^\setminus = 0$. So, the above system becomes

$$\begin{cases}
(1) \quad \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*) - \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, k \in \mathcal{H} \\
(2) \quad -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\
(3) \quad \sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*) - \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, f \in \mathcal{J} \\
(4) \quad -\Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, j \in \mathcal{J} \\
(5) \quad -\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^\setminus + \sum_{j\in\mathcal{J}} \Delta y_j^\setminus = 0
\end{cases} \tag{3.19}$$

Multiplying both sides of equation $F^{j.1}(\xi^*, b^*, e^*) = 0$ by Δy_j and using equation (4) in system (3.19), we get $\Delta y_j \cdot p^* = \alpha_j^* \Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*)$=0. Summing over j, we obtain $\sum_{j\in\mathcal{J}} \Delta y_j^C = -\sum_{j\in\mathcal{J}} \Delta y_j^\setminus \cdot p^{*\setminus}$. Multiplying equation (2) in system (3.19) by λ_h^*, summing over h, we obtain $\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^C = -\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^\setminus \cdot p^{*\setminus}$. Finally using equation (5) in system (3.19), we get $\sum_{h\in\mathcal{H}}\lambda_h^*\Delta x_h^C = \sum_{j\in\mathcal{J}}\Delta y_j^C$. Thus, using once again equation (5) in system (3.19), we get

$$\sum_{h\in\mathcal{H}}\lambda_h^*\Delta x_h = \sum_{j\in\mathcal{J}}\Delta y_j$$

From $F^{h.1}(\xi^*, b^*, e^*) = 0$ and equation (2) in system (3.19), we get $(\Delta x_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Ker} D_{x_h} u_h(x_h^*, x_{-h}^*, y^*)$. From equation (4) in system (3.19), we have that $(\Delta y_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \operatorname{Ker} D_{y_j} t_j(y_j^*, y_{-j}^*, x^*)$. Now, for every $h \in \mathcal{H}$ and for every $j \in \mathcal{J}$, define

$$v_h := \lambda_h^*\Delta x_h \text{ and } z_j := \Delta y_j \tag{3.20}$$

Thus, the vector $((x_h^*, v_h)_{h\in\mathcal{H}}, (y_j^*, z_j)_{j\in\mathcal{J}})$ satisfies the following conditions.

$$\sum_{h\in\mathcal{H}} v_h = \sum_{j\in\mathcal{J}} z_j \tag{3.21}$$

$$(v_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Ker} D_{x_h} u_h(x_h^*, x_{-h}^*, y^*) \tag{3.22}$$

$$(z_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \operatorname{Ker} D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) \tag{3.23}$$

Since $F^{j.1}(\xi^*, b^*, e^*) = F^{j.2}(\xi^*, b^*, e^*) = 0$ for every $j \in \mathcal{J}$, it follows that $t_j(y_j^*, y_{-j}^*, x_j^*; b_j^*) = 0$ for each $j \in \mathcal{J}$ and the gradients $(D_{y_j} t_j(y_j^*, y_{-j}^*, x_j^*))_{j\in\mathcal{J}}$ are positively collinear. Thus, we remark that from (3.21), (3.22) and (3.23), all the conditions of Assumption 3.18 are satisfied.

Multiplying both sides of equation (3) in system (3.19) by z_f, we get

$$\sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*)(z_f) = \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*)(z_f)$$

Since $\lambda_h^* \neq 0$ for all $h \in \mathcal{H}$, using the definition of v_h given in (3.20), it follows that for each $f \in \mathcal{J}$

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda_h^*} D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*)(z_f) = \sum_{j\in\mathcal{J}} \alpha_j^* z_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*)(z_f)$$

Summing over $f \in \mathcal{J}$, we get

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda_h^*} \sum_{f\in\mathcal{J}} D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*)(z_f) = \sum_{j\in\mathcal{J}} \alpha_j^* z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*)(z_f)$$

By Point 2 of Assumption 3.18, since $\lambda_h^* > 0$ for each $h \in \mathcal{H}$, we know that

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda_h^*} \sum_{f\in\mathcal{J}} D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*)(z_f) \leq 0$$

Thus, the equality above implies that

$$\sum_{j\in\mathcal{J}} \alpha_j^* z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*)(z_f) \leq 0$$

Since $\alpha_j^* > 0$ for all $j \in \mathcal{J}$, Point 1 of Assumption 3.18 implies that $z_j = 0$ for all $j \in \mathcal{J}$. Therefore, using the definition of z_j given in (3.20) we get $\Delta y_j = 0$ for all $j \in \mathcal{J}$. So, condition (3.21) becomes

$$\sum_{h \in \mathcal{H}} v_h = 0 \tag{3.24}$$

and equation (1) in system (3.19) becomes $\sum_{h \in \mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*) = 0$ for every $k \in \mathcal{H}$. Multiplying both sides by v_k, using the definition of v_h given in (3.20), one gets $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda_h^*} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) = 0$ for every $k \in \mathcal{H}$. Summing up $k \in \mathcal{H}$, we obtain $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda_h^*} \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) = 0$. By (3.22) and (3.24), all the conditions of Assumption 3.17 are satisfied, and so $v_h = 0$ for each $h \in \mathcal{H}$ since $\lambda_h^* > 0$. Thus, we get $\Delta x_h = 0$ for all $h \in \mathcal{H}$, and consequently $\Delta = 0$ which completes the proof. ■

3.6 Appendix A

Regular values and transversality

The theory of general economic equilibrium from a differentiable prospective is based on results from differential topology. First, we remind the definition of a regular value. Second, we summarize the results used in our analysis. These results, as well as generalizations on these issues, can be found for instance in Guillemin and Pollack (1974), Hirsch (1976), Mas-Colell (1985) and Villanacci et al. (2002).

