»Le present est plein de l’avenir, et chargé du passé«

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INHALTSVERZEICHNIS (Band I)

Davide Crippa (Venice)

LEIBNIZ'S PROOF OF THE IMPOSSIBILITY OF SQUARING THE HYPERBOLA[1]

1. Introduction

In his treatise "De quadratura arithmetica circuli ellipseos et hyperbolae" (A VII, 6, 51), Leibniz provides proof of the impossibility of finding an algebraic solution to the "universal quadrature of the circle, " which states that the relation between the arc and the chord cannot be expressed analytically or algebraically. For his proof, Leibniz crucially relied on the problem of dividing an arbitrary angle into an equal number of parts.[2] Based on Viete's posthumous treatise on angular sections, Leibniz assumed that the problem of general angular division was algebraically unsolvable; that is, it was impossible to reduce it to a finite-degree polynomial equation. He then argued that if the quadrature of the circle could be reduced to a finite-degree polynomial equation, the problem of dividing an angle into an arbitrary number of parts would also admit such a reduction, running into a contradiction.

Leibniz's proof that the quadrature of the hyperbola cannot be squared algebraically follows a similar structure to his proof of the quadrature of the circle. However, this proof has not been studied in Lützen and has only been sketched by Crippa.[3] This study aims to provide a thorough analysis of Leibniz's argument.

2. The impossibility theorem

After proving the impossibility of the "universal" or "general" quadrature of the circle, Leibniz enunciates a similar impossibility result that concerns the quadrature of the hyperbola as follows:

> In the same way, once a general relation between arcs and cords has been found, the universal section of the angle could be given by one equation of determinate degree; so once a general

1 This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101024431-LEGITIMATH.

2 Eberhard Knobloch: "Beyond Cartesian Limits: Leibniz's Passage from Algebraic to 'Transcendental' Mathematics", in: *Historia Mathematica,* 33/1 (2006), pp. 113–31, here pp. 127ff.; Jesper Lützen: "17th century arguments for the impossibility of the indefinite and the definite circle quadrature", in: *Revue d'Histoire Des Mathématiques*, 20/1 (2014), pp. 211–251, here p. 235; Crippa, Davide: *The Impossibility of Squaring the Circle in the 17th Century: A Debate Among Gregory, Huygens and Leibniz,* Cham: Springer International Publishing, 2019, pp. 146ff.

3 Lützen: "17th century Arguments"; Crippa: *The Impossibility*, p. 152ff.

> quadrature of the hyperbola (*generali inventa quadratura hyperbolae*), namely a relation between a number and its logarithm, has been found, any number of mean proportions could be found by one equation of determinate degree; which is absurd, as mathematicians know (...) thus it is impossible to find a general quadrature, or a construction applying to any given sector of the hyperbola, or of the circle and the ellipse, which is more geometrical than our own.[4]

For Leibniz, the "general" quadrature of the hyperbola refers to the quadrature of an arbitrary sector, namely a portion of the hyperbola included between the curve, one asymptote, and two parallels to the other asymptote (in the simplest case, we can take the rectangular hyperbola with equation $y = a/x$, such as in figure 1, where sector A is marked). In modern terms, Leibniz's impossibility theorem may be read as referring to the familiar integral $\int \frac{1}{x} = \ln(x)$, whose solution is represented by a transcendental function.

3. Logarithms

Leibniz's earliest knowledge of logarithms likely came from his trip to London in winter 1673, when he encountered the work of Mercator and Briggs.[5] During his time in Paris (1673–1676) he familiarized himself with the standard definition of logarithms and computational techniques through the works and friendship with Huygens, and the works by Pardies and Mercator, J. Gregory, Gregoire of St. Vincent, A. de Sarasa which are all referenced in many of his manuscripts from that period. In *De Quadratura Arithmetica* (A VII 6, 51), Leibniz dedicates at least two *scholia* to logarithms and their properties (scholium to proposition XIII; "definitio" and the rest of the scholium to proposition XLIII). He introduced logarithms using the standard early modern construction of matching an arithmetical with a geometrical progression (Briggs, *Arithmetica Logarithmica*), as follows:

numeri	1/16	1/8	1/4	1/2	1/1	2	4	8	A
Logs	-4	-3	-2	-1	0	1	2	3	B
Logs	-8	-6	-4	-2	0	2	4	6	C
Logs	0	1	2	3	4	5	6	7	D

