Notes on marriage markets with weak externalities*

María Haydée Fonseca-Mairena† Matteo Triossi‡

Abstract

We consider marriage markets with externalities. We focus on weak externalities, that is, markets in which each agent is primarily concerned about her partner. We formalize and prove the claim that weak externalities are not so significant in the marriage market: in this case the ω-core and the α-core coincide and are both non-empty. In addition, we show that, if we allow agents to block matchings without changing their mate, the results do not longer hold.

Keywords: Core, Matching, Marriage market, Stability, Weak externalities.

Economic Literature Classification Numbers: C71, C72, C78

*Triossi acknowledges financial support from Ca' Foscari University of Venice under project MAN.INS_TRIOSSI and from Ministerio Economía y Competitividad under projects MICIN Grants AEI/10.13039/501100011033 and AEI PID2020-118022GB-I0. The authors declare they do not have any conflict of interest.

†Corresponding author. Department of Economics and Management, Universidad Católica del Maule. San Miguel 3605, Talca, Chile. E-mail: mfonseca@ucm.cl.

‡Department of Management, Ca' Foscari University of Venice. Fondamenta San Giobbe, Cannaregio 873, 30121 Venice, Italy. E-mail: matteo.triossi@unive.it.

1 Introduction

We consider marriage markets with weak externalities in the sense that, even though agents have preferences over the full set of matchings they are primarily concerned with the partner they obtain. We focus on core and pairwise stable matchings. When there are externalities it matters whether agents outside of a deviating coalition react to a deviation. We consider two extreme cases. In in the first one, agents within a deviating coalition assume that agents outside of the coalition will not react to the deviation. In this case, we call agents optimistic. In the second case, the agents within a deviating coalition, say T, do not know how the agents outside T will react. They deviate if and only if no member of the coalition is worse off and at least one is strictly better off as a result of the deviation, independently on the behavior of the agents outside T. In this case we call agents prudent. If agents are optimistic pairwise stable matching may fail to exist under general externalities. If agents are prudent pairwise stable matchings exist under general externalities, but core stable matchings may fail to exist (Sasaki and Toda, 1996). Sasaki and Toda (1996) state that the externalities are not so significant in marriage markets with weak externalities. We formalize this idea and prove that in marriage markets with weak externalities:

1. It does not matter how optimistic or prudent agents are (Theorem 1), contrary to what happens in housing markets with weak externalities (Example 1).

2. To determine the core it is sufficient to check individual and pairwise blocking (Theorem 1), which generalizes Theorem 1 in Roth and Sotomayor (1992).

3. The core coincides with the core of an associated marriage market without externalities (Theorem 2), and it is nonempty (Proposition 1). These results do not hold in general allocation problems with weak externalities (see Example 6 in Fonseca-Mairena and Triossi, 2022).

4. Results 1-3 hold if we consider stronger blocking definitions (Proposition 2). This result and Theorem 1 in Fonseca-Mairena and Triossi (2022) imply that the core is implementable in Nash equilibrium in this setting.

We employ a concept of blocking in which, at least one agent belonging to a blocking coalition changes her mate. There are real-world situations in which agents can veto transactions in which they are not involved. An example are situations in which parents can veto or prearrange their children's marriages. It is also the case in some repugnance markets (see Roth 2007, 2015). We thus discuss dropping the assumption. We prove, by mean of an example, that, if we allow agents to block a matching without changing their mates, the previous results do not hold.

The structure of the paper is as follows. Section 2 introduces the model. Section 3 shows the results. Section 4 presents the conclusions.

