FURTHER DEVELOPMENTS ON THE POWER OF THE DOUBLE FREQUENCY DICKEY FULLER TEST ON UNIT ROOTS

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Abstract. In this paper we present some further investigations on the power of the Double Frequency Dickey Fuller test for unit root, recently proposed in literature to capture those situations where the time series might be affected by potential unknown structural breaks, asymmetrically located.
The use of Fourier function to approximate structural breaks has recently received large attention in unit root literature. The idea is that the Fourier approach allows capturing the behavior of a deterministic function form even if it is not periodic, working better than dummy variables, independent of the breaks are instantaneous or smooth and avoiding the problem of selecting the dates and the form of the breaks. The first attempts focused on the adoption of single frequency trig functions. More recently, it has been proposed an approach based on a double frequency in trig functions, which is more likely to capture also breaks that are asymmetrically located. Of this so-called Double Frequency Dickey Fuller test, it has been developed the asymptotic theory and, via simulations, its finite sample properties have been shown with respect to a variety of processes. To the best of our knowledge, however, no results have been presented with respect to the power of the Double Frequency Dickey Fuller test in case of occasional breaks data generating processes.
To address this issue we intend to conduct an extensive Monte Carlo experiment, concentrated on some occasional break data generating processes such as Mean Plus Noise and Markov Switching to evaluate the power of the test to distinguish also among this type of behavior.

1. Introduction

One of the most studied topics in the applied unit root time series literature is whether macroeconomic time series, in particular those considered by Nelson and Plosser (1982) are random walks or stationary processes around a level or a trend. The issue of stochastic versus deterministic trend has important practical policy implications. Until the empirical work of Nelson and Plosser (1982), the general view was that macroeconomic time series were stationary around a deterministic trend or level (Blanchard, 1981; Barro, 1976). However, after the introduction of Dickey and Fuller's tests for unit root (Dickey and Fuller, 1979, 1981), herafter DF and ADF, Nelson and Plosser (1982) find that with one exception, all historical time series have a unit root. This finding supports the real business cycle hypothesis and goes against the deterministic approach which separates business cycles from trend growth.

The paper by Nelson and Plosser (1982) started a long debate, with subsequent research. Phillips and Perron (1988) depart from the standard DF test assumptions of iid errors and developed a new test (PP test) that is robust to heterogeneity and serial correlation in the errors that has the same limiting distribution as ADF. From a different perspective, Sargan and Bhargava (1983) and Bhargava (1986) suggest tests in the Durbin Watson framework. Following Bhargava (1986), Schmidt and Phillips (1992) proposed a LM (Lagrange Multiplier) test whose power is argued to be larger than DF tests. Kwaitowski et al. (1992) proposed a stationarity test based on Lagrange Multiplier principle to a general error process similar to PP-type test. Leybourne and McCabe (1994) modify the KPSS test to form a stationarity test in the DF-type framework. Another line of research approaches the issue from a Bayesian perspective. In this regard it appears that the results concerning the stationarity of NP data differ with the choice of the prior.

However, as Perron (1989) pointed out, all these tests can be misleading if one does not account for the possibility of structural breaks in the time trend or level. His seminal paper opened an area of research to develop unit root tests that are robust to structural breaks or outliers in the data. This poses a serious problem for applied economists since the number duration and form of structural breaks may not be known. Moreover, detecting the number or the locations of the breaks may in turn cause an unknown pre-testing bias. A complicating factor can be also that a break occurring in a given year sometimes does not display its full impact immediately.

The first studies (Perron, 1989; Zivot and Andrews, 2002; Lee and Strazicich, 2003) use dummies to mimic structural breaks in the series. The drawback of this approach is that it generates too many nuisance parameters. This argument stimulate towards a different set of unit root and stationarity tests. Becker et al. (2006) develop tests which model any structural break of unknown form as a smooth process by means of the Fourier transforms. Several authors, starting from Gallant (1981), show that a Fourier approximation can often capture the behaviour of an unknown function, even if the function itself is not periodic. This testing framework requires only the specification of the proper frequency in the estimating equations thus reducing the number of estimated parameters. This ensures that, compared to dummy-based approaches, the tests have good size and power independently of the time or shape of the break. In this vein, there are recent proposals that generalize the original ideas of Becker et al. (2006), among which Enders and Lee (2012) who adopt the Fourier transform in a set-up where it is avoided the problem of selecting the dates, number, and form of breaks.