Definition 3.23 *Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function, assume $r \geq 1$. An element $y \in N$ is a regular value for f if for every $x^* \in f^{-1}(y)$, the differential mapping $Df(x^*)$ is onto.*

Theorem 3.24 *(Regular Value Theorem) Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function, assume $r \geq 1$. If $y \in N$ is a regular value for f, then*
(1) if $m < n$, $f^{-1}(y) = \emptyset$,
(2) if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M.

Corollary 3.25 *Let M, N be C^r manifolds of the same dimension. Let $f : M \to N$ be a C^r function. Assume $r \geq 1$. Let $y \in N$ a regular value for f such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of M.*

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 3.26 *(Transversality Theorem) Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function, assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, $y \in N$ is a regular value for f_ω, where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 3.27 *Let* (X, d) *and* (Y, d') *be two metric spaces. A function* $\pi : X \to Y$ *is proper if it is continuous and one among the following conditions holds true.*

(1) π *is closed and* $\pi^{-1}(y)$ *is compact for each* $y \in Y$*,*

(2) if K *is a compact subset of* Y*, then* $\pi^{-1}(K)$ *is a compact subset of* X*,*

(3) if $(x^n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $(\pi(x^n))_{n\in\mathbb{N}}$ *converges in* Y*, then* $(x^n)_{n\in\mathbb{N}}$ *has a converging subsequence in* X*.*

The conditions above are equivalent.

3.7 Appendix B

The computation of $D_{\xi,b,e}F(\xi^*, b^*, e^*)$ is described below. Vectors are treated as row matrices. The symbol "T" means transpose. 0 denotes the zero vector. With innocuous abuse of notation, the dimension of 0 is C or $C - 1$ depending on the dimension of the respective block of columns. $\mathbf{0}$ denotes the zero matrix. With innocuous abuse of notation, the size of $\mathbf{0}$ is $C \times C$ or $(C-1) \times (C-1)$ depending on the size of the respective block of rows and columns. $\widehat{I} := [I_{C-1}|0^T]_{(C-1)\times C}$ where I_{C-1} denotes the $(C-1) \times (C-1)$ identity matrix

$$
\begin{array}{c}
\\
D_{x_1}u_1(x_1,x_{-1},y)-\lambda_1 p \\
-p\cdot\left(x_1-e_1-\sum_{j\in\mathcal{J}} s_{j1}y_j\right) \\
\vdots \\
D_{x_h}u_h(x_h,x_{-h},y)-\lambda_h p \\
-p\cdot\left(x_h-e_h-\sum_{j\in\mathcal{J}} s_{jh}y_j\right) \\
\vdots \\
D_{x_H}u_H(x_H,x_{-H},y)-\lambda_H p \\
-p\cdot\left(x_H-e_H-\sum_{j\in\mathcal{J}} s_{jH}y_j\right) \\
p-\alpha_1 D_{y_1}t_1(y_1,y_{-1},x) \\
-t_1(y_1,y_{-1},x)+b_1 \\
\vdots \\
p-\alpha_j D_{y_j}t_j(y_j,y_{-j},x) \\
-t_j(y_j,y_{-j},x)+b_j \\
\vdots \\
p-\alpha_J D_{y_J}t_J(y_J,y_{-J},x) \\
-t_J(y_J,y_{-J},x)+b_J \\
\sum_{h\in\mathcal{H}} x_h^{\backslash}-\sum_{j\in\mathcal{J}} y_j^{\backslash}-\sum_{h\in\mathcal{H}} e_h^{\backslash}
\end{array}
\begin{array}{ccccccccccccccccccccccccccccccc}
x_1 & \lambda_1 & \cdots & x_h & \lambda_h & \cdots & x_H & \lambda_H & y_1 & \alpha_1 & \cdots & y_j & \alpha_j & \cdots & y_J & \alpha_J & e_1 & \cdots & e_h & \cdots & e_H & p^{\backslash} & b_1 & \cdots & b_j & \cdots & b_J \\
\end{array}
$$