2 The model

A one-to-one two-sided matching problem (commonly known as marriage market) is given by the triple (W, M, R) in which W and M are two nonempty sets of agents (usually called sets of women and men, respectively) such that $W \cap M = \emptyset$ and $|W| \cdot |M| \geq 2$. We denote by N the set of all agents, which is $N = W \cup M$. A matching is a one-to-one function $\mu : N \rightarrow N$ such that, for all $i \in N$, 1) $\mu^2(i) = i$, 2) $i \in W \Rightarrow \mu(i) \in M \cup \{i\}$, and 3) $i \in M \Rightarrow \mu(i) \in W \cup \{i\}$, where $\mu(i)$ denotes the mate of agent i assigned by the matching μ. By $R = (R_i)_{i \in N}$ we denote a preference profile in which each agent $i \in N$ has complete and transitive preference relation R_i defined over the set of matchings $\mathcal{M}$. Let P_i and I_i denote the asymmetric and symmetric parts, respectively, related to R_i. We say that (W, M, R) has **externalities** if there exists at least one agent who is not indifferent between two matchings in which she has the same partner. Formally, if there exists $i \in N$ and $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) = \mu'(i)$ and $\mu P_i \mu'$.

Definition 1 (Order preserving preferences) *The preference profile $R = (R_i)_{i \in N}$ is order preserving if for all $i \in N$:*

(1) *for all $\mu, \mu' \in \mathcal{M}$, $\mu I_i \mu' \Rightarrow \mu(i) = \mu'(i)$, and*

(2) *for all $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) \neq \mu'(i)$, $\mu P_i \mu' \Rightarrow \widetilde{\mu} P_i \widetilde{\mu}'$ for all $\widetilde{\mu}, \widetilde{\mu}' \in \mathcal{M}$ such that $\widetilde{\mu}(i) = \mu(i)$ and $\widetilde{\mu}'(i) = \mu'(i)$.*

In words, condition (1) means that each agent cannot be indifferent between two matchings that assign her different mates. Condition (2) means that each agent is concerned first of all about her partner, and then about the other pairings. Let $\mathcal{R}$ denote the set of order preserving preference profiles.[1] A problem (W, M, R) has **weak externalities** if all preferences are order preserving which is if $R \in \mathcal{R}$.

2.1 Two extreme cases of order preserving preferences

Order preserving preferences include two extreme cases: preferences without externalities, and strict order preserving preferences. Formally,

Definition 2 (Preferences without externalities) *The preference profile $R = (R_i)_{i \in N}$ is without externalities if for all $i \in N$:*

(1) *for all $\mu, \mu' \in \mathcal{M}$, $\mu I_i \mu' \Leftrightarrow \mu(i) = \mu'(i)$, and*

(2) *for all $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) \neq \mu'(i)$, $\mu P_i \mu' \Rightarrow \widetilde{\mu} P_i \widetilde{\mu}'$ for all $\widetilde{\mu}, \widetilde{\mu}' \in \mathcal{M}$ such that $\widetilde{\mu}(i) = \mu(i)$ and $\widetilde{\mu}'(i) = \mu'(i)$.*

Definition 3 (Strict order preserving preferences) *The preference profile $R = (R_i)_{i \in N}$ is strict order preserving if for all $i \in N$:*

(1) *for all $\mu, \mu' \in \mathcal{M}$, $\mu I_i \mu' \Leftrightarrow \mu = \mu'$, and*

(2) *for all $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) \neq \mu'(i)$, $\mu P_i \mu' \Rightarrow \widetilde{\mu} P_i \widetilde{\mu}'$ for all $\widetilde{\mu}, \widetilde{\mu}' \in \mathcal{M}$ such that $\widetilde{\mu}(i) = \mu(i)$ and $\widetilde{\mu}'(i) = \mu'(i)$.*

Let $\widetilde{\mathcal{R}}$ denote the set of preferences without externalities. Let $\mathcal{P}$ denote the set of strict order preserving preferences. By definition, $\widetilde{\mathcal{R}} \subsetneq \mathcal{R}$, $\mathcal{P} \subsetneq \mathcal{R}$, and $\widetilde{\mathcal{R}} \cap \mathcal{P} = \emptyset$.

[1] Order preserving preferences are also known as egocentric preferences (see Fonseca-Mairena and Triossi, 2022; Hong and Park, 2022).