4 Crippa: *The Impossibility*, p. 152. The original can be read in *De Quadratura Arithmetica* (A VII, 6, 51, p. 674): "nam, quemadmodum generali relatione inter arcum et latera inventa posset haberi sectio anguli universalis, per unam aequationem certi gradus; ita generali inventa quadratura hyperbolae sive relatione inter numerum et logarithmum, possent inveniri quotcunque mediae proportionales ope unius aequationis certi gradus, quod etiam absurdum esse, analyticis constat (. . .) Impossibilis est ergo quadratura generalis sive constructio serviens pro data qualibet parte Hyperbolae aut Circuli adeoque et Ellipseos, quae magis geometrica sit, quam nostra."

5 See: "Observata philosophica in itinere Anglicano sub initium anni 1673"; A VIII, 1, 1; "Observatio de logarithmis"; A VII, 8, 1.

Table 1. See A VII 6, 51, p. 622.

As it appears from the table above, the first row displays a sequence "A" of terms in a geometric progression, and the second, third and fourth rows display sequences of terms in an arithmetic progression, respectively B, C, D. The terms of the geometric progression are called by Leibniz "numbers" (*numeri*), and the corresponding terms of the arithmetic progressions are called "logarithms" (*logarithmi*) or "exponents" (*exponentes*). In order to distinguish the early-modern notion of logarithms from our own, we shall employ the notation: "$L(A) = X$" for "the logarithm of the number a is x."

It should be noted that early modern logarithms presented at least three outstanding differences with respect to our own logarithms. First, they were defined using two matching sequences and, as Leibniz's example shows, a number could have an infinity of logarithms depending on the chosen sequence. Moreover they were defined discretely, although the sequences can be made arbitrarily dense (A VII, 6, 51, p. 624). Leibniz also conceived a representation of the logarithmic relation on a continuous curve, namely the *linea logarithmica,* which corresponds to the graph of an exponential. Likely sources were Gaston Pardies '*Elemens de Géométrie* (1671), and James Gregory's *Geometriae pars universalis* (1668), both studied by Leibniz during his Parisian years. A *linea logarithmica* can be constructed pointwise by interpolating a net of points obtained by pairing a geometric progression and an arithmetic one. By inserting an increasing number of proportionals between two give points, the net can be made denser.[6] A continuous graph can be produced only by combining physical motions.[7]

Finally, among early-modern geometers there was no shared, general assumption that the logarithm of 1 is 0. For example, in sequence D of the table above, $L(^1/_{16}) = 0$. Postulating $L(1) = 0$ would become customary only after Euler.[8]

However, assuming $L(1) = 0$ had a clear computational advantage. It can be proved that, for every four numbers $A, B, C, D, \ldots$ belonging to a geometrical Progression we have that:

$A \times B = C \times D$ iff $L(A) + L(B) = L(C) + L(D)$, and

$\frac{A}{B} = \frac{C}{D}$ iff $L(A) - L(B) = L(C) - L(D)$.

This result is proven in Burn.[9] If we pose $L(1) = 0$, we obtain the well-known property of logarithms:

6 Leibniz observes that the point wise construction of the l*inea logarithmica* is the same as that of other transcendental curves, such as the quadratrix (A VII, 6, 51, p. 625).

7 Cf. Leibniz's construction of the logarithmica by motions in "De Curia Logarithmica"; A VII, 7, 67; and in G. W. Leibniz, "Schediasma de Resistentia Medii, & Motu projectorum gravium in medio resistente", *Acta eruditorum* (1689): fasc. 1–12.

8 See Robert P. Burn: "Alphonse Antonio de Sarasa and Logarithms", in: *Historia Mathematica* 28/1 (2001), pp. 1–17. https://doi.org/10.1006/hmat.2000.2295.