$$
\left(\begin{array}{ccccccccccccccccccccccccccccccc}
D^2_{x_1}u_1 & -p^{*T} & \cdots & D^2_{x_h x_1}u_1 & 0^T & \cdots & D^2_{x_H x_1}u_1 & 0^T & D^2_{y_1 x_1}u_1 & 0^T & \cdots & D^2_{y_j x_1}u_1 & 0^T & \cdots & D^2_{y_J x_1}u_1 & 0^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & -\lambda_1^*\hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
-p^* & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & s_{11}p^* & 0 & \cdots & s_{j1}p^* & 0 & \cdots & s_{J1}p^* & 0 & p^* & \cdots & 0 & \cdots & 0 & -(x_1^{*\backslash}-e_1^{*\backslash}-\sum_{j\in\mathcal{J}} s_{j1}y_j^{*\backslash}) & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
D^2_{x_1 x_h}u_h & 0^T & \cdots & D^2_{x_h}u_h & -p^{*T} & \cdots & D^2_{x_H x_h}u_h & 0^T & D^2_{y_1 x_h}u_h & 0^T & \cdots & D^2_{y_j x_h}u_h & 0^T & \cdots & D^2_{y_J x_h}u_h & 0^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & -\lambda_h^*\hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
0 & 0 & \cdots & -p^* & 0 & \cdots & 0 & 0 & s_{1h}p^* & 0 & \cdots & s_{jh}p^* & 0 & \cdots & s_{Jh}p^* & 0 & 0 & \cdots & p^* & \cdots & 0 & -(x_h^{*\backslash}-e_h^{*\backslash}-\sum_{j\in\mathcal{J}} s_{jh}y_j^{*\backslash}) & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
D^2_{x_1 x_H}u_H & 0^T & \cdots & D^2_{x_h x_H}u_H & -p^{*T} & \cdots & D^2_{x_H}u_H & 0^T & D^2_{y_1 x_H}u_H & 0^T & \cdots & D^2_{y_j x_H}u_H & 0^T & \cdots & D^2_{y_J x_H}u_H & 0^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & -\lambda_H^*\hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
0 & 0 & \cdots & 0 & 0 & \cdots & -p^* & 0 & s_{1H}p^* & 0 & \cdots & s_{jH}p^* & 0 & \cdots & s_{JH}p^* & 0 & 0 & \cdots & 0 & \cdots & p^* & -(x_h^{*\backslash}-e_h^{*\backslash}-\sum_{j\in\mathcal{J}} s_{jh}y_j^{*\backslash}) & 0 & \cdots & 0 & \cdots & 0 \\
-\alpha_1^*D^2_{x_1 y_1}t_1 & 0^T & \cdots & -\alpha_1^*D^2_{x_h y_1}t_1 & 0^T & \cdots & -\alpha_1^*D^2_{x_H y_1}t_1 & 0^T & -\alpha_1^*D^2_{y_1}t_1 & -D_{y_1}t_1^T & \cdots & -\alpha_1^*D^2_{y_j y_1}t_1 & 0^T & \cdots & -\alpha_1^*D^2_{y_J y_1}t_1 & 0^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
-D_{x_1}t_1 & 0 & \cdots & -D_{x_h}t_1 & 0 & \cdots & -D_{x_H}t_1 & 0 & -D_{y_1}t_1 & 0 & \cdots & -D_{y_j}t_1 & 0 & \cdots & -D_{y_J}t_1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\alpha_j^*D^2_{x_1 y_j}t_j & 0^T & \cdots & -\alpha_j^*D^2_{x_h y_j}t_j & 0^T & \cdots & -\alpha_j^*D^2_{x_H y_j}t_j & 0^T & -\alpha_j^*D^2_{y_1 y_j}t_j & 0^T & \cdots & -\alpha_j^*D^2_{y_j}t_j & -D_{y_j}t_j^T & \cdots & -\alpha_j^*D^2_{y_J y_j}t_j & 0^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
-D_{x_1}t_j & 0 & \cdots & -D_{x_h}t_j & 0 & \cdots & -D_{x_H}t_j & 0 & -D_{y_1}t_j & 0 & \cdots & -D_{y_j}t_j & 0 & \cdots & -D_{y_J}t_j & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\alpha_J^*D^2_{x_1 y_J}t_J & 0^T & \cdots & -\alpha_J^*D^2_{x_h y_J}t_J & 0^T & \cdots & -\alpha_J^*D^2_{x_H y_J}t_J & 0^T & -\alpha_J^*D^2_{y_1 y_J}t_J & 0^T & \cdots & -\alpha_J^*D^2_{y_j y_J}t_J & 0^T & \cdots & -\alpha_J^*D^2_{y_J}t_J & -D_{y_J}t_J^T & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \hat{I}^T & 0^T & \cdots & 0^T & \cdots & 0^T \\
-D_{x_1}t_J & 0 & \cdots & -D_{x_h}t_J & 0 & \cdots & -D_{x_H}t_J & 0 & -D_{y_1}t_J & 0 & \cdots & -D_{y_j}t_J & 0 & \cdots & -D_{y_J}t_J & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
\hat{I} & 0^T & \cdots & \hat{I} & 0^T & \cdots & \hat{I} & 0^T & -\hat{I} & 0^T & \cdots & -\hat{I} & 0^T & \cdots & -\hat{I} & 0^T & -\hat{I} & \cdots & -\hat{I} & \cdots & -\hat{I} & \mathbf{0} & 0^T & \cdots & 0^T & \cdots & 0^T
\end{array}\right)
$$

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