2.2 Two extreme cases of stability and core

Consider the situation in which a coalition of agents evaluates whether to deviate from a proposed matching. Since there are externalities, the payoff from the deviation will depend also on how the agents outside of the coalition will react. Next, we introduce two notions of blocking that reflect two extreme believes about those reactions. Consider a marriage market (W, M, R).

Definition 4 (ω-blocking) *The matching μ is ω-blocked by coalition $T \subseteq N$ if there exists μ' such that:*

(1) $\mu'(i) \in T$ *for all* $i \in T$*;*

(2) $\mu' R_i \mu$ *for all* $i \in T$*;*

(3) $\mu' P_i \mu$ *for some* $i \in T$*;*

(4) $\mu'(i) \neq \mu(i)$ *for some* $i \in T$*.*

In words, coalition T ω-blocks matching μ if, there exists a matching μ' in which the members of T are matched among themselves (Condition (1)), such that each agent of T is not worse off under μ' (Condition (2)), and at least one member of T is strictly better off under μ' (Condition (3)). Furthermore at least one agent in T has a different mate under μ' (Condition (4)). This definition of blocking is consistent with **optimistic** agents. Indeed, the members of coalition T, who propose matching μ', alternative to μ, expect that all agents outside T will follow the recommendation and match according to μ'.

Definition 5 (α-blocking) *The matching μ is α-blocked by coalition $T \subseteq N$ if there exists μ' such that:*

(1) $\mu'(i) \in T$ *for all* $i \in T$*;*

(2) $\widetilde{\mu} R_i \mu$ *for all* $\widetilde{\mu}$ *such that* $\widetilde{\mu}(i) = \mu'(i)$ *for all* $i \in T$*, for all* $i \in T$*;*

(3) $\widetilde{\mu} P_j \mu$ *for all* $\widetilde{\mu}$ *such that* $\widetilde{\mu}(i) = \mu'(i)$ *for all* $i \in T$*, for some* $j \in T$*.*

In words, coalition T α-blocks matching μ if, there exists a matching μ' in which the members of T are matched among themselves (Condition (1)), such that each agent of T is not worse off under all matchings in which the members of T get the same partner as under μ' (Condition (2)), and at least one member of T is strictly better off under all matchings in which the members of T get the same partner as under μ' (Condition (3)). This definition of blocking is consistent with **prudent** agents. Indeed, the agents of deviating coalition T do not have any information about how the agents outside T will react. Then, they consider all matchings consistent with the agreement taken by the coalition. They α-block matching μ only if in all those matchings some member is strictly better off without harming any of the other members of T.

A matching μ is in the ω-core (α-core) when it is not ω-blocked (α-blocked) by any coalition $T \subseteq N$. Let $\omega\mathcal{C}(R)$ and $\alpha\mathcal{C}(R)$ be the ω-core and the α-core, respectively. When the two coincide, we simply say core.

If there are externalities, $\omega\mathcal{C}(R) \subseteq \alpha\mathcal{C}(R)$. Moreover, the number of potential ω-blocking matching is very high and the ω-core can be empty (see Fonseca-Mairena and Triossi, 2022; Ehlers, 2018). Although the α-core is larger than the ω-core, it can be empty in marriage market with externalities (see Sasaki and Toda, 1996).

A matching μ is ω-individually rational (α-individually rational) when it is not ω-blocked (α-blocked) by any agent $i \in N$. A matching μ is ω-pairwise stable (α-pairwise stable) when it is not ω-blocked (α-blocked) by any pair $(w, m) \in W \times M$. A matching μ is ω-stable (α-stable) when it is ω-individually rational (α-individually rational) and ω-pairwise stable (α-pairwise stable). Let $\omega\mathcal{S}(R)$ and $\alpha\mathcal{S}(R)$ be the ω-stable set and the α-stable set, respectively. By definition $\omega\mathcal{S}(R) \subseteq \alpha\mathcal{S}(R)$ for all R. Moreover $\omega\mathcal{C}(R) \subseteq \omega\mathcal{S}(R)$ and $\alpha\mathcal{C}(R) \subseteq \alpha\mathcal{S}(R)$ for all R.