9 Burn: "Sarasa and Logarithms", p. 5.

$L(AB) = L(A) + L(B)$.

Another key concept for understanding Leibniz's proof of the impossibility of squaring the hyperbola is the notion of "dividing a ratio". In *De Quadratura Arithmetica* (A VII, 6, 51, *Scholium* to proposition XIII), Leibniz parallels the "division of the ratio" with the division of the angle, and claims that both problems stem from the most conceptually difficult subjects in geometry, namely "ratio" and "angle". As it appears in extant manuscripts, in particular A VII, 6, 51, p. 556) Leibniz's tacit reference could be to the conntroversy between Wallis and Hobbes on the subject of ratios and curvilinear angles.[10]

Manipulating ratios have a long history predating Leibniz's work, which goes back to Euclid and has been commonly used since the Middle Ages.[11] For instance, a ratio $(a{:}\,c)$ was customarily called the "duplicate" of a second, given ratio $(a{:}\,b)$ if $c$is the third proportional between a and b, and a ratio $(a{:}\,d)$ the "triplicate" of $(a{:}\,b)$ if d is the fourth proportional, namely if: $(a{:}\,b) :: (b{:}\,c) :: (c{:}\,d)$. Therefore, to "duplicate" a ratio meant to construct a third proportional, and to "triplicate" a ratio meant to construct a fourth proportional, if a third proportional is also given. If the terms of the proportion are line segments, these operations can be performed using a ruler and a compass. In algebraic terms, to duplicate the ratio $\frac{A}{B}$ meant to calculate the square of the ratio and to triplicate means to determine the cube of the ratio.[12]

Inversely, bisecting a ratio $(a{:}\,c)$ meant inserting one mean proportional between $a$and c, and to trisect a ratio means constructing two mean proportionals. If the ratio is taken between numbers or, for example in Cartesian geometry, between segments, then the n-th section of the ratio can be interpreted as the extraction of its n-th root.[13] Unlike the operation of duplicating or triplicating a ratio, the division of a ratio is not always constructible by ruler and compass but requires high-order curves, as Descartes showed in book three of the *Géométrie*.

Leibniz employs the terminology of composition and division of ratios in his *De Quadratura Arithmetica* (see for instance A VII 6, 51, p. 556ff). He may have drawn inspiration from Meibom and Wallis, who are also cited in the same passage.

Furthermore, because the insertion of mean proportionals between numbers amounts to constructing a geometric progression, by virtue of the definition of logarithms, dividing a ratio $(a{:}\,c)$ into an equal number of parts can be performed by

10 See also Douglas, M, Jesseph: *Squaring the Circle: The War Between Hobbes and Wallis,* University of Chicago Press, 1999; François Loget: "Wallis between Hobbes and Newton. The question of the horn angle in England", in: *Revue d'histoire des mathématiques,* 8/2 (2002), pp. 207–262.

11 Dudley E. Sylla: "Compounding ratios: Bradwardine, Oresme, and the first edition of Newton's Principa", in: E. Mendelsonn (ed.): *Transformation and Tradition in the Sciences, Essays in Honour of I Bernard Cohen*, 1984, pp. 11–43; Sabine Rommevaux-Tani: "Une théorie de la mesure des rapports dans le Chilias logarithmorum de Kepler (1624)", *Revue d'Histoire des Mathématiques,* 24/2, 6 (2018), pp. 107–20.

12 John Wallis: *Mathesis Universalis,* in *Opera Mathematica*, vol. I, Oxford, 1695, p. 156.

13 John Wallis: *Mathesis Universalis*, p. 156.

dividing the interval $L(c) - L(a)$ into the same number of parts, and finding the corresponding "number" in the geometric progression.

Just as the division of the angle is essential for Leibniz's proof of the impossibility of giving a universal quadrature of the circle, the division of the ratio is crucial for the impossibility of squaring the hyperbola.