Omay (2015) proposes a test that combines the methodologies of Becker et al. (2006) and Enders and Lee (2012) and considers the use of fractional frequency to improve the fitting. Cai and Omay (2022) propose a double Fourier frequency test that is able to capture breaks that are asymmetrically located.

For all these tests, the literature propose simulation studies to ascertain the size and power properties in finite samples. Most studies (among others, Enders and Lee, 2012; Cai and Omay, 2022) logistic smooth transition autoregressive (LSTAR) processes or exponential smooth transition autoregressive (ESTAR). To the best of our knowledge, none of them considers the case when the time series is generated by occasional break processes such as the Mean Plus Noise (Chen and Tiao, 1990; Engle and Smith, 1999) and Markov Switching (Hamilton, 1989) models that can exhibit a dependence pattern that be difficult to distinguish from a unit root one.

The research question is then to find out whether the Double Frequency Dickey-Fuller based tests have power versus occasional break data generating processes. The structure of the paper is as follows. In the second section, we will present the Double Frequency Dickey Fuller test. In the third section, we focus on occasional break processes. In the fourth section, we will present our Monte Carlo experiment and some conclusions.

2. Double Frequency Dickey Fuller Test

The modification of the DF test to account for a deterministic function d_t moves from the following AR(1) process with a deterministic trend

$$y_t = d_t + \theta y_{t-1} + \varepsilon_t \qquad t = 1, \dots, T \tag{1}$$

where the stationary term ε_t has variance σ^2, d_t is a deterministic function. If d_t is known, model (1) can be directly estimated and, in turn, the unit root hypothesis $H_0\colon \theta = 1$ can be tested. When d_t is unknown testing for unit root is problematic given the risk of misspecification of d_t. The idea underlying the DF test based on Fourier expansion is that it is often possible to approximate d_t using the Fourier expansions, as in Enders and Lee (2012):

$$d_t = \alpha_0 + \sum_{k=1}^{n} \alpha_k \left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{n} \beta_k \left(\frac{2\pi k t}{T}\right) \qquad n \leq T/2 \tag{2}$$

where n represents the number of cumulative frequencies included in the approximations and k represents a particular frequency. It is interesting to observe that in the absence of a nonlinear trend, all values $\alpha_k = \beta_k = 0$, so that the usual Dickey Fuller specification appears. Usually the number of frequencies n should be kept small to avoid overfitting; in particular, in the original idea of Enders and Lee (2012), the Fourier approximation is adopted for a single frequency (n=1) as follows

$$d_t = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi kt}{T}\right) + \beta \cos\left(\frac{2\pi kt}{T}\right) \tag{3}$$

that includes, via the first term in the sum where $i = 0{,}1$, both the intercept and the trend plus intercept versions and it also approximates, via the sinusoidal waves the smooth breaks. In expression (3), k is the frequency to be determined over a pre-given interval. However, as pointed by Omay (2015) the breaks caused by sudden geo political events and financial crisis are stochastically distributed and asymmetrically located. Following this logic, Cai and Omay (2022) relax the assumption that the frequency is identical and propose a more general set up where:

$$d_t^{Dfr} = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi k_s t}{T}\right) + \beta \cos\left(\frac{2\pi k_c t}{T}\right) \tag{4}$$

and within the framework of a DF unit root test, the model with optimal frequencies k_s and k_c is (Double Frequency Dickey Fuller, DFDF herafter):

$$y_t = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi k_s t}{T}\right) + \beta \cos\left(\frac{2\pi k_c t}{T}\right) + \theta y_{t-1} + \varepsilon_t, \tag{5}$$

and the test statistic for the unit root hypothesis $H_0\colon \theta = 1$ is:

$$\tau^{Dfr} = \frac{T(\hat{\theta}-1)}{\sqrt{T^2 \delta_{\hat{\theta}}^2}} \tag{6}$$

where $\hat{\theta}$ and $\delta_{\hat{\theta}}^2$ are OLS estimators of θ and standard errors. The asymptotic distribution of the test statistic τ^{Dfr} only depends on the frequencies k_s and k_c and the critical values are tabulated (Cai and Omay, 2022, table 1).