3 The results

Next we prove that in marriage markets with weak externalities the ω-core and the α-core coincide. Moreover, ω-stable and ω-core coincide as well as α-stable and α-core. Then, all relevant blocking coalitions have size two. That means that it does not matter whether agents are optimistic or prudent.

Theorem 1 *Let $R \in \mathcal{R}$. Then, $\omega\mathcal{S}(R) = \omega\mathcal{C}(R) = \alpha\mathcal{C}(R) = \alpha\mathcal{S}(R)$.*

Proof. Let $R \in \mathcal{R}$. By definition $\omega\mathcal{C}(R) \subseteq \omega\mathcal{S}(R) \subseteq \alpha\mathcal{S}(R)$ and $\omega\mathcal{C}(R) \subseteq \alpha\mathcal{C}(R) \subseteq \alpha\mathcal{S}(R)$. The proof of the claim is in two parts.

(i) We prove that $\mu \notin \omega\mathcal{C}(R) \Rightarrow \mu \notin \omega\mathcal{S}(R)$. Let $\mu \in \mathcal{M}$ be such that there exists a coalition $T \subseteq N$ which ω-blocks μ with some $\mu' \in \mathcal{M}$. By definition of ω-blocking, there exists some agent $i \in T$ such that $\mu'(i) \neq \mu(i)$ and $\mu'(i) \in T$. Let $\mu'(i) = j$. Since preferences are order preserving $\mu' P_i \mu$ and $\mu' P_j \mu$, then the pair (i, j) ω-blocks μ. Then $\omega\mathcal{S}(R) \subseteq \omega\mathcal{C}(R)$.

(ii) We prove that $\mu \notin \omega\mathcal{S}(R) \Rightarrow \mu \notin \alpha\mathcal{S}(R)$. Let $\mu \in \mathcal{M}$ be such that there exists a pair (i, j) which ω-blocks μ with some $\mu' \in \mathcal{M}$. Then $\mu(i) \neq j$. Also $\mu' P_i \mu$ and $\mu' P_j \mu$. Since preferences are order preserving $\widetilde{\mu}' P_i \mu$ for all $\widetilde{\mu}'$ such that $\widetilde{\mu}'(i) = j$ and $\widetilde{\mu}' P_j \mu$ for all $\widetilde{\mu}'$ such that $\widetilde{\mu}'(j) = i$. Then (i, j) α-blocks μ. Then $\alpha\mathcal{S}(R) \subseteq \omega\mathcal{S}(R)$.

We have $\alpha\mathcal{S}(R) = \omega\mathcal{S}(R) = \omega\mathcal{C}(R) \subseteq \alpha\mathcal{C}(R) \subseteq \alpha\mathcal{S}(R)$. Thus the claim follows from (i) and (ii). ■

Since $\widetilde{\mathcal{R}} \subsetneq \mathcal{R}$, Theorem 1 generalizes Theorem 1 in Roth and Sotomayor (1992) which proves that the core equals the set of stable matchings in marriage markets without externalities.

The findings of Theorem 1 do not extend to *housing markets* with weak externalities. In that environment, even under order preserving preferences, α-blocking is stronger than ω-blocking and blocking coalitions composed of three agents or more cannot be reduced to two agents blocking coalitions as proven in the following example (see also Hong and Park, 2022).