4. Squaring the hyperbola

Gregoire de S. Vincent (1584–1667), called "profondissimus geometra" by Leibniz (A VII, 6, 51, p. 12), was one of the first mathematicians to address the problem of squaring the hyperbola. In his work *Geometricum Quadraturae Circuli et Section Coni* (1647) he demonstrated several important properties of hyperbolic sectors. In particular, he demonstrated the following proposition:

> If points were taken in geometric progression along one asymptote of a hyperbola, and lines drawn through these points parallel to the other asymptote, then the areas between the parallel lines, bounded at one end by the asymptote and at the other by the hyperbola, were equal.[14]

Expanding on St. Vincent's work, Alphonse de Sarasa (1618–1667) clarified the significance of this discovery by relating it to the properties of logarithms (So*lutio problematis a R.P. Marinus Mersenno Minimo propositi* from 1649). In short, De Sarasa proved that if segments are described in geometric progression (i.e. forming a continuous proportion) on one asymptote of an equilateral hyperbola and lines are drawn to their extremes parallel to the other asymptote, then the areas of the hyperbolic sectors (namely the sectors formed by the given asymptote, a portion of the curve, and two successive parallels) are equal to one another.[15] This result grounds the relationship between hyperbolic areas and logarithms, as understood in the early modern period. In fact, an immediate corollary follows: The sequence formed by hyperbolic areas forms an arithmetic progression corresponding to the geometric progression formed by the sequence of their bases. Thus, the areas of these sectors are the logarithms of their respective bases. The hyperbola model provides a continuous representation of the logarithmic relation.

This is the connection between logarithms and the quadrature of the hyperbola that Leibniz places at the center of his impossibility proof.[16] Computing the area of a hyperbolic sector, as defined above, is equivalent to computing the logarithm of its base.

14 I quote here the paraphrasis given in Burn: "Sarasa and Logarithms", p. 2.

15 De Sarasa's result is examined in detail in Burn: "Sarasa and Logarithms", pp. 10–11.

16 Leibniz's knowledge of Gregoire of St Vincent's and De Sarasa's work is confirmed in various places, e.g. A VII 6, 51, p. 556, 632, and in particular p. 12: "Caeterum cum ex inventis profundissimi Geometrae P. Gregorii a S. Vincentio, P. Sarrasa Analogiam Logarithmorum ad spatia Hyperbolica ingeniose admodum deduxerit ."

5. Leibniz's impossibility proof

We can now analyze Leibniz's proof of the impossibility of finding a general quadrature for the hyperbola. Leibniz presumably followed the proof of the impossibility to find a universal quadrature of the circle, which will provide the layout for our analysis. We proceed by contradiction and assume that an algebraic formula exists that computes the area of any given hyperbolic sector (general quadrature of the hyperbola). If this were true, it would follow that the relation between the logarithm and its number is algebraic. Hence, let L denote, as before, the logarithm-relation, and assume $L(1) = 0$. We would then have that for any number x and the corresponding logarithm $L(x)$, there is an algebraic equation:

$$\Phi(x, L(x)) = 0 \qquad (1)$$

of a certain fixed degree. By hypothesis, we can replace $L(x)$ with any submultiple $1/\mathrm{n}L(x)$, where $n$is a positive integer. We shall obtain an equation:

$$\Phi(y, 1/\mathrm{n}L(x)) = 0 \qquad (2)$$

in the same degree (in fact, dividing the unknown by a known term does not change the degree), and where y denotes the unknown number corresponding to the n-th division of L(x).

However, as demonstrated in the previous section, dividing a logarithm into n equal parts ,i.e. dividing the interval $L(x) - L(1)$, upon the assumption that $L(1) = 0$, corresponds to inserting $n - 1$mean proportionals between the number x and a given unit, or constructing the n-th root of x. Bisecting $L(x)$ is equivalent to extracting the square root of x; trisecting $L(x)$ means to extract the (real) cube root of x, or to construct two mean proportionals (as shown in book 3 of Descartes' *Géométrie*); for $n = 5$, the constructing of four means is needed, which corresponds to the extraction of a quintic root. Hence, for any n:

$$\Phi(y, \frac{1}{n}L(x)) = \Phi(x^{\frac{1}{n}}, \frac{1}{n}L(x)) = 0 \qquad (3)$$