If a nonlinear trend is not actually present in the data, a standard unit root test, such as DF or ADF, is more powerful and there is no need of Fourier terms. So, before adopting the DFDF test with a predetermined frequency pair $(k_s,\ k_c)$, it recommended to test $H_0\colon linearity$ versus $H_1\colon non\ linearity$ via an adjusted F test:

$$F^{Dfr}(k_s, k_c) = \frac{\frac{SSR_0 - SSR_1(k_s, k_c)}{2}}{\frac{SSR_1(k_s, k_c)}{T-q}} \tag{7}$$

SSR_0 and $SSR_1(k_s, k_c)$ represent sum of squared residuals without and with Fourier components, q is the number of regressors. If H_0 is rejected, a functional form with Fourier components is suggested.

Selecting the double frequency is done with a grid search to find the optimal pair (k_s^*, k_c^*), through the minimization of the SSR. This leads to the modified F test:

$$F^{Dfr}(k_s^*, k_c^*) = max_{(k_s,k_c)} F^{Dfr}(k_s, k_c)$$

where $(k_s^*, k_c^*) = argmaxF^{Dfr}(k_s, k_c)$. Minimizing SSR is equivalent to maximizing the F^{Dfr} test statistic under the condition of maximum frequency k_{max} and a searching precision of Δk (critical values tabulated).

3. Occasional break processes

For all the above mentioned tests based on the Fourier approximation, the literature propose simulation studies to ascertain the size and power properties in finite samples. To the best of our knowledge, none of them considers occasional break processes in mean, such as Mean Plus Noise and Markov Switching whose patterns can be sometimes non easily distinguishable from strong dependent ones (Granger and Hyung, 2004).

The idea of occasional break processes is that the number of breaks that can occur in a specific period of time is somehow bounded. More formally, we assume, that the probability of breaks, p, converges to zero slowly as the sample size increases, i.e. $p \to 0$ as $T \to \infty$, yet $lim_{T\to\infty} Tp$ is a non-zero finite constant. This implies that letting p decrease with the sample size, realization tends to have just finite breaks.

The Mean Plus Noise model (Chen and Tiao, 1990; Engle and Smith, 1999) is a binomial model, characterized by sudden changes only

$$\begin{aligned} y_t &= m_t + \varepsilon_t, \\ m_t &= m_{t-1} + q_t \eta_t \end{aligned} \tag{8}$$

where ε_t is a noise variable, the occasional level shifts m_t are controlled by two variables q_t (date of breaks) and η_t (size of jump). η_t is an i.i.d. $N(0, \sigma_\eta^2)$ although the normality assumption can be dropped. $q_t$is assumed to be an i.i.d. sequence of Bernoulli random variables such that $P(q_t = 1) = p$.

The structural changes might also occur gradually, in this case a Markov switching model (Hamilton, 1989) is more appropriate to describe the behaviour of q_t. More in details, it is given s_t a latent random variable that can assume only values

0 or 1 and is assumed to be a Markov chain, with transition probability $p_{ij} = P(s_t = j|s_{t-1} = i)$. Then it is possible to use a switching model for q_t such that q_t=0 when s_t=0 and q_t=1 when s_t=1. In this specification a regime with s_t=1 represents a period of structural change, regardless of the value of s_{t-1}.

4. Monte Carlo experiment

The experiment investigates the performance of the DFDF and the adjusted F test. The DFDF test is applied using the optimal (k_s^*, k_c^*) frequencies identified in the implementation of the F test.

The sample size is T=50,150, 300, the number of simulations is 2000 and we consider the following occasional break data generating processes (DGPs):

1) Mean Plus Noise, where p = 0.005, 0.01, 0.05, 0.1, σ^2 =1, σ_η^2= 0.1
2) Markov Switching, where p, q==(0.95, 0.95); (0.95, 0.99); (0.99, 0.95); (0.99, 0.99), σ^2 =1, σ_η^2= 0.1. The initial state s_1 is generated by a Bernoulli random variable with p=0.5

The percentage of rejection of the null hypothesis of the DFDF and F tests, is an estimate of the power of both tests, that have been implemented in the version with linear trend as well as in the version with intercept only, i.e. constant level. The considered nominal sizes are 5% and 1%.