Example 1 *There are five agents and five houses. Let $H = \{h_1, h_2, h_3, h_4, h_5\}$ be the set of the houses. An assignment associates a house to each agent. The set of feasible assignments is $\mathcal{A}^f = \{a, b, c, e\}$ in which $a = (h_2, h_1, h_3, h_4, h_5)$, $b = (h_3, h_1, h_2, h_4, h_5)$,*

$c = (h_3, h_1, h_2, h_5, h_4)$, $e = (h_1, h_2, h_3, h_4, h_5)$. For example a assigns house h_2 to agent 1, house h_1 to agent 2, and house h_i to agent i for $i = 3, 4, 5$. The ω-blocking and the α-blocking are defined, in terms of assignments and similarly to definitions 4 and 5, respectively. Preferences are order preserving

$$R_1 : bP_1cP_1aP_1e; \qquad R_2 : bP_2aP_2cP_2e; \qquad R_3 : bP_3cP_3aP_3e;$$

$$R_4 : aP_4bP_4eP_4c; \qquad R_5 : aP_5bP_5eP_5c.$$

Assignments c and e are Pareto dominated by b. Assignment a is ω-blocked by $\{1, 2, 3\}$ employing b. Then $\omega\mathcal{C}(R) = \{b\}$ while $\alpha\mathcal{C}(R) = \{a, b\}$. In particular $\omega\mathcal{C}(R) \subsetneq \alpha\mathcal{C}(R)$. Also notice that $\omega\mathcal{C}(R) \subsetneq \omega\mathcal{S}(R) = \{a, b\}$, thus blocking coalitions formed by three agents cannot be reduced to blocking pairs.

Also, the claim of Theorem 1 does not longer hold for slight departures from order preserving preferences as shown in the next example.

Example 2 *Let $W = \{w_1, w_2\}$ and $M = \{m_1\}$. Let $\mu_1 = \{(w_1, m_1), w_2\}$, $\mu_2 = \{(w_2, m_1), w_1\}$, and $\mu_3 = \{w_1, w_2, m_1\}$. The preference profile $R \in \mathcal{R}$ is*

$$R_{w_1} : \mu_2 P_{w_1} \mu_3 P_{w_1} \mu_1; \qquad R_{w_2} : \mu_1 P_{w_2} \mu_3 P_{w_2} \mu_2; \qquad R_{m_1} : \mu_1 P_{m_1} \mu_2 P_{m_1} \mu_3.$$

Preferences are order preserving and $\omega\mathcal{S}(R) = \omega\mathcal{C}(R) = \alpha\mathcal{C}(R) = \alpha\mathcal{S}(R) = \{\mu_3\}$. We now modify the preferences of agent w_1 into $\widetilde{R}_{w_1} : \mu_2 P_{w_1} \mu_1 P_{w_1} \mu_3$. Notice that $\widetilde{R}_{w_1}$ is not order preserving. Let $\widetilde{R} = \left(\widetilde{R}_{w_1}, R_{w_2}, R_{w_3}\right)$. Then $\omega\mathcal{S}(\widetilde{R}) = \omega\mathcal{C}(\widetilde{R}) = \emptyset$ while $\alpha\mathcal{C}(\widetilde{R}) = \alpha\mathcal{S}(\widetilde{R}) = \{\mu_1\}$.

Finally, we show that the ω-core (and the α-core, by Theorem 1), of any order preserving preference profile is equal to the ω-core of an associated profile without externalities. For each $R \in \mathcal{R}$, let $R' = (R'_i)_{i \in N}$ be defined as follows, for every $i \in N$:

1. for all $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) \neq \mu'(i)$, $\mu P_i \mu' \Leftrightarrow \mu P'_i \mu'$;

2. for all $\mu, \mu' \in \mathcal{M}$ such that $\mu(i) = \mu'(i)$, $\mu I'_i \mu'$.

Observe that R' is a well defined profile of preferences without externalities because R is order preserving.

Theorem 2 *Let* $R \in \mathcal{R}$. *Then,* $\omega\mathcal{C}(R') = \omega\mathcal{C}(R)$.

Proof.

Let $R \in \mathcal{R}$. By Theorem 1 it is enough to prove that $\omega\mathcal{S}(R') = \omega\mathcal{S}(R)$.