The division of the ratio or the logarithm into $4,6 \ldots 2n$, thus an even number of parts (which corresponds to the insertion of $3,5 \ldots 2n - 1$ mean proportionals between two given quantities, with n>1) leads to algebraic equations which are in principle reducible using the geometric and algebraic techniques of Descartes' *Géométrie*.[17] Likewise the division into an odd number of parts may also lead to reducible equations in case of non-prime numbers. For example, the problem of dividing the ratio into 9 parts, i.e. of inserting 8 mean proportionals, yields a 9th degree equation that can be factored into a system of cubics. On the contrary, just as he did for the angular divisions, Leibniz might have taken for granted that the problem of dividing the ratio or logarithm into a prime number of parts (hence, the insertion of $2,4,6 \ldots 2n$ mean proportionals) led to an irreducible equation of corresponding degree. This result is not proven by Leibniz, but he may have been led to it upon reading Descartes' book three of the *Géométrie.* Descartes claimed to have given a

17 Marco Panza: "Rethinking Geometrical Exactness", *Historia Mathematica* 38/1 (2011), pp. 42–95, here p. 75.

general method to classify problems "more and more complex, ad infinitum" following the examples of the problems of inserting two and four mean proportionals (or dividing the angle into three or five parts.[18] In the case of the division of the ratio, Leibniz may have thought that an ever higher number of prime divisions would lead to equations whose degrees infinitely varied, depending on the number n (A VII 6, 51, p. 676). Therefore, for any prime n, the equation (3) could not be a polynomial equation in a fixed, finite degree. Hence, (1) was not a polynomial equation in a fixed, finite degree. This contradicts the initial assumption and concludes the proof.

6. Conclusions

As argued in Lützen and Crippa, Leibniz's impossibility results in the context of the quadrature of the circle and the hyperbola had the goal of securing that the positive solutions given in *De Quadratura Arithmetica*, namely those using infinite series, were the best solutions attainable. In other words, they excluded the existence of simpler solutions, i.e. algebraic ones. Thus, early modern impossibility results have quite a difference significance than contemporary one.

Although Gregory had previously worked on this topic before Leibniz,[19] the latter, in a clearer manner than Gregory, presented a distinction between two problems: that of determining the area quantitatively, and the problem of determining the nature (algebraic or not) of a relation between quantities, such as the relation between circular and linear quantities, like the arc of a circle and the chord or the tangent, or the relation between a logarithm and its number.

Leibniz's proofs of impossibility concern the impossibility of expressing the relation between a chord and the arc (namely circular functions) or the logarithms and its number (namely, logarithmic or exponential functions) in algebraic terms. In this sense, it can be understood by modern readers as stating the transcendental nature of logarithmic (or exponential) and circular functions.

It is worth noting that arcs and segments, as well as logarithms and numbers, represent pairs of non-homogeneous magnitudes. Even logarithms can be considered as non-homogeneous if they are represented by areas of hyperbolic sectors with respect to the base segments. Leibniz's impossibility proofs in *De Quadratura Arithmetica* demonstrate that Cartesian algebra is inadequate for studying relations that violate homogeneity requirements. By abandoning the requirement of dimensional homogeneity for his transcendental mathematics,[20] Leibniz appears to have contributed also in this way to the process of disentangling analysis from its geometrical underpinnings, known as "degeometrization of analysis".[21]

18 René Descartes: *The Geometry of René Descartes,* ed. by David E. Smith and Marcia L. Latham. La Salle: Open Court, 1952 (facsimile of the original edition, 1637), p. 240.

19 see Lützen: "17th century arguments for the impossibility of the indefinite and the definite circle quadrature" and Crippa: *The Impossibility*, chapter 2.

20 For a general overview of Leibniz's "transcendental mathematics", see Knobloch: "Beyond Cartesian Limits".

21 Henk J. M. Bos: *Redefining Geometrical Exactness*. Springer, 2000, p. 10.