The results are presented in the set of tables below (Tables 1-4). As we can see, the DFDF test confirms its excellent power properties in both occasional break DGPs even at the smallest sample size. Instead, the performance of the F test is low, in particular for Mean Plus Noise when p is low. This is not surprising, given that the smaller is the probability of jumps, the less the DPG shares nonlinear features. It also must be noticed that the power improves when p grows and in general with the sample size. This same pattern, although at a somewhat less evident extent, characterizes also the Markov Switching DGP.

Overall, our simulations confirm the very good power performance documented in the literature of the DFDF test, but cast some doubts on the power properties of the test F for occasional break DGPs. This issue is very crucial and should be considered with great care, given that the F test is preliminary to the DFDF test, hence a failure of the F test in rejecting the null hypothesis of linearity would imply a wrong use of the standard unit root test in place of the DFDF with consequent further wrong inference.

Table 1 – *Mean Plus Noise, percentage of rejection of null hypothesis (nominal size=5%).*

p	Linear Trend		Constant level	
	Test F	Test DFDF	Test F	Test DFDF
T=50				
0.005	0.002	1	0.004	1
0.01	0	1	0.004	1
0.05	0.002	1	0.008	1
0.1	0.004	1	0.028	1
T=150				
0.005	0	1	0.038	1
0.01	0.012	1	0.044	1
0.05	0.06	1	0.26	1
0.1	0.176	1	0.47	1
T=300				
0.005	0.02	1	0.12	1
0.01	0.072	1	0.216	1
0.05	0.4	1	0.738	1
0.1	0.614	1	0.874	1

Table 2 – *Markov Switching, percentage of rejection of null hypothesis (nominal size=5%).*

p,q	Linear Trend		Constant level	
	Test F	Test DFDF	Test F	Test DFDF
T=50				
0.95,0.95	0.042	1	0.09	0.984
0.95,0.99	0.014	1	0.04	0.994
0.99,0.95	0.056	1	0.15	0.976
0.99,0.99	0.034	1	0.094	0.994
T=150				
0.95,0.95	0.67	1	0.744	0.994
0.95,0.99	0.652	1	0.75	1
0.99,0.95	0.686	1	0.714	0.996
0.99,0.99	0.686	1	0.746	0.99
T=300				
0.95,0.95	0.92	1	0.904	0.998
0.95,0.99	0.952	1	0.928	0.998
0.99,0.95	0.918	1	0.92	0.994
0.99,0.99	0.94	1	0.926	0.998

Table 3 – *Mean Plus Noise, percentage of rejection of null hypothesis (nominal size=1%).*

p	Linear Trend		Constant level	
	Test F	Test DFDF	Test F	Test DFDF
T=50				
0.005	0	1	0	1
0.01	0	0.998	0	1
0.05	0.002	0.992	0.002	1
0.1	0.	0.998	0.008	1
T=150				
0.005	0	1	0.014	1
0.01	0.002	1	0.018	1
0.05	0.03	1	0.13	1
0.1	0.086	1	0.286	1
T=300				
0.005	0.012	1	0.072	1
0.01	0.044	1	0.144	1
0.05	0.28	1	0.618	1
0.1	0.492	1	0.78	1

Table 4 – *Markov Switching, percentage of rejection of null hypothesis (nominal size=1%).*

p,q	Linear Trend		Constant level	
	Test F	Test DFDF	Test F	Test DFDF
T=50				
0.95,0.95	0.008	0.984	0.09	0.984
0.95,0.99	0.006	1	0.04	0.994
0.99,0.95	0.012	1	0.15	0.976
0.99,0.99	0.01	1	0.094	0.994
T=150				
0.95,0.95	0.498	1	0.744	0.994
0.95,0.99	0.482	1	0.75	1
0.99,0.95	0.518	1	0.714	0.996
0.99,0.99	0.526	1	0.746	0.99
T=300				
0.95,0.95	0.862	1	0.904	0.998
0.95,0.99	0.878	1	0.928	0.998
0.99,0.95	0.846	1	0.92	0.994
0.99,0.99	0.88	1	0.926	0.998

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