First, we prove by contradiction that $\omega\mathcal{S}(R') \subseteq \omega\mathcal{S}(R)$. Assume that there exists $\mu \in \omega\mathcal{S}(R')$ such that $\mu \notin \omega\mathcal{S}(R)$. Then, there exists a pair (i, j) and $\mu' \in \mathcal{M}$ such that (i, j) ω-blocks μ with μ' under R. Since preferences are order preserving we have $\mu' P_i \mu$ which implies $\mu' P'_i \mu$, and $\mu' P_j \mu$ which implies $\mu' P'_j \mu$. Thus (i, j) ω-blocks μ with μ' under R' as well which contradicts the fact that $\mu \in \omega\mathcal{S}(R')$.

Second, we prove by contradiction that $\omega\mathcal{S}(R) \subseteq \omega\mathcal{S}(R')$. Assume that there exists $\mu \in \omega\mathcal{S}(R)$ such that $\mu \notin \omega\mathcal{S}(R')$. Then there exists a pair (i, j) and $\mu' \in \mathcal{M}$ such that (i, j) ω-blocks μ with μ' under R'. Since R' is a profile of preferences without externalities we have $\mu' P'_i \mu$ which implies that $\mu' P_i \mu$, and $\mu' P'_j \mu$ which implies that $\mu' P_j \mu$. Thus (i, j) ω-blocks μ with μ' under R as well which contradicts the fact that $\mu \in \omega\mathcal{S}(R)$. We conclude that $\omega\mathcal{S}(R') = \omega\mathcal{S}(R)$. ∎

From Theorem 1 and Theorem 2 it follows that $\omega\mathcal{C}(R') = \alpha\mathcal{C}(R)$ for all $R \in \mathcal{R}$. This does not hold in all *allocation problems* with order preserving preferences in which, in general, $\omega\mathcal{C}(R') \subsetneq \alpha\mathcal{C}(R)$ (Example 6 in Fonseca-Mairena and Triossi, 2022).

Theorem 2 implies that the ω-core is nonempty in marriage market with weak externalities.

Proposition 1 *Let* $R \in \mathcal{R}$. *Then,* $\omega\mathcal{C}(R) \neq \emptyset$.

The claim of Proposition 1 follows directly from Theorem 2 and Gale and Shapley (1962).

3.1 Stronger blocking concepts

We now prove that the previous results hold if we consider stronger blocking definitions.

Definition 6 (Strong ω-blocking) *The matching μ is strongly ω-blocked by coalition $T \subseteq N$ if there exists μ' such that:*

(1) $\mu'(i) \in T$ *for all* $i \in T$,

(2) $\mu' P_i \mu$ *for all* $i \in T$,

(3) $\mu'(i) \neq \mu(i)$ *for some* $i \in T$.

Definition 7 (Strong α-blocking) *The matching μ is strongly α-blocked by coalition $T \subseteq N$ if there exists μ' such that:*

(1) $\mu'(i) \in T$ *for all* $i \in T$,

(2) $\widetilde{\mu} P_i \mu$ *for all* $\widetilde{\mu}$ *such that* $\widetilde{\mu}(i) = \mu'(i)$ *for all* $i \in T$, *for all* $i \in T$.

A matching μ is in the weak ω-core (weak α-core) if it is not strongly ω-blocked (strongly α-blocked) by any coalition $T \subseteq N$. Let $\mathcal{W}\omega\mathcal{C}(R)$ and $\mathcal{W}\alpha\mathcal{C}(R)$ be the weak ω-core and the weak α-core, respectively. By definition $\mathcal{W}\omega\mathcal{C}(R) \subseteq \mathcal{W}\alpha\mathcal{C}(R)$ for all R.

Proposition 2 *Let $R \in \mathcal{R}$. Then, $\mathcal{W}\omega\mathcal{C}(R) = \mathcal{W}\alpha\mathcal{C}(R) = \alpha\mathcal{C}(R)$.*

Proof. Let $R \in \mathcal{R}$. The proof is in two parts.

(i) $\mathcal{W}\omega\mathcal{C}(R) = \mathcal{W}\alpha\mathcal{C}(R)$. By definition $\mathcal{W}\omega\mathcal{C}(R) \subseteq \mathcal{W}\alpha\mathcal{C}(R)$ for all $R \in \mathcal{R}$. Next we prove that $\mathcal{W}\alpha\mathcal{C}(R) \subseteq \mathcal{W}\omega\mathcal{C}(R)$. Let $\mu \in \mathcal{M}$ be such that there exists a coalition $T \subseteq N$ which strongly ω-blocks μ with some $\mu' \in \mathcal{M}$. By item (3) in the definition of strong ω-blocking there exists some $j \in T$ such that $\mu'(j) \neq \mu(j)$. Let $k = \mu'(j)$. Following the same arguments of item (i) in the proof of Theorem 1, we conclude that the coalition $\{j, k\}$ strongly α-blocks μ with μ', which implies $\mu \notin \mathcal{W}\alpha\mathcal{C}(R)$.

(ii) $\mathcal{W}\alpha\mathcal{C}(R) = \alpha\mathcal{C}(R)$. By definition $\alpha\mathcal{C}(R) \subseteq \mathcal{W}\alpha\mathcal{C}(R)$. We need to prove $\mathcal{W}\alpha\mathcal{C}(R) \subseteq \alpha\mathcal{C}(R)$. Let $\mu \in \mathcal{M}$ be such that there exists a coalition $T \subseteq N$ which α-blocks μ with some $\mu' \in \mathcal{M}$. By item (3) in the definition of α-blocking there exists some $j \in T$ such that $\mu'(j) \neq \mu(j)$. Let $k = \mu'(j)$. The proof is as in (i).

■

Since $\widetilde{\mathcal{R}} \cup \mathcal{P} \subsetneq \mathcal{R}$ Proposition 2 generalizes Proposition 4, item (a), in Fonseca-Mairena and Triossi (2022). This implies that the α-core (and thus the ω-core, the ω-stable, and the α-stable, by Theorem 1) is implementable in Nash equilibrium in marriage markets with weak externalities (see Theorem 1 in Fonseca-Mairena and Triossi, 2022).

Next we discuss some variants in the definition of blocking, for which weak externalities are significant in the marriage market.

3.2 Coalition veto power

Condition (4) of the definition of ω-blocking requires that at least one agent of the blocking coalition has a different partner in the new matching, i.e., $\mu'(i) \neq \mu(i)$ for some $i \in T$. It requires that a matching cannot be blocked through a coalition of agents which do not change their mates. For this reason we call the condition: *no veto power of the invariant coalition.*

However, there are environments in which some agents have control over pairings in which they are not directly involved. Relevant examples are situations in which parents can veto or arrange their children's marriage. It may also be the case for repugnant transaction. Alvin Roth (2015, p. 283) calls "a transaction repugnant if some people want to engage in it and other people don't want them to." In markets involving repugnant transactions the no veto power of the invariant coalition can also fail. Consider for example markets such as the ones for the sale of kidneys, slaves, children in adoption and surrogate mothers. In some situations, the legislation prohibits certain transactions, such as in the case of incest, nepotism, and the sale of liquor or cigarettes to minors. In these cases the legal environment imposes a restriction on the set of feasible matchings. However, in several situations, such as in the case of marriages between people of different social classes or casts, prearranged marriages, hiring and workplace discrimination based on sex and age, agents who do not change their mates can block certain matchings.

Next, we consider the impact of removing the no veto power of the invariant coalition condition.

Definition 8 (ω^*-blocking) *The matching μ is ω^*-blocked by the coalition $T \subseteq N$ if there exists μ' such that:*

1. *$\mu'(i) \in T$ for all $i \in T$,*

2. *$\mu' R_i \mu$ for all $i \in T$,*

3. *$\mu' P_i \mu$ for some $i \in T$.*

A matching μ is in the ω^*-core when it is not ω^*-blocked by any coalition $T \subseteq N$. For each $R \in \mathcal{R}$, let $\omega^*\mathcal{C}(R)$ be the ω^*-core.

By definition $\omega^*\mathcal{C}(R) \subseteq \omega\mathcal{C}(R) \subseteq \alpha\mathcal{C}(R)$ for all $R \in \mathcal{R}$. Example 3 shows that the inclusion is, in general, strict and that the ω^*-core can be empty in marriage markets with weak externalities.

Example 3 *Let $W = \{w_1, w_2\}$ and $M = \{m_1\}$. Let $\mu_1 = \{(w_1, m_1), w_2\}$, $\mu_2 = \{(w_2, m_1), w_1\}$, and $\mu_3 = \{w_1, w_2, m_1\}$. The preference profile $R \in \mathcal{R}$ is*

$$R_{w_1} : \mu_3 P_{w_1} \mu_2 P_{w_1} \mu_1; \qquad R_{w_2} : \mu_2 P_{w_2} \mu_3 P_{w_2} \mu_1; \qquad R_{m_1} : \mu_2 P_{m_1} \mu_3 P_{m_1} \mu_1.$$

We have $\omega^*\mathcal{C}(R) = \emptyset$ while $\omega\mathcal{S}(R) = \omega\mathcal{C}(R) = \alpha\mathcal{C}(R) = \alpha\mathcal{S}(R) = \{\mu_2\}$.

Example 3 illustrates that the no veto power of the invariant coalition is a necessary condition for the previous results to hold.

Then, even weak externalities are significant for the ω^*-core. First, the ω^*-core can be empty, second, the ω^*-core and the α-core can differ.[2] In particular if coalitions are allowed to blocks matchings without its members being involved in the changes required, markets with externalies cannot be reduced to market without externalities even when externalities are weak and core matchings may fail to exist if agents are not prudent.

[2]Notice that α-blocking satisfies the no veto power of the invariant coalition.

4 Conclusions

We have proved that in marriage market with weak externalities (1) it does not matter how optimistic or prudent agents are, (2) to determine the core it is sufficient to review individual and pairwise blocking, and (3) the core coincides with the core of the associated marriage market without externalities. Moreover these results hold if we consider stronger blocking definitions. We have compared these results with previous results in the literature regarding housing markets and allocation problems with weak externalities. We have shown that the no veto power of the invariant coalition is a necessary condition for the results.

References

[1] Ehlers, L., 2018. Strategy-proofness and essentially single-valued cores revisited. Journal of Economic Theory, 176, 393-407. https://doi.org/10.1016/j.jet.2018.04.008.

[2] Fonseca-Mairena, M.H., Triossi, M., 2022. Incentives and implementation in allocation problems with externalities. Journal of Mathematical Economics 99. https://doi.org/10.1016/j.jmateco.2021.102613.

[3] Gale, D., Shapley, L. S., 1962. College Admissions and the Stability of Marriage. T. Am. Math. Mon. 69, 9–15. https://doi.org/10.1080/00029890.1962.11989827.

[4] Hong, M., Park, J., 2022. Core and top trading cycles in a market with indivisible goods and externalities. Journal of Mathematical Economics 100. https://doi.org/10.1016/j.jmateco.2021.102627.

[5] Roth, A. E., 2007. Repugnance as a Constraint on Markets. Journal of Economic perspectives, 21(3), 37-58. https://pubs.aeaweb.org/doi/pdfplus/10.1257/jep.21.3.37.

[6] Roth, A. E., 2015. Who gets what–and why: the new economics of matchmaking and market design. Houghton Mifflin Harcourt.

[7] Roth, A. E., Sotomayor, M., 1992. Two-sided matching. Handbook of game theory with economic applications, 1, 485–541. https://doi.org/10.1016/S1574-0005(05)80019-0.

[8] Sasaki, H., Toda, M., 1996. Two-sided matching problems with externalities. J. Econ. Theory 70, 93–108. https://doi.org/10.1006/jeth.1996.0077.